Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 31, No. 1, (February 2026) pp. 67–77
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2026.31.1.67

Fuzzy c-spaces

C. Darsana, P. Sini

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 31, No. 1, February 2026

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 31, No. 1, (February 2026) pp. 67–77
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2026.31.1.67

@FMI

Fuzzy c-spaces

C. Darsana, P. Sini

Received 13 August 2025; Revised 15 September 2025; Accepted 4 December 2025

Abstract. In this paper, we introduce and investigate a new class of fuzzy structures termed fuzzy c-space by extending the classical notion of c-space into fuzzy contexts. We present the concept of fuzzy touching points and fuzzy t-closed sets. Furthermore, we define fuzzy c-continuous mapping and investigate some of its properties.

2020 AMS Classification: 05C40, 06B99, 03E72

Keywords: Fuzzy c-spaces, Fuzzy touching points, Fuzzy t-closed set, Fuzzy c-continuous functions.

Corresponding Author: C. Darsana (darsanadinesan@gmail.com)

1. Introduction

In this paper, we extend the concept of c-spaces to fuzzy contexts. The classical concepts of connectivity have found considerable use in image processing and analysis. In 1983, Börger [1] proposed a framework for connectivity, known as the theory of c-structures, by incorporating key ideas of connectedness from both topology and graph theory. Specifically, singletons are considered connected, and the union of connected sets sharing a common point is also connected. Several articles have been published in this area of research (See [2, 3, 4, 5, 6] for more details). However, this theory was largely confined to binary images and was set-oriented. In mathematics, both in theory and in practice, we often work with sets that have different structures. But sometimes, classical methods are not enough to deal with uncertainty. To overcome these challenges, Zadeh introduced fuzzy set theory in 1965 [7], which enabled the extension of classical mathematical structures into a fuzzy framework. Following this development, considerable research has also been carried out in complementary areas such as rough set theory, soft set theory, and related fields (See [8, 9, 10, 11, 12, 13, 14, 15, 16]).

A fuzzy set is a function from a universal set X to the interval $[0, 1]$, and a crisp set is a special case of a fuzzy set where the function takes only the values 0 or 1. Fuzzy set theory has found numerous applications in real-world problems. With its advent, conventional notions of connectedness have been broadened to fuzzy contexts. This naturally leads to the development of connectivity frameworks suitable for grayscale images, where a more nuanced theoretical structure is required.

The objective of this paper is to develop the theory of fuzzy c-spaces by extending the classical concept of c-spaces into the fuzzy setting. This generalization aims to provide a flexible framework for analyzing fuzzy connectedness that is applicable to both binary and grayscale images. The main advantage of fuzzy connectedness is in image segmentation, especially within medical imaging, where it facilitates the extraction of objects with vague or unclear boundaries.

In this work, we introduce the concept of fuzzy c-spaces and investigate several of their properties. This study is inspired by the work of Muscat and Buhagiar [3]. The motivation stems from the need to bridge the gap between classical c-space theory and fuzzy environments. Many real-world applications, such as medical imaging, object segmentation, and pattern recognition, require the identification of boundaries or transitions that are not strictly binary. In such cases, a fuzzy model of connectedness becomes essential.

A key component of our study is the introduction and analysis of fuzzy touching points. This plays an important role in several areas. They are particularly relevant in fuzzy boundary detection, which is critical for tasks such as edge detection in image processing. Additionally, fuzzy touching points are useful in understanding fuzzy continuity, where functions can be analyzed based on how they interact with such points.

2. Preliminaries

Throughout this paper, I will denote the unit interval [0,1] and X a nonempty set.

A *fuzzy set* in X is a function with domain X and values in I [17]. Let $f \in I^X$. Then the *support* of f denoted by $supp(f)$ is the set $\{x \in X : f(x) > 0\}$. Here $f(x)$ is called the *grade of membership* of x in f and X is the carrier of the fuzzy set f. If f takes only the values 0 and 1, then f is called a *crisp set* in X [17]. If $f(x) = a$ for all $x \in X$, then we denote the fuzzy set f by $\underline{a}$. Let f_1, $f_2 \in I^X$. Then we say that f_1 *is contained in* f_2 denoted by $f_1 \leq f_2$ [17], if $f_1(x) \leq f_2(x)$ for all $x \in X$. The *join* of f_1 and f_2, denoted by $f_1 \vee f_2$ [17], is defined as $(f_1 \vee f_2)(x) = max\{f_1(x), f_2(x)\}$. The *meet* of f_1 and f_2 is given by [17] $(f_1 \wedge f_2)(x) = min\{f_1(x), f_2(x)\}$. The *complement* of f, denoted by f' [17], is defined as $f'(x) = 1 - f(x)$ for all $x \in X$. Let $h : X \to Y$, $f \in I^X$, and $g \in I^Y$. Then $h(f)$ is a fuzzy set in Y defined as

$$h(f)(y) = \begin{cases} sup\{f(x) : x \in h^{-1}(y)\} & \text{if } h^{-1}(y) \neq \varnothing, \\ 0 & \text{if } \ h^{-1}(y) = \varnothing. \end{cases}$$

and $h^{-1}(g)$ is a fuzzy set in X, $h^{-1}(g)(x)$ defined by $g(h(x))$ for $x \in X$ [17]. A fuzzy point is a special type of fuzzy set that has a non zero membership value at exactly one element of the universal set. More precisely, a fuzzy set p in X is called a *fuzzy point* [17], if $p(x) = \begin{cases} a & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$ where $0 < a < 1$. We denote this fuzzy point by y_a. A fuzzy set p is called a crisp point if $p(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$ We denote this crisp point by y_1. A fuzzy point $p \in I^X$ is said to be a fuzzy point in a fuzzy set f if $p \leq f$.

3. Fuzzy c-spaces

Here, we introduce the notion of a fuzzy c-space.

Definition 3.1. Let X be a set and $\mathcal{F} \subseteq I^X$. Then $(X, \mathcal{F})$ is said to be a *fuzzy c-space*, if it satisfies the following conditions:
(i) $\underline{0} \in \mathcal{F}$,
(ii) if $\mathcal{A}$ is a family of fuzzy subsets in $\mathcal{F}$ with $\wedge\mathcal{A} \neq \underline{0}$, then $\vee\mathcal{A} \in \mathcal{F}$,
(iii) $x_1 \in \mathcal{F}$ for all $x \in X$.

Here $\mathcal{F}$ is called a *fuzzy c-structure* on X and elements of $\mathcal{F}$ are called *fuzzy connected sets*.
Here onwards, D_X denotes the set of all crisp points together with $\underline{0}$.
The following are some examples of fuzzy c-spaces.

Example 3.2. (1) Let X be any set. Then $\mathcal{F} = D_X$ is a fuzzy c-structure on X and is called the *trivial fuzzy c-structure*.
(2) Let X be any set and $\mathcal{F} = I^X$. Then $(X, \mathcal{F})$ is a fuzzy c-space called the *indiscrete fuzzy c-space*.
(3) Let (X, δ) be a fuzzy topological space. Then the collection of all fuzzy connected subsets of X forms a fuzzy c-structure on X.
(4) Let $\mathcal{F}$ be a fuzzy c-structure on X. Then $\mathcal{F} \cup \{\underline{1}\}$ is also a fuzzy c-structure on X, called the *Brunnian closure* of $\mathcal{F}$.
(5) Let $f \in I^X$. Then $\mathcal{F} = D_X \cup \{g \in I^X : f \leq g\}$ is a fuzzy c-structure on X, called the *fuzzy c-structure rooted at f*.

Theorem 3.3. *Let $(X, \mathcal{F})$ be a fuzzy c-space and $Y \subseteq X$. Then the collection $\mathcal{F}_Y = \{f \restriction_Y : f \in \mathcal{F}$ and $f(X \setminus Y) = \{0\}\}$ is a fuzzy c-structure on Y.*

Proof. It is clear that $\underline{0} \in \mathcal{F}_Y$. Let $f = y_1 \in \mathcal{F}$, where $y \in Y$ and $g = f \restriction_Y$. Then $g \in \mathcal{F}_Y$. Let $\{g_i : i \in I\}$ be a collection of fuzzy sets in $\mathcal{F}_Y$ with $\bigwedge_{i \in I} g_i \neq \underline{0}$. Then for each i, there exists $f_i \in \mathcal{F}$ such that $g_i = f_i \restriction_Y$ and $f_i(X \setminus Y) = \{0\}$. Now, $\bigwedge_{i \in I} g_i \neq \underline{0}$ implies $\bigwedge_{i \in I} f_i \neq \underline{0}$, it follows that $\bigvee_{i \in I} f_i \in \mathcal{F}$. Also, note that for each $x \in X \setminus Y$, $f_i(x) = 0$ for all $i \in I$, which implies $(\bigvee_{i \in I} f_i)(x) = 0$ for each $x \in X \setminus Y$. Thus $\bigvee_{i \in I} g_i$ can be written as $\bigvee_{i \in I} g_i = (\bigvee_{i \in I} f_i) \restriction_Y$. So $\bigvee_{i \in I} g_i \in \mathcal{F}_Y$. Hence the proof. □

Now, we define the concept of a sub fuzzy c-space.

Definition 3.4. Let $(X, \mathcal{F})$ be a fuzzy c-space and $Y \subseteq X$. Define $\mathcal{F}_Y = \{f \restriction_Y : f \in \mathcal{F}$ and $f(X \setminus Y) = \{0\}\}$. Then $(Y, \mathcal{F}_Y)$ is a fuzzy c-space called the *sub fuzzy c-space* of $(X, \mathcal{F})$ and $\mathcal{F}_Y$ is called the *sub fuzzy c-structure* on Y.

Definition 3.5. Let $(X, \mathcal{F})$ be a fuzzy c-space and $\mathcal{B} \subseteq I^X$. Then the intersection of all fuzzy c-structures containing $\mathcal{B}$, denoted by $\langle \mathcal{B} \rangle$ is a fuzzy c-structure on X and is called the *fuzzy c-structure generated by* $\mathcal{B}$. The elements of $\mathcal{B}$ are called*basic fuzzy connected sets.*

Definition 3.6. Let $(X, \mathcal{F})$ be a fuzzy c-space. Let K be a finite index set and $H : K \to \mathcal{F}$ be a function such that $H(k) \wedge H(k+1) \neq \underline{0}$ for each $k \in K$. Then $\{H(k) : k \in K\}$ is called a *chain.*
For our convenience, we denote $H(k)$ and $H(k+1)$ by H_k and H_{k+1}, respectively.

Let $\mathcal{G}$ be the collection of all $g \in I^X$ such that for any $g(x)$ and $g(y)$, there exists a chain $B : K \to \mathcal{B}$ such that each $B_k \leq g$, $k \in K$ and $B_{k'}(x) = g(x)$ and $B_{k''}(y) = g(y)$, together with D_X. Now, we claim that $\mathcal{G}$ is a fuzzy c-structure on X.

Let $\{g_j : j \in J\}$ be the collection of sets in $\mathcal{G}$ with $\bigwedge_{j \in J} g_j \neq \underline{0}$. Consider $\bigvee_{j \in J} g_j$. Let $\bigvee_{j \in J} g_j(x) = a$ and $\bigvee_{j \in J} g_j(y) = b$. Then there exists some g_1, g_2 with $g_1(x) = a$ and $g_2(y) = b$. Since $\bigwedge_{j \in J} g_j \neq \underline{0}$, there exists some $z \in X$ such that $g_j(z) \neq 0$ for all j. Consider $g_1(x)$ and $g_1(z)$. Then by our assumption, there exists a chain of basic fuzzy connected sets, say $B_1, B_2, \cdots, B_m$ contained in g_1 such that $B_i(x) = g_1(x) = a$ and $B_l(z) = g_1(z)$ for some $i, l \in \{1, 2, \cdots, m\}$. Similarly, consider $g_2(y)$ and $g_2(z)$. Then there exists a chain of basic fuzzy connected sets, say $B_1^{'}, B_2^{'}, \cdots, B_n^{'}$ contained in g_2 such that $B_i^{'}(y) = g_2(y) = b$ and $B_l^{'}(z) = g_2(z)$ for some $i^{'}$, $l^{'} \in \{1, 2, \cdots, n\}$. Observe that $B_l \wedge B_l^{'} \neq \underline{0}$, since $g_j(z) \neq 0$ for all $j \in J$. Consequently, there exists a chain of basic fuzzy connected sets $B_1, B_2, \cdots, B_m, B_1^{'}, B_2^{'}, \cdots, B_n^{'}$ such that $B_i(x) = \bigvee_{j \in J} g_j(x) = a$ and $B_i^{'}(y) = \bigvee_{j \in J} g_j(y) = b$. That is, $\bigvee_{j \in J} g_j \in \mathcal{G}$. Obviously, $\mathcal{B} \subseteq \mathcal{G}$. Also, $\mathcal{G} \subseteq \mathcal{F}$ for all $\mathcal{F}$ containing $\mathcal{B}$. Thus $\mathcal{G} = \langle \mathcal{B} \rangle$.
The following theorem summarizes the preceding observations.

Theorem 3.7. *Let $\mathcal{F}$ be a fuzzy c-structure on X generated by $\mathcal{B}$. Then any $f \in \mathcal{F} \setminus D_X$ is characterized by the condition that for any $f(x)$ and $f(y)$, there exists a chain $B : K \to \mathcal{B}$ such that each $B_k \leq f$, $k \in K$ and $B_{k'}(x) = f(x)$ and $B_{k''}(y) = f(y)$.*

4. Fuzzy Touching Points

In [3], Muscat and Buhagiar proposed the notion of touching points and examined its different features. Analogously, here, we introduce the concept of fuzzy touching points.

Definition 4.1. Let $(X, \mathcal{F})$ be a fuzzy c-space and $f \in I^X$. Then we say that *a fuzzy point p touches f*, if there exists $\underline{0} \neq f^{'} \in I^X$ with $f^{'} \leq f$ such that $p \vee f^{'} \in \mathcal{F}$.

The set of all fuzzy points touching f is denoted by $Ft(f)$.

Definition 4.2. We say that *two fuzzy sets f and g touch*, if there exists a fuzzy point $p \leq f \vee g$, which touches both f and g.

This is illustrated in the following example.

Example 4.3. Let $X = \{a, b, c\}$ and $\mathcal{F}$ be a fuzzy c-structure on X having base $\mathcal{B} = D_X \cup \{f_1, f_2\}$, where f_1 and f_2 are defined by $f_1(a) = 0, f_1(b) = 0.5, f_1(c) = 0.25$ and $f_2(a) = 0, f_2(b) = 0.5, f_2(c) = 0.5$. Consider the fuzzy sets f and g given by

$$f(a) = 0.25, f(b) = 0.5, f(c) = 0.5$$
$$g(a) = 0.25, g(b) = 0.25, g(c) = 1$$

Then $f \vee g$ is given by

$$(f \vee g)(a) = 0.25, (f \vee g)(b) = 0.5, (f \vee g)(c) = 1.$$

Let $p = b_{0.5} \leq f \vee g$. Take $f^{'}$ and $g^{'}$, where

$$f^{'}(a) = 0, f^{'}(b) = 0.25, f^{'}(c) = 0.5$$
$$g^{'}(a) = 0, g^{'}(b) = 0.25, g^{'}(c) = 1$$

Clearly, $f^{'} \leq f$ and $g^{'} \leq g$. Now, $(f^{'} \vee p)(a) = 0, (f^{'} \vee p)(b) = 0.5, (f^{'} \vee p)(c) = 0.5$ and $(g^{'} \vee p)(a) = 0, (g^{'} \vee p)(b) = 0.5, (g^{'} \vee p)(c) = 1$. Note that, $f^{'} \vee p$ and $g^{'} \vee p$ are fuzzy connected. Then f and g touch.

Remark 4.4. Let $(X, \mathcal{F})$ be a fuzzy c-space and f, g, and $k \in I^X$. If f touches g and g touches k, then f need not touch k. See the following example.

Example 4.5. Let $X = \{a, b, c, d\}$. Consider the fuzzy c-structure generated by $\mathcal{B} = D_X \cup \{f, g, k\}$, where

$$f(a) = 0.4, f(b) = 0.2, f(c) = 0, f(d) = 0$$
$$g(a) = 0, g(b) = 0.5, g(c) = 0.3, g(d) = 0$$
$$k(a) = 0, k(b) = 0, k(c) = 0.1, k(d) = 0.5$$

Here, f touches g and g touches k. But f does not touch k.

The following theorem establishes the condition under which f touches k.

Theorem 4.6. *Let $(X, \mathcal{F})$ be a fuzzy c-space and f, g, and $k \in I^X$. If f touches g and $g \leq k$, then f touches k.*

Proof. Since f touches g, there exists a fuzzy point $p \leq f \vee g$ that touches both f and g. Now, p touches g implies that there exists some $g^{'} \leq g$ such that $g^{'} \vee p$ is fuzzy connected. In addition, $g \leq k$ implies that $g^{'} \leq k$. It follows that p touches k, also $p \leq f \vee g \leq f \vee k$. Then f touches k. □

Theorem 4.7. *Let $\mathcal{F}$ be a fuzzy c-structure on X and $f \in \mathcal{F}$. Then p touches f if and only if $f \vee p$ is fuzzy connected.*

Proof. Suppose that p touches f. Then there exists a $f^{'}$ with $f^{'} \neq \underline{0}$ and $f^{'} \leq f$ such that $f^{'} \vee p$ is fuzzy connected. Since $f^{'} \leq f$, $f \wedge (f^{'} \vee p) \neq \underline{0}$ which implies that $f \vee (f^{'} \vee p)$ is fuzzy connected. Consequently, $f \vee p$ is fuzzy connected.

Conversely, suppose that $f \vee p$ is fuzzy connected. Then it is clear that p touches f. This completes the proof. □

The following theorems explore the properties of fuzzy touching points.

Theorem 4.8. *Let $(X, \mathcal{F})$ be a fuzzy c-space and f, $g \in I^X$.*

(1) *If $f \leq g$, then $Ft(f) \subseteq Ft(g)$.*
(2) *If f touches g, then g touches f.*
(3) *Let f, $g \in \mathcal{F}$. If f and g touch, then $f \vee g \in \mathcal{F}$.*

Proof. (1) Suppose that p is a fuzzy touching point of f. Then there exists some $\underline{0} \neq f^{'} \leq f$ such that $f^{'} \vee p \in \mathcal{F}$. Since $f^{'} \leq f \leq g$, p is a fuzzy touching point of g also.

(2) Trivial.

(3) Since f and g touches, there exists a fuzzy point $p \leq f \vee g$ which touches both f and g. Then $f \vee p$ and $g \vee p$ are fuzzy connected. Now, $(f \vee p) \wedge (g \vee p) \geq p \neq \underline{0}$. Thus $(f \vee p) \vee (g \vee p) \in \mathcal{F}$. So $f \vee g \in \mathcal{F}$. □

Proposition 4.9. *Let $(X, \mathcal{F})$ be a fuzzy c-space and f, $g \in I^X$ and $a \in I$.*
(1) If $f(x) = a \neq 0$ for some $x \in X$, then x_1 is a fuzzy touching point of f.
(2)If $f \leq g$, then $\bigvee Ft(f) \leq \bigvee Ft(g)$.
(3) $f \leq \bigvee Ft(f)$.
(4) If $f \in \mathcal{F}$, then every fuzzy point in f is a fuzzy touching point of f.
(5) If $f(x) \neq 0$ for all $x \in X$, then $\bigvee Ft(f) = \underline{1}$.

Proof. (1) Since $f(x) = a \neq 0$, we have $x_a \leq f$. Let $p = x_1$. Then $x_1 \vee x_a = x_1$ is fuzzy connected. Thus x_1 is a fuzzy touching point of f.

(2) By Theorem 4.8 (1), every fuzzy touching point of f is a fuzzy touching point of g. Then $\bigvee Ft(f) \leq \bigvee Ft(g)$.

(3) Let $x \in X$. If $f(x) = 0$, then $f(x) \leq \bigvee Ft(f)(x)$. Otherwise x_1 is a fuzzy touching point of f. Thus $f \leq \bigvee Ft(f)$.

(4) Let $p = x_a \leq f$. Then $p \vee f = x_a \vee f = f \in \mathcal{F}$. Thus every fuzzy point of f is a fuzzy touching point of f.

(5) Let $x \in X$. Then $f(x) = a$, where $a \neq 0$. Then $x_a \leq f$. Now, $x_1 \vee x_a = x_1 \in \mathcal{F}$. Thus x_1 is a fuzzy touching point of f. Consequently $\bigvee Ft(f) = \underline{1}$. □

Theorem 4.10. *Let $(X, \mathcal{F})$ be a fuzzy c-space and $f \in \mathcal{F}$, where $f(x) \neq 0$ for all $x \in X$. Then $\underline{1} \in \mathcal{F}$.*

Proof. Since $f \in \mathcal{F}$ and $f(x) \neq 0$ for all $x \in X$, $x_1 \wedge f \neq \underline{0}$. Then $x_1 \vee f \in \mathcal{F}$. Also, $f \leq x_1 \vee f$ implies $\bigwedge\limits_{x \in X} (x_1 \vee f) \neq \underline{0}$. Thus $\bigvee\limits_{x \in X} (x_1 \vee f) \in \mathcal{F}$. But $\bigvee\limits_{x \in X} (x_1 \vee f) = \underline{1}$. So $\underline{1} \in \mathcal{F}$. □

Definition 4.11. Let $(X, \mathcal{F})$ be a fuzzy c-space and $f \in I^X$. Then f is called *fuzzy t-closed*, if $\bigvee Ft(f) = f$. If f is fuzzy t-closed, then every fuzzy touching points f is contained in f.
Obviously, for any fuzzy c-space $(X, \mathcal{F})$, $\underline{1}$ and $\underline{0}$ are fuzzy t-closed.
We denote the set of all fuzzy points in f by $Fp(f)$.

Note that if $(X, \mathcal{F})$ is the trivial fuzzy c-space and $f \in I^X$, then f is a crisp subset of X if and only if f is fuzzy t-closed.

Theorem 4.12. *Let $(X, \mathcal{F})$ be a fuzzy c-space. If $f \in I^X$ is fuzzy t-closed, then f is a crisp set.*

Proof. Since f is fuzzy t-closed, the only fuzzy touching points of f are the fuzzy points in f. For $a < 1$, suppose that $f(x) = a \neq 0$. Consider $x_a \leq f$. Then x_1 is a fuzzy touching point of f, since $x_1 \vee x_a = x_1 \in \mathcal{F}$. Thus f is not t-closed, a contradiction. □

But the converse is not true. For example, consider the fuzzy c-structure $\mathcal{F} = D_X \cup \{f, g, k, l\}$ on X, where $X = \{a, b, c\}$ and

$$f(a) = 1, f(b) = 1, f(c) = 0;\;\; g(a) = 1, g(b) = 0, g(c) = 1;$$
$$k(a) = 0, k(b) = 1, k(c) = 1;\;\; l(a) = 1, l(b) = 1, l(c) = 1.$$

Let $p = c_1$. Then $f \vee p = \underline{1} \in \mathcal{F}$, which implies that p is a fuzzy touching point of f. But p is not a fuzzy point of f. Thus f is not fuzzy t-closed.

Remark 4.13. If $\{f_i : i \in I\}$ is a collection of all fuzzy t-closed sets in a fuzzy c-space $(X, \mathcal{F})$, then $\bigwedge_{i\in I} f_i$ is fuzzy t-closed.

Proof. Let $f = \bigwedge_{i\in I} f_i$. By Proposition 4.9 (3) , $f \leq \bigvee Ft(f)$. Since $f \leq f_i$ for all i, $\bigvee Ft(f) \leq \bigvee Ft(f_i)$. This implies that $\bigvee Ft(f) \leq f_i$ for all i, since f_i is fuzzy t-closed. It follows that $\bigvee Ft(f) \leq \bigwedge_{i\in I} f_i = f$. Then $f = \bigvee Ft(f)$. Thus f is fuzzy t-closed. □

Definition 4.14. The *fuzzy connective closure* of a fuzzy set f denoted by $\overline{f}$ is defined as the smallest fuzzy t-closed set containing f. That is, fuzzy connective closure of a fuzzy set f is the meet of all fuzzy t-closed sets that contain f.

Theorem 4.15. *Let $(X, \mathcal{F})$ be a fuzzy c-space and $f \in I^X$. Then*

(1) *f is fuzzy t-closed if and only if $\overline{f} = f$,*
(2) *$\bigvee Ft(f) \leq \overline{f}$.*

Proof. (1) Suppose that f is fuzzy t-closed. Then the smallest fuzzy t-closed set containing f is f itself, which implies $\overline{f} = f$.

Conversely, suppose that $\overline{f} = f$. That is, the meet of all fuzzy t-closed set containing f is f itself. Then f is fuzzy t-closed.

(2) Let $\mathbf{F}$ be a fuzzy t-closed set such that $f \leq \mathbf{F}$. Then $\bigvee Ft(\mathbf{F}) = \mathbf{F}$. Since $f \leq \mathbf{F}$, $\bigvee Ft(f) \leq \bigvee Ft(\mathbf{F})$ implies $\bigvee Ft(f) \leq \mathbf{F}$. This is true for all fuzzy t-closed set containing f. Thus we can write $\bigvee Ft(f) \leq \bigwedge\{\mathbf{F} : f \leq \mathbf{F}, \bigvee Ft(\mathbf{F}) = \mathbf{F}\} = \overline{f}$. So $\bigvee Ft(f) \leq \overline{f}$. □

Definition 4.16. A fuzzy set $f \in I^X$ is *fuzzy c-dense*, if $\bigvee Ft(f) = \underline{1}$.

In an indiscrete fuzzy c-space, every fuzzy set is fuzzy c-dense. For any fuzzy c-space $(X, \mathcal{F})$, if $f(x) \neq 0$ for all $x \in X$, then f is fuzzy c-dense.
Next, we present the concept of fuzzy connective space.

Definition 4.17. Let X be a set. A fuzzy connective structure on X is a fuzzy c-structure $\mathcal{F}$ satisfying the following conditions:

(i) if $\underline{0} \neq f \in \mathcal{F}$ and $\underline{0} \neq g \in \mathcal{F}$ with $f \vee g \in \mathcal{F}$, then there exists a fuzzy point $p \leq f \vee g$ such that $p \vee f \in \mathcal{F}$ and $p \vee g \in \mathcal{F}$,

(ii) if f, g, $k_i \in \mathcal{F}$ with meet $\underline{0}$ and $f \vee g \vee (\bigvee_{i \in I} k_i) \in \mathcal{F}$, then there exist $J \subseteq I$, $f \vee (\bigvee_{j \in J} k_j) \in \mathcal{F}$ and $g \vee (\bigvee_{i \in I \setminus J} k_i) \in \mathcal{F}$,
Then $(X, \mathcal{F})$ is called a *fuzzy connective space*.

Example 4.18. (1) The trivial fuzzy c-space (X, D_X) is a fuzzy connective space.
(2) The indiscrete fuzzy c-space is a fuzzy connective space.
(3) Let $X = \{a, b, c\}$. Then $\mathcal{F} = D_X \cup \{f, g, k\}$ is a fuzzy connective structure on X, where

$$f(a) = 1, f(b) = 0.4, f(c) = 0$$
$$g(a) = 0.4, g(b) = 1, g(c) = 0$$
$$k(a) = 1, k(b) = 1, k(c) = 0$$

The following is an example of a fuzzy c-structure that is not a fuzzy connective structure.

Example 4.19. Let $X = \{x, y, z, w\}$ and $\mathcal{F}$ be a fuzzy c-structure on X having base $\mathcal{B} = D_X \cup \{f, g, k\}$, where

$$f(x) = 0.5, f(y) = 0.8, f(z) = 0, f(w) = 0$$
$$g(x) = 0, g(y) = 0, g(z) = 0.4, g(w) = 0.7$$
$$k(x) = 0.5, k(y) = 0.8, k(z) = 0.4, k(w) = 0.7$$

But it is not a fuzzy connective structure. Here, $f \vee g = k \in \mathcal{F}$.

1. If $p = x_a$, $0 < a \leq 0.5$, then $p \vee f = f \in \mathcal{F}$ and $p \vee g \notin \mathcal{F}$.
2. If $p = y_a$, $0 < a \leq 0.8$, then $p \vee f = f \in \mathcal{F}$ and $p \vee g \notin \mathcal{F}$.
3. If $p = z_a$, $0 < a \leq 0.4$, then $p \vee f \notin \mathcal{F}$ and $p \vee g = g \in \mathcal{F}$.
3. If $p = w_a$, $0 < a \leq 0.7$, then $p \vee f \notin \mathcal{F}$ and $p \vee g = g \in \mathcal{F}$.

Then there does not exist a fuzzy point $p \leq f \vee g$ with $p \vee f \in \mathcal{F}$ and $p \vee g \in \mathcal{F}$.

5. Fuzzy c-continuity

In [3] Muscat J. and Buhagiar D. explore various aspects of c-continuous functions. In an analogous way, we define a fuzzy c-continuous mapping and examine its properties.

Definition 5.1. Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ be two fuzzy c-spaces. Then a function $h : X \to Y$ is called *fuzzy c-continuous*, if it maps fuzzy connected subsets of X into fuzzy connected subsets of Y.

Now, h is called a *fuzzy c-isomorphism*, if h is bijective and both h and h^{-1} are fuzzy c-continuous. A fuzzy c-isomorphism from X onto itself is called a *fuzzy c-automorphism*.

Example 5.2. Let $X = \{a, b, c\}$ and $\mathcal{F}$ be a fuzzy c-structure on X having base

$$\mathcal{B} = D_X \cup \{f, g, k\},$$

where

$$f(a) = 1, \ f(b) = 0.5, \ f(c) = 0;$$

$$g(a) = 0.5,\ g(b) = 0,\ g(c) = 1;$$
$$k(a) = 0,\ k(b) = 1,\ k(c) = 0.5.$$

If $h = I, (a,b,c)$ or (b,c,a), then h and h^{-1} map fuzzy connected sets to fuzzy connected sets. Thus h is a fuzzy c-automorphism on X.

Theorem 5.3. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ be two fuzzy c-spaces and $h : X \to Y$ be fuzzy c-continuous and $g \in I^X$. If p touches g, then $h(p)$ touches $h(g)$.*

Proof. Since p touches g, there exists a $\underline{0} \neq g^{'} \leq g$ such that $g^{'} \vee p \in \mathcal{F}$. Since h is fuzzy c-continuous, $h(g^{'} \vee p) \in \mathcal{F}'$. It follows that $h(g^{'}) \vee h(p) \in \mathcal{F}'$. As $g^{'} \leq g$, $g^{'}(x) \leq g(x)$ for all $x \in X$. Now, for all $y \in Y$,

$$h(g^{'})(y) = \bigvee\{g^{'}(x) : h(x) = y\} \leq \bigvee\{g(x) : h(x) = y\} = h(g)(y).$$

Then $h(g^{'}) \leq h(g)$. Thus $h(p)$ is a fuzzy touching point of $h(g)$. □

Theorem 5.4. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ be two fuzzy c-spaces and $h : X \to Y$ be fuzzy c-continuous. If $\mathbf{F}$ is a fuzzy t-closed set in $\mathcal{F}'$, then $h^{-1}(\mathbf{F})$ is fuzzy t-closed set in $\mathcal{F}$.*

Proof. Since $\mathbf{F}$ is fuzzy t-closed, $\bigvee Ft(\mathbf{F}) = \mathbf{F}$. Then every fuzzy touching points of $\mathbf{F}$ is contained in $\mathbf{F}$. Now, let p be a fuzzy touching point of $h^{-1}(\mathbf{F})$ and $p \nleq h^{-1}(\mathbf{F})$. Thus $h(p)$ is a fuzzy touching point of $\mathbf{F}$ and $h(p) \nleq \mathbf{F}$, which is a contradiction. As $\mathbf{F}$ is fuzzy t-closed by Theorem 4.12, $\mathbf{F}$ is a crisp set. So $h^{-1}(\mathbf{F})$ is also a crisp set. If $h^{-1}(\mathbf{F})(x) = 1$, then obviously x_1 is a fuzzy touching point of $h^{-1}(\mathbf{F})$. If $h^{-1}(\mathbf{F})(x) = 0$, then x_a cannot be a fuzzy touching point of $h^{-1}(\mathbf{F})$ for any $a \in (0,1]$. Hence $\bigvee Ft(h^{-1}(\mathbf{F})) = h^{-1}(\mathbf{F})$. Therefore the result holds. □

Proposition 5.5. (1) *Any function from a trivial fuzzy c-space is fuzzy c-continuous.*
(2) *Any function into an indiscrete fuzzy c-space is fuzzy c-continuous.*
(3) *Identity maps are fuzzy c-continuous.*

Proof. (1) Let h be a function from a trivial fuzzy c-space $(X, \mathcal{F})$ to some fuzzy c-space to $(Y, \mathcal{G})$. Then h maps $\underline{0}$ to $\underline{0}$ and crisp points in X to crisp points in Y. Thus h maps every fuzzy connected sets of X in to fuzzy connected sets of Y. So h is fuzzy c-continuous.

(2) Let h be a function from an arbitrary fuzzy c-space $(X, \mathcal{F})$ to an indiscrete fuzzy c-space to $(Y, \mathcal{G})$. Then every $f \in I^Y$ is fuzzy connected. Thus every fuzzy connected set of X mapped onto fuzzy connected sets in Y. So h is fuzzy c-continuous.

(3) Let h be the identity function on $(X, \mathcal{F})$. Then h maps every fuzzy connected set f to f itself. Thus h is fuzzy c-continuous. □

Theorem 5.6. *The composition of fuzzy c-continuous function is fuzzy c-continuous.*

Proof. Let $h : X \to Y$ and $h^{'} : Y \to Z$ be fuzzy c-continuous functions and f be a fuzzy connected subset of X. Then $h(f)$ is fuzzy connected in Y. Now, consider the composition of $h^{'}$ and h, $h^{'} \circ h(f) = h^{'}(h(f))$, which is fuzzy connected in Z. Thus the result is complete. □

The above result ensures that the class of fuzzy topological spaces, together with fuzzy continuous maps, is closed under function composition.

Example 5.7. Let $X = \{x, y, z\}$ and $Y = \{a, b, c\}$. Let $h : X \to Y$ be defined by

$$h(x) = a,\ h(y) = b,\ h(z) = c.$$

Consider the fuzzy c-structure $\mathcal{F} = D_X \cup \{f, g, k, l\}$ on X, where

$$f(x) = 0.7, f(y) = 0.7, f(z) = 0$$
$$g(x) = 1, g(y) = 0.7, g(z) = 0$$
$$k(x) = 1, k(y) = 1, k(z) = 0$$
$$l(x) = 0.7, l(y) = 1, l(z) = 0.$$

and $\mathcal{F}'$ is the fuzzy c-structure on Y generated by the base $\mathcal{B} = D_Y \cup \{f^{'}, g^{'}\}$, where

$$f^{'}(a) = 0.7, f^{'}(b) = 0.7, f^{'}(c) = 0$$
$$g^{'}(a) = 0, g^{'}(b) = 0.7, g^{'}(c) = 0.7$$

Since h maps fuzzy connected sets in X to fuzzy connected sets in Y, h is fuzzy c-continuous. Now, consider $Z = \{u, v, w\}$. Let $h^{'} : Y \to Z$ be defined by $h^{'}(a) = u$, $h^{'}(b) = w$, and $h^{'}(c) = v$. Let $\mathcal{F}''$ be the fuzzy c-structure on Z generated by the base $\mathcal{B} = D_Z \cup \{f^{''}, g^{''}\}$, where

$$f^{''}(u) = 0.7, f^{''}(v) = 0, f^{''}(w) = 0.7$$
$$g^{''}(u) = 0, g^{'}(v) = 0.7, g^{'}(w) = 0.7$$

Thus $h^{'}$ is also fuzzy c-continuous. Here, $h^{'} \circ h$ maps fuzzy connected subsets of X to fuzzy connected subsets of Z. So $h^{'} \circ h$ is also fuzzy c-continuous.

Remark 5.8. The inverse of a fuzzy c-continuous function need not be fuzzy c-continuous.
Consider 5.7, let $k^{'}$ be a fuzzy subset of Y given by $k^{'}(a) = 0, k^{'}(b) = 0.7, k^{'}(c) = 1$. Then $k^{'} \in \mathcal{F}'$, but $h^{-1}(k^{'}) = k^{'} \notin \mathcal{F}$. Therefore, h^{-1} is not fuzzy c-continuous.

6. Conclusion

In this paper, we have extended the classical theory of c-spaces into the fuzzy setting and discussed some properties of fuzzy c-spaces. We introduced and explored the concept of fuzzy touching points, which provide a richer understanding of fuzzy connectedness. The theory developed here not only bridges the gap between traditional c-space theory and fuzzy set theory but also opens new avenues for research in fuzzy topology, fuzzy logic systems, and image analysis.

References

[1] R. Börger, Connectivity spaces and component categories, Categorical Topology, Proceedings of the International Conference held at the University of Toledo (1983) 71–89.

[2] S. Dugowson, On connectivity spaces, Cahiers de Topologie et Géométrie Différentielle Catégoriques 51 (4) (2010) 282–315.

[3] J. Muscat and D. Buhagiar, Connective spaces, Memoirs of the Graduate School of Science and Engineering, Shimane University Series B, Mathematical Sciences 39 (2006) 1–13.

[4] J. Serra, Connectivity on complete lattices, Journal of Mathematical Imaging and Vision 9 (1998) 231–251.

[5] P. Sini, On completely homogeneous c-spaces, Heliyon 6 (12) (2020).

[6] A. K. Sruthi, Studies on *c*-spaces, PhD, University of Calicut, Kerala, India (2024).
[7] L. A. Zadeh, Fuzzy sets. Information and Control 8 (3) (1965) 338–353.
[8] R. Abu Gdairi, A. A. Nasef, M. A. El-Gayar and M. K. El-Bably, On fuzzy point applications of fuzzy topological spaces, International Journal of Fuzzy Logic and Intelligent Systems (2023) 162–72.
[9] M. K. El-Bably, R. A. Hosny and M. A. El-Gayar, Innovative rough set approaches using novel initial-neighborhood systems, Applications in medical diagnosis of Covid-19 variants, Inform. Sci. (2025) 122044.
[10] G. Senel, Analyzing the locus of soft spheres: Illustrative cases and drawings, European Journal of Pure and Applied Mathematics 11 (4)(2018) 946–957.
[11] G. Şenel, Lee Jeong Gon and K. Hur, Distance and similarity measures for octahedron sets and their application to MCGDM problems, Mathematics. 8 (10) (2020) 1–16.
[12] G. Şenel, Lee Jeong Gon, Y. B. Jun , Abbas Fadhil and K. Hur, Topological structures via interval-valued soft sets, Ann. Fuzzy Math. Inform. 22 (2) (2021) 133–169.
[13] G. Şenel, Lee Jeong Gon, J. Kim , D. H. Yang and K. Hur, Cubic crisp sets and their application to topology, Ann. Fuzzy Math. Inform. 21 (3) (2021) 227–265.
[14] K. Hur, Lee Jeong Gon, and J. Kim, Octahedron topological spaces, Ann. Fuzzy Math. Inform. 22 (1) (2021) 77–101.
[15] G. Şenel, A new approach to hausdorff space theory via the soft sets. Mathematical Problems in Engineering. 9 (2016) 1–6.
[16] G. Şenel, An innovatie algorithm based on octahedron sets via multi-criteria decision making, Symmetry 16 (2024) 1107.
[17] Sunil Mathew, J. N Mordeson and D. S. Malik, Fuzzy graph theory, Springer International Publishing (2008).

C. Darsana (darsanadinesan@gmail.com)
Department of Mathematics, University of Calicut, postal code 673635, Malappuram, Kerala, India

P. Sini (sini@uoc.ac.in)
Department of Mathemathics, University of Calicut, postal code 673635, Malappuram, Kerala, India