

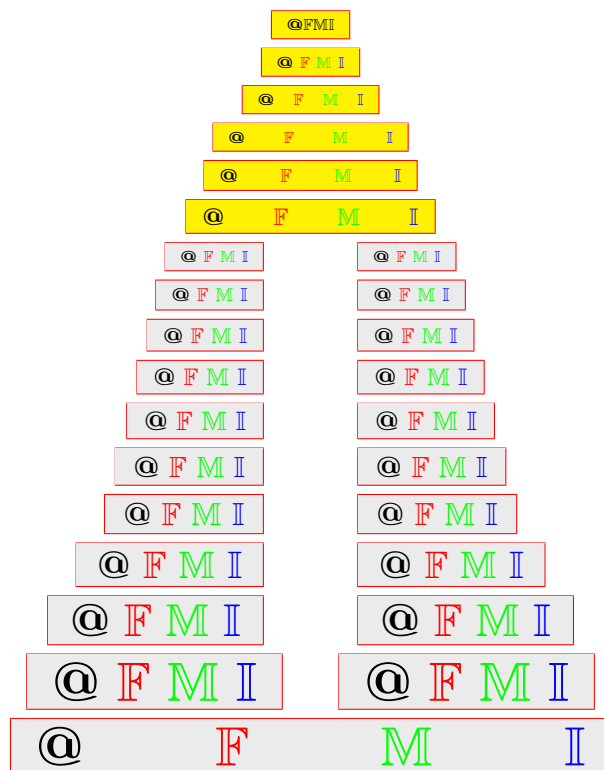
Annals of **F**uzzy **M**athematics and **I**nformatics  
 Volume 31, No. 3, (June 2026) pp. 187–197  
 ISSN: 2093–9310 (print version)  
 ISSN: 2287–6235 (electronic version)  
<http://www.afmi.or.kr>  
<https://doi.org/10.30948/afmi.2026.31.3.187>

@FMI

© Research Institute for Basic  
 Science, Wonkwang University  
<http://ribs.wonkwang.ac.kr>

## An introduction to the theory of Hom–Hilbert algebras

YOUNG BAE JUN



Reprinted from the  
 Annals of Fuzzy Mathematics and Informatics  
 Vol. 31, No. 3, June 2026

## An introduction to the theory of Hom–Hilbert algebras

YOUNG BAE JUN

Received 9 March 2026; Revised 29 March 2026; Accepted 9 April 2026

---

**ABSTRACT.** The concept of (commutative, bounded) Hom-Hilbert algebras is introduced, and examples are provided to illustrate it. Fundamental properties of the (commutative and bounded) Hom-Hilbert algebra are investigated, and a method for deriving the generalized Hilbert algebra from the Hom-Hilbert algebra is presented, and vice versa. A process is given in which every commutative Hom-Hilbert algebras form a poset structure. The join-semilattice is induced based on the commutative Hom-Hilbert algebra, and the meet-semilattice is induced based on the bounded Hom-Hilbert algebra. The commutative residuated lattice based on the bounded commutative Hom-Hilbert algebra is constructed.

2020 AMS Classification: 03G10, 03G25, 06F35

**Keywords:** (Commutative, bounded) Hom-Hilbert algebra, (Generalized) Hilbert algebra, (Meet, join) semilattice, Residuated lattice.

**Corresponding Author:** Young Bae Jun ([skywine@gmail.com](mailto:skywine@gmail.com))

---

### 1. INTRODUCTION

**H**om-type algebras are generalizations of classical algebraic structures (like associative algebras, Lie algebras, groups, etc.) in which the defining identities are “twisted” by a linear self-map, typically denoted  $\alpha$ . As an application of Hom-type algebras, it appears primarily in mathematical physics as an algebraic framework for quantum deformations and twisted symmetries in mathematical physics, and as generalized structures in Poisson/symplectic and algebraic geometry. In algebraic structures, the Hom-type theory is applied to groups, Lie-algebras, etc. (See [1, 2, 3, 4, 5]).

This paper focuses on the study of Hom-type algebras on generalized Hilbert algebras. We are going to deal with the following items.

- (1) Making the (commutative, bounded) Hom-Hilbert algebra: We define the concept of (commutative, bounded) Hom-Hilbert algebras.
- (2) Finding examples for the (commutative, bounded) Hom-Hilbert algebra: We give examples to illustrates the (commutative, bounded) Hom-Hilbert algebra.
- (3) Examining the fundamental properties of the (commutative and bounded) Hom-Hilbert algebra: We examine several properties on the (commutative, bounded) Hom-Hilbert algebra.
- (4) Suggesting a way to induce the Hom-Hilbert algebra from the generalized Hilbert algebra, and vice versa: We suggest a way to derive a generalized Hilbert algebra from a Hom-Hilbert algebra, and vice versa.
- (5) Forming the poset, semilattice structures, and residuated lattice from a Hom-Hilbert algebra: We show that every commutative Hom-Hilbert algebra forms the poset structure. We induce the join-semilattice based on the commutative Hom-Hilbert algebra, and we induce the meet-semilattice based on the bounded Hom-Hilbert algebra. We construct the commutative residuated lattice based on the bounded commutative Hom-Hilbert algebra.

## 2. PRELIMINARIES

A *Hilbert algebra* (See [6]) is defined as a triple  $(H, *, 1)$ , where  $*$  is a binary operation and  $1$  is a constant, satisfying the following three axioms for all  $x, y, z \in H$ :

- (HA1)  $x * (y * x) = 1$ ,
- (HA2)  $(x * (y * z)) * ((x * y) * (x * z)) = 1$ ,
- (HA3)  $x * y = 1, y * x = 1 \implies x = y$ .

A *generalized Hilbert algebra* (briefly, *g-Hilbert algebra*) (See [7]) is defined as a triple  $(H, *, 1)$  where  $*$  is a binary operation and  $1$  is a constant, satisfying the following three axioms for all  $x, y, z \in H$ :

- (gH1)  $1 * x = x$ ,
- (gH2)  $x * x = 1$ ,
- (gH3)  $x * (y * z) = y * (x * z)$ ,
- (gH4)  $x * (y * z) = (x * y) * (x * z)$ .

Every Hilbert algebra is a *g-Hilbert algebra*, but not vice versa (See [7]).

A *g-Hilbert algebra*  $(H, *, 1)$  is said to be *commutative* (See [7]), if it satisfies:

$$(2.1) \quad (\forall x, y \in H)((x * y) * y = (y * x) * x).$$

## 3. HOM-HILBERT ALGEBRAS

As a Hom-type generalization of a classical *g-Hilbert algebra*, we introduce the concept of Hom-Hilbert algebras.

**Definition 3.1.** A *Hom-Hilbert algebra* is a quadruple  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$ , where  $H$  is a nonempty set,  $\rightarrow$  is a binary operation on  $H$ ,  $1 \in H$  is the unit and  $\alpha$  is a self-map (called the twisting map) of  $H$ , satisfying the following axioms for all  $x, y, z \in H$ :

- (HH1)  $\alpha(1) \rightarrow x = x$ ,
- (HH2)  $\alpha(x) \rightarrow x = 1$ ,
- (HH3)  $\alpha(x) \rightarrow (\alpha(y) \rightarrow z) = \alpha(y) \rightarrow (\alpha(x) \rightarrow z)$ ,

$$(HH4) \alpha(x) \rightarrow (\alpha(y) \rightarrow z) = \alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow z).$$

**Example 3.2.** (1) Let  $H = \{1, a, b\}$  be a set with the Cayley table for  $\rightarrow$ :

| $\rightarrow$ | 1 | a | b |
|---------------|---|---|---|
| 1             | 1 | a | b |
| a             | 1 | 1 | 1 |
| b             | 1 | a | 1 |

Define the twisting map  $\alpha : H \rightarrow H$  by  $\alpha(1) = 1$ ,  $\alpha(a) = a$  and  $\alpha(b) = a$ . It is routine to check that  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a Hom-Hilbert algebra.

(2) Let  $H$  be the infinite real interval  $[0, 1]$  and consider the Gödel implication  $\rightarrow$ , that is,

$$(\forall x, y \in H) \left( x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases} \right).$$

Define the twisting map

$$\alpha : H \rightarrow H, x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1. \end{cases}$$

Then  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a Hom-Hilbert algebra.

Define a binary relation  $\leq_\alpha$  by

$$(\forall x, y \in H) (x \leq_\alpha y \iff \alpha(x) \rightarrow y = 1).$$

**Proposition 3.3.** *Every Hom-Hilbert algebra  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  satisfies the following conditions: for all  $x, y, z \in H$ ,*

$$(3.1) \quad x \leq_\alpha x, x \leq_\alpha 1,$$

$$(3.2) \quad x \leq_\alpha y, y \leq_\alpha z \implies x \leq_\alpha z,$$

$$(3.3) \quad y \leq_\alpha z \implies \alpha(x) \rightarrow y \leq_\alpha \alpha(x) \rightarrow z,$$

$$(3.4) \quad x \leq_\alpha \alpha(y) \rightarrow x,$$

$$(3.5) \quad x \leq_\alpha y \implies \alpha(y) \rightarrow z \leq_\alpha \alpha(x) \rightarrow z,$$

$$(3.6) \quad \alpha(x) \rightarrow y \leq_\alpha \alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(x) \rightarrow z).$$

*Proof.* The axiom (HH2) immediately means  $x \leq_\alpha x$ . If we put  $y = x$  and  $z = x$  in (HH4), then

$$\begin{aligned} \alpha(x) \rightarrow 1 &\stackrel{(HH2)}{=} \alpha(x) \rightarrow (\alpha(x) \rightarrow x) = \alpha(\alpha(x) \rightarrow x) \rightarrow (\alpha(x) \rightarrow x) \\ &\stackrel{(HH2)}{=} \alpha(1) \rightarrow 1 \stackrel{(HH1)}{=} 1. \end{aligned}$$

Thus  $x \leq_\alpha 1$ . So (3.1) is valid.

Suppose  $x \leq_\alpha y$  and  $y \leq_\alpha z$ . Then  $\alpha(x) \rightarrow y = 1$  and  $\alpha(y) \rightarrow z = 1$ . Thus

$$\begin{aligned}\alpha(x) \rightarrow z &\stackrel{(HH1)}{=} \alpha(1) \rightarrow (\alpha(x) \rightarrow z) \\ &= \alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow z) \\ &\stackrel{(HH4)}{=} \alpha(x) \rightarrow (\alpha(y) \rightarrow z) \\ &= \alpha(x) \rightarrow 1 \stackrel{(3.1)}{=} 1.\end{aligned}$$

So  $x \leq_\alpha z$ , which shows (3.2).

Suppose  $y \leq_\alpha z$ . Then  $\alpha(y) \rightarrow z = 1$ . Thus

$$\alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow z) \stackrel{(HH4)}{=} \alpha(x) \rightarrow (\alpha(y) \rightarrow z) = \alpha(x) \rightarrow 1 \stackrel{(3.1)}{=} 1.$$

So  $\alpha(x) \rightarrow y \leq_\alpha \alpha(x) \rightarrow z$ , which proves (3.3). For every  $x, y \in H$ , we have

$$\alpha(x) \rightarrow (\alpha(y) \rightarrow x) \stackrel{(HH3)}{=} \alpha(y) \rightarrow (\alpha(x) \rightarrow x) \stackrel{(HH2)}{=} \alpha(y) \rightarrow 1 \stackrel{(3.1)}{=} 1.$$

Hence  $x \leq_\alpha \alpha(y) \rightarrow x$  which proves (3.4).

Suppose  $x \leq_\alpha y$ . then  $\alpha(x) \rightarrow y = 1$ . Thus

$$\begin{aligned}\alpha(x) \rightarrow (\alpha(y) \rightarrow z) &\stackrel{(HH4)}{=} \alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow z) \\ &= \alpha(1) \rightarrow (\alpha(x) \rightarrow z) \\ &\stackrel{(HH1)}{=} \alpha(x) \rightarrow z.\end{aligned}$$

It follows that

$$\begin{aligned}\alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(x) \rightarrow z) &= \alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(x) \rightarrow (\alpha(y) \rightarrow z)) \\ &\stackrel{(HH3)}{=} \alpha(x) \rightarrow (\alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(y) \rightarrow z)) \\ &\stackrel{(HH2)}{=} \alpha(x) \rightarrow 1 \stackrel{(3.1)}{=} 1.\end{aligned}$$

So  $\alpha(y) \rightarrow z \leq_\alpha \alpha(x) \rightarrow z$ , i.e., (3.5) is valid. For every  $x, y, z \in H$ , we have

$$\begin{aligned}\alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(x) \rightarrow z)) \\ &\stackrel{(HH3)}{=} \alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow z)) \\ &\stackrel{(HH4)}{=} \alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(x) \rightarrow (\alpha(y) \rightarrow z)) \\ &\stackrel{(HH3)}{=} \alpha(x) \rightarrow (\alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(y) \rightarrow z)) \\ &\stackrel{(HH2)}{=} \alpha(x) \rightarrow 1 \stackrel{(3.1)}{=} 1,\end{aligned}$$

Hence  $\alpha(x) \rightarrow y \leq_\alpha \alpha(\alpha(y) \rightarrow z) \rightarrow (\alpha(x) \rightarrow z)$ , i.e., (3.6) is valid.  $\square$

We can see that the binary relation  $\leq_\alpha$  is a preorder on  $H$  with 1 as greatest element (See (3.1) and (3.2)).

In Example 3.2, we have

$$\leq_\alpha = \{(1, 1), (a, 1), (a, a), (a, b), (b, 1), (b, a), (b, b)\}.$$

Then  $a \leq_\alpha b$  and  $b \leq_\alpha a$ , but  $a \neq b$ . Thus the binary relation  $\leq_\alpha$  does not satisfy the antisymmetry.

The theorem below is a way to derive a  $g$ -Hilbert algebra from a Hom-Hilbert algebra.

**Theorem 3.4.** *Let  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  be a Hom-Hilbert algebra. If we define  $x * y = \alpha(x) \rightarrow y$  for  $x, y \in H$ , then  $(H, *, 1)$  is a  $g$ -Hilbert algebra.*

*Proof.* Let  $x, y, z \in H$ . Then  $1 * x = \alpha(1) \rightarrow x \stackrel{(\text{HH1})}{=} x$ ,  $x * x = \alpha(x) \rightarrow x \stackrel{(\text{HH2})}{=} 1$ ,

$$\begin{aligned} x * (y * z) &= \alpha(x) \rightarrow (y * z) = \alpha(x) \rightarrow (\alpha(y) \rightarrow z) \\ &\stackrel{(\text{HH3})}{=} \alpha(y) \rightarrow (\alpha(x) \rightarrow z) \\ &= \alpha(y) \rightarrow (x * z) = y * (x * z), \end{aligned}$$

and

$$\begin{aligned} x * (y * z) &= \alpha(x) \rightarrow (y * z) = \alpha(x) \rightarrow (\alpha(y) \rightarrow z) \\ &\stackrel{(\text{HH4})}{=} \alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow z) \\ &= \alpha(x * y) \rightarrow (x * z) \\ &= (x * y) * (x * z). \end{aligned}$$

Thus  $(H, *, 1)$  is a  $g$ -Hilbert algebra.  $\square$

We form a Hom-Hilbert algebra from a  $g$ -Hilbert algebra.

**Theorem 3.5.** *Let  $(H, *, 1)$  be a  $g$ -Hilbert algebra and let  $\alpha : H \rightarrow H$  be any bijective map. Define a binary operation  $\rightarrow$  in  $H$  by  $x \rightarrow y := \alpha^{-1}(x) * y$  for  $x, y \in H$ . Then the quadruple  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a Hom-Hilbert algebra.*

*Proof.* Let  $x, y, z \in H$ . Since  $\alpha$  is bijective, we get

$$\alpha(x) \rightarrow y = \alpha^{-1}(\alpha(x)) * y = x * y.$$

Then  $\alpha(1) \rightarrow x = 1 * x \stackrel{(\text{gH1})}{=} x$ ,  $\alpha(x) \rightarrow x = x * x \stackrel{(\text{gH2})}{=} 1$ , and

$$\begin{aligned} \alpha(x) \rightarrow (\alpha(y) \rightarrow z) &= \alpha(x) \rightarrow (y * z) = x * (y * z) \stackrel{(\text{gH3})}{=} y * (x * z) \\ &= \alpha(y) \rightarrow (x * z) = \alpha(y) \rightarrow (\alpha(x) \rightarrow z). \end{aligned}$$

Thus  $(\text{HH1})$ ,  $(\text{HH2})$ , and  $(\text{HH3})$  are valid. Finally, we have

$$\begin{aligned} \alpha(x) \rightarrow (\alpha(y) \rightarrow z) &= \alpha(x) \rightarrow (y * z) = x * (y * z) \\ &\stackrel{(\text{gH4})}{=} (x * y) * (x * z) \\ &= (\alpha(x) \rightarrow y) * (\alpha(x) \rightarrow z) \\ &= \alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow z). \end{aligned}$$

So  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a Hom-Hilbert algebra.  $\square$

The example below illustrates Theorem 3.5.

**Example 3.6.** Let  $H = \{0, 1, a, b\}$  be a set with the Cayley table for  $*$ :

| $*$ | 0 | a | b | 1 |
|-----|---|---|---|---|
| 0   | 1 | 1 | 1 | 1 |
| a   | b | 1 | b | 1 |
| b   | a | a | 1 | 1 |
| 1   | 0 | a | b | 1 |

Then  $(H, *, 1)$  is a  $g$ -Hilbert algebra. Define a map  $\alpha : H \rightarrow H$  by  $\alpha(1) = 1$ ,  $\alpha(a) = b$ ,  $\alpha(b) = a$  and  $\alpha(0) = 0$ . Now, we apply our formula to generate the twisted Hom-implication presented by  $x \rightarrow y := \alpha^{-1}(x) * y$  and its Cayley table is as follows:

| $\rightarrow$ | 0 | a | b | 1 |
|---------------|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 |
| a             | a | a | 1 | 1 |
| b             | b | 1 | b | 1 |
| 1             | 0 | a | b | 1 |

Thus  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a Hom-Hilbert algebra.

**Definition 3.7.** A Hom-Hilbert algebra  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is said to be *commutative*, if it satisfies:

$$(3.7) \quad (\forall x, y \in H)(\alpha(\alpha(x) \rightarrow y) \rightarrow y = \alpha(\alpha(y) \rightarrow x) \rightarrow x).$$

**Example 3.8.** Let  $H = \{1, a\}$  and define the binary operation  $\rightarrow$  with the following Cayley table:

| $\rightarrow$ | 1 | a |
|---------------|---|---|
| 1             | 1 | 1 |
| a             | 1 | a |

Let  $\alpha : H \rightarrow H$  be defined by  $\alpha(1) = a$  and  $\alpha(a) = 1$ . Then  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a commutative Hom-Hilbert algebra.

**Proposition 3.9.** Every commutative Hom-Hilbert algebra  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  satisfies for all  $x, y \in H$ :

$$(3.8) \quad x \leq_{\alpha} y \implies y = \alpha(\alpha(y) \rightarrow x) \rightarrow x,$$

$$(3.9) \quad x \leq_{\alpha} \alpha(\alpha(x) \rightarrow y) \rightarrow y.$$

*Proof.* Suppose  $x \leq_{\alpha} y$ . Then  $\alpha(x) \rightarrow y = 1$ . Thus

$$\alpha(\alpha(y) \rightarrow x) \rightarrow x \stackrel{(3.7)}{=} \alpha(\alpha(x) \rightarrow y) \rightarrow y = \alpha(1) \rightarrow y \stackrel{(\text{HH1})}{=} y.$$

Also, we have

$$\alpha(x) \rightarrow (\alpha(\alpha(x) \rightarrow y) \rightarrow y) \stackrel{(3.7)}{=} \alpha(\alpha(x) \rightarrow y) \rightarrow (\alpha(x) \rightarrow y) \stackrel{(\text{HH2})}{=} 1.$$

So  $x \leq_{\alpha} \alpha(\alpha(x) \rightarrow y) \rightarrow y$ . □

**Theorem 3.10.** If  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a commutative Hom-Hilbert algebra, then  $(H, \leq_{\alpha})$  is a poset.

*Proof.* Since the binary relation  $\leq_\alpha$  is a preorder on  $H$ , it is sufficient to show that the binary relation  $\leq_\alpha$  satisfies the antisymmetry. Suppose  $x \leq_\alpha y$  and  $y \leq_\alpha x$ , i.e.,  $\alpha(x) \rightarrow y = 1$  and  $\alpha(y) \rightarrow x = 1$ . Then

$$y \stackrel{(3.8)}{=} \alpha(\alpha(y) \rightarrow x) \rightarrow x = \alpha(1) \rightarrow x \stackrel{(HH1)}{=} x.$$

Thus  $(H, \leq_\alpha)$  is a poset.  $\square$

In a Hom-Hilbert algebra  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$ , we define the twisted join, denoted as  $\vee_\alpha$ , as follows:

$$(\forall x, y \in H)(x \vee_\alpha y = \alpha(\alpha(x) \rightarrow y) \rightarrow y).$$

It is clear that if  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a commutative Hom-Hilbert algebra, then  $x \vee_\alpha y = y \vee_\alpha x$ .

**Theorem 3.11.** *If  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is a commutative Hom-Hilbert algebra, then the map*

$$f : H \rightarrow H, x \mapsto x \vee_\alpha y$$

*is  $\leq_\alpha$ -monotone, and  $x \vee_\alpha y$  is the least upper bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ , and  $(H, \vee_\alpha, \leq_\alpha)$  is a join-semilattice with top element 1.*

*Proof.* Suppose  $y \leq_\alpha z$ . Then  $\alpha(z) \rightarrow x \leq_\alpha \alpha(y) \rightarrow x$  by (3.5), which implies from (3.5) again that

$$f(y) = y \vee_\alpha x = \alpha(\alpha(y) \rightarrow x) \rightarrow x \leq_\alpha \alpha(\alpha(z) \rightarrow x) \rightarrow x = z \vee_\alpha x = f(z).$$

Thus  $f$  is  $\leq_\alpha$ -monotone.

Let  $x, y \in H$ . Then  $x \leq_\alpha x \vee_\alpha y$  by (3.9). Also,  $y \leq_\alpha y \vee_\alpha x \stackrel{(3.7)}{=} x \vee_\alpha y$ . Thus  $x \vee_\alpha y$  is an upper bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ . Let  $z \in H$  be an upper bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ , i.e.,  $x \leq_\alpha z$  and  $y \leq_\alpha z$ . Then  $\alpha(\alpha(z) \rightarrow x) \rightarrow x = z$  by (3.8). Thus

$$x \vee_\alpha y = \alpha(\alpha(x) \rightarrow y) \rightarrow y = \alpha(\alpha(y) \rightarrow x) \rightarrow x \leq_\alpha \alpha(\alpha(z) \rightarrow x) \rightarrow x = z.$$

So  $x \vee_\alpha y$  is the least upper bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ . Since  $(H, \leq_\alpha)$  is a poset (see Theorem 3.10),  $(H, \vee_\alpha, \leq_\alpha)$  is a join-semilattice, and 1 is the top element since  $x \vee_\alpha 1 = \alpha(\alpha(x) \rightarrow 1) \rightarrow 1 = 1$ .  $\square$

**Definition 3.12.** A Hom-Hilbert algebra  $\mathcal{H} := (H, \rightarrow, 1, \alpha)$  is said to be *bounded*, if there exists an element  $0 \in H$  such that  $0 \leq_\alpha x$ , i.e.,  $\alpha(0) \rightarrow x = 1$ , for all  $x \in H$ .

The bounded Hom-Hilbert algebra will be denoted as the quintuple  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$ .

**Example 3.13.** Let  $H = \{0, 1, a\}$  be a set with the Cayley table for  $\rightarrow$ :

| $\rightarrow$ | 0 | a | 1 |
|---------------|---|---|---|
| 0             | 1 | 1 | 1 |
| a             | 0 | 1 | 1 |
| 1             | 0 | a | 1 |

Define the twisting map  $\alpha : H \rightarrow H$  by  $\alpha(1) = 1$ ,  $\alpha(a) = 0$ , and  $\alpha(0) = 0$ . It is routine to check that  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  is a bounded Hom-Hilbert algebra.



Let  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  be a bounded Hom-Hilbert algebra. Define a unary operation

$$\neg_\alpha : H \rightarrow H, \quad x \mapsto \alpha(x) \rightarrow 0.$$

**Proposition 3.14.** *Every bounded Hom-Hilbert algebra  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  satisfies the following assertions for all  $x, y \in H$ :*

$$(3.10) \quad x \vee_\alpha 0 = x,$$

$$(3.11) \quad \neg_\alpha(0) = 1, \quad \neg_\alpha(1) = 0,$$

$$(3.12) \quad \alpha(0) \rightarrow \neg_\alpha(x) = 1,$$

$$(3.13) \quad \neg_\alpha(\neg_\alpha(x)) = x,$$

$$(3.14) \quad x \leq_\alpha y \implies \neg_\alpha(y) \leq_\alpha \neg_\alpha(x).$$

*Proof.* (3.10): If we take  $y = 0$  in (3.7), then

$$x \vee_\alpha 0 = \alpha(\alpha(x) \rightarrow 0) \rightarrow 0 = \alpha(\alpha(0) \rightarrow x) \rightarrow x = \alpha(1) \rightarrow x \stackrel{(HH1)}{=} x$$

for all  $x \in H$ .

$$(3.11): \neg_\alpha(0) = \alpha(0) \rightarrow 0 \stackrel{(HH2)}{=} 0 \text{ and } \neg_\alpha(1) = \alpha(1) \rightarrow 0 \stackrel{(HH1)}{=} 0.$$

(3.12): For every  $x \in H$ , we get

$$\begin{aligned} \alpha(0) \rightarrow \neg_\alpha(x) &= \alpha(0) \rightarrow (\alpha(x) \rightarrow 0) \stackrel{(HH3)}{=} \alpha(x) \rightarrow (\alpha(0) \rightarrow 0) \\ &\stackrel{(HH2)}{=} \alpha(x) \rightarrow 1 \stackrel{(3.1)}{=} 1. \end{aligned}$$

(3.13): If we take  $y = 0$  in (3.7), then

$$\begin{aligned} \neg_\alpha(\neg_\alpha(x)) &= \alpha(\alpha(x) \rightarrow 0) \rightarrow 0 = \alpha(\alpha(0) \rightarrow x) \rightarrow x \\ &= \alpha(1) \rightarrow x \stackrel{(HH1)}{=} x. \end{aligned}$$

(3.14): Suppose  $x \leq_\alpha y$ . Then  $\alpha(x) \rightarrow y = 1$ . Thus

$$\begin{aligned} \alpha(\neg_\alpha(y)) \rightarrow \neg_\alpha(x) &= \alpha(\alpha(y) \rightarrow 0) \rightarrow (\alpha(x) \rightarrow 0) \\ &\stackrel{(HH3)}{=} \alpha(x) \rightarrow (\alpha(\alpha(y) \rightarrow 0) \rightarrow 0) \\ &= \alpha(x) \rightarrow (\neg_\alpha(\neg_\alpha(y))) \\ &\stackrel{(3.13)}{=} \alpha(x) \rightarrow y = 1. \end{aligned}$$

So  $\neg_\alpha(y) \leq_\alpha \neg_\alpha(x)$ . □

Let  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  be a bounded Hom-Hilbert algebra. Using the twisted join  $\vee_\alpha$ , we define the twisted meet, denoted by  $\wedge_\alpha$ , as follows:

$$(\forall x, y \in H)(x \wedge_\alpha y = \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y))).$$

Using the Commutativity of the twisted join  $\vee_\alpha$ , it is clear that if  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  is a bounded commutative Hom-Hilbert algebra, then  $x \wedge_\alpha y = y \wedge_\alpha x$  for all  $x, y \in H$ .

**Theorem 3.15.** *Let  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  be a bounded Hom-Hilbert algebra. Then the twisted meet  $x \wedge_\alpha y$  is the greatest lower bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ , and  $(H, \wedge_\alpha, \leq_\alpha)$  is a meet-semilattice with the bottom element 0.*

*Proof.* Using Theorem 3.11, we know that  $\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$  is the least upper bound of  $\neg_\alpha(x)$  and  $\neg_\alpha(y)$ , i.e.,  $\neg_\alpha(x) \leq_\alpha \neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$  and  $\neg_\alpha(y) \leq_\alpha \neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$ . It follows from (3.14) and (3.13) that

$$x \wedge_\alpha y = \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)) \leq_\alpha \neg_\alpha(\neg_\alpha(x)) = x$$

and  $x \wedge_\alpha y = \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)) \leq_\alpha \neg_\alpha(\neg_\alpha(y)) = y$ . Then  $x \wedge_\alpha y$  is a lower bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ . Let  $z$  be any other lower bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ , i.e.,  $z \leq_\alpha x$  and  $z \leq_\alpha y$ . Then  $\neg_\alpha(x) \leq_\alpha \neg_\alpha(z)$  and  $\neg_\alpha(y) \leq_\alpha \neg_\alpha(z)$  by (3.14). Thus  $\neg_\alpha(z)$  is an upper bound of  $\neg_\alpha(x)$  and  $\neg_\alpha(y)$ . So  $\neg_\alpha(x) \vee_\alpha \neg_\alpha(y) \leq_\alpha \neg_\alpha(z)$  since  $\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$  is the least upper bound of  $\neg_\alpha(x)$  and  $\neg_\alpha(y)$ . Using (3.13) and (3.14) yields  $z = \neg_\alpha(\neg_\alpha(z)) \leq_\alpha \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)) = x \wedge_\alpha y$ . Hence  $x \wedge_\alpha y$  is the greatest lower bound of  $x$  and  $y$  with respect to the partial order  $\leq_\alpha$ . Since  $(H, \leq_\alpha)$  is a poset (See Theorem 3.10),  $(H, \wedge_\alpha, \leq_\alpha)$  is a meet-semilattice with the bottom element 0.  $\square$

The combination of Theorem 3.10, Theorem 3.11 and Theorem 3.15 yields the following theorem.

**Theorem 3.16.** *If  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  is a bounded commutative Hom-Hilbert algebra, then the quadruple  $(H, \leq_\alpha, \vee_\alpha, \wedge_\alpha)$  is a bounded lattice.*

**Proposition 3.17.** *Every bounded commutative Hom-Hilbert algebra  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  satisfies the following items: for all  $x, y, z \in H$ ,*

$$(3.15) \quad \neg_\alpha(x) \vee_\alpha y = \alpha(x) \rightarrow y,$$

$$(3.16) \quad z \wedge_\alpha x \leq_\alpha y \iff z \leq_\alpha \alpha(x) \rightarrow y,$$

$$(3.17) \quad x \wedge_\alpha (y \vee_\alpha z) = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z),$$

$$(3.18) \quad x \vee_\alpha (y \wedge_\alpha z) = (x \vee_\alpha y) \wedge_\alpha (x \vee_\alpha z).$$

*Proof.* (3.15): For every  $x, y \in H$ , we have

$$\begin{aligned} \neg_\alpha(x) \vee_\alpha y &= \alpha(\alpha(\neg_\alpha(x)) \rightarrow y) \rightarrow y \\ &\stackrel{(3.7)}{=} \alpha(\alpha(y) \rightarrow \neg_\alpha(x)) \rightarrow \neg_\alpha(x) \\ &= \alpha(\alpha(y) \rightarrow (\alpha(x) \rightarrow 0)) \rightarrow (\alpha(x) \rightarrow 0) \\ &\stackrel{(HH3)}{=} \alpha(\alpha(x) \rightarrow (\alpha(y) \rightarrow 0)) \rightarrow (\alpha(x) \rightarrow 0) \\ &= \alpha(\alpha(x) \rightarrow \neg_\alpha(y)) \rightarrow (\alpha(x) \rightarrow 0) \\ &\stackrel{(HH4)}{=} \alpha(x) \rightarrow (\alpha(\neg_\alpha(y)) \rightarrow 0) \\ &= \alpha(x) \rightarrow \neg_\alpha(\neg_\alpha(y)) \\ &\stackrel{(3.13)}{=} \alpha(x) \rightarrow y. \end{aligned}$$

(3.16): If  $z \wedge_\alpha x \leq_\alpha y$ , then  $\neg_\alpha(\neg_\alpha(z) \vee_\alpha \neg_\alpha(x)) \leq_\alpha y$ . Thus

$$\neg_\alpha(y) \stackrel{(3.14)}{\leq_\alpha} \neg_\alpha(\neg_\alpha(\neg_\alpha(z) \vee_\alpha \neg_\alpha(x))) \stackrel{(3.13)}{=} \neg_\alpha(z) \vee_\alpha \neg_\alpha(x).$$

So

$$\begin{aligned}
1 &= \alpha(\neg_\alpha(y)) \rightarrow (\neg_\alpha(z) \vee_\alpha \neg_\alpha(x)) \\
&\stackrel{(3.15)}{=} \alpha(\neg_\alpha(y)) \rightarrow (\alpha(z) \rightarrow \neg_\alpha(x)) \\
&\stackrel{(HH3)}{=} \alpha(z) \rightarrow (\alpha(\neg_\alpha(y)) \rightarrow \neg_\alpha(x)) \\
&\stackrel{(3.15)}{=} \alpha(z) \rightarrow (\neg_\alpha(\neg_\alpha(y)) \vee_\alpha \neg_\alpha(x)) \\
&\stackrel{(3.13)}{=} \alpha(z) \rightarrow (y \vee_\alpha \neg_\alpha(x)) \\
&= \alpha(z) \rightarrow (\neg_\alpha(x) \vee_\alpha y) \\
&\stackrel{(3.15)}{=} \alpha(z) \rightarrow (\alpha(x) \rightarrow y).
\end{aligned}$$

Hence  $z \leq_\alpha \alpha(x) \rightarrow y$ .

Conversely, suppose  $z \leq_\alpha \alpha(x) \rightarrow y$ . Let  $\mathbf{b} = z \wedge_\alpha x$ . Then  $\mathbf{b} \leq_\alpha z$  and  $\mathbf{b} \leq_\alpha x$ . Thus  $\mathbf{b} \leq_\alpha \alpha(x) \rightarrow y$  since  $\leq_\alpha$  is transitive. So

$$\begin{aligned}
1 &= \alpha(\mathbf{b}) \rightarrow (\alpha(x) \rightarrow y) \stackrel{(HH4)}{=} \alpha(\alpha(\mathbf{b}) \rightarrow x) \rightarrow (\alpha(\mathbf{b}) \rightarrow y) \\
&= \alpha(1) \rightarrow (\alpha(\mathbf{b}) \rightarrow y) \stackrel{(HH1)}{=} \alpha(\mathbf{b}) \rightarrow y,
\end{aligned}$$

Hence  $z \wedge_\alpha x = \mathbf{b} \leq_\alpha y$ .

(3.17): Let  $x, y, z \in H$ . Since  $x \wedge_\alpha y \leq_\alpha x$  and  $x \wedge_\alpha y \leq_\alpha y \leq_\alpha y \vee_\alpha z$ , the transitivity of  $\leq_\alpha$  gives  $x \wedge_\alpha y \leq_\alpha y \vee_\alpha z$ . Then  $x \wedge_\alpha y \leq_\alpha x \wedge_\alpha (y \vee_\alpha z)$ . By the similar way,  $x \wedge_\alpha z \leq_\alpha x \wedge_\alpha (y \vee_\alpha z)$ . It follows that

$$(x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z) \leq_\alpha x \wedge_\alpha (y \vee_\alpha z).$$

Let  $R = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)$ . Then  $x \wedge_\alpha y \leq_\alpha R$  and  $x \wedge_\alpha z \leq_\alpha R$ . Thus  $y \leq_\alpha \alpha(x) \rightarrow R$  and  $z \leq_\alpha \alpha(x) \rightarrow R$  by (3.16), i.e.,  $\alpha(x) \rightarrow R$  is an upper bound of  $y$  and  $z$ . So  $y \vee_\alpha z \leq_\alpha \alpha(x) \rightarrow R$  which is equivalent to

$$x \wedge_\alpha (y \vee_\alpha z) \leq_\alpha R = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z).$$

Hence  $x \wedge_\alpha (y \vee_\alpha z) = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)$ .

(3.18): It is obtained in a similar way to (3.17).  $\square$

**Theorem 3.18.** *If  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  is a bounded commutative Hom-Hilbert algebra, then the quadruple  $(H, \leq_\alpha, \vee_\alpha, \wedge_\alpha)$  is a distributive bounded lattice.*

*Proof.* This is an immediate result of Theorem 3.16 and Proposition 3.17.  $\square$

**Lemma 3.19.** *Let  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  be a bounded commutative Hom-Hilbert algebra. Then  $(H, \wedge_\alpha, 1)$  and  $(H, \vee_\alpha, 0)$  are commutative monoids. If we define  $x \Rightarrow y := \alpha(x) \rightarrow y$  for all  $x, y \in H$ , then  $z \wedge_\alpha x \leq_\alpha y \iff z \leq_\alpha x \Rightarrow y$ .*

*Proof.* For every  $x, y, z \in H$ , let  $\mathbf{c} = (x \wedge_\alpha y) \wedge_\alpha z$ . Then  $\mathbf{c}$  is a lower bound of  $x \wedge_\alpha y$  and  $z$ . Since  $\mathbf{c} \leq_\alpha x \wedge_\alpha y$ , it is also a lower bound of  $x$  and  $y$ . Thus  $\mathbf{c} \leq_\alpha x$ ,  $\mathbf{c} \leq_\alpha y$  and  $\mathbf{c} \leq_\alpha z$ . Also,  $\mathbf{c} \leq_\alpha (y \wedge_\alpha z)$ , since  $\mathbf{c}$  is a lower bound of  $y$  and  $z$ , so  $(x \wedge_\alpha y) \wedge_\alpha z = \mathbf{c} \leq_\alpha x \wedge_\alpha (y \wedge_\alpha z)$ . By applying the exact same logic in reverse, we get  $x \wedge_\alpha (y \wedge_\alpha z) \leq_\alpha (x \wedge_\alpha y) \wedge_\alpha z$ . Using the antisymmetry of  $\leq_\alpha$  yields  $x \wedge_\alpha (y \wedge_\alpha z) = (x \wedge_\alpha y) \wedge_\alpha z$ . It is clear that  $x \wedge_\alpha 1 \leq_\alpha x$ . Since

$x \leq_\alpha 1$  and  $x \leq_\alpha x$ , i.e.,  $x$  is a lower bound  $x$  and 1, we have  $x \leq_\alpha x \wedge_\alpha 1$ . Thus  $x \wedge_\alpha 1 = x$ . Since the twisted meet  $\wedge_\alpha$  is commutative, it also follows that  $1 \wedge_\alpha x = x$ . So  $(H, \wedge_\alpha, 1)$  is a commutative monoid. By the similar way, we can verify that  $(H, \vee_\alpha, 0)$  is a commutative monoid. If  $x \Rightarrow y \stackrel{\text{def}}{=} \alpha(x) \rightarrow y$  for all  $x, y \in H$ , then  $z \wedge_\alpha x \leq_\alpha y \iff z \leq_\alpha x \Rightarrow y$  for all  $x, y, z \in H$  by (3.16).  $\square$

By comprehensively considering the above discussion, we derive the following theorem.

**Theorem 3.20.** *If  $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$  is a bounded commutative Hom-Hilbert algebra, then  $(H, \leq_\alpha, \wedge_\alpha, \vee_\alpha, \Rightarrow, 0, 1)$  is a commutative residuated lattice.*

#### REFERENCES

- [1] A. A. A. Agboola, M. A. Ibrahim, A. O. Adeniji and S. A. Adebisi, On fundamental properties of Hom-groups, J. Mahani Math. Res. 14 (2) (2025) 81–97. <https://doi.org/10.22103/jmmr.2024.24063.1697>
- [2] Z. Chebel, H. Adimi and H. Bouremel, Hom-actions and class equation for Hom-groups, Journal of Geometry and Physics 207 (2025) 105371. <https://doi.org/10.1016/j.geomphys.2024.105371>
- [3] M. Hassanzadeh, Lagrange’s theorem for Hom-groups, Rocky Mountain Journal of Mathematics 49 (3) (2019) 773–787.
- [4] A. Makhlof and S. D. Silvestrov, Hom-algebra structures, Journal of Generalized Lie Theory and Applications 2 (2) (2008) 51–64. <https://www.hilarispublisher.com/open-access/homalgebra-structures-1736-4337-2-115.pdf>
- [5] Y. Sheng and D. Chen, Hom-Lie2-algebras, Journal of Algebra 376 (2013) 174–195. <http://dx.doi.org/10.1016/j.jalgebra.2012.11.032>
- [6] A. Diego, Sur les algèbres de Hilbert, Collection de Logique Mathématique, Serie A, 21, Gauthier-Villars, Paris 1966.
- [7] R. A. Borzooei and J. Shohani, On generalized Hilbert algebras, Italian Journal of Pure and Applied Mathematic 29 (2012) 71–86.

YOUNG BAE JUN (skywine@gmail.com)

Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea ORCID iD <https://orcid.org/0000-0002-0181-8969>