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Abstract. In this paper, we introduce the notion of general fuzzy group rings as an extension of classical group rings in the framework of fuzzy algebra. We also define fuzzy group rings induced from fuzzy subgroups and investigate their fundamental algebraic properties. Several structural results concerning these fuzzy group rings are established. In particular, we introduce the concept of a quotient general fuzzy group ring and utilize it and prove the corresponding isomorphism theorems. Furthermore, a comparative analysis between general fuzzy group rings and fuzzy group rings induced by fuzzy subgroups is presented, highlighting the essential distinctions between these two constructions. Additional related results are obtained to illustrate the behavior and applicability of the proposed structures. The results extend certain classical concepts of group ring theory to the fuzzy setting and contribute to the development of fuzzy algebraic systems.

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Keywords: General fuzzy group ring; Fuzzy subgroup; Quotient general fuzzy group ring; Isomorphism theorems; Group rings.

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1. INTRODUCTION

The theory of fuzzy sets, introduced by Zadeh [14], provides a mathematical framework for handling uncertainty through graded membership functions. This idea has been successfully incorporated into algebraic structures, leading to the development of fuzzy groups, rings, and related systems. The seminal work of Rosenfeld [12] initiated the study of fuzzy groups, establishing the basic properties of fuzzy subgroups. Subsequent investigations, including Liu's work on fuzzy invariant subgroups and fuzzy ideals [4]. The homomorphism of fuzzy groups [11], and the systematic treatments of fuzzy algebra by Mordeson and Malik [6] and Mordeson, Bhutani, and Rosenfeld [7], significantly expanded the theory. Applications and foundational aspects of fuzzy systems have also been discussed by Negoita and Ralescu [9].

On the other hand, classical algebra provides the structural foundation for these developments. Standard texts such as Dummit and Foote [2] and Herstein [3] present the fundamental theory of groups and rings, while the structure of group rings and their algebraic properties are studied extensively in Passman [10]. Representation-theoretic aspects of finite groups are treated in Musili [8]. Group rings $R[G]$ play an important role in algebra by linking group theory with ring theory and serving as a useful tool in the study of representations and module structures.

The extension of classical algebraic constructions to fuzzy settings has attracted considerable attention in recent years. In particular, fuzzy analogues of associative algebras and their homomorphism properties have been investigated by Yang and Zhou [13]. More recently, Abraham et al. [1] introduced new perspectives on fuzzy group algebras, further motivating the study of fuzzy generalizations of group-theoretic constructions.

This paper investigates fuzzy group rings from two perspectives. First, general fuzzy group rings are considered, defined axiomatically on the classical group ring $R[G]$. Second, induced fuzzy group rings are studied, obtained naturally from fuzzy subgroups of G. Although these two constructions are related, they are not equivalent and exhibit different structural properties. The notion of a quotient general fuzzy group ring is develop and establish corresponding isomorphism theorems. A comparative analysis between general and induced fuzzy group rings is also presented, highlighting their structural differences.

These results extend classical group ring theory to the fuzzy setting and contribute to the ongoing development of fuzzy algebraic structures, providing a foundation for further investigations involving homomorphisms, quotient constructions and related algebraic properties.

2. Preliminaries

In this section, we recall some basic definitions and results related to fuzzy sets, fuzzy subgroups, rings and group rings that will be used throughout the paper.

Definition 2.1. [14] Let X be a nonempty set. A **fuzzy subset** μ of X is defined as a mapping $\mu : X \to [0, 1]$, where $\mu(x)$ represents the degree of membership of the element $x \in X$.

Definition 2.2. [12, 14] For $\alpha \in [0, 1]$, the **α-level set** (or α-cut) and the **support** of a fuzzy set μ are respectively defined as $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ and $\text{supp}(\mu) = \{x \in X \mid \mu(x) > 0\}$.

Definition 2.3. [12] Let G be a group. A fuzzy subset $\mu : G \to [0, 1]$ is called a **fuzzy subgroup** of G if for all $x, y \in G$, $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$.

A fuzzy subgroup μ is called a **normal fuzzy subgroup** of G, if it satisfies $\mu(xyx^{-1}) = \mu(y)$.

Definition 2.4. [4] Let R be a ring. A fuzzy subset $\mu : R \to [0,1]$ is called a **fuzzy subring** of R if for all $x, y \in R$, $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.5. [4] Let R be a ring. A fuzzy subset $\mu : R \to [0,1]$ is called a **fuzzy ideal** of R if for all $x, y \in R$, $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$.

Definition 2.6. [10] Let R be a ring and G a group. The **group ring** $R[G]$ is defined as the set of all formal sums $\sum_{g\in G} r_g g$, where $r_g \in R$ and only finitely many coefficients r_g are nonzero.

Addition and multiplication in $R[G]$ are defined as:

$$\sum_{g\in G} r_g g + \sum_{g\in G} s_g g = \sum_{g\in G} (r_g + s_g) g \text{ and}$$
$$(\sum_{g\in G} r_g g)(\sum_{g\in G} s_g g) = \sum_{g\in G} \big(\sum_{hk=g} r_h s_k\big) g.$$

With these operations, $R[G]$ form a ring. It is commutative if and only if R is commutative and G is abelian. Moreover, if R has a multiplicative identity 1_R and e is the identity element of the G, then the multiplicative identity element of $R[G]$, denoted by $1_{R[G]}$, is given by $1_R \cdot e$.

3. Fuzzy Group Rings

This section consists of three subsections. In the first subsection, the notion of general fuzzy group rings is defined and some examples are constructed. In the second subsection we study some of their structural properties. In the third subsection we study the concept of homomorphism of general fuzzy group rings and establish three fundamental isomorphism theorems of general fuzzy group rings.

3.1. General Fuzzy Group Rings.

Throughout this section, R denotes a ring with identity and G denotes a group.

Definition 3.1. A fuzzy set $\nu : R[G] \to [0,1]$ is called a **general fuzzy group ring (GFGR)** of $R[G]$ if for all $x, y \in R[G]$ the following conditions holds:
(GF1) **Fuzzy additivity**: $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$.
(GF2) **Fuzzy multiplicativity**: $\nu(xy) \geq \min\{\nu(x), \nu(y)\}$.
(GF3) **Symmetry**: $\nu(-x) = \nu(x)$.
(GF4) **Identity dominance**: $\nu(1_{R[G]}) \geq \nu(x), \forall x \in R[G]$.
(GF5) **Compatibility with group inversion**: $\nu(g^{-1}) = \nu(g), \forall g \in G$.

Remark 3.2. Let $\nu : R[G] \to [0,1]$ be a general fuzzy group ring. Then

- $\nu(0) = \sup\{\nu(x) : x \in R[G]\}$
- $\nu(0) \geq \nu(1_{R[G]}) \geq \nu(x), \forall x \in R[G]$.

Indeed from (GF1) to (GF3), we have $\nu(0) \geq \nu(x), \forall x \in R[G]$, so
$\nu(0) = \sup\{\nu(x) : x \in R[G]\}$. In particular, we have $\nu(0) \geq \nu(1_{R[G]})$.
Further, from (GF4), $\nu(1_{R[G]}) \geq \nu(x), \forall x \in R[G]$. Combining these inequalities gives

$$\nu(0) \geq \nu(1_{R[G]}) \geq \nu(x), \forall x \in R[G].$$

Remark 3.3. Conditions (GF1) to (GF4) ensure that a GFGR defines a fuzzy subring of $R[G]$, while condition (GF5) guarantees compatibility with the natural embedding of the group G into the group ring $R[G]$. The following is an immediate result from the **Definition 3.1**.

Theorem 3.4. *Every general fuzzy group ring is a fuzzy subring of* $R[G]$.

Example 3.5. Let R be a ring and H be a subgroup of G containing the identity element e. Define a fuzzy set $\nu : R[G] \to [0,1]$ by

$$\nu\left(\sum_{g\in G} r_g g\right) = \begin{cases} 1, & \text{if } r_g = 0 \text{ for all } g \notin H \\ 0, & \text{otherwise .} \end{cases}$$

The condition "$r_g = 0$ for all $g \notin H$" is exactly the condition that $\sum_{g\in G} r_g g \in R[H]$, so equivalently,

$$\nu(x) = \begin{cases} 1, & \text{if } x \in R[H] \\ 0, & \text{if } x \notin R[H]. \end{cases}$$

This mean ν is just the characteristic function of the subgroup ring $R[H]$.
Also, $supp(\nu) = \{x \in R[G] | \nu(x) > 0\} = R[H]$. Thus ν is a general fuzzy group ring whose support is precisely the subgroup ring $R[H]$.

Example 3.6. Let $R = \mathbb{Z}$ and $G = \{e, a\}$ with $a^2 = e$. Define fuzzy set $\nu : R[G] \to [0,1]$ by

$$\nu(r_e e + r_a a) = \begin{cases} 1, & \text{if } r_a = 0, \\ 0, & \text{if } r_a \neq 0. \end{cases}$$

Then ν satisfies (GF1) to (GF5) and hence defines a general fuzzy group ring.
(Note that Example 3.6 is a special case of Example 3.5 on taking $H = \{e\}$)

Example 3.7. Let J be the Jacobson radical of the group ring $R[G]$.
Define the fuzzy set $\nu : R[G] \to [0,1]$ as:

$$\nu(x) = \begin{cases} 1, & \text{if } x \notin J, \\ \alpha, & \text{if } x \in J, \end{cases}$$

where $0 < \alpha < 1$. Since J is a two-sided ideal of $R[G]$ and contains no units, we verify (GF1)-(GF5):

- (GF1) For $x, y \in R[G]$, if $x, y \in J$ then $x + y \in J$ and $\nu(x + y) = \alpha$. If at least one of $x, y \notin J$, then $\min\{\nu(x), \nu(y)\} = \alpha$ and $\nu(x + y) \in \{\alpha, 1\} \geq \alpha$. If both $x, y \notin J$, then $x+y \notin J$, hence $\nu(x+y) = 1$. Thus $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$.
- (GF2) For $x, y \in R[G]$, if $x \in J$ or $y \in J$, then $xy \in J$ so $\nu(xy) = \alpha$. If both $x, y \notin J$, then $xy \notin J$, hence $\nu(xy) = 1$. Thus $\nu(xy) \geq \min\{\nu(x), \nu(y)\}$.
- (GF3) Since J is an ideal, $x \in J \iff -x \in J$, hence $\nu(-x) = \nu(x)$.
- (GF4) As $1_{R[G]} \notin J$, $\nu(1_{R[G]}) = 1 \geq \nu(x)$ for all $x \in R[G]$.
- (GF5) For all $g \in G$, both g and g^{-1} are units, hence not in J. Thus $\nu(g) = \nu(g^{-1}) = 1$.

Hence ν satisfies (GF1) to (GF5) and defines a general fuzzy group ring on $R[G]$.

Remark 3.8. Let us now define fuzzy set $\nu : R[G] \to [0,1]$ in the following two components
(i) $\theta : R \to [0,1]$ be a normal fuzzy subring of R and
(ii) $\mu : G \to [0,1]$ is a fuzzy subgroup of G. Then for $x = \sum_{g \in G} r_g g \in R[G]$, the fuzzy set ν is defined as

$$\nu(x) = \begin{cases} \min_{r_g \neq 0}\{\theta(r_g), \mu(g)\}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

Let $x = \sum r_g g$ and $y = \sum s_g g \in R[G]$. Since θ is a normal fuzzy subring and μ is a fuzzy subgroup, we verify:

(GF1) For $x+y = \sum (r_g + s_g)g$, we have $\theta(r_g + s_g) \geq \min\{\theta(r_g), \theta(s_g)\}$. Hence each term in $x+y$ has value at least $\min\{\nu(x), \nu(y)\}$, so $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$.
(GF2) For $xy = \sum r_g s_h(gh)$, $\theta(r_g s_h) \geq \min\{\theta(r_g),\, \theta(s_h)\}$ and $\mu(gh) \geq \min\{\mu(g), \mu(h)\}$. Thus each term in xy has value at least $\min\{\nu(x), \nu(y)\}$, giving $\nu(xy) \geq \min\{\nu(x), \nu(y)\}$.
(GF3) Since $\theta(-r_g) = \theta(r_g)$, it follows that $\nu(-x) = \nu(x)$.
(GF4) For $1_{R[G]} = 1 \cdot e$, $\nu(1_{R[G]}) = \min\{\theta(1), \mu(e)\} = \theta(1)$. As $\theta(1) \geq \theta(r_g)$ and $\mu(e) = 1 \geq \mu(g)$, we get $\nu(1_{R[G]}) \geq \nu(x)$ for all x.
(GF5) For $g \in G$, $\nu(g) = \min\{\theta(1), \mu(g)\}$ and $\nu(g^{-1}) = \min\{\theta(1), \mu(g^{-1})\}$. Since $\mu(g^{-1}) = \mu(g)$, we have $\nu(g^{-1}) = \nu(g)$.

Hence ν satisfies (GF1) to (GF5), so it defines a general fuzzy group ring.

Example 3.9. Let $R = \mathbb{Z}$ and $G = \mathbb{Z}$. Define fuzzy set $\theta : R \to [0,1]$ and $\mu : G \to [0,1]$ as

$$\theta(r) = \tfrac{1}{1+|r|}, \text{ for all } r \in R \text{ and } \mu(g) = \tfrac{1}{1+|g|}, \text{ for all } g \in G.$$

Now, define fuzzy set $\nu : R[G] \to [0,1]$ by

$$\nu(x) = \begin{cases} \min_{r_g \neq 0}\{\theta(r_g), \mu(g)\}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

Since θ and μ decrease with absolute value and satisfy $|a \pm b| \leq |a| + |b|$, we verify:

(GF1) For $x + y = \sum (r_g + s_g)g$, we have $\theta(r_g + s_g) \geq \min\{\theta(r_g), \theta(s_g)\}$. Hence each term of $x + y$ has membership at least $\min\{\nu(x), \nu(y)\}$, so $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.
(GF2) For $xy = \sum r_g s_h(g + h)$, $\theta(r_g s_h) \geq \min\{\theta(r_g), \theta(s_h)\}$ and $\mu(g + h) \geq \min\{\mu(g), \mu(h)\}$. Thus $\nu(xy) \geq \min\{\nu(x), \nu(y)\}$.
(GF3) Since $|-r| = |r|$, we have $\theta(-r) = \theta(r)$, hence $\nu(-x) = \nu(x)$.
(GF4) For $1_{R[G]} = 1 \cdot 0$, $\nu(1) = \min\left\{\frac{1}{2}, 1\right\} = \frac{1}{2}$. For all x, $\nu(x) \leq \frac{1}{2}$, hence $\nu(1_{R[G]}) \geq \nu(x)$.
(GF5) For $g \in \mathbb{Z}$, $g^{-1} = -g$ and $\mu(g) = \mu(-g)$, hence $\nu(g) = \nu(g^{-1})$.

Therefore, ν satisfies (GF1) to (GF5) and defines a general fuzzy group ring.

Example 3.10. Let $\mu : G \to [0,1]$ be a fuzzy subgroup of G. Define $\nu : R[G] \to [0,1]$ by

$$\nu\left(\sum_{g \in G} r_g g\right) = \min\{\mu(g) | r_g \neq 0\} \text{ and } \nu(0) = 1.$$

Since μ is a fuzzy subgroup, it satisfies $\mu(gh) \geq \min\{\mu(g), \mu(h)\}$ and $\mu(g^{-1}) = \mu(g)$. We verify:

(GF1) Let $x = \sum r_g g$ and $y = \sum s_g g$. The support of $x + y$ is contained in the union of supports of x and y. Hence every g occurring in $x + y$ satisfies $\mu(g) \geq \min\{\nu(x), \nu(y)\}$, so $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

(GF2) For $xy = \sum r_g s_h(gh)$, we have $\mu(gh) \geq \min\{\mu(g), \mu(h)\}$. Hence each term in xy has membership at least $\min\{\nu(x), \nu(y)\}$, giving $\nu(xy) \geq \min\{\nu(x), \nu(y)\}$.

(GF3) Since $\mu(g^{-1}) = \mu(g)$, the membership values are unchanged under inversion, hence $\nu(-x) = \nu(x)$.

(GF4) For $1_{R[G]} = e$, we have $\nu(1_{R[G]}) = \mu(e)$. Since $\mu(e) \geq \mu(g)$ for all $g \in G$, it follows that $\nu(1_{R[G]}) \geq \nu(x)$ for all $x \in R[G]$.

(GF5) For $g \in G$, we have $\nu(g) = \mu(g) = \mu(g^{-1}) = \nu(g^{-1})$.

Thus ν satisfies (GF1) to (GF5) and hence defines a fuzzy group ring on $R[G]$.

In this general framework:

- (i) Fuzziness may depend on coefficients.
- (ii) Fuzziness may mix ring and group structure.
- (iii) Not determined solely by subgroup data.

For example: Consider the general fuzzy group ring $\nu : R[G] \to [0, 1]$ defined by

$$\nu\left(\sum_{g\in G} r_g g\right) = \min\{\theta(r_g) | r_g \in R\backslash\{0\}\} \text{ and } \nu(0) = 1.$$

for a fuzzy ideal θ of R. This depends entirely on coefficients, not support.
Hence: General theory is coefficient-sensitive.

3.2. **Structural Properties of General Fuzzy Group Rings.** In this section, the internal structure of general fuzzy group rings through level subrings, support theory, homomorphic behavior are investigated. structure.

Definition 3.11. Let $\nu : R[G] \to [0, 1]$ be a general fuzzy group ring and let $\alpha \in (0, 1]$. The α**-level set of** ν (or the **level set** ν_α **of** ν) is defined as

$$\nu_\alpha = \{x \in R[G] \mid \nu(x) \geq \alpha\}.$$

Theorem 3.12. *If ν is a general fuzzy group ring, then for each $\alpha \in (0, 1]$, the level set ν_α of ν is a subring of $R[G]$.*

Proof. Let $x, y \in \nu_\alpha$. Then $\nu(x) \geq \alpha$ and $\nu(y) \geq \alpha$.
By (GF1), $\nu(x + y) \geq \min\{\nu(x), \nu(y)\} \geq \alpha$, so $x + y \in \nu_\alpha$.
Also, by (GF2), $\nu(xy) \geq \min\{\nu(x), \nu(y)\} \geq \alpha$, hence $xy \in \nu_\alpha$.
Again, by (GF3), $\nu(-x) = \nu(x) \geq \alpha$, so $-x \in \nu_\alpha$.
Further, (GF4), $\nu(1_{R[G]}) \geq \nu(x) \geq \alpha$, thus $1_{R[G]} \in \nu_\alpha$.
Therefore, ν_α is closed under addition, multiplication, and additive inverses, and contains the Identity element. Hence level set is a subring of $R[G]$. □

Theorem 3.13. *A fuzzy set $\nu : R[G] \to [0, 1]$ is a general fuzzy group ring if and only if:*

(1) *The family $\{\nu_\alpha : \alpha \in (0, 1]\}$ forms a descending chain of subrings of $R[G]$, i.e.,*

$$\alpha_1 \leq \alpha_2 \Rightarrow \nu_{\alpha_2} \subseteq \nu_{\alpha_1};$$

(2) *$\nu(0) = 1$;*
(3) *$\nu(g^{-1}) = \nu(g)$ for all $g \in G$.*

Moreover, in this case, $\nu(x) = \sup\{\alpha \in (0, 1] : x \in \nu_\alpha\}$.

Proof. ($\Rightarrow$) **Necessity** Assume that ν is a general fuzzy group ring. Then by **Theorem 3.12** each ν_α is a subring. Next, let $\alpha_1 \leq \alpha_2$ and $x \in \nu_{\alpha_2}$. Then $\nu(x) \geq \alpha_2 \geq \alpha_1$, so $x \in \nu_{\alpha_1}$. Hence $\nu_{\alpha_2} \subseteq \nu_{\alpha_1}$. From **Remark 3.2** we have $\nu(0) = \sup\{\nu(x) : x \in R[G]\} = 1$.
Further, $\nu(g^{-1}) = \nu(g)$ for all $g \in G$ follows from (GF5).
For any $x \in R[G]$, $x \in \nu_\alpha \Leftrightarrow \nu(x) \geq \alpha$. Hence, $\nu(x) = \sup\{\alpha \in (0, 1] : x \in \nu_\alpha\}$.

($\Leftarrow$) **Sufficient** Assume that conditions (1)–(3) hold. We prove that ν satisfies (GF1) to (GF5). **For (GF1) and (GF2)** Let $x, y \in R[G]$ and let $\alpha = \min\{\nu(x), \nu(y)\}$. Then $x, y \in \nu_\alpha$. Since each ν_α is a subring, closure under addition and multiplication implies that $x + y, xy \in \nu_\alpha$, i.e.,

$$\nu(x + y) \geq \alpha = \min\{\nu(x), \nu(y)\}, \quad \nu(xy) \geq \alpha = \min\{\nu(x), \nu(y)\}.$$

For (GF3) Let $x \in R[G]$ such that $\nu(x) = \alpha$. Then $x \in \nu_\alpha$. Since ν_α is closed under additive inverse, so $-x \in \nu_\alpha$ implies that $\nu(-x) \geq \alpha = \nu(x)$. Replacing x with $-x$, we get $\nu(x) \geq \nu(-x)$ gives that $\nu(-x) = \nu(x)$. Thus (GF3) holds.
For (GF4) Let $x \in R[G]$ and let $\alpha = \nu(x)$. Then $x \in \nu_\alpha$ and since ν_α is a subring, so it contain $1_{R[G]}$. Hence $\nu(1_{R[G]}) \geq \alpha = \nu(x)$. Thus (GF4) holds.
For (GF5) This follows from condition (3). Thus the fuzzy set ν satisfies (GF1)–(GF5), hence it is a general fuzzy group ring. ☐

Corollary 3.14. *Let $\nu : R[G] \to [0, 1]$ is a general fuzzy group ring. Then ν is completely determined by its family of level cut subrings $\{\nu_\alpha : \alpha \in (0, 1]\}$. In fact, for every $x \in R[G]$, $\nu(x) = \sup\{\alpha \in (0, 1] : x \in \nu_\alpha\}$.*

Proof. This is an immediate consequence of **Theorem 3.13**. Since each ν_α is a subring and the family is descending, knowing all the ν_α uniquely specifies which $x \in R[G]$ belong to which level, and hence uniquely reconstructs $\nu(x)$ for every $x \in R[G]$. ☐

Definition 3.15. Let $R[G]$ be a group ring, and let $\nu_1, \nu_2 : R[G] \to [0, 1]$ be two general fuzzy group rings on $R[G]$. The intersection of ν_1 and ν_2, denoted by $\nu = \nu_1 \cap \nu_2$, is defined as:

$$\nu(x) = \min\{\nu_1(x), \nu_2(x)\}, \forall x \in R[G].$$

Theorem 3.16. *The intersection of two general fuzzy group rings is again a general fuzzy group ring.*

Proof. Let $\nu_1, \nu_2 : R[G] \to [0, 1]$ be two general fuzzy group rings on $R[G]$.
Consider $x, y \in R[G], g \in G$ be any elements, then

$$\begin{aligned}
(\nu_1 \cap \nu_2)(x + y) &= \min\{\nu_1(x + y), \nu_2(x + y)\} \\
&\geq \min\{min\{\nu_1(x), \nu_1(y)\}, min\{\nu_2(x), \nu_2(y)\}\} \\
&= \min\{min\{\nu_1(x), \nu_2(x)\}, min\{\nu_1(y), \nu_2(y)\}\} \\
&= \min\{(\nu_1 \cap \nu_2)(x), (\nu_1 \cap \nu_2)(y)\}.
\end{aligned}$$

and

$$\begin{aligned}(\nu_1 \cap \nu_2)(xy) &= \min\{\nu_1(xy), \nu_2(xy)\}\\ &\geq \min\{min\{\nu_1(x), \nu_1(y)\}, min\{\nu_2(x), \nu_2(y)\}\}\\ &= \min\{min\{\nu_1(x), \nu_2(x)\}, min\{\nu_1(y), \nu_2(y)\}\}\\ &= \min\{(\nu_1 \cap \nu_2)(x), (\nu_1 \cap \nu_2)(y)\}.\end{aligned}$$

Also, $(\nu_1 \cap \nu_2)(-x) = \min\{\nu_1(-x), \nu_2(-x)\} = \min\{\nu_1(x), \nu_2(x)\} = (\nu_1 \cap \nu_2)(x)$. and $(\nu_1 \cap \nu_2)(1_{R[G]}) = \min\{\nu_1(1_{R[G]}), \nu_2(1_{R[G]})\} \geq \min\{\nu_1(x), \nu_2(x)\} = (\nu_1 \cap \nu_2)(x)$. Also, $(\nu_1 \cap \nu_2)(g^{-1}) = \min\{\nu_1(g^{-1}), \nu_2(g^{-1})\} = \min\{\nu_1(g), \nu_2(g)\} = (\nu_1 \cap \nu_2)(g)$. Hence $(\nu_1 \cap \nu_2)$ is a general fuzzy group ring. □

Theorem 3.17. *The α-level cut of intersection of two general fuzzy group rings is equal to the intersection of α-level cut of two general fuzzy group rings, i.e.,*

$$(\nu_1 \cap \nu_2)_\alpha = (\nu_1)_\alpha \cap (\nu_2)_\alpha, \text{ where } \alpha \in (0, 1].$$

Proof. Let $x \in R[G]$ be any element, then $x \in (\nu_1 \cap \nu_2)_\alpha \Leftrightarrow (\nu_1 \cap \nu_2)(x) \geq \alpha$ $\Leftrightarrow \min\{\nu_1(x), \nu_2(x)\} \geq \alpha \Leftrightarrow \nu_1(x) \geq \alpha$ and $\nu_2(x) \geq \alpha \Leftrightarrow x \in (\nu_1)_\alpha$ and $x \in (\nu_2)_\alpha$ $\Leftrightarrow x \in (\nu_1)_\alpha \cap (\nu_1)_\alpha$. Thus,

$$(\nu_1 \cap \nu_2)_\alpha = (\nu_1)_\alpha \cap (\nu_2)_\alpha.$$

□

Definition 3.18. The **support** of a general fuzzy group ring ν is defined by

$$\mathrm{Supp}(\nu) = \{x \in R[G] \mid \nu(x) > 0\}.$$

Theorem 3.19. *The* $\mathrm{Supp}(\nu)$ *of a general fuzzy group ring is a subring of $R[G]$.*

Proof. Let $x, y \in \mathrm{Supp}(\nu)$. Then $\nu(x) > 0$ and $\nu(y) > 0$.
By (GF1), $\nu(x+y) \geq \min\{\nu(x), \nu(y)\} > 0$. By (GF2), $\nu(xy) \geq \min\{\nu(x), \nu(y)\} > 0$. By (GF3), $\nu(-x) = \nu(x) > 0$. Also, by (GF4), $1_{R[G]} \in \mathrm{Supp}(\nu)$.
Thus the support is a subring. □

Next, we define the notion of quotient general fuzzy group ring:

Definition 3.20. Let $(R[G], \nu)$ be a general fuzzy group ring (GFGR) and let I be an ideal of the group ring $R[G]$. Suppose that $\nu : R[G] \to [0, 1]$ satisfies

$$\nu(x + k) = \nu(x), \quad \forall x \in R[G],\ k \in I.$$

Define a fuzzy set $\widetilde{\nu} : R[G]/I \to [0, 1]$ by $\widetilde{\nu}(x + I) = \nu(x), \quad x \in R[G]$.
Then $(R[G]/I, \widetilde{\nu})$ is called the **quotient general fuzzy group ring** of $(R[G], \nu)$ modulo I.

Proposition 3.21. *The fuzzy set $\widetilde{\nu}$ defined above is well-defined and makes $R[G]/I$ a general fuzzy group ring, i.e., $(R[G]/I, \widetilde{\nu})$ is a quotient general fuzzy group ring.*

Proof. First we show that $\widetilde{\nu}$ is well-defined. Suppose $x + I = y + I$.
Then $x - y \in I$, so $x = y + k$ for some $k \in I$. By the given condition, $\nu(x) = \nu(y + k) = \nu(y)$. Hence $\widetilde{\nu}(x + I) = \widetilde{\nu}(y + I)$. Thus $\widetilde{\nu}$ is well-defined.

Next we verify the GFGR properties. Let $x, y \in R[G], g \in G$. Then we have

(GF1)

$$\begin{aligned}\widetilde{\nu}((x+I)+(y+I)) &= \widetilde{\nu}(x+y+I)\\ &= \nu(x+y)\\ &\geq \min\{\nu(x),\nu(y)\}\\ &= \min\{\widetilde{\nu}(x+I),\widetilde{\nu}(y+I)\}.\end{aligned}$$

(GF2)

$$\begin{aligned}\widetilde{\nu}((x+I)(y+I)) &= \widetilde{\nu}(xy+I)\\ &= \nu(xy)\\ &\geq \min\{\nu(x),\nu(y)\}\\ &= \min\{\widetilde{\nu}(x+I),\widetilde{\nu}(y+I)\}.\end{aligned}$$

(GF3) $\widetilde{\nu}(-(x+I)) = \widetilde{\nu}(-x+I) = \nu(-x) = \nu(x) = \widetilde{\nu}(x+I)$.
(GF4) $\widetilde{\nu}(1_{R[G]}+I) = \nu(1_{R[G]}) \geq \nu(x) = \widetilde{\nu}(x+I)$.
(GF5) $\widetilde{\nu}(g^{-1}+I) = \nu(g^{-1}) = \nu(g) = \widetilde{\nu}(g+I)$.

Hence all the axioms of a general fuzzy group ring are satisfied. Therefore $(R[G]/I, \widetilde{\nu})$ is a quotient GFGR. □

Remark 3.22. The condition $\nu(x+k) = \nu(x)$ for all $x \in R[G]$ and $k \in I$ is necessary and sufficient for the fuzzy membership function ν to induce a well-defined fuzzy structure on the quotient ring $R[G]/I$.

Example 3.23. Let $R = \mathbb{Z}$ and G be a finite group. Consider the group ring $\mathbb{Z}[G]$ with a fuzzy membership function $\nu : \mathbb{Z}[G] \to [0,1]$ satisfying the axioms of a general fuzzy group ring. Let I be an ideal of $\mathbb{Z}[G]$ such that $\nu(x+k) = \nu(x)$ for all $x \in \mathbb{Z}[G]$ and $k \in I$. Then the induced function $\widetilde{\nu}(x+I) = \nu(x)$ defines a quotient general fuzzy group ring $(\mathbb{Z}[G]/I, \widetilde{\nu})$.
First we show that $\widetilde{\nu}$ is well-defined:
Let $x+I = y+I \Rightarrow y = x+k$ for some $k \in I$, and hence $\nu(y) = \nu(x)$.
Next, we verify for (GF1)-(GF5):
(GF1)

$$\widetilde{\nu}((x+I)+(y+I)) = \nu(x+y) \geq \min\{\nu(x),\nu(y)\} = \min\{\widetilde{\nu}(x+I),\widetilde{\nu}(y+I)\}.$$

(GF2)

$$\widetilde{\nu}((x+I)(y+I)) = \nu(xy) \geq \min\{\nu(x),\nu(y)\} = \min\{\widetilde{\nu}(x+I),\widetilde{\nu}(y+I)\}.$$

(GF3)

$$\widetilde{\nu}(-(x+I)) = \nu(-x) = \nu(x) = \widetilde{\nu}(x+I).$$

(GF4)

$$\widetilde{\nu}(1+I) = \nu(1) \geq \nu(x) = \widetilde{\nu}(x+I).$$

(GF5) For $g \in G$,

$$\widetilde{\nu}(g^{-1}+I) = \nu(g^{-1}) = \nu(g) = \widetilde{\nu}(g+I).$$

Hence all GF axioms are inherited from ν, and $(\mathbb{Z}[G]/I, \widetilde{\nu})$ is a quotient general fuzzy group ring.

3.3. **Homomorphisms of General Fuzzy Group Rings.** In this section, the impact of general fuzzy group ring under GFGR-homomorphism is analyzed, followed by the development of isomorphism theorems. Throughout this section, a general fuzzy group ring ν on $R[G]$ by $(R[G], \nu)$.

Definition 3.24. Let $(R[G], \nu_1)$ and $(S[H], \nu_2)$ be general fuzzy group rings. Then a map $\phi : (R[G], \nu_1) \to (S[H], \nu_2)$ is called a GFGR-homomorphism if:

(1) $\phi : R[G] \to S[H]$ is a unital ring homomorphism (i.e., ϕ is a ring homomorphism with identity preserving map) and
(2) $\nu_2(\phi(x)) \geq \nu_1(x)$ for all $x \in R[G]$.

To avoid confusion, we denote a GFGR-homomorphism by $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$, while its underlying ring-homomorphism is written as $\phi : R[G] \to S[H]$.

Remark 3.25. (1) A GFGR-homomorphism $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ is called a GFGR
-monomorphism from $(R[G], \nu_1)$ to $(S[H], \nu_2)$ if $\phi : R[G] \to S[H]$ is an injective;
(2) A GFGR-homomorphism $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ is called a GFGR-epimorphism from $(R[G], \nu_1)$ to $(S[H], \nu_2)$ if $\phi : R[G] \to S[H]$ is a surjective;
(3) A GFGR-homomorphism $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ is called a GFGR-isomorphism from $(R[G], \nu_1)$ to $(S[H], \nu_2)$ if $\phi : R[G] \to S[H]$ is a bijection.

Remark 3.26. (1) For all $x \in R[G]$, $\nu_2(\phi(x)) = \sup\{\nu_1(\phi^{-1}(y)) : y = \phi(x)\}$ if the mapping $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ is a GFGR-homomorphism;
(2) For all $x \in R[G]$, $\nu_2(\phi(x)) = \nu_1(x)$ if the mapping $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ is a GFGR-isomorphism.

Example 3.27. The identity map $\bar{i} : (R[G], \nu_1) \to (R[G], \nu_1)$ defined by $i(x) = x, \forall x \in R[G]$ is a GFGR-homomorphism, for $\nu_1(i(x)) = \nu_1(x) \geq \nu_1(x)$.

Example 3.28. Let $R = \mathbb{Z}_4, S = \mathbb{Z}_2$, $G = \{e, g, g^2\}$ with $g^3 = e, H = \{e, h, h^2\}$ with $h^3 = e$.
Define the natural ring homomorphism $\pi : R \to S$ by $\pi(a) = a(mod 2) = \overline{a}$, for $a \in R$ and the group homomorphism $\theta : G \to H$ by $\theta(g) = h$. Extend these to a group ring homomorphism

$$\phi : R[G] \to S[H] \quad \text{by} \quad \phi(a + bg + cg^2) = \overline{a} + \overline{b}h + \overline{c}h^2.$$

Now, it is easy to verify that ϕ is a unital ring homomorphism.
Define fuzzy sets $\nu_1 : R[G] \to [0, 1]$ and $\nu_2 : S[H] \to [0, 1]$ by

$$\nu_1(a + bg + cg^2) = \begin{cases} 1, & \text{if } a = b = c = 0, \\ 0.8, & \text{if } b = c = 0,\ a \neq 0, \\ 0.5, & \text{if otherwise,} \end{cases}$$

$$\nu_2(a + bh + ch^2) = \begin{cases} 1, & \text{if } a = b = c = 0, \\ 0.9, & \text{if } b = c = 0,\ a \neq 0, \\ 0.6, & \text{if otherwise.} \end{cases}$$

Let $x = a + bg + cg^2 \in R[G]$. Then $\phi(a + bg + cg^2) = \overline{a} + \overline{b}h + \overline{c}h^2$.
Case(1): When $b = c = 0$, then $x = a$, $\nu_1(x) = 0.8$ and $\phi(x) = \overline{a}$.

$$\text{So, } \nu_2(\phi(x)) = 0.9. \text{ Thus } \nu_2(\phi(x)) \geq \nu_1(x).$$

Case(2): When otherwise, then $\nu_1(x) = 0.5$ and $\nu_2(\phi(x)) = 0.6$. Thus $\nu_2(\phi(x)) \geq \nu_1(x)$.
Hence the induced mapping $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ is a GFGR-homomorphism.

Theorem 3.29. *Let G be a group and N a normal subgroup of G. Let $\psi : G \to G/N$ be the natural group homomorphism defined by $\psi(g) = gN$ for all $g \in G$. Let $\phi : R[G] \to R[G/N]$ be the group ring homomorphism induced by ψ.*

If ν_1 is a general fuzzy group rings on $R[G]$ and ν_2 is a general fuzzy group ring on $R[G/N]$ defined by $\nu_2(y) = \sup\{\nu_1(x) | \quad \phi(x) = y\}, \quad y \in R[G/N]$, then the mapping $\overline{\phi} : (R[G], \nu_1) \to (R[G/N], \nu_2)$ is a GFGR-homomorphism.

Proof. Since N is a normal subgroup of G, the natural mapping $\psi : G \to G/N, \psi(g) = gN$ is a surjective group homomorphism. Hence it induces a group ring homomorphism
$\phi : R[G] \to R[G/N]$ defined by linear extension

$$\phi\big(\sum_{g \in G} a_g g\big) = \sum_{g \in G} a_g \psi(g) = \sum_{g \in G} a_g (gN).$$

Thus ϕ is a ring homomorphism.

Let $x \in R[G]$. By definition of ν_2, $\nu_2(\phi(x)) = \sup\{\nu_1(z) | \quad \phi(z) = \phi(x)\}$.

Since x belongs to the set $\{z \in R[G] | \quad \phi(z) = \phi(x)\}$, it follows that $\nu_2(\phi(x)) \geq \nu_1(x)$.

Hence the fuzzy compatibility condition holds and therefore, $\overline{\phi} : (R[G], \nu_1) \to (R[G/N], \nu_2)$ is a GFGR-homomorphism.

□

Proposition 3.30. *The composite of two general fuzzy group ring homomorphisms is again a general fuzzy group homomorphism.*

Proof. Let $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ and $\bar{\psi} : (S[H], \nu_2) \to (T[K], \nu_3)$ be two general fuzzy group ring homomorphisms. Then $\phi : R[G] \to S[H]$ and $\psi : S[H] \to T[K]$ are unital ring-homomorphisms. Which implies $\psi \circ \phi : R[G] \to T[K]$ is also a unital ring homomorphism. Further, $\nu_3((\psi \circ \phi)(x)) = \nu_3(\psi(\phi(x))) \geq \nu_2(\phi(x)) \geq \nu_1(x)$. Hence $\overline{\psi \circ \phi} : (R[G], \nu_1) \to (T[K], \nu_3)$ is a general fuzzy group ring homomorphism. □

Definition 3.31. [5] A *category* $\mathcal{C}$ consists of the following data:

- A class of objects, denoted by $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of morphisms (or arrows) $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For each triple of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

assigning to each pair (g, f) a composite $g \circ f$.

- For each object $A \in \mathrm{Ob}(\mathcal{C})$, an identity morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

These satisfy the following axioms:

(1) (Associativity) For all morphisms $f : A \to B$, $g : B \to C$, and $h : C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(2) (Identity) For every morphism $f : A \to B$,

$$1_B \circ f = f = f \circ 1_A.$$

Proposition 3.32. *The collection of all general fuzzy group rings together with GFGR-homomorphisms forms a category.*

Proof. Let **GFGR** denote the class whose objects are all general fuzzy group rings $(R[G], \nu)$, where $R[G]$ is a group ring and $\nu : R[G] \to [0, 1]$ is a general fuzzy group ring. The morphisms are GFGR-homomorphisms as defined in **Definition 3.24**; that is, a mapping

$$\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$$

such that:

(i) $\phi : R[G] \to S[H]$ is a unital ring homomorphism,
(ii) $\nu_2(\phi(x)) \geq \nu_1(x)$ for all $x \in R[G]$.

For every object $(R[G], \nu)$, the identity mapping

$$\bar{i} : (R[G], \nu) \to (R[G], \nu), \quad i(x) = x,$$

is a GFGR-homomorphism since it is a unital ring homomorphism and satisfies $\nu(i(x)) = \nu(x)$ for all $x \in R[G]$ (by **Example 3.27**). Hence, identity morphisms exist.

Let

$$\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2), \quad \bar{\psi} : (S[H], \nu_2) \to (T[K], \nu_3)$$

be GFGR-homomorphisms. Then $\psi \circ \phi : R[G] \to T[K]$ is a unital ring homomorphism, and for all $x \in R[G]$,

$$\nu_3((\psi \circ \phi)(x)) = \nu_3(\psi(\phi(x))) \geq \nu_2(\phi(x)) \geq \nu_1(x).$$

Thus, $\overline{\psi \circ \phi}$ is also a GFGR-homomorphism. Therefore, morphisms are closed under composition (by **Proposition 3.30**).

Finally, associativity of composition follows from the associativity of function composition of the underlying ring homomorphisms, and the identity laws

$$\bar{i} \circ \bar{\phi} = \bar{\phi}, \quad \bar{\psi} \circ \bar{i} = \bar{\psi}$$

hold for all morphisms.

Hence, all category axioms are satisfied, and the collection of all general fuzzy group rings together with GFGR-homomorphisms forms a category. □

Theorem 3.33. *Let $(R[G], \nu_1)$ and $(S[H], \nu_2)$ be general fuzzy group rings, and let $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ be a surjective GFGR-homomorphism. Then:*

(1) *$\phi(\nu_1)$ is a general fuzzy group ring on $S[H]$.*
(2) *$\phi^{-1}(\nu_2)$ is a general fuzzy group ring on $R[G]$.*

Proof. Let $\phi(\nu_1)(y) = \sup\{\nu_1(x) \mid \phi(x) = y\}$ for $y \in S[H]$, and $\phi^{-1}(\nu_2)(x) = \nu_2(\phi(x))$ for $x \in R[G]$.

(1) $\phi(\nu_1)$ **is a GFGR:**
For all $y_1, y_2 \in S[H]$, there exists $x_1, x_2 \in R[G]$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Using the GFGR axioms of ν_1:

$$\begin{aligned}
\phi(\nu_1)(y_1+y_2) &= \sup\{\nu_1(x) \mid \ \phi(x) = y_1+y_2\} \\
&= \sup\{\nu_1(x) \mid \ \phi(x) = \phi(x_1)+\phi(x_2) = \phi(x_1+x_2)\} \\
&= \sup\{\nu_1(x_1+x_2) \mid \ \phi(x_1+x_2) = y_1+y_2\} \\
&\geq \sup\{\min\{\nu_1(x_1), \nu_1(x_2)\} \mid \ \phi(x_1) = y_1 \text{ and } \phi(x_2) = y_2\} \\
&= \min\{\sup\{\nu_1(x_1) \mid \ \phi(x_1) = y_1\}, \sup\{\nu_1(x_2) \mid \ \phi(x_2) = y_2\}\} \\
&= \min\{\phi(\nu_1)(y_1), \phi(\nu_1)(y_2)\}.
\end{aligned}$$

Similarly, we can show that $\phi(\nu_1)(y_1y_2) \geq \min\{\phi(\nu_1)(y_1), \phi(\nu_1)(y_2)\}$.
Also, $\phi(\nu_1)(-y_1) = \phi(\nu_1)(y_1)$, $\phi(\nu_1)(1_{S[H]}) \geq \phi(\nu_1)(y_1)$ and $\phi(\nu_1)(g^{-1}) = \phi(\nu_1)(g)$.
Hence, $\phi(\nu_1)$ satisfies (GF1)–(GF5).
Hence $\phi(\nu_1)$ is a general fuzzy group ring on $S[H]$.

(2) $\phi^{-1}(\nu_2)$ **is a GFGR:**
For all $x, y \in R[G]$, using the GFGR axioms of ν_2:

$$\begin{aligned}
\phi^{-1}(\nu_2)(x+y) &= \nu_2(\phi(x+y)) \\
&= \nu_2(\phi(x)+\phi(y)) \\
&\geq \min\{\nu_2(\phi(x)), \nu_2(\phi(y))\} \\
&= \min\{\phi^{-1}(\nu_2)(x), \phi^{-1}(\nu_2)(y)\}.
\end{aligned}$$

Similarly, we can show that $\phi^{-1}(\nu_2)(xy) \geq \min\{\phi^{-1}(\nu_2)(x), \phi^{-1}(\nu_2)(y)\}$.
Also, $\phi^{-1}(\nu_2)(-x) = \phi^{-1}(\nu_2)(x)$, $\phi^{-1}(\nu_2)(1_{R[G]}) \geq \phi^{-1}(\nu_2)(x)$ and $\phi^{-1}(\nu_2)(g^{-1}) = \phi^{-1}(\nu_2)(g)$.
Thus, $\phi^{-1}(\nu_2)$ also satisfies (GF1)–(GF5).
Hence $\phi^{-1}(\nu_2)$ is a general fuzzy group ring on $R[G]$. □

Definition 3.34. Let $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ be a GFGR-homomorphism as defined in **Definition 3.24**, then the **kernel of** $\bar{\phi}$ is denoted by $ker(\bar{\phi})$ and defined as the fuzzy set obtained from ν_1 on restriction to $ker(\phi)$, i.e., $ker(\bar{\phi}) = \nu_1|_{ker(\phi)}$.

Theorem 3.35. *Let $\bar{\phi} : (R[G], \nu_1) \to (S[H], \nu_2)$ be a GFGR-homomorphism as defined in **Definition 3.24**. Then:*

(1) *The $ker(\bar{\phi})$ is a fuzzy ideal of $R[G]$;*
(2) *If $\bar{\phi}$ is injective, then $\nu_1(x) \leq (\nu_2 \circ \phi)(x)$ for all $x \in R[G]$.*
(3) *Suppose $\bar{\phi}$ is surjective and that $\nu_1(x+k) = \nu_1(x)$ for all $x \in R[G]$, $k \in ker(\phi)$. Then there exists a unique fuzzy set ν_2 on $S[H]$ such that $\nu_2(y) = \nu_1(x)$ whenever $\phi(x) = y$.*

Proof. **1**. By definition $ker(\bar{\phi}) = \nu_1|_{ker(\phi)}$, where $ker(\phi) = \{x \in K[G] : \phi(x) = 0\}$.
Since ϕ is a ring-homomorphism, $ker(\phi)$ is an ordinary ideal of $K[G]$.
Let $x, y \in ker(\phi)$ Then
$\phi(x+y) = \phi(x)+\phi(y) = 0 \Rightarrow x+y \in ker(\phi)$ and $\phi(rx) = \phi(xr) = 0$ for all $r \in R[G]$.
Using (GF1)-(GF3) for ν_1:

$\nu_1(x+y) \geq \min\{\nu_1(x), \nu_1(y)\}$; $\nu_1(rx) \geq \min\{\nu_1(r), \nu_1(x)\}$ and $\nu_1(-x) = \nu_1(x)$. Taking restriction $\nu_1|_{ker(\phi)}$ we have

$$ker(\bar{\phi})(x+y) \geq \min\{ker(\bar{\phi})(x), ker(\bar{\phi})(y)\}.$$

$$ker(\bar{\phi})(rx) \geq \min\{ker(\bar{\phi})(r), ker(\bar{\phi})(x)\} \text{ and}$$
$$ker(\bar{\phi})(xr) \geq \min\{ker(\bar{\phi})(x), ker(\bar{\phi})(r)\}.$$

$ker(\bar{\phi})(-x) = ker(\bar{\phi})(x)$. Hence $ker(\bar{\phi})$ is a fuzzy ideal of $R[G]$.

2. From **Remark 3.25 (1)** $\bar{\phi}$ is injective implies ϕ is injective. As $\bar{\phi}$ is a GFGR-homomorphism, we have $\nu_2(\phi(x)) \geq \nu_1(x)$ for all $x \in R[G]$. But $(\nu_2 \circ \phi)(x) = \nu_2(\phi(x))$.
Thus $\nu_1(x) \leq (\nu_2 \circ \phi)(x)$ for all $x \in R[G]$.

3. From **Remark 3.25 (2)** $\bar{\phi}$ is surjective. Let $y \in S[H]$, choose $x \in R[G]$ such that $\phi(x) = y$ and define $\nu_2(y) = \nu_1(x)$.

ν_2 **is Well defined:** Suppose that $\phi(x_1) = \phi(x_2) = y$. Then $\phi(x_1 - x_2) = \phi(x_1) - \phi(x_2) = 0$. So $x_1 - x_2 \in ker(\phi)$. Hence $x_1 = x_2 + k$ for some $k \in ker(\phi)$. By given hypothesis $\nu_1(x_1) = \nu_1(x_2 + k) = \nu_1(x_2)$. Hence ν_2 is well-defined.

For Uniqueness: Let μ be another fuzzy set on $S[H]$ satisfying $\mu(y) = \nu_1(x)$, whenever $\phi(x) = y$. Since ϕ is surjective, every $y \in S[H]$ arises in this way. Therefore, $\mu(y) = \nu_2(y), \forall y \in S[H]$. Thus $\mu = \nu_2$, proving uniqueness. □

Theorem 3.36. *Let $\bar{\phi} : (R[G], \nu_1) \longrightarrow (S[H], \nu_2)$ be a surjective GFGR-homomorphism, and let $\phi : R[G] \to S[H]$ be the underlying unital ring homomorphism. Suppose that*

$$\nu_1(x+k) = \nu_1(x) \quad \textit{for all } x \in R[G],\ k \in \ker(\phi). \tag{3.1}$$

Then

(1) *There exists a well-defined fuzzy set $\widetilde{\nu} : R[G]/\ker(\phi) \to [0,1]$ given by*
$$\widetilde{\nu}(x + \ker(\phi)) = \nu_1(x),$$
which makes $(R[G]/\ker(\phi), \widetilde{\nu})$ a general fuzzy group ring.

(2) *The induced mapping $\widetilde{\psi} : (R[G]/\ker(\phi), \widetilde{\nu}) \longrightarrow (S[H], \nu_2)$ defined by*
$$\widetilde{\psi}(x + \ker(\phi)) = \phi(x)$$
is a GFGR-isomorphism.

Consequently, $(R[G]/\ker(\phi), \widetilde{\nu}) \cong (S[H], \nu_2)$ as general fuzzy group rings.

Proof. **For (1)** To show that the fuzzy set $\widetilde{\nu} : R[G]/\ker(\phi) \to [0,1]$ defined by

$$\widetilde{\nu}(x + \ker(\phi)) = \nu_1(x)$$

is a general fuzzy group ring on $R[G]/\ker(\phi)$. We first show that $\widetilde{\nu}$ is well-defined. Suppose $x + \ker(\phi) = y + \ker(\phi)$. Then $x - y \in \ker(\phi)$, so $x = y + k$ for some $k \in \ker(\phi)$. By **assumption** (3.1), $\nu_1(x) = \nu_1(y+k) = \nu_1(y) \Rightarrow \widetilde{\nu}(x + \ker(\phi)) = \widetilde{\nu}(y + \ker(\phi))$.

Hence $\widetilde{\nu}$ is well-defined.

Let $x, y \in R[G], g \in G$. Then we have
(GF1)
$$\begin{aligned}
\widetilde{\nu}((x+\ker(\phi))+(y+\ker(\phi))) &= \widetilde{\nu}(x+y+\ker(\phi)) \\
&= \nu_1(x+y) \\
&\geq \min\{\nu_1(x), \nu_1(y)\} \\
&= \min\{\widetilde{\nu}(x+\ker(\phi)), \widetilde{\nu}(y+\ker(\phi))\}.
\end{aligned}$$
(GF2)
$$\begin{aligned}
\widetilde{\nu}((x+\ker(\phi))(y+\ker(\phi))) &= \widetilde{\nu}(xy+\ker(\phi)) \\
&= \nu_1(xy) \\
&\geq \min\{\nu_1(x), \nu_1(y)\} \\
&= \min\{\widetilde{\nu}(x+\ker(\phi)), \widetilde{\nu}(y+\ker(\phi))\}.
\end{aligned}$$
(GF3) $\widetilde{\nu}(-(x+\ker(\phi))) = \widetilde{\nu}(-x+\ker(\phi)) = \nu_1(-x) = \nu_1(x) = \widetilde{\nu}(x+\ker(\phi))$.
(GF4) $\widetilde{\nu}(1_{R[G]}+\ker(\phi)) = \nu_1(1_{R[G]}) \geq \nu_1(x) = \widetilde{\nu}(x+\ker(\phi))$.
(GF5) $\widetilde{\nu}(g^{-1}+\ker(\phi)) = \nu_1(g^{-1}) = \nu_1(g) = \widetilde{\nu}(g+\ker(\phi))$.

Thus $(R[G]/\ker(\phi), \widetilde{\nu})$ is a general fuzzy group ring.

For (2) Define $\widetilde{\psi} : (R[G]/\ker(\phi), \widetilde{\nu}) \longrightarrow (S[H], \nu_2)$ by $\widetilde{\psi}(x+\ker(\phi)) = \phi(x)$. If $x+\ker(\phi) = y+\ker(\phi)$, then $x-y \in \ker(\phi)$ and so $\phi(x) = \phi(y)$ which imply $\widetilde{\psi}$ is well-defined.

$$\begin{tikzcd}
(R[G], \nu_1) \arrow[r, "\bar{\phi}"] \arrow[d, "\widetilde{\varepsilon}"] & (S[H], \nu_2) \\
(R[G]/\ker(\phi), \widetilde{\nu}) \arrow[ur, dashed, "\widetilde{\psi}"'] &
\end{tikzcd}$$

Since ϕ is surjective, $\bar{\phi}$ is surjective. If $\widetilde{\psi}(x+\ker(\phi)) = 0$, then $\phi(x) = 0$, so $x \in \ker(\phi)$, hence $x+\ker(\phi) = 0+\ker(\phi)$. Thus $\widetilde{\psi}$ is injective.

Finally, $\nu_2(\widetilde{\psi}(x+\ker(\phi))) = \nu_2(\phi(x)) \geq \nu_1(x) = \widetilde{\nu}(x+\ker(\phi))$.
Since $\widetilde{\psi}$ is bijective, it is a GFGR-isomorphism. □

Theorem 3.37 (**Second Isomorphism Theorem for GFGR).** *Let $(R[G], \nu)$ be a general fuzzy group ring and let I and J be ideals of $R[G]$. Assume that the fuzzy membership function ν is invariant under addition by elements of J, that is, $\nu(x+k) = \nu(x), \quad \forall x \in R[G],\ k \in J$.*
Then the following statements hold:

(1) *$(I+J)/J$ is an ideal of the quotient ring $R[G]/J$.*
(2) *There exists a ring isomorphism $\Psi : (I+J)/J \longrightarrow I/(I \cap J)$ defined by*
$$\Psi((a+b)+J) = a+(I \cap J), \qquad a \in I, b \in J.$$
(3) *The induced fuzzy group ring structures on the quotient rings are given by*

$\nu_{(I+J)/J}(x+J) = \nu(x), x \in I+J$ *and* $\nu_{I/(I\cap J)}(x+(I\cap J)) = \nu(x), x \in I$.

Under these quotient general fuzzy group rings, the mapping Ψ is a GFGR-isomorphism.

Proof. Since I and J are ideals of $R[G]$, their sum $I+J$ is also an ideal of $R[G]$. Hence $(I+J)/J$ is an ideal of the quotient ring $R[G]/J$.
Define $\Psi : (I+J)/J \longrightarrow I/(I\cap J)$ by $\Psi((a+b)+J) = a+(I\cap J)$, $a \in I, b \in J$.
Suppose $(a_1+b_1)+J = (a_2+b_2)+J$ with $a_1, a_2 \in I, b_1, b_2 \in J$. Then $(a_1+b_1)-(a_2+b_2) \in J$, which implies $a_1 - a_2 \in J$. Since $a_1 - a_2 \in I$ as well, we have $a_1 - a_2 \in I \cap J$. Thus $a_1 + I\cap J = a_2 + I \cap J$. Hence Ψ is well-defined.
Next, let $a_i \in I, b_i \in J$ $(i = 1, 2)$

$$\begin{aligned}
\Psi\big(((a_1+b_1)+(a_2+b_2))+J\big) &= \Psi\big((a_1+a_2+b_1+b_2)+J\big)\\
&= (a_1+a_2)+(I\cap J)\\
&= (a_1+(I\cap J))+(a_2+(I\cap J))\\
&= \Psi\big((a_1+b_1)+J\big)+\Psi\big((a_2+b_2)+J\big)
\end{aligned}$$

$$\begin{aligned}
\Psi\big((a_1+b_1)(a_2+b_2)+J\big) &= \Psi\big((a_1a_2+a_1b_2+b_1a_2+b_1b_2)+J\big)\\
&= (a_1a_2)+(I\cap J)[\because a_1b_2, b_1a_2, b_1b_2 \in J]\\
&= (a_1+(I\cap J))(a_2+(I\cap J))\\
&= \Psi\big((a_1+b_1)+J\big)\Psi\big((a_2+b_2)+J\big).
\end{aligned}$$

Hence Ψ is a ring homomorphism. The mapping Ψ is clearly surjective.
If $\Psi((a+b)+J) = I\cap J$, then $a \in I \cap J$. Hence $(a+b)+J = J$, which implies that kernel is trivial. Therefore Ψ is injective and hence is a ring isomorphism.
Define $\nu_{(I+J)/J}(x+J) = \nu(x), x \in I+J$ and $\nu_{I/(I\cap J)}(x+I\cap J) = \nu(x), x \in I$.
If $x+J = y+J$, then $x-y \in J$. By the invariance condition we have $\nu(x) = \nu(y)$.
Thus the quotient fuzzy sets are well defined. Finally

$$\nu_{I/(I\cap J)}(\Psi((a+b)+J)) = \nu(a) = \nu(a+b) = \nu_{(I+J)/J}((a+b)+J).$$

Hence Ψ preserves the fuzzy membership values and therefore is a GFGR-isomorphism. □

Theorem 3.38 (**Third Isomorphism Theorem for GFGR**). *Let $(R[G], \nu)$ be a general fuzzy group ring and let I and J be ideals of $R[G]$ such that $I \subseteq J$. Assume that the fuzzy membership function ν is invariant under addition by elements of J, that is,*

$$\nu(x+k) = \nu(x), \qquad \forall x \in R[G],\ k \in J.$$

Define the quotient fuzzy sets

$$\nu_{R[G]/I}(x+I) = \nu(x), x \in R[G], \text{ and } \nu_{J/I}(x+I) = \nu(x), x \in J.$$

Then

(1) *$(J/I, \nu_{J/I})$ is a general fuzzy group ring and an ideal of the GFGR $(R[G]/I, \nu_{R[G]/I})$*
(2) *The quotient $\big((R[G]/I)/(J/I), \widetilde{\nu}\big)$ is a general fuzzy group ring, where*

$$\widetilde{\nu}\big((x+I)+(J/I)\big) = \nu(x).$$

(3) *The mapping* $\Phi : (R[G]/I)/(J/I) \longrightarrow R[G]/J$ *defined by*

$$\Phi\big((x+I)+(J/I)\big) = x+J.$$

is a GFGR-isomorphism.
Consequently, $((R[G]/I)/(J/I), \widetilde{\nu}) \cong ((R[G]/J, \nu_{R[G]/J})$ *as general fuzzy group rings.*

Proof. Since $I \subseteq J$, the quotient J/I is an ideal of the quotient ring $R[G]/I$. The invariance condition ensures that the quotient fuzzy sets $\nu_{R[G]/I}$ and $\nu_{J/I}$ are well-defined.

Define $\Phi : (R[G]/I)/(J/I) \longrightarrow R[G]/J$ by $\Phi\big((x+I)+(J/I)\big) = x+J$. Suppose $(x+I)+(J/I) = (y+I)+(J/I)$. Then $x-y \in J$, which implies $x+J = y+J$. Hence Φ is well-defined. Next, let $x, y \in R[G]$. Then

$$\begin{aligned}\Phi\big((x+I)+(J/I)+(y+I)+(J/I)\big) &= \Phi\big((x+y+I)+(J/I)\big)\\ &= x+y+J\\ &= (x+J)+(y+J)\\ &= \Phi\big((x+I)+(J/I)\big)+\Phi\big((y+I)+(J/I)\big)\end{aligned}$$

$$\begin{aligned}\Phi\big(((x+I)+(J/I))((y+I)+(J/I))\big) &= \Phi\big(((xy+I)+(J/I))\big)\\ &= xy+J\\ &= (x+J)(y+J)\\ &= \Phi\big((x+I)+(J/I)\big)\Phi\big((y+I)+(J/I)\big).\end{aligned}$$

Thus Φ is a ring homomorphism. The mapping Φ is clearly surjective. Its kernel is given by

$$Ker(\Phi) = \{(x+I)+(J/I) | x \in J\} = J/I.$$

For the fuzzy membership functions we obtain

$$\nu_{R[G]/J}\big(\Phi((x+I)+(J/I))\big) = \nu(x) = \widetilde{\nu}((x+I)+(J/I)).$$

Therefore Φ preserves the fuzzy structure and hence is a GFGR-isomorphism. □

4. Induced Fuzzy Group Rings

Now, the notion of fuzzy group ring $R[\mu]$, induced by a fuzzy subgroup μ on G, is defined.

Definition 4.1. Let G be a finite group and $\mu : G \to [0,1]$ a fuzzy subgroup on G. The fuzzy group ring induced by μ is denoted by $R[\mu]$ is a fuzzy set $R[\mu] : R[G] \to [0,1]$ defined by

$$R[\mu](x) = \begin{cases} \min\{\mu(g) : g \in supp(x)\}, & \text{if } supp(x) \neq \varnothing \\ 1, & \text{if } supp(x) = \varnothing. \end{cases}$$

where $x = \sum_{g\in G} r_g g \in R[G]$, $supp(x) = \{g \in G : r_g \neq 0\}$.
In short we call $R[\mu]$ as an **induced fuzzy group ring (IFGR)**.

Remark 4.2. Note that in $R[\mu]$ coefficients do not influence membership.

Theorem 4.3. *$R[\mu]$ is a fuzzy subring of $R[G]$.*

Proof. Let $x = \sum_{g\in G} r_g g$ and $y = \sum_{g\in G} s_g g$ be any two element of the group ring $R[G]$.

$$\begin{aligned}
R[\mu](x+y) &= R[\mu]\left[\sum_{g\in G} r_g g + \sum_{g\in G} s_g g\right] = R[\mu]\left[\sum_{g\in G}(r_g+s_g)g\right]\\
&= \min\{\mu(g) : g \in supp(x+y)\}[\because supp(x+y) \subseteq supp(x) \cup supp(y)]\\
&\geq \min\{\min\{\mu(g) : g \in supp(x)\}, \min\{\mu(g) : g \in supp(y)\}\}\\
&= \min\{R[\mu](x), R[\mu](y)\}.
\end{aligned}$$

Also, for $x = \sum\limits_{g\in G} r_g g, y = \sum\limits_{h\in G} s_h h \in R[G]$, we have

$$\begin{aligned}
R[\mu](xy) &= R[\mu]\left[\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} s_h h\right)\right]\\
&= R[\mu]\left[\sum_{t\in G}\left(\sum_{g\in G}(r_g s_{g^{-1}t})\right)t\right], \text{ where } t = gh\\
&= R[\mu]\left[\sum_{t\in G} m_t t\right] \text{, where } m_t = \sum_{g\in G}(r_g s_{g^{-1}t})\\
&= \min\{\mu(t) : t \in supp(xy)\}\\
&= \min\{\mu(gg^{-1}t) : t \in supp(xy)\}\\
&\geq \left[\min\{\min\{\mu(g), \mu(g^{-1}t)\} : t \in supp(xy)\}\right]\\
&= [\min\{\min\{\mu(g), \mu(h)\} : g \in supp(x), h \in supp(y)] [\because of *]\\
&\geq \min\{\min\{\mu(g) : g \in supp(x)\}, \min\{\mu(h) : h \in supp(y)\}\}\\
&= \min\{R[\mu](x), R[\mu](y)\}.
\end{aligned}$$

[* Since $supp(xy) \subseteq supp(x)supp(y)$. This means that if $t \in supp(xy)$, then $m_t \neq 0$, so there exists $g \in G$ such that $r_g \neq 0$ and $s_{g^{-1}t} \neq 0$, Hence $g \in supp(x)$ and $h \in supp(y)$].
This concludes the proof that fuzzy group ring is a fuzzy subring of $R[G]$. □

Theorem 4.4. *Every induced fuzzy group ring is a general fuzzy group ring. But the converse need not be true.*

Example 4.5. Let $R = \mathbb{R}$. $G = \mathbb{Z}$ group of integers under addition. Define the fuzzy set $\nu : R[G] \to [0,1]$ by

$$\nu\left(\sum_{g\in G} r_g g\right) = \min\{|r_g||r_g \neq 0\} \text{ and } \nu(0) = 1.$$

We verify (GF1)-(GF4):

(GF1) Let $x = \sum r_g g$ and $y = \sum s_g g$. For $x + y = \sum(r_g + s_g)g$, the smallest nonzero coefficient magnitude is controlled by those of x and y, hence

$$\nu(x+y) \geq \min\{\nu(x), \nu(y)\}.$$

(GF2) For $xy = \sum r_g s_h(g+h)$, coefficients are products $r_g s_h$, whose magnitudes preserve the minimal bound, hence

$$\nu(xy) \geq \min\{\nu(x), \nu(y)\}.$$

(GF3) Since $|-r_g| = |r_g|$, we have $\nu(-x) = \nu(x)$.

(GF4) For $1_{\mathbb{R}[G]} = 1 \cdot 0$, we have $\nu(1) = 1$, hence

$$\nu(1_{\mathbb{R}[G]}) \geq \nu(x) \quad \forall x \in \mathbb{R}[G].$$

Thus ν satisfies (GF1)-(GF4) but depends on coefficients. It cannot arise from any fuzzy subgroup of G. Hence IFGR $\subsetneqq$ GFGR.

Example 4.6. Let $R = \mathbb{R}$ be the ring of real numbers and $G = \{1, a, a^2\}$ with $a^3 = 1$ be a cyclic group of order 3. Consider the fuzzy set μ on G defined by

$$\mu(g) = \begin{cases} 1, & \text{if } g = 1 \\ 0.6, & \text{if } g = a, a^2 \end{cases}$$

It is straightforward to verify that μ is a fuzzy subgroup on G. Also, the fuzzy set $R[\mu] : R[G] \to [0,1]$ defined by

$$R[\mu](x) = \begin{cases} 1, & \text{if } supp(x) = \varnothing \text{ or } supp(x) = \{1\} \\ 0.6, & \text{otherwise .} \end{cases}$$

where $x = r_1.1 + r_2.a + r_3.a^2 \in R[G]$ is an induced fuzzy group ring $R[\mu]$ on $R[G]$.

Example 4.7. Let $R = \mathbb{R}$ and $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ be the symmetric group on three elements. Let $A_3 = \{e, (123), (132)\}$ be the alternating subgroup of S_3.
Define a fuzzy set μ on G by

$$\mu(g) = \begin{cases} 1, & \text{if } g = e \\ 0.8, & \text{if } g \in \{(123), (132)\} \\ 0.5, & \text{if } g \in \{(12), (13), (23)\}. \end{cases}$$

It is straightforward to check that μ is a fuzzy subgroup on G.
For $x = \sum_{g \in S_3} r_g g \in R[G]$. The fuzzy set $R[\mu]$ on $R[G]$ is given by

$$R[\mu](x) = \begin{cases} 1, & \text{if } supp(x) = \varnothing \text{ or } \{e\} \\ 0.8, & \text{if } supp(x) \subseteq A_3 \text{ and } supp(x) \neq \{e\} \\ 0.5, & \text{if } supp(x) \text{ contains a transposition.} \end{cases}$$

Now, it is easy to verify that $R[\mu]$ defines a fuzzy group ring on the group ring $R[G]$.

Proposition 4.8. *For $\alpha \in (0,1]$. The level-cut set of a fuzzy group ring $R[\mu]$ induced from fuzzy subgroup μ is the group ring of the corresponding α-cut set of fuzzy subgroup μ, i.e.,*

$$(R[\mu])_\alpha = R[\mu_\alpha].$$

Proof. Let $\alpha \in (0,1]$ and $x = \sum_{g \in G} a_g g \in R[\mu]$, we have:

$$\begin{aligned} x \in (R[\mu])_\alpha &\Leftrightarrow R[\mu](x) \geq \alpha \\ &\Leftrightarrow \min\{\mu(g) : a_g \neq 0\} \geq \alpha \\ &\Leftrightarrow \mu(g) \geq \alpha \;\text{ for all }\; a_g \neq 0 \;\text{ in }\; x = \sum_{g \in G} a_g g \\ &\Leftrightarrow g \in \mu_\alpha, \;\text{ for all }\; a_g \neq 0 \;\text{ in }\; x = \sum_{g \in G} a_g g \\ &\Leftrightarrow x = \sum_{g \in G} a_g g \in R(\mu_\alpha). \end{aligned}$$

Hence, $(R[\mu])_\alpha = R[\mu_\alpha]$. □

Proposition 4.9. *If μ and ν two fuzzy subgroups on a group G with $\mu \subseteq \nu$, then on the group ring $R[G]$ the induced fuzzy group rings satisfy, $R[\mu] \subseteq R[\nu]$.*

Proof. By Definition, $\mu \subseteq \nu$ gives $\mu(g) \leq \nu(g)$ for every $g \in G$.
For any $x = \sum_{g \in G} a_g g \in R[G]$, we have

$$\begin{aligned} R[\mu](x) &= R[\mu]\left(\sum_{g \in G} a_g g\right) \\ &= \min\{\mu(g) : a_g \neq 0\} \\ &\leq \min\{\nu(g) : a_g \neq 0\} \\ &= R[\nu](x). \end{aligned}$$

Hence, $R[\mu] \subseteq R[\nu]$. □

Proposition 4.10. *If μ and ν are two fuzzy subgroups defined on a group G, then $R[\mu \cap \nu] = R[\mu] \cap R[\nu]$ as the induced fuzzy group rings on $R[G]$.*

Proof. For fuzzy subgroups μ and ν on group G, $\mu \cap \nu$ is also a fuzzy subgroups of G (see **Theorem 1 of** [12]). For any $x = \sum_{g \in G} a_g g \in R[G]$, we have

$$\begin{aligned} R[\mu \cap \nu](x) &= R[\mu \cap \nu]\left(\sum_{g \in G} a_g g\right) \\ &= \min\{(\mu \cap \nu)(g) : a_g \neq 0\} \\ &= \min\left[\min\{\mu(g), \nu(g)\} : a_g \neq 0\right] \\ &= \min\left\{\min\{\mu(g) : a_g \neq 0\}, \min\{\nu(g) : a_g \neq 0\}\right\} \\ &= \min\{R[\mu](x), R[\nu](x)\} \\ &= (R[\mu] \cap R[\nu])(x). \end{aligned}$$

Hence, $R[\mu \cap \nu] = R[\mu] \cap R[\nu]$. □

Theorem 4.11. *Let $H \leq G$ be a subgroup and μ a fuzzy subgroup of G. Then the restriction of $R[\mu]$ to $R[H] \subseteq R[G]$ defines a fuzzy group ring $R[\mu_H]$ on $R[H]$.*

Proof. Let $\mu_H = \mu|_H$ be the restriction of the fuzzy subgroup μ to H, define by $\mu_H(h) = \mu(h)$, for all $h \in H$. Since $H \leq G$, it follows directly that μ_H satisfies the defining conditions of fuzzy subgroup on H. Since $R[H]$ is a subring of the group ring $R[G]$, it is closed under addition, multiplication, and scalar multiplication.

Let $x, y \in R[H]$. Then $x, y \in R[G]$ also. As $R[\mu]$ is a fuzzy group ring on $R[G]$, we have

$$R[\mu](x+y) \geq \min\{R[\mu](x), R[\mu](y)\} \text{ and } R[\mu](xy) \geq \min\{R[\mu](x), R[\mu](y)\}.$$

Restricting to $R[H]$ and noting that $R[\mu_H](x) = R[\mu](x)$, for all $x \in R[H]$, we obtain

$$R[\mu_H](x+y) \geq \min\{R[\mu_H](x), R[\mu_H](y)\} \text{ and }$$
$$R[\mu_H](xy) \geq \min\{R[\mu_H](x), R[\mu_H](y)\}.$$

Hence $R[\mu_H]$ is a fuzzy group ring on $R[H]$. □

Theorem 4.12. *The fuzzy group ring μ is the largest fuzzy ring on $R[G]$ compatible with the fuzzy subgroup ν.*

Proof. Let $\eta : R[G] \to [0,1]$ be any fuzzy ring such that $\eta(g) \leq \nu(g)$ for all $g \in G$.
For any $x = \sum r_g g \in R[G]$,

$$\eta(x) \leq \min\{\eta(g) \mid r_g \neq 0\} \leq \min\{\nu(g) \mid r_g \neq 0\} = \mu(x).$$

Thus $\eta \leq \mu$, proving that μ is maximal. □

Theorem 4.13. *Let $f : R[G_1] \to R[G_2]$ be a group ring homomorphism induced by a group homomorphism $\psi : G_1 \to G_2$. Then the inverse image under f of an IFGR $R[\nu]$ of $R[G_2]$ is an IFGR of $R[G_1]$, where ν is a fuzzy subgroup on G_2.*

Proof. Let $\psi : G_1 \to G_2$ be a group homomorphism. Then the induced group ring homomorphism $f : R[G_1] \to R[G_2]$ is defined by:

$$f(\textstyle\sum_{g \in G_1} a_g g) = \sum_{g \in G_1} a_g \psi(g), \text{ for all } \sum_{g \in G_1} a_g g \in R[G_1].$$

Let $R[\nu]$ be an IFGR on $R[G_2]$ corresponding to the fuzzy subgroup ν on G_2. Then:

$$\begin{aligned} f^{-1}(R[\nu])\left(\sum_{g \in G_1} a_g g\right) &= R[\nu]\left(f\left(\sum_{g \in G_1} a_g g\right)\right) \\ &= R[\nu]\left(\sum_{g \in G_1} a_g \psi(g)\right) \\ &= \min\{\nu(\psi(g)) : a_g \neq 0\} \\ &= \min\{\psi^{-1}(\nu)(g) : a_g \neq 0\} \\ &= R[\psi^{-1}(\nu)]\left(\sum_{g \in G_1} a_g g\right). \end{aligned}$$

Therefore, $f^{-1}(R[\nu]) = R[\psi^{-1}(\nu)]$. Since $\psi^{-1}(\nu)$ is a fuzzy subgroup on G_1 (see Theorem (2.1) of [11]), it follows that $R[\psi^{-1}(\nu)]$ is an IFGR on $R[G_1]$. Hence, $f^{-1}(R[\nu])$ is an IFGR on $R[G_1]$. □

Theorem 4.14. *Let $f : R[G_1] \to R[G_2]$ be a group ring homomorphism induced by a group homomorphism $\psi : G_1 \to G_2$, and μ be a fuzzy subgroup on G_1. Then the image of the IFGR $R[\mu]$ under f is an IFGR $R[\psi(\mu)]$ of $R[G_2]$.*

Proof. Let $\psi : G_1 \to G_2$ be a group homomorphism. Then the induced group ring homomorphism $f : R[G_1] \to R[G_2]$ is defined by:

$$f(\textstyle\sum_{g\in G_1} a_g g) = \sum_{g\in G_1} a_g \psi(g), \text{ for all } \sum_{g\in G_1} a_g g \in R[G_1].$$

Let $R[\mu]$ be an IFGR on $R[G_1]$ corresponding to the fuzzy subgroup μ of G_1. Then:

$$\begin{aligned}
f(R[\mu])\left(\sum_{g\in G_2} a_g g\right) &= \sup\left\{R[\mu]\left(\sum_{h\in G_1} b_h h\right) : f\left(\sum_{h\in G_1} b_h h\right) = \sum_{g\in G_2} a_g g\right\}\\
&= \sup\left\{\min\{\mu(h) : b_h \neq 0\} : \sum_{h\in G_1} b_h \psi(h) = \sum_{g\in G_2} a_g g\right\}\\
&= \sup\left\{\min\{\mu(h) : b_h \neq 0\} : a_g = \sum_{h\in G_1, \psi(h)=g} b_h, \forall g \in G_2\right\}\\
&= \sup\left\{\min\{\mu(h) : a_g \neq 0\},\ \text{where } \psi(h) = g\right\}\\
&= \min\left\{\sup\{\mu(h) : \psi(h) = g\} : a_g \neq 0\right\}\\
&= \min\left\{\psi(\mu)(g) : a_g \neq 0\right\}\\
&= R[\psi(\mu)]\left(\sum_{g\in G_2} a_g g\right).
\end{aligned}$$

Therefore, $f(R[\mu]) = R(\psi(\mu))$. Since $\psi(\mu)$ is a fuzzy subgroup on G_2 (see Theorem (2.2) of [11]), it follows that $R[\psi(\mu)]$ is an IFGR on $R[G_2]$, i.e., $f[R(\mu)]$ is an IFGR of $R[G_2]$. □

5. Conclusion

This paper introduces the concepts of general fuzzy group rings and fuzzy group rings induced from fuzzy subgroups, providing a framework that extends the classical theory of group rings to the fuzzy algebraic setting. The fundamental definitions and constructions of these structures were established, and several structural properties were investigated to understand their algebraic behavior.

The notion of a quotient general fuzzy group ring was defined and utilized to formulate and prove the fundamental isomorphism theorems for general fuzzy group rings. These results demonstrate that many classical algebraic principles can be suitably extended to the fuzzy context. Furthermore, a detailed comparison between general fuzzy group rings and fuzzy group rings induced from fuzzy subgroups was carried out, highlighting the essential differences between the two structures and clarifying their respective roles within fuzzy algebra.

The results obtained in this work contribute to the development of fuzzy algebraic structures and provide a basis for further exploration of fuzzy extensions of other algebraic systems. Future research may focus on investigating additional properties of fuzzy group rings, studying their homomorphic images, and exploring possible applications in related areas of algebra and fuzzy mathematical modeling.

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