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Existence of solutions for fuzzy fractional impulsive dynamic equations on time scale

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Abstract. In this paper, we consider a class of fuzzy fractional impulsive dynamic equations on time scales of the form

$$\begin{aligned} \delta_H^\alpha y(t) &= p(t)\cdot y(t) + f(t, y(t)),\ t \in [0, T], \\ y(0) &= y_0, \\ y(t_i^+) &= y(t_i^-) + I(t_i, y(t_i)),\ i \in \{1, \ldots, n\}. \end{aligned}$$

Using Krasnosel′skii and Sadovskii′s fixed point theorems, we investigated and established the existence and uniqueness of solutions of this class of equations.

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Corresponding Author: S. K. Samaila(samailask@fuotuoke.edu.ng)

1. Introduction

Mathematical modeling of some phenomena becomes more realistic and suitable by non-continuous dynamical equations, so, in this regard, it is necessary to consider both continuous and discrete models for such problems. These equations can be interpreted by idea of time scales, which was introduced for the first time in 1988 by Stefan Hilger [1] (for more details see [2]). The time scales calculus is a unification of the continuous and discrete analysis, which describes the difference and differential equations together as well as allowing us to deal with combining equations of two differential and difference equations simultaneously (See, for example, [2, 3, 4, 5]). The theory of dynamic equations on time scales has many interesting applications in control theory, mathematical economics, mathematical biology, engineering and technology (See [2, 6, 7, 8, 9]). In some cases, there exists uncertainty, ambiguity or vague factors in such problems, and fuzzy theory and interval analysis are powerful tools for modeling these equations on time scales. In [10], authors introduced and considered the notions of delta derivative and delta integral to fuzzy valued functions on time scales. These definitions may accurately describe fuzzy dynamic processes where time may flow continuously and discretely at different stages in the model; in other words, these concepts are useful in modeling fuzzy start-stop processes as seen in [11], [12] and the references therein.

Shahidi and Khastan [13]introduced the concept of fractional derivative for fuzzy functions on time scales. The presented fuzzy fractional derivative is a natural extension of the generalized Hukuhara derivative. In this paper, our aim is to lies within the investigation of the existence and uniqueness of solution for the class of fuzzy fractional impulsive dynamic equation of the form

$$\begin{array}{rcl} \delta_H^\alpha y(t) &=& p(t)\cdot y(t) + f(t,y(t)),\ t\in[0,T], \\ y(0) &=& y_0, \\ y(t_i^+) &=& y(t_i^-) + I(t_i, y(t_i)),\ i\in\{1,\dots,n\}, \end{array} \tag{1.1}$$

where $y(t_i^\pm) = \lim\limits_{t\to t_i^\pm} y(t)$, $y(t_i) = y(t_i^-)$, $i\in\{1,\dots,n\}$ and $\alpha\in(0,1]$

- **(G1):** $p:\mathbb{T}\to\mathbb{R}, p\in\mathcal{R}, y_0\in F(\mathbb{R})$,
- **(G2):** $f\in\mathcal{C}(\mathbb{T}\times F(\mathbb{R}))$,
- **(G3):** there exist positive functions q and r, defined on $[0,T]$ such that

$$\begin{array}{rcl} \xi &=& \max\limits_{t\in[0,T]} \int\limits_0^t |e_p(t,\sigma(s))| q(s)\Delta s < \infty, \\ \eta &=& \max\limits_{t\in[0,T]} \int\limits_0^t |e_p(t,\sigma(s))| r(s)\Delta s < \infty, \\ D(t^{\alpha-1}\cdot f(t,x), t^{\alpha-1}\cdot f(t,y)) &\le& r(t)D(x,y), \\ D(t^{\alpha-1}\cdot f(t,y),\widetilde{0}) &\le& q(t) + r(t)D(y,\widetilde{0}),\ t\in\mathbb{T}, \quad x,y\in F(\mathbb{R}), \end{array}$$

- **(G4):** there exists a positive constant A such that $D(I(t,x),I(t,y))\le AD(x,y)$, $x,y\in F(\mathbb{R})$, $t\in[0,T]$.

Here $F(\mathbb{R})$ is the set of real fuzzy numbers, $D(\cdot,\cdot)$ is the Hausdorff distance and $\widetilde{0}$ is the zero element of $F(\mathbb{R})$. We investigate the problem (1.1) for existence and uniqueness of the solutions. To the best of our knowledge, there is a gap in the references for investigations of the equation (1.1).

The paper is organized as follows. In the next section, we give some auxiliary results needed for the proof of the main results. In Section 3, we formulate and prove the main results. A conclusion is made in Section 4. Throughout this work, we assume a good knowledge on time scale calculus and fuzzy dynamic calculus on time scales (for more details we refer the reader to the books [14] and [15]).

2. Preliminaries

In this section, we will give some basic facts regarding the fuzzy fractional differentiation and integration on time scales. The exposition follows the paper [13]. Let

$\mathbb{T}$ be a time scale with delta differentiation operator and forward jump operator σ and Δ, respectively.

Definition 2.1 ([13])**.** Let $f : \mathbb{T} \to F(\mathbb{R})$. We say that the fuzzy function f is *fuzzy fractional differentiable of order* α *at* t, if there is an element $\delta_H^\alpha f(t) \in \mathbb{R}_F$ such that for any $\epsilon > 0$ there exists a $\delta > 0$ for which

(i) if $f(t+h) \ominus_H f(\sigma(t))$ and $f(\sigma(t)) \ominus_H f(t-h)$ exist, for any $h > 0$ with $t-h, t+h \in U_{\mathbb{T}}(t,\delta)$, then we have

$$\begin{aligned} D\big((f(t+h) \ominus_H f(\sigma(t)))\, t^{1-\alpha}, \delta_H f(t)(h-\mu(t))\big) &\leq \epsilon(h-\mu(t)), \\ D\big((f(\sigma(t)) \ominus_H f(t-h))\, t^{1-\alpha}, \delta_H^\alpha f(t)(h+\mu(t))\big) &\leq \epsilon(h+\mu(t)). \end{aligned}$$

(ii) if $f(\sigma(t)) \ominus_H f(t+h$ and $f(t-h) \ominus_H f(\sigma(t))$ exist, for any $h > 0$ with $t-h, t+h \in U_{\mathbb{T}}(t,\delta)$, then we have

$$\begin{aligned} D\big((f(\sigma(t)) \ominus_H f(t+h))\, t^{1-\alpha}, \delta_H^\alpha f(t)(\mu(t)-h)\big) &\leq \epsilon(h-\mu(t)), \\ D\big((f(t-h) \ominus_H f(\sigma(t)))\, t^{1-\alpha}, \delta_H^\alpha f(t)(\mu(t)+h)\big) &\leq \epsilon(h+\mu(t)). \end{aligned}$$

The number $\delta_H^\alpha f(t)$ is said to be the α*-derivative of* f *at* t. We say that f is α_1-*differentiable at* t if case 1 holds. In this case, we will write $\delta_H^{\alpha_1} f(t)-$differentiable. We say that f is α_2-*differentiable at* t if the case 2 holds and we will write $\delta_H^{\alpha_2} f(t)-$differentiable. If $A \subset \mathbb{T} \cap (0,\infty)$, we will say that f is α*-differentiable on* A, if it is α-differentiable at any point of the set A.

Definition 2.2 ([16])**.** A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive*, provided $1 + \mu(t)p(t) \neq 0, \forall\ t \in \mathbb{T}^\kappa$. Where $\mu(t) := \sigma(t) - t$ (the grainness function) and the $\kappa-$operator $\mathbb{T}^k$ is given by

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}), & \text{if } \sup \mathbb{T} \leq \infty \\ \mathbb{T}, \text{ otherwise.} \end{cases}$$

The next theorem shows that the α_1-derivative and α_2-derivative are additive operators in terms of the $\kappa-$operator.

Theorem 2.3 ([14])**.** *Let* $f, g : \mathbb{T} \to F(\mathbb{R})$ *be* α*-differentiable at* $t \in \mathbb{T}^\kappa$ *and* $\lambda \in \mathbb{R}$. *Then*

(1) $f + g : \mathbb{T} \to F(\mathbb{R})$ *is* α*-differentiable at* t *and*

$$\delta_H^\alpha (f+g)(t) = \delta_H^\alpha f(t) + \delta_H^\alpha g(t),$$

(2) $\lambda \cdot f : \mathbb{T} \to F(\mathbb{R})$ *is* α*-differentiable at* t *and*

$$\delta_H(\lambda \cdot f)(t) = \lambda \cdot \delta_H^\alpha f(t).$$

Definition 2.4 ([16])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous*, if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exists at all left-dense points in $\mathbb{T}$.

In this article, we denote the set of all regressive and rd-continuous functions on $\mathbb{T}$ to be $\mathfrak{R}$.

Definition 2.5 ([17]). The *exponential function* e_p on time scales for $p \in \mathfrak{R}$ is defined by: for all $s, t \in \mathbb{T}$,

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right).$$

Where $\xi_h(z)$ is the cylinder transformation defined by $\xi_h(z) = \frac{1}{h}\log(1+zh)$.

Definition 2.6 ([13]). Let $f : \mathbb{T} \to F(\mathbb{R})$ be a regulated function and $a, b \in \mathbb{T}$, $a, b > 0$. Then a *fuzzy fractional integral* (shortly α-integral), $t > 0$, is defined by

$$\int f(t)\delta_H^\alpha t = \int t^{\alpha-1} \cdot f(t)\delta_H t \quad \text{and} \quad \int_a^b f(t)\delta_H^\alpha t = \int_a^b t^{\alpha-1} \cdot f(t)\delta_H t.$$

Next we, present some basic properties of the α-integral as follows.

Theorem 2.7 ([13]). *Let $a, b, c \in \mathbb{T}$, $a, b, c > 0$, $\lambda \in \mathbb{R}$ and $f, g : \mathbb{T} \to F(\mathbb{R})$ be rd-continuous. Then*

(1) $\int_a^b (f(t) + g(t))\delta_H^\alpha t = \int_a^b f(t)\delta_H^\alpha t + \int_a^b g(t)\delta_H^\alpha t$,
(2) $\int_a^b \lambda \cdot f(t)\delta_H^\alpha t = \lambda \cdot \int_a^b f(t)\delta_H^\alpha t$,
(3) $\int_a^b \lambda \cdot f(t)\delta_H^\alpha t = \int_a^c \lambda \cdot f(t)\delta_H^\alpha t + \int_c^b \lambda \cdot f(t)\delta_H^\alpha t$,
(4) $\int_a^a \lambda \cdot f(t)\delta_H^\alpha t = \widetilde{0}$.

Theorem 2.8 ([14]). *Let $f : \mathbb{T} \to F(\mathbb{R})$ be continuous and $c \in F(\mathbb{R})$. Then*

(1) $F(t) = c + \int_a^t \lambda \cdot f(\tau)\delta_H^\alpha \tau$, $t \in \mathbb{T}$, $t > 0$, *is α_1-differentiable and* $\delta_H^\alpha F(t) = f(t)$,
(2) $F(t) = c \ominus_H \int_a^t \lambda \cdot f(\tau)\delta_H^\alpha \tau$, $t \in \mathbb{T}$, $t > 0$, *is α_2-differentiable and* $\delta_H^\alpha F(t) = -f(t)$. *Here $\ominus_H$ denotes the H-difference (for more details see [15]).*

Throughout this paper we will consider the case of α_1-derivative and α_1-integral. To prove our main results, we will use the following fixed point theorems.

Theorem 2.9 (Krasnosel'skii Fixed Point Theorem). *Let M be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that*

(1) $A : M \to \mathcal{B}$ *is completely continuous,*
(2) $B : M \to \mathcal{B}$ *is a contraction,*
(3) $x, y \in M$ *implies* $Ax + By \in M$.

Then the map $A + B$ has a fixed point in M.

Theorem 2.10 (Sadovskii's Fixed Point Theorem). *Let P be a condensing operator on a Banach space X. If $P(D) \subset D$ for a convex,closed and bounded set D of X, then P has a fixed point in D.*

3. Major Section

Define

$$
\begin{aligned}
t_{n+1} &= T,\\
J_0 &= [0, t_1],\\
J_k &= (t_k, t_{k+1}], \quad k \in \{1, \ldots, n\},\\
PC &= \Big\{ y : [0, T] \to F(\mathbb{R}), \quad y \in \mathcal{C}(J_k), \quad \exists y(t_k^{\pm}),\\
&\qquad y(t_k^-) = y(t_k), \quad k \in \{1, \ldots, n\} \Big\},\\
PC^1 &= \left\{ y : [0, T] \to F(\mathbb{R}), \quad y \in \mathcal{C}^1(J_k), \quad k \in \{1, \ldots, n\} \right\}.
\end{aligned}
$$

The set PC is a Banach space endowed with the supremum norm

$$\|u\| = \max_{k \in \{0, \ldots, n\}} \|u\|_k,$$

where

$$\|u\|_k = \sup_{t \in J_k} D(u(t), \widetilde{0}), \quad k \in \{0, \ldots, n\}.$$

For $u, v \in PC$, when we write $\|u - v\|$ we have in mind

$$\|u - v\| = \max_{k \in \{0, \ldots, n\}} \|u - v\|_k,$$

where

$$\|u - v\|_k = \sup_{t \in J_k} D(u(t), v(t)), \ k \in \{0, \ldots, n\}.$$

Theorem 3.1. *The function $y \in PC^1$ is a solution of the equation (1.1) if and only if $y \in PC$ satisfies the following integral equation*

$$
\begin{aligned}
y(t) &= \int_0^t e_p(t, \sigma(\tau)) \cdot f(\tau, y(\tau)) \delta_H^\alpha \tau + e_p(t, 0) \cdot y_0\\
&\quad + \sum_{\{k : t_k < t\}} e_p(t, t_k) \cdot I(t_k, y(t_k)).
\end{aligned} \tag{3.1}
$$

Proof. Suppose $y \in PC^1$ is a solution of the equation (1.1). Then for $t \in J_0$, we have

$$
\begin{aligned}
\delta_H^\alpha y(t) &= p(t) \cdot y(t) + f(t, y(t)), \quad t \in J_0,\\
y(0) &= y_0.
\end{aligned} \tag{3.2}
$$

For its solution, we have

$$y(t) = e_p(t, 0) \cdot y_0 + \int_0^t e_p(t, \sigma(\tau)) \cdot f(\tau, y(\tau)) \delta_H^\alpha \tau, \ t \in J_0$$

and

$$y(t_1) = e_p(t_1, 0) \cdot y_0 + \int_0^{t_1} e_p(t_1, \sigma(\tau)) \cdot f(\tau, y(\tau)) \delta_H^\alpha \tau.$$

For $t \in J_1$, we get the following IVP

$$\begin{aligned} \delta_H^\alpha y(t) &= p(t)\cdot y(t) + f(t, y(t)), \quad t \in J_1, \\ y(t_1^+) &= e_p(t_1, 0)\cdot y_0 + \int_0^{t_1} e_p(t_1, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau \\ &\quad + I(t_1, y(t_1)). \end{aligned}$$

For its solution we have the following representation.

$$\begin{aligned} y(t) &= e_p(t, t_1)\cdot\Bigg(e_p(t_1, 0)\cdot y_0 + \int_0^{t_1} e_p(t_1, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau \\ &\quad + I(t_1, y(t_1))\Bigg) + \int_{t_1}^{t} e_p(t, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau \\ &= e_p(t, 0)\cdot y_0 + \int_0^{t_1} e_p(t, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau \\ &\quad + e_p(t, t_1)\cdot I(t_1, y(t_1)) + \int_{t_1}^{t} e_p(t, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau \\ &= e_p(t, 0)\cdot y_0 + \int_0^{t} e_p(t, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau \\ &\quad + e_p(t, t_1)\cdot I(t_1, y(t_1)). \end{aligned}$$

Continuing this integration by parts representation process, we get (3.1). Suppose $y \in PC$ satisfies (3.1). Then for $t \in J_0$, we have

$$y(t) = e_p(t, 0)\cdot y_0 + \int_0^{t} e_p(t, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau. \tag{3.3}$$

We δ_H^α-differentiate the equation (3.3) and we find

$$\begin{aligned} \delta_H^\alpha y(t) &= (p(t)e_p(t, 0))\cdot y_0 + e_p(\sigma(t), \sigma(t))\cdot f(t, y(t)) \\ &\quad + p(t)\cdot\int_0^{t} e_p(t, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau \\ &= p(t)\cdot\left(e_p(t, 0)\cdot y_0 + \int_0^{t} e_p(t, \sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha \tau\right) \\ &\quad + f(t, y(t)) \\ &= p(t)\cdot y(t) + f(t, y(t)),\ t \in J_0. \end{aligned}$$

Moreover, by equation (3.3), we find

$$(3.4)\quad \begin{aligned} y(t_1^-) &= e_p(t_1^-,0)\cdot y_0 + \int_0^{t_1^-} e_p(t_1^-,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha \tau \\ &= e_p(t_1^+,0)\cdot y_0 + \int_0^{t_1^+} e_p(t_1^+,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha \tau. \end{aligned}$$

For $t \in J_1$, we have

$$(3.5)\quad \begin{aligned} y(t) &= e_p(t,0)\cdot y_0 + \int_0^{t} e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha \tau \\ &\quad + e_p(t,t_1)\cdot I(t_1,y(t_1)). \end{aligned}$$

The last equation we δ_H^α-differentiate and we arrive at

$$\begin{aligned} \delta_H^\alpha y(t) &= p(t)\cdot (e_p(t,0)\cdot y_0) + e_p(\sigma(t),\sigma(t))\cdot f(t,y(t)) \\ &\quad + p(t)\cdot \int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha \tau \\ &\quad + p(t)\cdot (e_p(t,t_1)\cdot I(t_1,y(t_1))) \\ &= p(t)\cdot y(t) + f(t,y(t)), \quad t\in J_1. \end{aligned}$$

We put $t = t_1^+$ into (3.5) and using (3.4), we obtain

$$\begin{aligned} y(t_1^+) &= e_p(t_1^+,0)\cdot y_0 + \int_0^{t_1^+} e_p(t_1^+,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha \tau \\ &\quad + I(t_1,y(t_1)) \\ &= y(t_1^-) + I(t_1,y(t_1)). \end{aligned}$$

Substitution and differentiating again, we have $y \in PC^1$ and it satisfies (1.1). This completes the proof.

□

Next, we define self transformations P, Q and R on PC as follows;

$$\begin{aligned} Py(t) &= \int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha \tau, \\ Qy(t) &= e_p(t,0)\cdot y_0 + \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,y(t_k)), \\ Ry(t) &= Py(t) + Qy(t),\ t\in[0,T]. \end{aligned}$$

Lemma 3.2. *Suppose* $(G1)-(G3)$. *Then* $P: PC \to PC$ *is completely continuous.*

Proof. Note that $P : PC \to PC$. Take $y \in PC$ arbitrarily. Then applying ($G3$), we arrive at

$$\begin{aligned} D(Py(t),\widetilde{0}) &= D\left(\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^{\alpha}\tau,\widetilde{0}\right) \\ &= D\left(\int_0^t \left(\tau^{\alpha-1}e_p(t,\sigma(\tau))\right)\cdot f(\tau,y(\tau))\delta_H\tau,\widetilde{0}\right) \\ &\le \int_0^t |e_p(t,\sigma(\tau))| D(t^{\alpha-1}\cdot f(\tau,y(\tau)),\widetilde{0})\Delta\tau \\ &\le \int_0^t |e_p(t,\sigma(\tau))| \left(q(\tau)+r(\tau)D(y(\tau),\widetilde{0})\right)\Delta\tau \\ &= \int_0^t |e_p(t,\sigma(\tau))| q(\tau)\Delta\tau + \int_0^t |e_p(t,\sigma(\tau))| r(\tau)D(y(\tau),\widetilde{0})\Delta\tau \\ &\le \xi + \left(\int_0^t |e_p(t,\sigma(\tau))| r(\tau)\Delta\tau\right)\|y\| \\ &\le \xi + \eta\|y\|, \quad t\in[0,T]. \end{aligned}$$

Thus we find $\|Py\| \le \xi + \eta\|y\|$.

Let $\{y_n\}_{n\in\mathbb{N}} \subset PC$ be such that

$$y_n \to y, \quad \text{as} \quad n \to \infty,$$

where $y \in PC$. Then we have: for $t \in [0,T]$,

$$\begin{aligned} & D(Py_n(t),Py(t)) \\ &= D\left(\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y_n(\tau))\delta_H^{\alpha}\tau, \int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^{\alpha}\tau\right) \\ &= D\left(\int_0^t \left(\tau^{\alpha-1}e_p(t,\sigma(\tau))\right)\cdot f(\tau,y_n(\tau))\delta_H^{\alpha}\tau, \int_0^t \left(\tau^{\alpha-1}e_p(t,\sigma(\tau))\right)\cdot f(\tau,y(\tau))\delta_H\tau\right) \\ &\le \int_0^t |e_p(t,\sigma(\tau))| D(\tau^{\alpha-1}\cdot f(\tau,y_n(\tau)),\tau^{\alpha-1}\cdot f(\tau,y(\tau)))\Delta\tau \\ &\le \int_0^t |e_p(t,\sigma(\tau))| r(\tau)D(y_n(\tau),y(\tau))\Delta\tau \\ &\le \left(\int_0^t |e_p(t,\sigma(\tau))| r(\tau)\Delta\tau\right)\|y_n-y\| \\ &\le \eta\|y_n-y\| \end{aligned}$$

and

$$\|Py_n - Py\| \le \eta\|y_n - y\|.$$

Thus $P : PC \to PC$ is continuous. Now, we take $y \in PC$ arbitrarily and $t_1, t_2 \in [0, T]$. Without loss of generality, suppose that $t_1 < t_2$. Then

$$\begin{aligned} D(Py(t_2), Py(t_1)) &= D\Big(\int_0^{t_2} e_p(t_2, \sigma(\tau)) \cdot f(\tau, y(\tau))\delta_H^\alpha \tau, \\ &\qquad \int_0^{t_1} e_p(t_1, \sigma(\tau)) \cdot f(\tau, y(\tau))\delta_H^\alpha \tau\Big) \end{aligned}$$

$$\begin{aligned} &= D\Big(\int_0^{t_1} (e_p(t_2, \sigma(\tau)) - e_p(t_1, \sigma(\tau))) \cdot f(\tau, y(\tau))\delta_H^\alpha \tau, \\ &\qquad \int_{t_1}^{t_2} e_p(t_2, \sigma(\tau)) \cdot f(\tau, y(\tau))\delta_H^\alpha \tau\Big) \\ &= D\Big(\int_0^{t_1} \big(\tau^{\alpha-1} e_p(t_2, \sigma(\tau)) - e_p(t_1, \sigma(\tau))\big) \cdot f(\tau, y(\tau))\delta_H \tau, \\ &\qquad \int_{t_1}^{t_2} \big(\tau^{\alpha-1} e_p(t_2, \sigma(\tau))\big) \cdot f(\tau, y(\tau))\delta_H^\alpha \tau\Big) \end{aligned}$$

$$\begin{aligned} &\leq |e_p(t_2, t_1) - 1| \int_0^{t_1} |e_p(t_1, \sigma(\tau))| D(\tau^{\alpha-1} \cdot f(\tau, y(\tau)), \widetilde{0})\Delta\tau \\ &\quad + \int_{t_1}^{t_2} |e_p(t_2, \sigma(\tau))| D(\tau^{\alpha-1} \cdot f(\tau, y(\tau)), \widetilde{0})\Delta\tau \\ &\to 0, \quad \text{as} \quad t_1 \to t_2. \end{aligned}$$

Now, using the Arzela-Ascoli theorem, we conclude that $P : PC \to PC$ is completely continuous. This completes the proof. □

Lemma 3.3. *Suppose that (G1)-(G4) hold. Let also,*

$$A = \max_{t \in [0,T]} \sum_{k=1}^{n} |e_p(t, t_k)| \leq \zeta < 1. \tag{3.6}$$

Then $Q : PC \to PC$ is a contraction.

Proof. Note that $e_p(t,t_k)$, $k \in \{1,\dots,n\}$, is a continuous function. Then $Q : PC \to PC$. Let now, $y, z \in PC$ be arbitrarily chosen. Then we have

$$\begin{aligned}
D(Qy(t), Qz(t)) &= D\Bigg(e_p(t,0)\cdot y_0 + \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k)),\\
&\qquad e_p(t,0)\cdot y_0 + \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, z(t_k))\Bigg)\\
&= D\left(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k)), \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, z(t_k))\right)\\
&\leq \sum_{\{k:t_k<t\}} |e_p(t,t_k)| D(I(t_k, y(t_k)), I(t_k, z(t_k)))\\
&\leq A \sum_{\{k:t_k<t\}} |e_p(t,t_k)| D(y(t_k), z(t_k))\\
&\leq A \sum_{\{k:t_k<t\}} |e_p(t,t_k)| \|y-z\|\\
&\leq \zeta\|y-z\|, \quad t\in[0,T],
\end{aligned}$$

where $\|Qy - Qz\| \leq \zeta\|y-z\|$. Then $Q : PC \to PC$ is a contraction. This completes the proof. □

Theorem 3.4. *Assume that* $\gamma = \max_{t\in[0,T]} \sum_{k=1}^{n} |e_p(t,t_k)| D(I(t_k,0),\widetilde{0}) < \infty$. *Let (G1)-(G4) and (3.6) hold. Let also,* ξ, γ, η, A *and* S *be positive numbers satisfying the inequality*

$$\xi + \gamma + \max_{t\in[0,T]} |e_p(t,0)| D(y_0,\widetilde{0}) + S\left(\eta + A + \max_{t\in[0,T]} \sum_{k=1}^{n} |e_p(t,t_k)|\right) \leq S. \tag{3.7}$$

Then the equation (1.1) has a solution y *such that* $\|y\| \leq S$.

Proof. Define the set $C = \{y \in PC : \|y\| \leq S\}$. By Lemma 3.2, it follows that $P : PC \to PC$ is completely continuous. By Lemma 3.3, it follows that $Q : PC \to PC$ is a contraction. Now let $y, z \in C$ be arbitrarily chosen. Then $\|y\| \leq S$, $\|z\| \leq S$ and

$$\begin{aligned}
&D(Py(t) + Qz(t), \widetilde{0})\\
&= D\Bigg(\int_0^t e_p(t,\sigma(s))\cdot f(s,y(s))\delta_H^\alpha s + \sum_{k=1}^{n} e_p(t,t_k)\cdot I(t_k,(t_k)) + e_p(t,0)\cdot y_0, \widetilde{0}\Bigg)\\
&= D[\int_0^t \left(s^{\alpha-1} e_p(t,\sigma(s))\right)\cdot f(s,y(s))\delta_H s + \sum_{k=1}^{n} e_p(t,t_k)\cdot I(t_k, z(t_k)) + e_p(t,0)\cdot y_0, \widetilde{0}]\\
&\leq D[\int_0^t \left(s^{\alpha-1} e_p(t,\sigma(s))\right)\cdot f(s,y(s))\delta_H s, \widetilde{0}] + D\Bigg(\sum_{k=1}^{n} e_p(t,t_k)\cdot I(t_k, z(t_k)), \widetilde{0}\Bigg)\\
&\quad + D\Big(e_p(t,0)\cdot y_0, \widetilde{0}\Big)
\end{aligned}$$

$$\leq \int_0^t |e_p(t,\sigma(s))| D(s^{\alpha-1}\cdot f(s,y(s)),\widetilde{0})\Delta s + \sum_{k=1}^n |e_p(t,t_k)| D(I(t_k,z(t_k)),\widetilde{0})$$
$$+|e_p(t,0)|D(y_0,\widetilde{0})$$
$$\leq \int_0^t |e_p(t,\sigma(s))| D(s^{\alpha-1}\cdot f(s,y(s)),\widetilde{0})\Delta s + \sum_{k=1}^n |e_p(t,t_k)| (D(I(t_k,z(t_k)),I(t_k,0))$$
$$+D(I(t_k,0),\widetilde{0})) + |e_p(t,0)|D(y_0,\widetilde{0})$$
$$\leq \int_0^t |e_p(t,\sigma(s))| \left(q(s)+r(s)D(y(s),\widetilde{0})\right)\Delta s + \sum_{k=1}^n |e_p(t,t_k)| D(I(t_k,z(t_k)),I(t_k,0))$$
$$+\sum_{k=1}^n |e_p(t,t_k)| D(I(t_k,0),\widetilde{0}) + |e_p(t,0)|D(y_0,\widetilde{0})$$
$$\leq \xi + \eta\|y\| + A\sum_{k=1}^n |e_p(t,t_k)| D(z(t_k),\widetilde{0}) + \gamma + |e_p(t,0)|D(y_0,\widetilde{0})$$
$$\leq \xi + \gamma + \eta S + |e_p(t,0)|D(y_0,\widetilde{0}) + \left(A\sum_{k=1}^n |e_p(t,t_k)|\right) S$$
$$= \xi + \gamma + |e_p(t,0)|D(y_0,\widetilde{0}) + S\left(\eta + A\sum_{k=1}^n |e_p(t,t_k)|\right)$$
$$\leq \xi + \gamma + \max_{t\in[0,T]} |e_p(t,0)|D(y_0,\widetilde{0}) + S\left(\eta + A\max_{t\in[0,T]}\sum_{k=1}^n |e_p(t,t_k)|\right)$$
$$\leq S, \quad t\in[0,T].$$

Thus

$$\|Py+Qz\| \leq S, \quad t\in[0,T],$$

i.e., $Py+Qz \in C$. So the Krasnosel'skii fixed point theorem, it follows that the operator $P+Q$ has a fixed point in C. Now, using Theorem 3.1, we conclude that the equation (1.1) has a solution in the set C. This completes the proof. □

Theorem 3.5. *Assume* (G1)-(G4), (3.6), (3.7), *Theorem 3.7 and* $\eta+\zeta<1$. *Then the equation* (1.1) *has a unique solution.*

Proof. By Theorem 3.4, it follows that the equation (1.1) has at least one solution in the set C. Suppose that the equation (1.1) has two solutions y and z. Then

$$D(Ry(t),Rz(t)) = D\Bigg(\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha\tau + e_p(t,0)\cdot y_0$$
$$+\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,y(t_k)),$$
$$\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,z(\tau))\delta_H^\alpha\tau + e_p(t,0)\cdot y_0$$
$$+\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,z(t_k))\Bigg)$$
$$= D\Bigg(\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha\tau + \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,y(t_k)),$$
$$\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,z(\tau))\delta_H^\alpha\tau + \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,z(t_k))\Bigg)$$
$$= D\Bigg(\int_0^t e_p(t,\sigma(\tau))\cdot (f(\tau,y(\tau))\ominus_H f(\tau,z(\tau)))\delta_H^\alpha\tau,$$

$$\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot (I(t_k,z(t_k))\ominus_H I(t_k,y(t_k)))\Big)$$
$$\leq D\Big(\int_0^t e_p(t,\sigma(\tau))\cdot (f(\tau,y(\tau))\ominus_H f(\tau,z(\tau)))\delta_H^\alpha \tau, \widetilde{0}\Big)$$
$$+D\Big(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot (I(t_k,y(t_k))\ominus_H I(t_k,z(t_k))), \widetilde{0}\Big)$$
$$= D\Big(\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha \tau, \int_0^t e_p(t,\sigma(\tau))\cdot f(\tau,z(\tau))\delta_H^\alpha \tau\Big)$$
$$+D\Big(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,y(t_k)), \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,z(t_k))\Big)$$
$$= D\Big(\int_0^t \left(\tau^{\alpha-1}e_p(t,\sigma(\tau))\right)\cdot f(\tau,y(\tau))\delta_H \tau,$$
$$\int_0^t \left(\tau^{\alpha-1}e_p(t,\sigma(\tau))\right)\cdot f(\tau,z(\tau))\delta_H \tau\Big)$$
$$+D\Big(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,y(t_k)), \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k,z(t_k))\Big)$$
$$\leq \int_0^t |e_p(t,\sigma(\tau))| D(\tau^{\alpha-1}\cdot f(\tau,y(\tau)), \tau^{\alpha-1}\cdot f(\tau,z(\tau)))\Delta\tau$$
$$+\sum_{\{k:t_k<t\}} |e_p(t,t_k)| D(I(t_k,y(t_k)), I(t_k,z(t_k)))$$
$$\leq \int_0^t |e_p(t,\sigma(\tau))| r(\tau) D(y(\tau),z(\tau))\Delta\tau$$
$$+A\sum_{\{k:t_k<t\}} |e_p(t,t_k)| D(y(t_k),z(t_k))$$
$$\leq \eta\|y-z\| + \zeta\|y-z\|$$
$$= (\eta+\zeta)\|y-z\|$$
$$< \|y-z\|, \quad t\in[0,T].$$

Thus $\|y-z\| = \|Ry-Rz\| < \|y-z\|$, which is a contradiction. This completes the proof. □

Suppose

(G5): There exist an increasing functions $\ell_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$D(I_k(t_k,y(t_k)),\widetilde{0}) \leq l_k(D(y(t_k),\widetilde{0})), \quad k\in\{0,\dots,n\},$$

and $\lim\limits_{r\to\infty} \frac{l_k(r)}{r} = 0, \quad k\in\{0,\dots,n\}$.

(G6): For each $r>0$ there exists a continuous function $h_r : [0,T]\to(0,\infty)$ such that

$$D(t^{\alpha-1}\cdot f(t,y),\widetilde{0}) \leq h_k(t) \quad \text{for each} \quad (t,y)\in[0,T]\times F(\mathbb{R}), \quad D(y,\widetilde{0})\leq r,$$

and $\liminf_{r\to\infty} \frac{1}{r}\int_0^T h_r(s)\Delta s = \beta_1 < \infty$.

Theorem 3.6. *Suppose* (G1), (G2), (G4)–(G6) *and*

$$\beta_1 \max_{(t,\tau)\in[0,T]\times[0,T]} |e_p(t,\sigma(\tau))| < 1. \tag{3.8}$$

Then the equation (1.1) has at least one solution.

Proof. We shall proof this theorem in three phases or cases. Let $R : PC \to PC$.

Case I : We will prove that R is continuous. Let $\{y_m\}_{m\in\mathbb{N}}$ be a sequence of elements of PC such that $y_m \to y$, as $m \to \infty$, $y \in PC$. Then

$$
\begin{aligned}
&D(Ry_m(t), Ry(t))\\
&= D\Big[\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau, y_m(\tau))\delta_H^\alpha\tau + e_p(t,0)\cdot y_0\\
&+ \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y_m(t_k)) \int_0^t e_p(t,\sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha\tau\\
&+ e_p(t,0)\cdot y_0 + \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k))\Big]\\
&= D\bigg(\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau, y_m(\tau))\delta_H^\alpha\tau\\
&+ \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y_m(t_k)) \int_0^t e_p(t,\sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha\tau\\
&+ \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k))\bigg)\\
&= D\bigg(\int_0^t e_p(t,\sigma(\tau))\cdot (f(\tau, y_m(\tau)) \ominus_H f(\tau, y(\tau)))\delta_H^\alpha\tau,\\
&\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot (I(t_k, y(t_k)) \ominus_H I(t_k, y_m(t_k)))\bigg)\\
&\le D\bigg(\int_0^t e_p(t,\sigma(\tau))\cdot (f(\tau, y_m(\tau)) \ominus_H f(\tau, y(\tau)))\delta_H^\alpha\tau, \widetilde{0}\bigg)\\
&+ D\bigg(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot (I(t_k, y_m(t_k)) \ominus_H I(t_k, y(t_k))), \widetilde{0}\bigg)\\
&= D\bigg(\int_0^t e_p(t,\sigma(\tau))\cdot f(\tau, y_m(\tau))\delta_H^\alpha\tau, \int_0^t e_p(t,\sigma(\tau))\cdot f(\tau, y(\tau))\delta_H^\alpha\tau\bigg)\\
&+ D\bigg(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y_m(t_k)), \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k))\bigg)\\
&= D\bigg(\int_0^t \big(\tau^{\alpha-1}e_p(t,\sigma(\tau))\big)\cdot f(\tau, y_m(\tau))\delta_H\tau, \int_0^t \big(\tau^{\alpha-1}e_p(t,\sigma(\tau))\big)\cdot f(\tau, y(\tau))\delta_H^\alpha\tau\bigg)\\
&+ D\bigg(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y_m(t_k)), \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k))\bigg)\\
&\le \int_0^t |e_p(t,\sigma(\tau))| D(\tau^{\alpha-1}\cdot f(\tau, y_m(\tau)), \tau^{\alpha-1}\cdot f(\tau, y(\tau)))\Delta\tau\\
&+ \sum_{\{k:t_k<t\}} |e_p(t,t_k)| D(I(t_k, y_m(t_k)), I(t_k, y(t_k)))\\
&\le \max_{(t,\tau)\in[0,T]\times[0,T]} |e_p(t,\sigma(\tau))| \int_0^t D(\tau^{\alpha-1}\cdot f(\tau, y_m(\tau)), \tau^{\alpha-1}\cdot f(\tau, y(\tau)))\Delta\tau\\
&+ A \sum_{\{k:t_k<t\}} \max_{t\in[0,T]} |e_p(t,t_k)| D(y_m(t_k), y(t_k)) \to 0, \quad \text{as} \quad m\to\infty, \quad t\in[0,T].
\end{aligned}
$$

Thus $\|Ry_m - Ry\| \to 0$ as $m \to \infty$.

Case II : For each constant $r > 0$, denote $B_r = \{y \in PC : \|y\| \leq r\}$. Note that B_r is a bounded, closed and convex set in PC. We will prove that there exists a number $r > 0$ such that $R(B_r) \subseteq B_r$. Suppose the contrary, i.e., for each $r > 0$ there exists a function $y_r \in B_r$ such that $D(R(y_r(t)), \widetilde{0}) > r$ for some $t \in [0, T]$. Then

$$\begin{aligned}
r &< D(R(y_r(t)), \widetilde{0}) \\
&= D\bigg(\int_0^t e_p(t, \sigma(\tau)) \cdot f(\tau, y_r(\tau)) \delta_H^\alpha \tau + e_p(t, 0) \cdot y_0 \\
&+ \sum_{\{k: t_k < t\}} e_p(t, t_k) \cdot I(t_k, y_r(t_k)), \widetilde{0} \bigg) \\
&\leq D\bigg(\int_0^t e_p(t, \sigma(\tau)) \cdot f(\tau, y_r(\tau)) \delta_H^\alpha \tau, \widetilde{0} \bigg) + D(e_p(t, 0) \cdot y_0, \widetilde{0}) \\
&+ D\bigg(\sum_{\{k: t_k < t\}} e_p(t, t_k) \cdot I(t_k, y_r(t_k)), \widetilde{0} \bigg) \\
&= D\bigg(\int_0^t \left(\tau^{\alpha-1} e_p(t, \sigma(\tau)) \right) \cdot f(\tau, y_r(\tau)) \delta_H \tau, \widetilde{0} \bigg) + D(e_p(t, 0) \cdot y_0, \widetilde{0}) \\
&+ D\bigg(\sum_{\{k: t_k < t\}} e_p(t, t_k) \cdot I(t_k, y_r(t_k)), \widetilde{0} \bigg) \\
&\leq \int_0^t |e_p(t, \sigma(\tau))| D(\tau^{\alpha-1} \cdot f(\tau, y_r(\tau)), \widetilde{0}) \Delta \tau + |e_p(t, 0)| D(y_0, \widetilde{0}) \\
&+ \sum_{\{k: t_k < t\}} |e_p(t, t_k)| D(I(t_k, y_r(t_k)), \widetilde{0}) \\
&\leq \max_{t \in [0,T]} |e_p(t, 0)| D(y_0, \widetilde{0}) + \int_0^t |e_p(t, \sigma(\tau))| D(\tau^{\alpha-1} \cdot f(\tau, y_r(\tau)), \widetilde{0}) \Delta \tau \\
&+ \sum_{\{k: t_k < t\}} |e_p(t, t_k)| D(I(t_k, y_r(t_k)), \widetilde{0}) \\
&\leq \max_{t \in [0,T]} |e_p(t, 0)| D(y_0, \widetilde{0}) + \max_{(t,\tau) \in [0,T] \times [0,T]} |e_p(t, \sigma(\tau))| \int_0^t h_r(s) \Delta s \\
&+ \sum_{\{k: t_k < t\}} \max_{t \in [0,T]} |e_p(t, t_k)| l_k \left(D(y(t_k), \widetilde{0}) \right) \\
&\leq \max_{t \in [0,T]} |e_p(t, 0)| D(y_0, \widetilde{0}) \\
&+ \max_{(t,\tau) \in [0,T] \times [0,T]} |e_p(t, \sigma(\tau))| \int_0^t h_r(s) \Delta s + \sum_{\{k: t_k < t\}} \max_{t \in [0,T]} |e_p(t, t_k)| l_k(r).
\end{aligned}$$

Thus

$$\begin{aligned}
1 &< \max_{t \in [0,T]} |e_p(t, 0)| \tfrac{D(y_0, \widetilde{0})}{r} \\
&+ \max_{(t,\tau) \in [0,T] \times [0,T]} |e_p(t, \sigma(\tau))| \tfrac{1}{r} \int_0^t h_r(s) \Delta s \\
&+ \sum_{\{k: t_k < t\}} \max_{t \in [0,T]} |e_p(t, t_k)| \tfrac{l_k(r)}{r}
\end{aligned}$$

and taking limits as $r \to \infty$, we find

$$1 \leq \max_{t\in[0,T]} |e_p(t,0)| \lim_{r\to\infty} \frac{D(y_0,\widetilde{0})}{r}$$
$$+ \max_{(t,\tau)\in[0,T]\times[0,T]} |e_p(t,\sigma(\tau))| \lim_{r\to\infty} \frac{1}{r}\int_0^t h_r(s)\Delta s$$
$$+ \sum_{\{k:t_k<t\}} \max_{t\in[0,T]} |e_p(t,t_k)| \lim_{r\to\infty} \frac{l_k(r)}{r}$$
$$= \beta_1 \max_{(t,\tau)\in[0,T]\times[0,T]} |e_p(t,\sigma(\tau))|,$$

which is a contradiction to our assumption and consequently, $\exists\ r_1 > 0$ such that $R(B_{r_1}) \subseteq B_{r_1}$.

Case III : Here, we are to show that $R(B_{r_1})$ is an equi-continuous family of functions. To this end, Let $t^1, t^2 \in [0,T]$ be arbitrarily chosen. Without loss of generality, suppose $t^1 < t^2$. Then

$$D(R(y)(t^2), R(y)(t^1))$$
$$= D[\int_0^{t^2} e_p(t^2,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha\tau + e_p(t^2,0)\cdot y_0$$
$$+ \sum_{\{k:t_k<t^2\}} e_p(t^2,t_k)\cdot I(t_k,y(t_k)), \int_0^{t^1} e_p(t^1,\sigma(\tau)) f(\tau,y(\tau))\delta_H^\alpha\tau$$
$$+ e_p(t^1,0)\cdot y_0 + \sum_{\{k:t_k<t^1\}} e_p(t^1,t_k)\cdot I(t_k,y(t_k))]$$
$$= D\Bigg(\int_0^{t^1} \big(e_p(t^2,\sigma(\tau)) - e_p(t^1,\sigma(\tau))\big)\cdot f(\tau,y(\tau))\delta_H^\alpha\tau$$
$$+ \int_{t^1}^{t^2} e_p(t^2,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha\tau + \big(e_p(t^2,0) - e_p(t^1,0)\big)\cdot y_0$$
$$+ \sum_{\{k:t_k<t^1\}} \big(e_p(t^2,t_k) - e_p(t^1,t_k)\big)\cdot I(t_k,y(t_k)) + \sum_{\{k:t^1\leq t_k<t^2\}} e_p(t^2,t_k)\cdot I(t_k,y(t_k)), \widetilde{0}\Bigg)$$
$$\leq D\Bigg(\int_0^{t^1} \Big(\tau^{\alpha-1}\big(e_p(t^2,\sigma(\tau)) - e_p(t^1,\sigma(\tau))\big)\Big)\cdot f(\tau,y(\tau))\delta_H\tau, \widetilde{0}\Bigg)$$
$$+ D[\int_{t^1}^{t^2} e_p(t^2,\sigma(\tau))\cdot f(\tau,y(\tau))\delta_H^\alpha\tau, \widetilde{0}]$$
$$+ D\Big(\big(e_p(t^2,0) - e_p(t^1,0)\big)\cdot y_0, \widetilde{0}\Big) + D\Bigg(\sum_{\{k:t_k<t^1\}} \big(e_p(t^2,t_k) - e_p(t^1,t_k)\big)\cdot I(t_k,y(t_k)), \widetilde{0}\Bigg)$$
$$+ D\Bigg(\sum_{\{k:t^1\leq t_k<t^2\}} e_p(t^2,t_k)\cdot I(t_k,y(t_k)), \widetilde{0}\Bigg)$$
$$\leq \int_0^{t^1} \big|e_p(t^2,\sigma(\tau)) - e_p(t^1,\sigma(\tau))\big| D(\tau^{\alpha-1}\cdot f(\tau,y(\tau)), \widetilde{0})\Delta\tau$$
$$+ \int_{t^1}^{t^2} \big|e_p(t^2,\sigma(\tau))\big| D(f(\tau,y(\tau)),\widetilde{0})\Delta\tau + \big|e_p(t^2,0) - e_p(t^1,0)\big| D(y_0,\widetilde{0})$$
$$+ \sum_{\{k:t_k<t^1\}} \big|e_p(t^2,t_k) - e_p(t^1,t_k)\big| D(I(t_k,y(t_k)),\widetilde{0}) + \sum_{\{k:t^1\leq t_k<t^2\}} \big|e_p(t^2,t_k)\big| D(I(t_k,y(t_k)),\widetilde{0})$$

$\to 0, \quad \text{as} \quad t^1 \to t^2.$
From this and the Arzela-Ascoli theorem, we conclude that $R : PC \to PC$ is completely continuous and it is condensing. Thus, Sadovskii's fixed point theorem, it follows that R has a fixed point in PC, which is a solution of the equation (1.1). □

Theorem 3.7. *Suppose* ($G1$), ($G2$), ($G4$), ($G6$), (3.6) *and* $A = \sum_{\{k:t_k<t\}} \max_{t\in[0,T]} |e_p(t,t_k)| < 1$. *Then the equation* (1.1) *has at least one solution.*

Proof. As we have proved in Theorem 3.6, there exists a positive number r such that $R(B_r) \subseteq B_r$ and the operator Q is a compact operator on B_r. Let now, $y, z \in PC$. Then

$$\begin{aligned}
&D(Py(t), Pz(t))\\
&= D\Big(e_p(t,0)\cdot y_0 + \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k)), e_p(t,0)\cdot y_0\\
&+ \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, z(t_k))\Big)\\
&= D\Big(\sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, y(t_k)), \sum_{\{k:t_k<t\}} e_p(t,t_k)\cdot I(t_k, z(t_k))\Big)\\
&\leq \sum_{\{k:t_k<t\}} |e_p(t,t_k)| D(I(t_k, y(t_k)), I(t_k, z(t_k)))\\
&\leq A \sum_{\{k:t_k<t\}} \max_{t\in[0,T]} |e_p(t,t_k)| D(y(t_k), z(t_k))\\
&\leq A \sum_{\{k:t_k<t\}} \max_{t\in[0,T]} |e_p(t,t_k)| \|y - z\|, \quad t \in [0,T].
\end{aligned}$$

Thus

$$\begin{aligned}
&\|Py - Pz\|\\
&\leq A \sum_{\{k:t_k<t\}} \max_{t\in[0,T]} |e_p(t,t_k)| \|y - z\| < \|y - z\|.
\end{aligned}$$

Consequently $R = P + Q$ is a condensing operator on B_r. So by the Krasnosel'skii fixed point theorem, it follows that the equation (1.1) has a solution in PC. This completes the proof. □

4. Conclusions

The authors [13] introduced a new concept for fuzzy fractional differentiation and integration on time scales. Using this concept, we investigated a class of fuzzy fractional impulsive dynamic equations on arbitrary time scales. We establish existence and uniqueness of the solutions of the considered equations using the Krasnosel'skii and Sadovskii fixed point theorem. In our future study, stability of solutions using fixed point techniques will be investigated, leading to the understanding of perturbation effects on the solutions, which is essential for ill-posed problems.

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Svetlin G. Georgiev (svetlingeorgiev1@gmail.com)
Sorbonne University, Paris, France

Alhaji J. Alkali (alkali4real2012@gmail.com)
Department of Mathematics, Ahmadu Bello University, Zaria

Sylvanus K. Samaila (samailask@fuootuoke.edu.ng)
Department of Mathematics and Statistics, Federal University Otuoke