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Bipolar fuzzy Sheffer stroke ideals of Sheffer stroke BCK-algebras

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Abstract. In this paper, we introduce the concepts of $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideals of an $SSBCK$-algebra, give some of their examples and study their properties. Furthermore, we investigate properties of $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideals under $SSBCK$-homomorphisms between $SSBCK$-algebras. Next, we investigate the Cartesian product of $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideals.

2020 AMS Classification: 06F05, 03G25, 03G10

Keywords: $SSBCK$-algebra, $SSBCK$-homomomorphism, $BFSS$-[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideal, The image and preimage of $BFSS$-[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideal, The Cartesian product of $BFSS$-[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideals.

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1. Introduction

By using the notion of Sheffer stroke operations introduced by Sheffer [1], Oner et al. [2] [resp. [3], [4], [5], [6], [7] and [8]] applied it to basic algebras [resp. Hilbert algebras, UP-algebras, BCK-algebras, BE-algebras, BH-algebras and KU algebras]. In particular, Oner et al. [9] studied fuzzy anti ideals.

We intend to conduct training by structuring our paper as follows: In Section 3, we introduce the concepts of $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-,

$BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideal and study their properties along with several examples. Moreover, we obtain a relationship between $BFSSA$-[resp. $BFSSAPI$-, $BFSSAI$- and $BFSSC$-]ideal and its level set (See Yheorem 3.19). Section 4 is devoted to investigating the Cartesian product of $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-and $BFSSAC$-]ideals.

2. Preliminaries

In this section, I recall basic fundamental concepts of Sheffer stroke operations, Sheffer stroke BCK-algebras, various fuzzy ideals, bipolar fuzzy sets and three Results ([5, 10]).

Definition 2.1 ([10])**.** Let $(X, \circ)$ be a groupoid. Then $\circ$ is said to be *Sheffer stroke operation* on X, if it satisfies the following conditions: for all $x, y, z \in X$,
(S_1) (Commutativity) $x \circ y = y \circ x$,
(S_2) (Absorption) $(x \circ x) \circ (x \circ y) = x$,
(S_3) $x \circ [(y \circ z) \circ (y \circ z)] = [(x \circ y) \circ (x \circ y)] \circ z$,
(S_4) (Absorption) $[x \circ ((x \circ x) \circ (y \circ y))] \circ [x \circ ((x \circ x) \circ (y \circ y))] = x$.

From (S_2), it is obvious that

$$(x \circ x) \circ (x \circ x) = x \text{ for each } x \in X. \tag{2.1}$$

Definition 2.2 ([5])**.** Let $\circ$ be a Sheffer stroke operation on a set X. Then a *Sheffer stroke BCK-algebra* (briefly, $SSBCK$-algebra) is a structure $(X, \circ, 0)$ of type $(2, 0)$ with the constant $0 \in X$ satisfying the following axioms hold: for all $x, y, z \in X$,
($sBCK_1$) $[(((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ (z \circ z))) \circ (((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ (z \circ z)))] \circ (z \circ (y \circ y)) = 0 \circ 0$,
($sBCK_2$) $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$ and $(y \circ (x \circ x)) \circ y \circ (x \circ x)) = 0$ imply $x = y$.

Result 2.3 (Lemma 3.1, [5])**.** *Let X be an $SSBCK$-algebra. Then the followings hold: for all $x, y, z \in X$,*
(1) $(x \circ (x \circ x)) \circ (x \circ x) = x$,
(2) $(x \circ (x \circ x)) \circ (x \circ (x \circ x)) = 0$,
(3) $x \circ (((x \circ (y \circ y)) \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (y \circ y))) = 0 \circ 0$,
(4) $(0 \circ 0) \circ (x \circ x) = x$,
(5) $x \circ 0 = 0 \circ 0$,
(6) $(x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0)) = x$,
(7) $(0 \circ (x \circ x)) \circ (0 \circ (x \circ x)) = 0$,
(8) $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$,
(9) $((x \circ (((y \circ (z \circ z)) \circ (y \circ (z \circ z))))) \circ ((y \circ (x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (y \circ (x \circ (z \circ z)) \circ (x \circ (z \circ z))))) = 0 \circ 0$,
(10) $((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (y \circ y) = 0 \circ 0$.

Result 2.4 (Lemma 3.2, [5])**.** *Let $(X, \circ, 0)$ be an $SSBCK$-algebra. A binary relation $\leq$ is defined on X as follows:*

$$x \leq z \text{ if and only if } (x \circ (z \circ z)) \circ (x \circ (z \circ z)) = 0.$$

Then the binary relation $\leq$ is a partial order on X such that $0 \leq x$ for each $x \in X$. Moreover, we have $y \leq (x \circ (y \circ y))$ and $x \leq z$ imply $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq (z \circ (y \circ y)) \circ (z \circ (y \circ y))$ for all $x, y, z \in X$.

Let X be an $SSBCK$-algebra. Then $1 = 0 \circ 0$ is the greatest element and $0 = 1 \circ 1$ is the least element of X.

Result 2.5 (Proposition 3.1, [5]). *Let $(X, \circ, 0)$ be an $SSBCK$-algebra. Then the following hold: for all $x, y, z \in X$,*
(1) $x \leq z$ *implies* $(y \circ (z \circ z)) \circ (y \circ (z \circ z)) \leq (y \circ (x \circ x)) \circ (y \circ (x \circ x))$,
(2) $((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z) = ((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (y \circ y)$,
(3) $((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \leq z$ *if and only if* $((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \leq y$,
(4) $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq x$,
(5) $x \leq y \circ (x \circ x)$,
(6) $x \leq (x \circ (y \circ y)) \circ (y \circ y)$,
(7) *if* $x \leq y$, *then* $z \circ (x \circ x) \leq z \circ (y \circ y)$.

Definition 2.6. Let $(X, \circ, 0)$ be an $SSBCK$-algebra and I a non-empty subset of X. Then I is called a:
(i) *Sheffer stoke ideal* (in briefly, SS-ideal) of X ([9]), if it satisfies the conditions:
(SSI_1) $0 \in I$,
(SSI_2) $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in I$ and $y \in I$ imply $x \in I$,
(ii) *Sheffer stroke positive implicative ideal* (in briefly, $SSPI$-ideal) of X ([8]), if it satisfies the following conditions: for $x, y, z \in X$,
(SSI_1) $0 \in I$,
(SSPI_2) $[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \circ [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \in I$ and $(y \circ (z \circ z)) \circ (y \circ (z \circ z)) \in I$ imply $(x \circ (z \circ z)) \circ (x \circ (z \circ z)) \in I$,
(iii) *Sheffer stroke implicative ideal* (in briefly, SSI-ideal) of X ([9]), it satisfies the following conditions: for all $x, y, z \in X$,
(SSI_1) $0 \in I$,
(SSII_2) $[((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x)))) \circ (z \circ z)]$
$\circ [((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x)))) \circ (z \circ z)] \in I$ and $z \in I$ imply $x \in I$,
(iv) *Sheffer stroke sub-implicative ideal* (in briefly, $SSSI$-ideal of X ([9]), if it satisfies the following conditions: for all $x, y, z \in X$,
(SSI_1) $0 \in I$,
(SSSI_2) $[((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)] \circ [((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)] \in I$ and $z \in I$ imply $(y \circ y \circ (x \circ x))) \circ (y \circ y \circ (x \circ x))) \in I$,
(v) *Sheffer stroke commutative ideal* (in briefly, SSC-ideal of X ([8]), if it satisfies the following conditions: for all $x, y, z \in X$,
(SSI_1) $0 \in I$,
(SSC_2) $[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \circ [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \in I$ and $z \in I$ imply $[x \circ ((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))] \circ [x \circ ((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))] \in I$.

Let $(a_j)_{j \in J}$ be a family of members of $[0, 1]$. Then we will denote $\bigvee_{j \in J} a_j$ and $\bigwedge_{j \in J} a_j$ as follows:

$$\bigvee_{j \in J} a_j = sup_{j \in J} a_j \text{ and } \bigwedge_{j \in J} = inf_{j \in J} a_j,$$

where J is an index set.

For a non-empty set X, a mapping $A : X \to [0,1]$ is called a *fuzzy set* in X (See [11]). The *empty fuzzy set* and the *whole fuzzy set* in X, denoted by 0_X and 1_X, are respectively, defined as follows: for all $x \in X$,

$$0_X(x) = 0 \text{ and } 1_X(x) = 1.$$

For $A \in [0,1]^X$ and $(A_j)_{j\in J} \subset [0,1]^X$, the *complement* A^c of A, the *union* $\bigcup_{j\in J} A_j$ and the *intersection* $\bigcap_{j\in J} A_j$ are defined as follows: for all $x \in X$,

$$A^c(x) = 1 - A(x),\ (\bigcup_{j\in J} A_j)(x) = \bigvee_{j\in J} A_j(x) \text{ and } (\bigcap_{j\in J} A_j)(x) = \bigwedge_{j\in J} A_j(x),$$

where $[0,1]^X$ denotes the set of all fuzzy sets in X.

Definition 2.7 ([9])**.** Let $(X, \circ, 0)$ be an $SSBCK$-algebra and $A \in [0,1]^X$. Then A is called a:

(i) *fuzzy Sheffer stroke subalgebra* (in briefly, $FSSS$-algebra) of X, if $A((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \geq A(x) \wedge A(y)$ for all $x, y \in X$,

(ii) *fuzzy Sheffer stroke anti subalgebra* (in briefly, $FSSAS$-algebra) of X, if $A((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \leq A(x) \vee A(y)$ for all $x, y \in X$.

Definition 2.8 ([9])**.** Let $(X, \circ, 0)$ be an $SSBCK$-algebra and $I \in [0,1]^X$. Then I is called a:

(i) *fuzzy Sheffer stroke ideal* (in briefly, FSS-ideal) of X, if it satisfies the following conditions: for all $x, y \in X$,

(FSSI_1) $I(0) \geq I(x)$,

(FSSI_2) $I(x) \geq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge I(y)$,

(ii) *fuzzy Sheffer stroke anti ideal* (in briefly, $FSSA$-ideal) of X, if it satisfies the following conditions: for all $x, y \in X$,

(FSSI_1) $I(0) \geq I(x)$,

(FSSAI_2) $I(x) \leq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \vee I(y)$,

(iii) *fuzzy Sheffer stroke sub-implicative ideal* (in briefly, $FSSSI$-ideal) of X, if it satisfies the following conditions: for all $x, y \in X$,

(FSSI_1) $I(0) \geq I(x)$,

(FSSSI_2) $I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))$
$\geq I([((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]) \wedge I(z)$,

(iv) *fuzzy Sheffer stroke implicative ideal* (in briefly, $FSSI$-ideal) of X, if it satisfies the following conditions: for all $x, y \in X$,

(FSSI_1) $I(0) \geq I(x)$,

(FSSI_2) $I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))$
$\geq I([((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]) \wedge I(z)$,

Definition 2.9 ([12])**.** Let X be a non-empty set. Then a pair $A = (A^-, A^+)$ is called a *bipolar-valued fuzzy set* or *bipolar fuzzy set* in X, if $A^+ : X \to [0,1]$ and $A^- : X \to [-1,0]$ are mappings. The *bipolar fuzzy empty set* [resp. the *bipolar fuzzy whole set*] in X (See [13]), denoted by $0_{bp} = (0^-_{bp}, 0^+_{bp})$ [resp. $1_{bp} = (1^-_{bp}, 1^+_{bp})$], is a bipolar fuzzy set in X defined as follows: for all $x \in X$,

$$0^+_{bp}(x) = 0 = 0^-_{bp} \text{ [res.} 1^+_{bp}(x) = 1 \text{ and } 1^-_{bp}(x) = -1].$$

We will denote the set of all bipolar fuzzy sets in X as $BPF(X)$.

See [14, 15] for the definitions of the equality, the inclusion, the compliment, the intersection and the union on $BPF(X)$.

3. Bipolar fuzzy Sheffer stroke ideals of BCK-algebras

I define the concepts of bipolar fuzzy ideals of Sheffer stroke BCK-algebras and study some of their various properties. In particular, we obtain relationships between some bipolar fuzzy Sheffer stroke ideals and their (lower) level sets (See Theorem 3.19).

We define the order $\leq$ on $[-1,0]\times[0,1]$ by: for $(s_0,t_0),(s_1,t_1)\in[-1,0]\times[0,1]$,

$$(s_0,t_0)\leq(s_1,t_1) \text{ if and only if } s_0\leq s_1,\ t_0\leq t_1.$$

Definition 3.1. Let $(X,\circ,0)$ be an $SSBCK$-algebra and $A\in[0,1]^X$. Then A is called a:

(i) *bipolar fuzzy Sheffer stroke subalgebra* (in briefly, $BFSSS$-algebra) of X, if for all $x,y\in X$,

$$A((x\circ(y\circ y))\circ(x\circ(y\circ y)))\geq A(x)\wedge A(y),\ i.e.,$$
$$A^-((x\circ(y\circ y))\circ(x\circ(y\circ y)))\leq A^-(x)\vee A^-(y),$$
$$A^+((x\circ(y\circ y))\circ(x\circ(y\circ y)))\geq A^+(x)\wedge A^+(y),$$

(ii) *bipolar fuzzy Sheffer stroke anti subalgebra* (in briefly, $BFSSAS$-algebra) of X, if for all $x,y\in X$,

$$A((x\circ(y\circ y))\circ(x\circ(y\circ y)))\leq A(x)\vee A(y),\ i.e.,$$
$$A^-((x\circ(y\circ y))\circ(x\circ(y\circ y)))\geq A^-(x)\wedge A^-(y),$$
$$A^+((x\circ(y\circ y))\circ(x\circ(y\circ y)))\leq A^+(x)\vee A^+(y).$$

Definition 3.2. Let $(X,\circ,0)$ be an $SSBCK$-algebra and $I\in[0,1]^X$. Then I is called a:

(i) *bipolar fuzzy Sheffer stroke ideal* (in briefly, $BFSS$-ideal) of X, if it satisfies the following conditions: for all $x,y\in X$,

(BFSSI_1) $I(0)\geq I(x)$, i.e., $I^-(x)\leq I^-(0)$, $I^+(0)\geq I^+(x)$,

(BFSSI_2) $I(x)\geq I((x\circ(y\circ y))\circ(x\circ(y\circ y)))\wedge I(y)$, i.e.,

$I^-(x)\leq I^-((x\circ(y\circ y))\circ(x\circ(y\circ y)))\vee I^-(y)$,

$I^+(x)\geq I^+((x\circ(y\circ y))\circ(x\circ(y\circ y)))\wedge I^+(y)$,

(ii) *bipolar fuzzy Sheffer stroke anti ideal* (in briefly, $BFSSA$-ideal) of X, if it satisfies the following conditions: for all $x,y\in X$,

(BFSSAI_1) $I(0)\leq I(x)$, i.e., $I^-(0)\geq I^-(x)$, $I^+(0)\leq I^+(x)$,

(BFSSAI_2) $I(x)\leq I((x\circ(y\circ y))\circ(x\circ(y\circ y)))\vee I(y)$, i.e.,

$I^-(x)\geq I((x\circ(y\circ y))\circ(x\circ(y\circ y)))\wedge I^-(y)$,

$I^+(x)\leq I^+((x\circ(y\circ y))\circ(x\circ(y\circ y)))\vee I^+(y)$,

(iii) *bipolar fuzzy Sheffer stroke sub-implicative ideal* (in briefly, $BFSSSI$-ideal) of X, if it satisfies the following conditions: for all $x,y\in X$,

(BFSSI_1) $I(0)\geq I(x)$,

(BFSSSI_2) $I((y\circ(y\circ(x\circ x)))\circ(y\circ(y\circ(x\circ x))))$

$\geq I([((x\circ(x\circ(y\circ y)))\circ(x\circ(x\circ(y\circ y))))\circ(z\circ z)]$

$\circ[((x\circ(x\circ(y\circ y)))\circ(x\circ(x\circ(y\circ y))))\circ(z\circ z)])\wedge I(z)$,

(iv) *bipolar fuzzy Sheffer stroke anti sub-implicative ideal* (in briefly, $BFSSASI$-ideal) of X, if it satisfies the following conditions: for all $x, y \in X$,

(BFSSAI$_1$) $I(0) \leq I(x)$,

(BFSSASII$_2$) $I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))$
$\leq I([((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]) \vee I(z)$,

(v) *bipolar fuzzy Sheffer stroke positive implicative ideal* (in briefly, $BFSSPI$-ideal) of X, if it satisfies the following conditions: for all $x, y \in X$,

(BFSSI$_1$) $I(0) \geq I(x)$,

(BFSSPI$_2$) $I((x \circ (z \circ z))) \circ (x \circ (z \circ z))))$
$\geq I([((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (z \circ z)]) \wedge I((y \circ (z \circ z)) \circ (y \circ (z \circ z)))$,

(vi) *bipolar fuzzy Sheffer stroke anti positive implicative ideal* (in briefly, $BFSSAPI$-ideal) of X, if it satisfies the following conditions: for all $x, y \in X$,

(BFSSAI$_1$) $I(0) \leq I(x)$,

(BFSSAPII$_2$) $I((x \circ (z \circ z))) \circ (x \circ (z \circ z))))$
$\leq I([((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (z \circ z)]) \vee I((y \circ (z \circ z)) \circ (y \circ (z \circ z)))$,

(vii) *bipolar fuzzy Sheffer stroke implicative ideal* (in briefly, $BFSSI$-ideal) of X, if it satisfies the following conditions: for $x, y, z \in X$,

(BFSSI$_1$) $I(0) \geq I(x)$,

(BFSSII$_2$) $I(x) \geq I([(x \circ (y \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (y \circ (x \circ x))) \circ (z \circ z)]) \wedge I(z)$,

(viii) *bipolar fuzzy Sheffer stroke anti implicative ideal* (in briefly, $BFSSAI$-ideal) of X, if it satisfies the following conditions: for $x, y, z \in X$,

(BFSSAI$_1$) $I(0) \leq I(x)$,

(BFSSAI$_2$) $I(x) \leq I([(x \circ (y \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (y \circ (x \circ x))) \circ (z \circ z)]) \vee I(z)$,

(ix) *bipolar fuzzy Sheffer stroke commutative ideal* (in briefly, $BFSSC$-ideal) of X, if it satisfies the following conditions: for all $x, y, z \in X$,

(BFSSI$_1$) $I(0) \geq I(x)$,

(BFSSC$_2$) $I([x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))]$
$\circ[x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))])$
$\geq I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]$
$\circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) \wedge I(z)$,

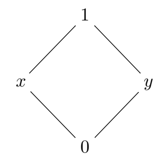
(x) *bipolar fuzzy Sheffer stroke anti commutative ideal* (in briefly, $BFSSAC$-ideal) of X, if it satisfies the following conditions: for all $x, y, z \in X$,

(BFSSAI$_1$) $I(0) \leq I(x)$,

(BFSSAC$_2$) $I([x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))]$
$\circ[x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))])$
$\leq I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]$
$\circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) \vee I(z)$.

It is obvious that if I is a $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$, $BFSSAPI$-, $BFSSI$-, $BFSAI$-, $BFSSC$- and $BFSSAC$-]ideal of X, then I^+ is a fuzzy SS-[resp. SSA-, $SSSI$-, $SSASI$-, $SSPI$, $SSAPI$, SSI- $SSAI$-, SSC-and $SSAC$-]ideal of X.

Example 3.3. (1) (See Example 3.1, [5]) Consider set $X = \{0, x, y, 1\}$ with the following Hasse diagram:



Let $(X, \circ, 0)$ be the $SSBCK$-algebra having the binary operation $\circ$ on X with the following Cayley table:

$\circ$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Table 3.1

For $(s_0, t_0), (s_1, t_1) \in [-1, 0] \times [0, 1]$ such that $(s_0, t_0) \geq (s_1, t_1)$, consider the bipolar fuzzy set $A = (A^-, A^+)$ in X defined by:

$$A(0) = (s_0, t_0),\ A(x) = A(y) = A(1) = (s_1, t_1).$$

Then clearly, A is a $BFSS$-subalgebra of X.

(2) Let $(X, \circ, 0)$ be the $SSBCK$-algebra given in Example 3.3 (1) and consider the bipolar fuzzy set $B = (B^-, B^+)$ in X defined by:

$$B(0) = (s_0, t_0),\ B(x) = B(y) = B(1) = (s_1, t_1),$$

where $(s_0, t_0), (s_1, t_1) \in [-1, 0] \times [0, 1]$ such that $(s_0, t_0) \leq (s_1, t_1)$. Then clearly, B is a $BFSSA$-subalgebra of X.

(3) Consider the bipolar fuzzy set A in X given in Example 3.3 (1). Then it is clear that A is a $BFSS$-[resp. $BFSSSI$-, $BFSSPI$-, $BFSSI$- and $BFSSC$-]ideal of X.

(4) Let B be the bipolar fuzzy set in X given in Example 3.3 (2). Then it is obvious that B is a $BFSSA$-[resp. $BFSSASI$-, $BFSSAPI$-, $BFSSAI$-and $BFSSAC$-]ideal of X.

Proposition 3.4. *If A is a $BFSS$-[resp. $BFSSA$-]subalgebra of an $SSBCK$-algebra X, then $A(0) \leq A(x)$[resp. $A(0) \leq A(x)$] for all $x \in X$.*

Proof. Suppose A is a $BFSS$-subalgebra of X and let $x \in X$. Then we have
$A(0) = A((x \circ (x \circ x)) \circ (x \circ (x \circ x)))$ [By Result 2.3 (2)]
$\geq A(x) \vee A(x)$ [By the hypothesis]
$= A(x)$.
Now suppose A is a $BFSSA$-subalgebra of X and let $x \in X$. Then we have
$A(0) = A((x \circ (x \circ x)) \circ (x \circ (x \circ x)))$ [By Result 2.3 (2)]
$\leq A(x) \vee A(x)$ [By the hypothesis]
$= A(x)$. □

Proposition 3.5. (1) *Every* $BFSS$-[resp, $BFSSA$-]*ideal of an* $SSBCK$*-algebra* X *is order reversing* [resp. preserving].

(2) *Every* $BFSS$-[resp, $BFSSA$-]*ideal of an* $SSBCK$*-algebra* X *is a* $BFSS$-[resp. $BFSSA$-]*subalgebra of* X.

Proof. (1) Let I be a $BFSS$-ideal of X and suppose $x \leq y$ in X. Then we have
$I(x) \geq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge I(y)$, [By ($\mathrm{BFSSI}_2$)]
$= I(0) \wedge I(y)$ [By the hypothesis]
$= I(y)$. [By (BFSSI_1)]
Thus I is order reversing.

Now let I be a $BFSSA$-ideal of X and suppose $x \leq y$ in X. Then we have
$I(x) \leq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \vee I(y)$, [By ($\mathrm{BFSSAI}_2$)]
$= I(0) \vee I(y)$ [By the hypothesis]
$= I(y)$. [By (BFSSAI_1)]
Thus I is order preserving.

(2) Let I be a $BFSS$-ideal of X and $x, y \in X$. Then we get
$I(x \circ (y \circ y)) \circ (x \circ (y \circ y))) \geq I(x)$ [By Result 2.5 (4) and (1)]
$\geq I(x \circ (y \circ y)) \circ (x \circ (y \circ y))) \vee I(y)$
[By (BFSSI_2)]
$\leq I(x) \wedge I(y)$.
Thus I is a $BFSS$-subalgebra of X.

Now let I be a $BFSSA$-ideal of X and $x, y \in X$. Then we get
$I(x \circ (y \circ y)) \circ (x \circ (y \circ y))) \leq I(x)$ [By Result 2.5 (4) and (1)]
$\leq I(x \circ (y \circ y)) \circ (x \circ (y \circ y))) \vee I(y)$
[By (BFSSAI_2)]
$\leq I(x) \vee I(y)$.
Thus I is a $BFSSA$-subalgebra of X. □

A $BFSS$-[resp. $BFSSA$-]subalgebra of X may not be a $BFSS$-[resp. $BFSSA$-]ideal of X.

Example 3.6. (1) Consider the $SSBCK$-algebra $(X, \circ, 0)$ given in Example 3.3 (1) and for $(s_0, t_0), (s_1, t_1), (s_2, t_2) \in [-1, 0] \times [0, 1]$ such that $(s_0, t_0) > (s_1, t_1) > (s_2, t_2)$, the bipolar fuzzy I in X as follows:

$$I(0) = (s_0, t_0),\ I(x) = I(y) = (s_1, t_1) \text{ and } I(1) = (s_2, t_2).$$

Then we can easily show that I is a $BFSS$-subalgebra. On the other hand,

$$I^+(1) = t_2 < t_1 = I^+((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))),$$
$$I^-(1) = t_2 > t_1 = I^-((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))).$$

Thus I is not a $BFSS$-ideal of X.

(2) Let $(X, \circ, 0)$ be the $SSBCK$-algebra given in Example 3.3 (1) and I the bipolar fuzzy set in X as follows:

$$I(0) = (s_0, t_0),\ I(x) = I(y) = (s_1, t_1) \text{ and } I(1) = (s_2, t_2),$$

where $(s_0, t_0), (s_1, t_1), (s_2, t_2) \in [-1, 0] \times [0, 1]$ such that $(s_0, t_0) < (s_1, t_1) < (s_2, t_2)$. Then we can easily show that I is a $BFSSA$-subalgebra. On the other hand,

$$I^+(1) = t_2 > t_1 = I^+((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))),$$

$$I^-(1) = t_2 < t_1 = I^-((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))).$$

Thus I is not a $BFSSA$-ideal of X.

Proposition 3.7. *Every $BFSSPI$-[resp. $BFSSAPI$-]ideal of an $SSBCK$-algebra X is a $BFSS$-[resp. $BFSSA$-]ideal of X, but the converse is not true.*

Proof. Let I be a $BFSSPI$-ideal of X. Then clearly, $I(0) \geq I(x)$ for all $x \in X$. Thus I satisfies the condition (BFSSI$_1$).

Now let $x, y \in X$. Then we have
$I((x \circ (0 \circ 0))) \circ (x \circ (0 \circ 0))))$
$\geq I([((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (0 \circ 0)]$
$\circ[((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (0 \circ 0)]) \wedge I((y \circ (0 \circ 0)) \circ (y \circ (0 \circ 0)))$
[In (BFSSPI$_2$), putting $z = 0$]
Thus by Result 2.7 (6), we get

$$I(x) \geq I((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \wedge I(y).$$

So I satisfies the condition (BFSSPI$_2$). Hence I is a $BFSS$-ideal of X.

Let I be a $BFSSAPI$-ideal of X. Then it is obvious that $I(0) \leq I(x)$ for all $x \in X$. Thus I satisfies the condition (BFSSAI$_1$).

Let $x, y \in X$. Then we have
$I((x \circ (0 \circ 0))) \circ (x \circ (0 \circ 0))))$
$\leq I([((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (0 \circ 0)]$
$\circ[((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \circ (0 \circ 0)]) \vee I((y \circ (0 \circ 0)) \circ (y \circ (0 \circ 0)))$
[In (BFSSAPI$_2$), putting $z = 0$]
Thus by Result 2.7 (6), we get

$$I(x) \leq I((x \circ (y \circ y))) \circ (x \circ (y \circ y)))) \vee I(y).$$

So I satisfies the condition (BFSSAPI$_2$). Hence I is a $BFSSA$-ideal of X.

See Example 3.8 for the converse. □

Example 3.8 (See Example 4.23, [9])**.** (1) Let $(X, \circ, 0)$ be the $SSBCK$-algebra $(X, \circ, 0)$ such that the binary operation $\circ$ on X has the Cayley table (See Table 3.2):

$\circ$	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	1	v	1	1	v	v	1	v
y	1	1	u	1	u	1	u	u
z	1	1	1	t	1	t	t	t
t	1	v	u	1	z	v	u	z
u	1	1	x	x	t	t	t	t
v	1	1	u	t	u	t	x	x
1	1	v	u	t	z	y	x	0

Table 3.2

Consider the bipolar fuzzy set $I = (I^-, I^+)$ in X defined as follows:

$$I(0) = I(z) = (s_0, t_0),\ I(x) = I(y) = I(t) = I(u) = I(v) = I(1) = (s_1, t_1),$$

where $(s_0, t_0) > (s_1, t_1)$ in $[-1, 0] \times [0, 1]$.
Then we can easily see that I is a $BFSS$-ideal of X. However we have
$t_1 = I^+(x) = I^+(x \circ (z \circ z)) \circ (x \circ (z \circ z))$
$< t_0 = I^+(z) \wedge I^+(0)$
$= I^+([((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]$
$\circ[((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]) \wedge I^+((0 \circ (z \circ z)) \circ (0 \circ (z \circ z)))$,
$s_1 = I^-(x) = I^-(x \circ (z \circ z)) \circ (x \circ (z \circ z))$
$> s_0 = I^-(z) \vee I^-(0)$
$= I^-([((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]$
$\circ[((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]) \vee I^-((0 \circ (z \circ z)) \circ (0 \circ (z \circ z)))$,
Thus I is not a $BFSSPI$-ideal of X.

(2) In (1), consider the bipolar fuzzy set $I = (I^-, I^+)$ in X defined as follows:

$$I(0) = I(z) = (-t_0, t_0),\ I(x) = I(y) = I(t) = I(u) = I(v) = I(1) = (-t_1, t_1),$$

where $(s_0, t_0) < (s_1, t_1)$ in $[-1, 0] \times [0, 1]$.
Then we can easily see that I is a $BFSSA$-ideal of X. However we have
$t_1 = I^+(x) = I^+(x \circ (z \circ z)) \circ (x \circ (z \circ z))$
$> t_0 = I^+(z) \wedge I^+(0)$
$= I^+([((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]$
$\circ[((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]) \wedge I^+((0 \circ (z \circ z)) \circ (0 \circ (z \circ z)))$,
$s_1 = I^-(x) = I^-(x \circ (z \circ z)) \circ (x \circ (z \circ z))$
$< s_0 = I^-(z) \vee I^-(0)$
$= I^-([((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]$
$\circ[((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (z \circ z)]) \vee I^-((0 \circ (z \circ z)) \circ (0 \circ (z \circ z)))$.
Thus I is not a $BFSSAPI$-ideal of X.

Proposition 3.9. *Every* $BFSSI$-[resp. $BFSSAI$-]*ideal of an* $SSBCK$*-algebra* X *is a* $BFSS$-[resp. $BFSSA$-]*ideal of* X*, but the converse is not true.*

Proof. Let I be a $BFSSI$-ideal of X. Then clearly, $I(0) \geq I(x)$ for all $x \in X$. Thus I satisfies the condition (BFSSI_1).

Let $x, z \in X$. Then we have
$I(x) \geq I([(x \circ (x \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (x \circ (x \circ x))) \circ (z \circ z)]) \wedge I(z)$
[In (BFSSII_2), putting $y = x$]
$= I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \wedge I(z)$.
Thus I satisfies the condition (BFSSII_2). So I is a $BFSS$-ideal of X.

Now let I be a $BFSSAI$-ideal of X. Then it is clear that $I(0) \leq I(x)$ for all $x \in X$. Thus I satisfies the condition (BFSSAI_1).

Let $x, z \in X$. Then we have
$I(x) \leq I([(x \circ (x \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (x \circ (x \circ x))) \circ (z \circ z)]) \vee I(z)$
[In ($\mathrm{BFSSAII}_2$), putting $y = x$]
$= I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \vee I(z)$.
Thus I satisfies the condition ($\mathrm{BFSSAII}_2$). So I is a $BFSSA$-ideal of X.

See Example 3.10 for the converse. □

Example 3.10. (1) Let I be a bipolar fuzzy set $SSBCK$-algebra X in X given by Example 3.15 (1). Then I is a $BFSS$-ideal but not a $BFSSSI$-ideal of X.

(2) Let I be a bipolar fuzzy set in given by Example 3.15 (2). Then I is a $BFSSA$-ideal but not a $BFSSASI$-ideal of X.

Theorem 3.11. *Let I be a $BFSS$-[resp. $BFSSA$-]ideal of an $SSBCK$-algebra X. Then I is a $BFSSI$-[resp. $BFSSAI$-]ideal of X if and only if it satisfies the following conditions: for all $x, y \in X$,*

$$I(x) \geq I((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x))))$$

$$[resp.\ I(x) \leq I((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x))))].$$

Proof. Suppose I be a $BFSS$-ideal of X and satisfy the following condition:

$$I(x) \geq I((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x)))) \text{ for all } x, y \in X. \tag{3.1}$$

Suppose the following inequality hods: for all $x, y, z \in X$,
$I((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x))))$
$\geq I([(x \circ (y \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (y \circ (x \circ x))) \circ (z \circ z)]) \wedge I(z)$.
Then by (3.1),
$I(x)$
$\geq I([(x \circ (y \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (y \circ (x \circ x))) \circ (z \circ z)]) \wedge I(z)$.
Thus I satisfies the condition ($BFSSII_2$). So I is a $BFSSI$-ideal of X.

The proof of the converse is obvious.

Suppose I be a $BFSSA$-ideal of X and satisfy the following condition:

$$I(x) \leq I((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x)))) \text{ for all } x, y \in X. \tag{3.2}$$

Suppose the following inequality hods: for all $x, y, z \in X$,
$I((x \circ (y \circ (x \circ x))) \circ (x \circ (y \circ (x \circ x))))$
$\leq I([(x \circ (y \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (y \circ (x \circ x))) \circ (z \circ z)]) \vee I(z)$.
Then by (3.2),
$I(x)$
$\leq I([(x \circ (y \circ (x \circ x))) \circ (z \circ z)] \circ [(x \circ (y \circ (x \circ x))) \circ (z \circ z)]) \vee I(z)$.
Thus I satisfies the condition ($BFSSAII_2$). So I is a $BFSSAI$-ideal of X.

The proof of the converse is easy. □

Proposition 3.12. *Every $BFSSC$-[resp. $BFSSAC$-]ideal of an $SSBCK$-algebra X is a $BFSS$-[resp. $BFSSA$-]ideal of X, but the converse is not true.*

Proof. Let I be a $BFSSC$-ideal of X. Then clearly, $I(0) \geq I(x)$ for all $x \in X$. Thus I satisfies the condition ($BFSSI_1$).

Let $x, y \in X$. Then we have
$I(x)$
$= I([x \circ (0 \circ ((0 \circ (x \circ x))) \circ (0 \circ (x \circ x))))]$
$\circ[x \circ (0 \circ ((0 \circ (x \circ x))) \circ (0 \circ (x \circ x))))])$ [By Result 2.3 (7) and (6)]
$\geq I([((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (y \circ y)]$
$\circ[((x \circ (0 \circ 0))) \circ (x \circ (0 \circ 0))) \circ (y \circ y)]) \wedge I(y)$ [By ($BFSSC_2$)
$= I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge I(y)$. [By Result 2.3 (6)]
Thus I satisfies the condition ($BFSSAC_2$). So I is

Now let I be a $BFSSAC$-ideal of X. Then clearly, $I(0) \leq I(x)$ for all $x \in X$. Thus I satisfies the condition ($BFSSAI_1$).

Let $x, y \in X$. Then we have

$I(x)$
$= I([x \circ (0 \circ ((0 \circ (x \circ x))) \circ (0 \circ (x \circ x))))]$
$\circ[x \circ (0 \circ ((0 \circ (x \circ x))) \circ (0 \circ (x \circ x))))])$ [By Result 2.3 (7) and (6)]
$\leq I([((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (y \circ y)]$
$\circ[((x \circ (0 \circ 0))) \circ (x \circ (0 \circ 0))) \circ (y \circ y)]) \vee I(y)$ [By (BFSSAC$_2$)
$= I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge I(y)$. [By Result 2.3 (6)]
See Example 3.13 for the converses. □

Example 3.13. (1) Let I be a bipolar fuzzy set in given by Example 3.15 (1). Then I is a $BFSS$-ideal but not a $BFSSC$-ideal of X.

(2) Let I be a bipolar fuzzy set in given by Example 3.15 (2). Then I is a $BFSSA$-ideal but not a $BFSSAI$-ideal of X.

Proposition 3.14. *Let I be a $BFSS$-*[resp. *$BFSSA$-*]*ideal of an $SSBCK$-algebra X. If I is a $BFSSC$-*[resp. *$BFSSAC$-*]*ideal of X if and only if it satisfies the following condition: for all $x, y \in X$,*

(3.3)
$$I([x\circ(y\circ((y\circ(x\circ x)))\circ(y\circ(x\circ x))))]\circ[x\circ(y\circ((y\circ(x\circ x)))\circ(y\circ(x\circ x))))]) \geq I((x\circ(y\circ y))\circ(x\circ(y\circ y)))$$

(3.4)
$$[resp. I([x\circ(y\circ((y\circ(x\circ x)))\circ(y\circ(x\circ x))))]\circ[x\circ(y\circ((y\circ(x\circ x)))\circ(y\circ(x\circ x))))]) \leq I((x\circ(y\circ y))\circ(x\circ(y\circ y)))].$$

Proof. Suppose I is a $BFSSC$-ideal of X and let $x, y \in X$. Then we have
$I([x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))]$
$\circ[x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))])$
$\geq I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (0 \circ 0)]$
$\circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (0 \circ 0)]) \wedge I(0)$
[In (BFSSC$_2$, putting $z = 0$]
$= I(((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \wedge I(0)$
[By Result 2.3 (6)]
$= I(((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))))$.
[By (BFSSI$_1$)]
Thus (3.3) holds.

Conversely, suppose a $BFSS$-ideal I of X satisfies (3.3) and let $x, y, z \in X$. Then
$I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]$
$\circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) \wedge I(z)$
$\leq I((x \circ (y \circ y)) \circ (x \circ (y \circ y)))$ [since I is a $BFSS$-ideal of X]
$\leq I([x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))]$
$\circ[x \circ (y \circ ((y \circ (x \circ x))) \circ (y \circ (x \circ x))))])$. [By (3.3)]
Thus (BFSSC$_2$) holds. So I is a $BFSSC$-ideal of X. □

Proposition 3.15. *Let I be a $BFSS$-*[resp. *$BFSSA$-*]*ideal of an $SSBCK$-algebra X. If $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \geq z$* [resp. $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq z$] *in X, then $I(x) \geq I(y) \wedge I(z)$* [resp. $I(x) \leq I(y) \vee I(z)$] *for all $x, y, z \in X$.*

Proof. Let I be a $BFSS$-ideal of an X and suppose $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \geq z$ in X. Then by Proposition 3.5 (1) and hypothesis,

$$I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \leq I(z), \tag{3.5}$$

Thus we have

$I(x) \geq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge I(y)$
$\geq I(z) \wedge I(y)$. [By (3.5)]

Now let I be a $BFSSA$-ideal of an X and suppose $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq z$ in X. Then by Proposition 3.5 (1) and hypothesis,

$$I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \leq I(z), \tag{3.6}$$

Thus we get
$I(x) \leq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge I(y)$
$\leq I(z) \vee I(y)$. [By (3.6)] □

Theorem 3.16. *Let X be an $SSBCK$-algebra and $I \in [0,1]^X$. Then I is a $BFSS$-*[resp. $BFSSSI$-, $BFSSPI$-, $BFSSI$- and $BFSSC$-]*ideal of X if and only if I^c is a $BFSSA$-*[resp. $BFSSASI$-, $BFSSAPI$-, $BFSSAI$- and $BFSSAC$-]*ideal of X.*

Proof. Suppose I is a $BFSS$-ideal of X and $x \in X$. Then by ($BFSSI_1$), we get

$$I^c(0) = 1 - I(0) = 1 - I(x) = I^c(x).$$

Thus the condition ($BFSSASII_1$) holds.

Now let $x, y \in X$. Then we have
$I^c(x) = 1 - I(x)$
$\leq 1 - I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge I(y)$ [By ($BFSSI_2$)]
$\leq (1 - I^c((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \wedge (1 - I^c(y))$
$= I^c((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \vee (1 - I^c(y))$.
Thus the condition ($BFSSASII_2$) holds. So $I^c$is a $BFSS$-ideal of X. The proof of the converse can be proved easily.

Suppose I is a $BFSSSI$-ideal of X and $x \in X$. Then by ($BFSSSI_1$), we get

$$I^c(0) = 1 - I(0) = 1 - I(x) = I^c(x).$$

Thus the condition ($BFSSASII_1$) holds.

Now let $x, y, z \in X$. Then we have
$I^c((y \circ y \circ x \circ x))) \circ (y \circ y \circ x \circ x))))$
$= 1 - I((y \circ y \circ x \circ x))) \circ (y \circ y \circ x \circ x))))$
$\leq 1 - I([((x \circ x \circ (y \circ y))) \circ (x \circ x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ x \circ (y \circ y))) \circ (x \circ x \circ (y \circ y)))) \circ (z \circ z)]) \wedge I(z)$
$= 1 - (1 - I^c([((x \circ x \circ (y \circ y))) \circ (x \circ x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ x \circ (y \circ y))) \circ (x \circ x \circ (y \circ y)))) \circ (z \circ z)]) \wedge (1 - I^c(z))$
$= I^c([((x \circ x \circ (y \circ y))) \circ (x \circ x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ x \circ (y \circ y))) \circ (x \circ x \circ (y \circ y)))) \circ (z \circ z)]) \vee I^c(z)$.
Thus the condition ($BFSSASII_2$) holds. So I^c is a $BFSSSAI$-ideal of X.

The proof of the converse can be easily shown.

We can see the remainders similarly. □

Proposition 3.17. *Let X be an $SSBCK$-algebra and $I \in [0,1]^X$. If I is a $BFSS$-*[resp. $BFSSPI$-, $BFSSI$- and $BFSSC$-]*anti ideal of X, then A_I is an SS-*[resp. $SSPI$-, SSI-and SSC-]*ideal of X, where $A_I = \{x \in X : I(x) = I(0)\}$.*

Proof. Suppose I is a $BFSSA$-ideal of X. Then clearly, $0 \in A_I$. Thus A_I satisfies the condition (SSI_1).

Now suppose $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in A_I$ and $y \in A_I$. Then clearly,

$$I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) = I(0) = I(y).$$

Thus by (BFSSAI$_2$),

$$I(x) \leq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \vee I(y) = I(0).$$

By (BFSSAI$_1$), $I(x) \geq I(0)$. So $I(x) = I(0)$, i.e., $x \in A_I$. Hence A_I satisfies the condition (SSI$_2$). Therefore A_I is an SS-ideal of X.

Suppose I is a $BFSSAC$-aideal of X. Then clearly, A_I satisfies the condition (SSI$_1$). Now suppose the following conditions are satisfied: for all $c, y, z \in X$,
$[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \circ [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \in A_I$ and $z \in A_I$. Then we get
$I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \circ [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) = I(0)$ and $I(z) = I(0)$. Thus by the hypothesis, we have

$$\begin{aligned} & I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \\ & \circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) \\ & \leq I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \\ & \circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) \vee I(z) \\ & = I(0). \end{aligned}$$

By the condition (BFSSAI$_1$),

$$\begin{aligned} & I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \\ & \circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) \\ & \geq I(0). \end{aligned}$$

So

$$\begin{aligned} & I([((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \\ & \circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)]) \\ & = I(0). \end{aligned}$$

Hence we get

$$\begin{aligned} & [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \\ & \circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (z \circ z)] \in A_I. \end{aligned}$$

Therefore A_I is an SSC-ideal of X. We can similarly prove the remainders. □

Definition 3.18. Let X be a set, $A \in [0,1]^X$ and $(s,t) \in [-1,0] \times [0.1]$. Then

(i) the (s,t)-*level set* of A, denoted by $[A]_{(s,t)}$, is defined as follows:

$$[A]_{(s,t)} = \{x \in X : A(x) \geq t\} = \{x \in X : A_-(x) \leq s, \ A_+(x) \geq t\}.$$

(ii) the *lower* (s,t)-*level set* of A, denoted by $[A]^{(s,t)}$, is defined as follows:

$$[A]^{(s,t)} = \{x \in X : A(x) \leq t\} = \{x \in X : A_-(x) \geq s, \ A_-(x) \leq t\}.$$

It is obvious that $[A]^{1_{bp}} = X$ and $[A]_{(s,t)} \cup [A]^{(s,t)} = X$ for $(s,t) \in [-1,0] \times [0,1]$. Moreover, if $(s_1,t_1) \leq (s_2,t_2)$, then $[A]_{(s_2,t_2)} \supset [A]_{(s_1,t_1)}$ and $[A]^{(s_1,t_1)} \subset [A]^{(s_2,t_2)}$.

Theorem 3.19. *Let X be an $SSBCK$-algebra and $I \in [0,1]^X$. Then I is a $BFSSA$-* [resp. *$BFSSAPI$-, $BFSSAI$- and $BFSSAC$-*]*ideal of X if and only if $[I]_{(s,t)}$* [resp. $[I]^{(s,t)}$] *is an SS-*[resp. *$SSPI$-, SSI-and SSC-*]*ideal of X for all $(s,t) \in [-1,0] \times [0,1]$ with $(s,t) \leq I(0)$* [resp. $(s,t) \geq I(0)$].

Proof. Suppose I is a $BFSSA$-ideal of X and let $t \in [0,1]$ with $(s,t) \geq I(0)$, i.e., $s \leq I^-(0), t \geq I^+(0)$. Then clearly, $0 \in [I]^{(s,t)}$. Now suppose $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in$

$[I]^{(s,t)}$ and $y \in [I]^{(s,t)}$ for all $x, y \in X$. Then $I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \leq (s,t)$, $I(y) \leq (s,t)$. Thus

$$I(x) \leq I((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \vee I(y) \leq (s,t).$$

So $x \in [I]^{(s,t)}$. Hence $[I]^{(s,t)}$ is an SS-ideal of X.

Conversely, $[I]^{(s,t)}$ is an SS-ideal of X. Assume that the condition (FSSI_1) does not hold. Then there is $x_0 \in X$ such that $I^-(0) < I^-(x_0)$, $I^+(0) > I^+(x_0)$. Let $(s_0, t_0) = (\frac{1}{2}(I^-(0) + I^-(x_0)), \frac{1}{2}(I^+(0) + I^+(x_0))$. Then

$$0 \geq I^-(x_0) > s_0 \geq I^-(0) \geq -1,\ 0 \leq I^+(x_0) < t_0 \leq I^+(0) \leq 1.$$

Thus $0 \notin [I]^{(s_0,t_0)}$. This is a contradiction. So the condition (FSSI_1) holds.

Now assume that the condition (FSSI_2) does not hold. Then there are $x_0, y_0 \in X$ such that $I(x_0) > I((x_0 \circ (y_0 \circ y_0)) \circ (x_0 \circ (y_0 \circ y_0))) \vee I^-(y_0)$. Let $(s_0, s_0) \in [-1, 0] \times [0.1]$ be the following equalities:

$$s_0 = \frac{1}{2}(I^-(x_0) + I^-((x_0 \circ (y_0 \circ y_0)) \circ (x_0 \circ (y_0 \circ y_0))) \vee I^-(y_0)),$$

$$t_0 = \frac{1}{2}(I^+(x_0) + I^+((x_0 \circ (y_0 \circ y_0)) \circ (x_0 \circ (y_0 \circ y_0))) \wedge I^+(y_0)).$$

Then $s_0 > I^-(x_0)$ and $0 \geq I^-((x_0 \circ (y_0 \circ y_0)) \circ (x_0 \circ (y_0 \circ y_0))) \wedge I^-(y_0) > s_0 \geq -1$.

$t_0 < I^+(x_0)$ and $0 \leq I^+((x_0 \circ (y_0 \circ y_0)) \circ (x_0 \circ (y_0 \circ y_0))) \vee I^+(y_0) < s_0 \leq 1$. Thus $I((x_0 \circ (y_0 \circ y_0)) \circ (x_0 \circ (y_0 \circ y_0)))$, $I(y_0) < (s_0, t_0)$. So $(x_0 \circ (y_0 \circ y_0)) \circ (x_0 \circ (y_0 \circ y_0))$, $y_0 \notin [I]^{s_0}$ but $x_0 \in [I]^{(s_0,t_0)}$. This is a contradiction. So the condition (FSSI_2) holds. Hence I is a $BFSS$-ideal of X.

The proofs of the remainders can easily seen. □

Proposition 3.20. (1) *Every $BFSSSI$-*[resp. *$BFSSASI$-*]*ideal of an $SSBCK$-algebra X is order reversing* [resp. preserving].

(2) *Every $BFSSSI$-*[resp. *$BFSSASI$-*]*ideal of an $SSBCK$-algebra X is a $BFSS$-*[resp. *$BFSSA$-*]*ideal of X.*

(3) *Let I be a $BFSSS$-*[resp. *$BFSSA$-*]*ideal of an $SSBCK$-algebra X. If I satisfies the following condition: for all $x, y \in X$,*

$$(3.7)\quad I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x)))) \geq I((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))$$

$$(3.8)\quad [resp.\ I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x)))) \leq I((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))],$$

then I is a $BFSSSI$-[resp. $BFSSASI$-]ideal of X.

Proof. (1) Let I be a $BFSSSI$-ideal of X and suppose $x \leq z$ in X. Then we have

$I(x)$
$\geq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \wedge I(z)$
[In (BFSSSI_2), putting $y = x$]
$= I([(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]$
$\circ[(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]) \wedge I(z)$
[By (S_3)]
$= I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \wedge I(z)$ [By Result 2.3 (2) and (6)]
$= I(0) \wedge I(z)$ [Since $x \leq z$, $(x \circ (z \circ z)) \circ (x \circ (z \circ z)) = 0$]
$= I(z)$. [By (BFSSSI_1)]

Now let I be a $BFSSASI$-ideal of X and suppose $x \leq z$ in X. Then we have
$I(x)$
$\leq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \vee I(z)$
[In (BFSSASI$_2$), putting $y = x$]
$= I([(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]$
$\circ[(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]) \vee I(z)$
[By (S$_3$)]
$= I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \vee I(z)$ [By Result 2.3 (2) and (6)]
$= I(0) \vee I(z)$ [Since $x \leq z$, $(x \circ (z \circ z)) \circ (x \circ (z \circ z)) = 0$]
$= I(z)$. [By (BFSSASI$_1$)]

(2) Let I be a $BFSSS$-ideal of X. Then clearly, $I(0) \geq I(x)$ for all $X \in X$ by (BFSSI$_1$). Now let $x, z \in X$. Then we get
$I(x)$
$\geq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \wedge I(z)$
[In (BFSSSI$_2$), putting $y = x$]
$= I([(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]$
$\circ[(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]) \wedge I(z)$
[By (S$_3$)]
$= I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \wedge I(z)$ [By Result 2.3 (2) and (6)]
Thus I satisfies (BFSSI$_2$). So I is a $BFSS$-ideal of X.

Now let I be a $BFSSASI$-ideal of X. Then clearly, $I(0) \leq I(x)$ for all $X \in X$ by (BFSSASI$_1$). Now let $x, z \in X$. Then we get
$I(x)$
$\leq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \vee I(z)$
[In (BFSSASI$_2$), putting $y = x$]
$= I([(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]$
$\circ[(((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (z \circ z))]) \vee I(z)$
[By (S$_3$)]
$= I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \vee I(z)$ [By Result 2.3 (2) and (6)]
Thus I satisfies (BFSSASI$_2$). So I is a $BFSSA$-ideal of X.

(3) Suppose I satisfies the condition (3.7) and let $x, y, z \in X$. Then we have
$I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))$
$\geq I((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))$ [By (3.7)]
$\geq I([((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]) \wedge I(z)$. [By (BFSSI$_2$)]
Thus I is a $BFSSSI$-ideal of X.

Now suppose I satisfies the condition (3.8) and let $x, y, z \in X$. Then we have
$I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))$
$\leq I((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))$ [By (3.8)]
$\leq I([((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]$
$\circ[((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (z \circ z)]) \vee I(z)$. [By (BFSSASI$_2$)]
Thus I is a $BFSSASI$-ideal of X. □

Proposition 3.21. *Every* $BFSSSI$-[resp. $BFSSASI$-]*ideal of an* $SSBCK$*-algebra* X *is a* $BFSS$-[resp. $BFSSAS$-]*algebra.*

Proof. Let I be a $BFSSASI$-ideal of X and let $x, z \in X$. Then by Result 2.5 (4), we have

$[((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)] \circ [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)]$
$\leq ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\leq x.$

Thus by Proposition 3.20 (1),

$$I([((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)] \circ [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)]) \geq I(x). \tag{3.9}$$

On the other hand, we get

$I((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x))))$
$= I(x)$
$\geq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ [((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \wedge I(z)$
[In ($\mathrm{BFSSASI}_2$), putting $y = x$]
$= I((x \circ z(\circ z)) \circ (x \circ z(\circ z))) \wedge I(z).$

So we have

$I((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\geq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ [((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \wedge I(z).$

Hence $I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \geq I(x) \wedge I(z)$. Therefore I is a $BFSSS$-algebra.

Now let I be a $BFSSASI$-ideal of X and let $x, z \in X$. Then by Result 2.5 (4), we have

$[((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)] \circ [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)]$
$\leq ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\leq x.$

Thus by Proposition 3.20 (1),

$$I([((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)] \circ [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (z \circ z)]) \leq I(x). \tag{3.10}$$

On the other hand, we get

$I((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x))))$
$= I(x)$
$\leq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ [((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \vee I(z)$
[In ($\mathrm{BFSSASI}_2$), putting $y = x$]
$= I((x \circ z(\circ z)) \circ (x \circ z(\circ z))) \vee I(z).$

So we have

$I((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\leq I([((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]$
$\circ [((x \circ (x \circ (x \circ x))) \circ (x \circ (x \circ (x \circ x)))) \circ (z \circ z)]) \vee I(z).$

Hence $I((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \leq I(x) \vee I(z)$. Therefore I is a $BFSSAS$-algebra. □

A $BFSS$-[resp. $BFSSAS$-]algebra of X may not be a $BPSSSI$-[resp. $BFSSASI$-]ideal of X (See Example 3.22).

Example 3.22. Consider the $SSBCK$-algebra $(X, \circ, 0)$ given in Example 3.3 (1). Let I the bipolar fuzzy set in X defined as follows:

$$I(0) = (s_0, t_0),\ I(x) = I(y) = (s_1, t_1),\ I(1) = (s_2, t_2),$$

where $(s_0, t_0) > (s_1, t_1) > (s_2, t_2)$ in $[-1, 0] \times [0, 1]$.
Then I is a $BFSS$-algebra of X. But we have
$I^-((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1))))$
$= I^-(1) = s_2 > s_1$
$= I^-([((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]$
$\circ[((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]) \vee I^-(x)$,
$I^+((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1))))$
$= I^+(1) = t_2 < t_1$
$= I^+([((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]$
$\circ[((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]) \wedge I^+(x)$.
Thus I does not satisfy the condition of a $BFSSS$-algebra of X. So I is not a $BFSSSI$-ideal of X.

Now let $(X, \circ, 0)$ be the $SSBCK$-algebra given in Example 3.3 (1). Consider the bipolar fuzzy set I in X defined as follows:

$$I(0) = (s_0, t_0),\ I(x) = I(y) = (s_1, t_1),\ I(1) = (s_2, t_2),$$

where $(s_0, t_0) < (s_1, t_1) < (s_2, t_2)$ in $[-1, 0] \times [0, 1]$.
Then I is a $BFSSAS$-algebra of X. But we have
$I^-((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1))))$
$= I^-(1) = s_2 < s_1$
$= I^-([((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]$
$\circ[((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]) \wedge I^-(x)$.
$I^+((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1))))$
$= I^+(1) = t_2 > t_1$
$= I^+([((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]$
$\circ[((1 \circ (1 \circ (1 \circ 1))) \circ (1 \circ (1 \circ (1 \circ 1)))) \circ (x \circ x)]) \vee I^+(x)$.
Thus I does not satisfy the condition of a $BFSSAS$-algebra of X. So I is not a $BFSSASI$-ideal of X.

Definition 3.23 ([9])**.** An $SSBCK$-algebra X is said to be *implicative*, if for all $x, y \in X$,

$$x \circ (x \circ (y \circ y)) = y \circ (y \circ (x \circ x)).$$

Proposition 3.24. *Let X be an implicative $SSBCK$-algebra X. Then every $BFSS$-*[resp. *$BFSSA$-*]*ideal of X is a $BFSSSI$-*[resp. *$BFSSASI$-*]*ideal of X.*

Proof. Let I be a $BFSS$-ideal of X. Then by ($BFSSI_1$), $I(0) \geq I(x)$ for all $x \in X$. Thus I satisfies the condition ($BFSSI_1$).

Now let $x, y, z \in X$. Then
$I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))$
$\geq I([(((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))) \circ (z \circ z)]$
$\circ[(((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))) \circ (z \circ z)]) \wedge I(z)$
[By ($BFSSSII_2$)]
$= I([(((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))) \circ (z \circ z)]$

$\circ[(((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))) \circ (z \circ z)]) \wedge I(z)$.
[By Definition 3.23]
So I satisfies the condition (BFSSSII$_2$). Hence I is a $BFSSSI$-ideal of X.

Now let I be a $BFSSA$-ideal of X. Then by (BFSSAI$_1$), $I(0) \leq I(x)$ for all $x \in X$. Thus I satisfies the condition (BFSSAI$_1$).

Let $x, y, z \in X$. Then
$I((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))$
$\leq I([(((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))) \circ (z \circ z)]$
$\circ[(((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x))))) \circ (z \circ z)]) \vee I(z)$
[By (BFSSASII$_2$)]
$= I([(((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))) \circ (z \circ z)]$
$\circ[(((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))))) \circ (z \circ z)]) \vee I(z)$.
[By Definition 3.23]
So I satisfies the condition (BFSSASII$_2$). Hence I is a $BFSSASI$-ideal of X. □

The following is an immediate consequence of Proposition 3.20 (2) and Proposition 3.24.

Theorem 3.25. *Let X be an implicative $SSBCK$-algebra X and $I \in [0,1]^X$. Then I is a $BFSSA$-*[resp. *$BFSSA$-*]*ideal of X if and only if it is a $BFSSASI$-*[resp. *$BFSSASI$-*]*ideal of X.*

4. The images and the pre-images of bipolar fuzzy sets under a homomorphism of $SSBCK$-algebras

In this section, we introduce the notions of the image and the pre-image under a mapping and study their properties of $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BPSSI$-, $BFSSAPI$-, $BFSSC$- and $BFSSAC$-]ideals under $SSBCK$-homomorphism. Furthermore, we investigate the Cartesian product of $BFSS$-[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BPSSI$-, $BFSSAPI$-, $BFSSC$- and $BFSSAC$-]ideals.

Definition 4.1 ([9]). Let $(X, \circ_A, 0_A)$ and $(Y, \circ_B, 0_B)$ be two $SSBCK$-algebras. Then a mapping $f : X \to Y$ is called a *Sheffer stroke BCK-homomorphism* (in briefly, $SSBCK$-homomorphism), if it satisfies the following condition: for all $x, y \in X$,

$$f(x \circ_A y) = f(x) \circ_B f(y).$$

Furthermore, f is called a:

(i) *Sheffer stroke BCK-epimorphism* (in briefly, $SSBCK$-epimorphism), if it is an $SSBCK$-homomorphism and surjective,

(ii) *Sheffer stroke BCK-isomorphism* (in briefly, $SSBCK$-isomorphism), if it is both an $SSBCK$-epimorphism and injective,

(iii) *Sheffer stroke BCK-endomorphism* (in briefly, $SSBCK$-endomorphism), if $X = Y$ and it is an $SSBCK$-homomorphism,

(iv) *Sheffer stroke BCK-automorphism* (in briefly, $SSBCK$-automorphism), if $X = Y$ and it is an $SSBCK$-isomorphism.

The set of all $SSBCK$-homomophisms from X to Y is denoted by $Hom(X, Y)$.

Definition 4.2. Let X, Y be non-empty sets, $f : X \to Y$ a mapping and $A = (A^-, A^+) \in BPF(X)$, $B = (B^-, B^+) \in BPF(Y)$. Then

(i) the *image of A under f*, denoted by $f(A)$, is a bipolar fuzzy set in Y defined as follows: for all $y \in Y$,

$$f(A)(y) = (f(A)^-, f(A)^+)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \varnothing \\ \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where

$$f(A)^-(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} A^-(x) & \text{if } f^{-1}(y) \neq \varnothing \\ \\ -1 & \text{otherwise,} \end{cases} \tag{4.2}$$

$$f(A)^+(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A^+(x) & \text{if } f^{-1}(y) \neq \varnothing \\ \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

(ii) the *preimage of B under f*, denoted by $B \circ f$, is a bipolar fuzzy set in X defined as follows: for all $x \in X$,

$$(B \circ f)(x) = B(f(x)). \tag{4.4}$$

Proposition 4.3. *Let $(X, \circ, 0)$ and $(Y, \circ^{'}, 0^{'})$ be $SSBCK$-algebras. If B is a $BFSS$-*[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BPSSI$-, $BFSSAPI$-, $BFSSC$- and $BFSSAC$-]*ideal of Y. Then $f^{-1}(B)$ is is a $BFSS$-*[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BPSSI$-, $BFSSAPI$-, $BFSSC$- and $BFSSAC$-]*ideal of X for all $f \in Hom(X, Y)$.*

Proof. Let $f \in Hom(X, Y)$ and suppose B is a $BFSS$-ideal of Y. Then by (BFSSI$_1$),

$$f^{-1}(B)(0) = B(f(0)) = B(0^{'}) \geq B(f(x)) = f^{-1}(B)(x) \text{ for all } x \in X.$$

Thus $f^{-1}(B)$ satisfies the condition (BFSSI$_1$).

Now let $x, y \in X$. Then we have
$f^{-1}(B)((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge f^{-1}(B)(y)$
$= B(f((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge B(f(y))$
$= B((f(x) \circ^{'} (f(y) \circ^{'} f(y))) \circ^{'} (f(x) \circ^{'} (f(y) \circ^{'} f(y)))) \wedge B(f(y))$
[Since f is an $SSBCK$-homomorphism]
$\leq B(f(x))$ [By(BFSSI$_2$)]
$= f^{-1}(B)(x)$.
Thus $f^{-1}(B)$ satisfies the condition (BFSSI$_2$). So $f^{-1}(B)$ is a $BFSS$-ideal of X.

Suppose B is a $BFSSA$-ideal of Y. Then by (BFSSAI$_1$),

$$f^{-1}(B)(0) = B(f(0)) = B(0^{'}) \leq B(f(x)) = f^{-1}(B)(x) \text{ for all } x \in X.$$

Thus $f^{-1}(B)$ satisfies the condition (BFSSAI$_1$).

Now let $x, y \in X$. Then we have

$f^{-1}(B)((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \vee f^{-1}(B)(y)$
$= B(f((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \wedge B(f(y))$
$= B((f(x) \circ^{'} (f(y) \circ^{'} f(y))) \circ^{'} (f(x) \circ^{'} (f(y) \circ^{'} f(y)))) \vee B(f(y))$
[Since f is an $SSBCK$-homomorphism]
$\geq B(f(x))$ [By (BFSSAI$_2$)]
$= f^{-1}(B)(x)$.

Thus $f^{-1}(B)$ satisfies the condition (BFSSAI$_2$). So $f^{-1}(B)$ is a $BFSSA$-ideal of X.

The proof of the remainders are easy. □

Proposition 4.4. *Let $(X, \circ, 0)$, $(Y, \circ^{'}, 0^{'})$ be $SSBCK$-algebras, $f : X \to Y$ an $SSBCK$-epimorphism and $B \in BPF(Y)$. If $f^{-1}(B)$ is a $BFSS$-*[resp. *$BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-* and *$BFSSAC$-*]*ideal of X, then B is a $BFSS$-*[resp. *$BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-* and *$BFSSAC$-*]*ideal of Y.*

Proof. Suppose $f^{-1}(B)$ is a $BFSSPI$-ideal of X and let $x \in Y$. Since f is surjective, there is $a \in X$ such that $x = f(a)$. Then we get

$B(x) = B(f(a))$ [Since $x = f(a)$]
$= f^{-1}(B)(a)$
$\leq f^{-1}(B)(0)$ [By (BFSSI$_1$)]
$= B(f(0))$
$= B(0^{'})$. [Since f is an $SSBCK$-homomorphism]

Thus B satisfies the condition (BFSSI$_1$)

Now let $x, y, z \in Y$. Since f is surjective, there are $a, b, c \in X$ such that $x = f(a),\ y = f(b),\ z = f(c)$. Then we have

$B((x \circ^{'} (z \circ^{'} z)) \circ^{'} (x \circ^{'} (z \circ^{'} z)))$
$= B((f(a) \circ^{'} (f(c) \circ^{'} f(c))) \circ^{'} (f(a) \circ^{'} (f(c) \circ^{'} f(c))))$
[Since $x = f(a),\ y = f(b),\ z = f(c)$]
$= B(f((a \circ (c \circ c)) \circ (a \circ (c \circ c))))$ [Since f is an $SSBCK$-homomorphism]
$= f^{-1}(B)((a \circ (c \circ c)) \circ (a \circ (c \circ c)))$
$\geq f^{-1}(B)([((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]$
$\circ[((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]) \wedge f^{-1}(B)((b \circ (c \circ c)) \circ (b \circ (c \circ c)))$
[By (BFSSPII$_2$)
$= B(f(([((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]$
$\circ[((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]) \wedge B(f(((b \circ (c \circ c)) \circ (b \circ (c \circ c)))$
$= B([((f(a) \circ^{'} (f(b) \circ^{'} f(b))) \circ^{'} (f(a) \circ^{'} (f(b) \circ^{'} f(b)))) \circ^{'} (f(c) \circ^{'} f(c)]$
$\circ^{'}[((f(a) \circ^{'} (f(b) \circ^{'} f(b))) \circ^{'} (f(a) \circ^{'} (f(b) \circ^{'} f(b)))) \circ^{'} (f(c) \circ^{'} f(c)])$
$\wedge B((f(b) \circ^{'} (f(c) \circ^{'} f(c)))) \circ^{'} (f(b) \circ^{'} (f(c) \circ^{'} f(c))))$
[Since f is an $SSBCK$-homomorphism]
$= B([((x \circ^{'} (y \circ^{'} y)) \circ^{'} (x \circ^{'} (y \circ^{'} y))) \circ^{'} (z \circ^{'} z)]$
$\circ^{'}[((x \circ^{'} (y \circ^{'} y)) \circ^{'} (x \circ^{'} (y \circ^{'} y))) \circ^{'} (z \circ^{'} z)]) \wedge B((y \circ^{'} (z \circ^{'} z) \circ^{'} (y \circ^{'} (z \circ^{'} z))$.

Thus B satisfies the condition (BFSSPII$_2$). So B is a $BFSSPI$-ideal of X.

Now suppose $f^{-1}(B)$ is a $BFSSAPI$-ideal of X and let $x \in Y$. Since f is surjective, there is $a \in X$ such that $x = f(a)$. Then we get
$B(x) = B(f(a))$ [Since $x = f(a)$]
$= f^{-1}(B)(a)$
$\geq f^{-1}(B)(0)$ [By (BFSSAI$_1$)]
$= B(f(0))$
$= B(0^{'})$. [Since f is an $SSBCK$-homomorphism]
Thus B satisfies the condition (BFSSAI$_1$)

Now let $x, y, z \in Y$. Since f is surjective, there are $a, b, c \in X$ such that $x = f(a),\ y = f(b),\ z = f(c)$. Then we have
$B((x \circ^{'} (z \circ^{'} z)) \circ^{'} (x \circ^{'} (z \circ^{'} z)))$
$= B((f(a) \circ^{'} (f(c) \circ^{'} f(c))) \circ^{'} (f(a) \circ^{'} (f(c) \circ^{'} f(c))))$
[Since $x = f(a),\ y = f(b),\ z = f(c)$]
$= B(f((a \circ (c \circ c)) \circ (a \circ (c \circ c))))$ [Since f is an $SSBCK$-homomorphism]
$= f^{-1}(B)((a \circ (c \circ c)) \circ (a \circ (c \circ c)))$
$\leq f^{-1}(B)([((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]$
$\circ[((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]) \vee f^{-1}(B)((b \circ (c \circ c)) \circ (b \circ (c \circ c)))$
[By (BFSSAPII$_2$)
$= B(f(([((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]$
$\circ[((a \circ (b \circ b))) \circ (a \circ (b \circ b)))) \circ (c \circ c)]) \vee B(f(((b \circ (c \circ c)) \circ (b \circ (c \circ c)))$
$= B([((f(a) \circ^{'} (f(b) \circ^{'} f(b))) \circ^{'} (f(a) \circ^{'} (f(b) \circ^{'} f(b)))) \circ^{'} (f(c) \circ^{'} f(c)]$
$\circ^{'}[((f(a) \circ^{'} (f(b) \circ^{'} f(b))) \circ^{'} (f(a) \circ^{'} (f(b) \circ^{'} f(b)))) \circ^{'} (f(c) \circ^{'} f(c)])$
$\vee B((f(b) \circ^{'} (f(c) \circ^{'} f(c)))) \circ^{'} (f(b) \circ^{'} (f(c) \circ^{'} f(c))))$
[Since f is an $SSBCK$-homomorphism]
$= B([((x \circ^{'} (y \circ^{'} y)) \circ^{'} (x \circ^{'} (y \circ^{'} y))) \circ^{'} (z \circ^{'} z)]$
$\circ^{'}[((x \circ^{'} (y \circ^{'} y)) \circ^{'} (x \circ^{'} (y \circ^{'} y))) \circ^{'} (z \circ^{'} z)]) \vee B((y \circ^{'} (z \circ^{'} z) \circ^{'} (y \circ^{'} (z \circ^{'} z))$.
Thus B satisfies the condition (BFSSAPII$_2$). B is a $BFSSAPI$-ideal of X.

Also, we can prove the remainders. □

Definition 4.5. Let X be a non-empty set and $A \in BPF(X)$. Then we say that *A has the sup property*, if for any subset T of X, there is $t_0 \in T$ such that

$$\bigvee_{t \in T} A(t) = (\bigwedge_{t \in T} A^{-}(t), \bigvee_{t \in T} A^{+}(t)) = A(t_0).$$

Proposition 4.6. *Let* $(X, \circ, 0)$, $(Y, \circ^{'}, 0^{'})$ *be $SSBCK$-algebras, $f : X \to Y$ an $SSBCK$-epimorphism and $A \in BPF(X)$ have the sup property. If A is a $BFSS$-*[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$- and $BFSSAC$-]*ideal of X, then $f(A)$ is a $BFSS$-*[resp. $BFSSA$-, $BFSSSI$-, $BFSSASI$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$- and $BFSSAC$-]*ideal of Y.*

Proof. Suppose A has the sup property and is a $BFSSPI$-ideal of X. Let $B = f(A)$ and $x \in Y$. Note that $0 \in f^{-1}(0^{'})$ and $f^{-1}(x)$ is a subset of X. Then we have
$B(0^{'}) = \bigvee_{t \in f^{-1}(0^{'})} A(t)$
$= A(0)$ [Since A has the sup property and $0 \in f^{-1}(0^{'})$]

$\geq A(x)$ for all $a \in X$. [By (BFSSI$_1$)]
Thus $B(0^{'}) \geq \bigvee_{t \in f_{-1}(x)} A(t) = B(x)$. So $B = f(A)$ satisfies the condition (BFSSI$_1$).

Now let $x, y, z \in Y$. Since f is surjective, there are $a, b, c \in X$ such that

$$x = f(a), y = f(b), z = f(c), \text{ i.e., } a \in f^{-1}(x), b \in f^{-1}(y), c \in f^{-1}(z).$$

In particular, a, b, c satisfy the following equalities:
$A([((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)]$
$\circ[((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)])$
$= \bigvee_{t \in f^{-1}([((x\circ'(y\circ'y))\circ'(x\circ'(y\circ'y)))\circ'(z\circ'z)]\circ'[((x\circ'(y\circ'y))\circ'(x\circ'(y\circ'y)))\circ'(z\circ'z)])} A(t)$,
$A((b \circ (c \circ c)) \circ (b \circ (c \circ c))) = \bigvee_{t \in f^{-1}((y\circ'(z\circ'z))\circ'(y\circ'(z\circ'z)))} A(t)$,
$A((a \circ (c \circ c)) \circ (a \circ (c \circ c))) = \bigvee_{t \in f^{-1}((x\circ'(z\circ'z))\circ'(x\circ'(z\circ'z)))} A(t)$.
Then we have
$B((x \circ^{'} (z \circ^{'} z)) \circ^{'} (x \circ^{'} (z \circ^{'} z)))$
$= \bigvee_{t \in f^{-1}((x\circ'(z\circ'z))\circ'(x\circ'(z\circ'z)))} A(t)$
$= A((a \circ (c \circ c)) \circ (a \circ (c \circ c)))$
$\geq A([((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)]$
$\circ[((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)]) \wedge A((b \circ (c \circ c)) \circ (b \circ (c \circ c)))$
[By (BFSSPII$_2$)]
$= \bigvee_{t \in f^{-1}([((x\circ'(y\circ'y))\circ'(x\circ'(y\circ'y)))\circ'(z\circ'z)]\circ'[((x\circ'(y\circ'y))\circ'(x\circ'(y\circ'y)))\circ'(z\circ'z)])} A(t)$
$\wedge \bigvee_{t \in f^{-1}((x\circ'(z\circ'z))\circ'(x\circ'(z\circ'z)))} A(t)$
$= B([((x \circ^{'} (y \circ^{'} y)) \circ^{'} (x \circ^{'} (y \circ^{'} y))) \circ^{'} (z \circ^{'} z)]$
$\circ^{'}[((x \circ^{'} (y \circ^{'} y)) \circ^{'} (x \circ^{'} (y \circ^{'} y))) \circ^{'} (z \circ^{'} z)])$
$\wedge B((x \circ^{'} (z \circ^{'} z)) \circ^{'} (x \circ^{'} (z \circ^{'} z)))$.
Thus $f(A)$ satisfies the condition (BFSSPII$_2$). So $f(A)$ is a $BFSSPI$-ideal of Y.

Now suppose A has the sup property and is a $BFSSAPI$-ideal of X. Let $B = f(A)$ and $x \in Y$. Then we have
$B(0^{'}) = \bigvee_{t \in f^{-1}(0^{'})} A(t)$
$= A(0)$ [Since A has the sup property and $0 \in f^{-1}(0^{'})$]
$\leq A(x)$ for all $a \in X$. [By (BFSSAI$_1$)]
Thus $B(0^{'}) \leq \bigvee_{t \in f_{-1}(x)} A(t) = B(x)$. So $B = f(A)$ satisfies the condition (BFSSAI$_1$).

Now let $x, y, z \in Y$. Since f is surjective, there are $a, b, c \in X$ such that

$$x = f(a), y = f(b), z = f(c), \text{ i.e., } a \in f^{-1}(x), b \in f^{-1}(y), c \in f^{-1}(z).$$

In particular, a, b, c satisfy the following equalities:
$A([((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)]$
$\circ[((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)])$
$= \bigvee_{t \in f^{-1}([((x\circ'(y\circ'y))\circ'(x\circ'(y\circ'y)))\circ'(z\circ'z)]\circ'[((x\circ'(y\circ'y))\circ'(x\circ'(y\circ'y)))\circ'(z\circ'z)])} A(t)$,
$A((b \circ (c \circ c)) \circ (b \circ (c \circ c))) = \bigvee_{t \in f^{-1}((y\circ'(z\circ'z))\circ'(y\circ'(z\circ'z)))} A(t)$,
$A((a \circ (c \circ c)) \circ (a \circ (c \circ c))) = \bigvee_{t \in f^{-1}((x\circ'(z\circ'z))\circ'(x\circ'(z\circ'z)))} A(t)$.
Then we have
$B((x \circ^{'} (z \circ^{'} z)) \circ^{'} (x \circ^{'} (z \circ^{'} z)))$
$= \bigvee_{t \in f^{-1}((x\circ'(z\circ'z))\circ'(x\circ'(z\circ'z)))} A(t)$
$= A((a \circ (c \circ c)) \circ (a \circ (c \circ c)))$
$\leq A([((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)]$

$\circ[((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \circ (c \circ c)]) \vee A((b \circ (c \circ c)) \circ (b \circ (c \circ c)))$
[By (BFSSAPII$_2$)]
$= \bigvee_{t \in f^{-1}([((x \circ' (y \circ' y)) \circ' (x \circ' (y \circ' y))) \circ' (z \circ' z)] \circ' [((x \circ' (y \circ' y)) \circ' (x \circ' (y \circ' y))) \circ' (z \circ' z)])} A(t)$
$\vee \bigvee_{t \in f^{-1}((x \circ' (z \circ' z)) \circ' (x \circ' (z \circ' z)))} A(t)$
$= B([((x \circ' (y \circ' y)) \circ' (x \circ' (y \circ' y))) \circ' (z \circ' z)]$
$\circ' [((x \circ' (y \circ' y)) \circ' (x \circ' (y \circ' y))) \circ' (z \circ' z)])$
$\vee B((x \circ' (z \circ' z)) \circ' (x \circ' (z \circ' z)))$.
Thus $f(A)$ satisfies the condition (BFSSAPII$_2$). So $f(A)$ is a $BFSSAPI$-ideal of Y. □

Let $(X, \circ_X, 0_X, \leq_X), (Y, \circ_Y, 0_Y, \leq_Y)$ be two $SSBCK$-algebras. Then two binary relation $\leq$ and $\circ$ on $X \times Y$ is defined as follows: for all $(x_1, x_2), (y_1, y_2)$,

$$(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } x_1 \leq_X y_1,\ x_2 \leq_Y y_2,$$

$$(x_1, x_2) \circ (y_1, y_2) = (x_1 \circ_X y_1, x_2 \circ_Y y_2).$$

Thus $(X \times Y, \circ, (0_X, 0_Y), \leq)$ is a partial order $SSBCK$-algebra. Furthermore, for each $A \in BPF(X)$ and $A \in BPF(Y)$, the Cartesian product of A, B, denoted by $A \times B = (A^- \times B^-, A^+ \times B^+)$, is a bipolar fuzzy set in $X \times Y$ defined as follows: for all $(x, y) \in X \times Y$,

$$(A \times B)(x, y) = A(x) \wedge B(y) \text{ (See [16])}.$$

Proposition 4.7. *Let $(X, \circ_X, 0_X, \leq_X), (Y, \circ_Y, 0_Y, \leq_Y)$ be two $SSBCK$-algebras and $A \in BPF(x), B \in BPF(Y)$. If A is a $BFSS$-*[resp. *$BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-* and *$BFSSAC$-*]*ideal of X and B a $BFSS$-*[resp. *$BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-* and *$BFSSAC$-*]*ideal of Y, then $A \times B$ is a $BFSS$-*[resp. *$BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$-* and *$BFSSAC$-*]*ideal of $X \times Y$*

Proof. Suppose A is an $BFSSI$-ideal of X, B a $BFSSI$-ideal of Y and let $(x, y) \in X \times Y$. Then (BFSSI$_1$), we get

$$(A \times B)(0_X, 0_Y) = A(0_X) \wedge B(0_Y) \geq A(x) \wedge B(y) = (A \times B)(x, y).$$

Thus $A \times B$ satisfies the condition (BFSSI$_1$).
Now let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times Y$. Then we have
$(A \times B)(x_1, x_2) = A(x_1) \wedge B(x_2)$
$\geq (A([(x_1 \circ_X (y_1 \circ_X (x_1 \circ_X x_1))) \circ_X (z_1 \circ_X z_1)]$
$\circ_X [(x_1 \circ_X (y_1 \circ_X (x_1 \circ_X x_1))) \circ_X (z_1 \circ_X z_1)]) \wedge A(z_1))$
$\wedge (B([(x_2 \circ_Y (y_2 \circ_Y (x_2 \circ_Y x_2))) \circ_Y (z_2 \circ_Y z_2)]$
$\circ_Y [(x_2 \circ_Y (y_2 \circ_Y (x_2 \circ_Y x_2))) \circ_Y (z_2 \circ_Y z_2)]) \wedge B(z_2))$
[By (BFSSII$_2$)]
$= (A \times B)([(x_1, x_2) \circ ((y_1, y_2) \circ (x_1, x_2) \circ (x_1, x_2)))) \circ ((z_1, z_2) \circ (z_1, z_2))]$
$\circ [(x_1, x_2) \circ (_1, y_2) \circ (x_1, x_2) \circ (x_1, x_2))) \circ ((z_1, z_2) \circ (z_1, z_2))])$
$\wedge (A \times B)(z_1, z_2)$.
Thus $A \times B$ satisfies the condition (BFSSII$_2$). So $A \times B$ is a $BFSSI$-ideal of $X \times Y$. Also, we can show the remainders. □

The following is an immediate consequence of Proposition 4.7.

Corollary 4.8. *Let X be an $SSBCK$-algebra and $A \in BPF(X)$. If A is a $BFSS$-*[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$- and , $BFSSAC$-]*ideal of X, then $A \times A$ is a $BFSS$-*[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$- and $BFSSAC$-]*ideal of $X \times X$.*

5. Conclusions

We introduced the concepts of $BFSS$-[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$- and $BFSSAC$-]ideals of an $SSBCK$-algebra and obtained some of their properties. Moreover, we studied the properties of $BFSS$-[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$- and $BFSSAC$-]ideals under $SSBCK$-homomorphisms between $SSBCK$-algebras. Furthermore, we investigate the behaviors of Cartesian product of $BFSS$-[resp. $BFSSA$-, $BFSSPI$-, $BFSSAPI$-, $BFSSI$-, $BFSSAI$-, $BFSSC$- and $BFSSAC$-]ideals.

In the future, we will study the Sheffer stroke BCK-algebras in the sense of another fuzzy concepts.

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