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Abstract.
In this paper, we introduce the concept of interval-valued bipolar fuzzy graphs and obtain some of its basic properties, and give some examples. Also, we define an isomorphism and an isometric between interval-valued bipolar fuzzy graphs and discuss with some of their properties.

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Keywords: Interval-valued bipolar fuzzy set, Interval-valued bipolar fuzzy graph, Complete interval-valued bipolar fuzzy graph, Self complementary, Isomorphism, Automorphism, Isometry.

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1. Introduction

Graph theory has numerous applications to problems in computer science, electrical engineering, system analysis, economics, networking routing and transportation, etc. To overcome containing various kinds of uncertainty, in 1965, Zadeh [1] had introduced the concept of a fuzzy set as the generalization of a crisp set. Also he [2] had defined the notion of interval-valued fuzzy sets playing basic roles in many fields of pure and applied science (See [3]). In 1975, Rosenfeld [4] introduced the concept of fuzzy graphs considered by the fuzzy relations between fuzzy sets and developed the structures of fuzzy graphs. After then, Bhattacharya [5] dealt with some properties on fuzzy graphs. Mordeson and Peng [6] introduced some operations on fuzzy graphs. In particular, Dey et al. [7] dealt with genetic algorithms for solving fuzzy shortest path problems (See [8, 9, 10, 11, 12] for further research papers). Akram and Dudek [13] introduced the concept of interval-valued fuzzy graphs and studied some of its properties. Also they dealt with some operations on interval-valued fuzzy graphs. After that time, Pal et al. [14] studied further results related to interval-valued fuzzy graphs. Talebi and Rashmalou [15] investigated properties of isomorphism and the complements on interval-valued fuzzy graphs (See [16, 17] for further research papers).

In 1994, Zhang [18] introduced the concept of a bipolar fuzzy set (Refer to [19, 20, 21]) as a generalization of fuzzy sets. Bosc and Pivert [22] said "Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. However, negative statements express what is impossible, rejected, or forbidden." After then, Jun and Park [23], Jun et al. [24] and Lee [25] applied bipolar fuzzy sets to BCK/BCI-algebras. Recently, Kim et al. [26] studied neighborhood systems in bipolar fuzzy topological spaces. Lee and Hur [27] dealt with various properties of bipolar fuzzy relations. In particular, Akram [28, 29] introduced the notion of bipolar fuzzy graphs and regular bipolar fuzzy graphs and obtained some properties of self complementary and self weak complementary strong bipolar fuzzy graph. Talebi et al [30] investigated some operations on bipolar fuzzy graph.

The motivation behind introducing interval-valued bipolar fuzzy graphs (IVBFGs) arises from the need to model systems that exhibit both positive and negative influences with inherent uncertainty. Many real-world problems involve duality (positive/negative) and imprecision, which cannot be effectively captured by classical or even interval-valued fuzzy graphs alone. IVBFGs provide a more flexible framework for representing such complex relationships.

Potential application areas of IVBFGs include decision support systems, social network analysis, medical diagnosis systems, recommendation systems, and risk assessment, where both positive and negative aspects coexist and uncertainty is prevalent. The main contributions of this paper are summarized as follows:

- We introduce the concept of interval-valued bipolar fuzzy graphs (IVBFGs) and establish their fundamental properties.
- We define basic operations and structural properties of IVBFGs.
- We suggest possible future research directions in this domain.

The purpose of our research is to study the structure of a graph based on an interval-valued bipolar fuzzy set. To do this, the research is conducted in the following order: In Section 2, we recall some basic concepts for graphs and interval-valued bipolar fuzzy sets. In Section 3, we define an interval-valued bipolar fuzzy graph and some operations on interval-valued bipolar fuzzy graphs, and deal with some of their basic properties and give some examples. In Section 4, we introduce the notions of isomorphisms and isometrics between interval-valued bipolar fuzzy graphs, and study some of their properties.

2. Preliminaries

In this section, we review some definitions of undirected graphs needed in next sections. Also, we list the concept of bipolar fuzzy set, the complement of a bipolar fuzzy set, the inclusion between two bipolar fuzzy sets, the union and the intersection of two bipolar fuzzy sets. Throughout this paper, we will denote the $[0, 1]$ and $[-1, 0]$ as I and $-I$, respectively. Also, we will denote the set of all closed subintervals of I and $-I$ as $[I]$ and $[-I]$, respectively.

Definition 2.1 ([31]). (i) A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^* such that every edge corresponds to a two-element subset of V. We use the notation xy instead of $\{x, y\}$. In this case, the number of vertices [resp. of edges] in V is called the order [resp. size] of G^* and is denoted by $|V|$ [resp. $|E|$]. In particular, a graph G^*, we denote the set of vertices and edges of G^* as $V(G^*)$ and $E(G^*)$, respectively.

(ii) $G^* = (V, E)$ is called a (p, q) graph, if $|V| = p$ and $|E| = q$.

(iii) Two vertices x and y in an undirected graph G^* are said to be adjacent in G^*, if (x, y) is an edge of G^*.

(vi) A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices.

Definition 2.2 ([31]). Let $G^* = (V, E)$ and $H^* = (W, F)$ be two graphs. Then $H^* = (W, F)$ is called a subgraph of $G^* = (V, E)$, if $W \subset V$ and $F \subset E$.

Definition 2.3 ([31]). Let $G^* = (V, E)$ be a graph. Then $\overline{G^*}$ is called the complement of G^*, if $V(\overline{G^*}) = V$ and $e \in E(\overline{G^*})$ if and only if $e \notin E$.

It is clear that two vertices are adjacent in $\overline{G^*}$ if and only if they are nor adjacent in G^*.

Definition 2.4 ([31]). Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs. Then the Cartesian product of G_1^* and G_2^*, denoted by $G^* = G_1^* \times G_2^* = (V, E)$, is a graph defined as follows:

(i) $V = V_1 \times V_2$,

(ii) $E = \{(x, x_2)(x, y_2) : x \in V_1,\ x_2y_2 \in E_2\} \cup \{(x_1, y)(x_2, y) : y \in V_2,\ x_1y_1 \in E_1\}$.

Definition 2.5 ([31]). Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs. Then the composition of G_1^* and G_2^*, denoted by $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, is the graph, where

$$E^0 = E \cup \{(x_1, x_2)(y_1, y_2) : x_1y_1 \in E_1,\ x_2 \neq y_2\}$$

and E is defined in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

Definition 2.6 ([31]). Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs.

(i) The union of G_1^* and G_2^*, denoted by $G_1^* \cup G_2^*$, is the simple graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. In fact, $G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$.

(ii) The join of G_1^* and G_2^*, denoted by $G_1^* + G_2^*$, is the simple graph with the vertex set $V_1 \cup V_2$ such that $V_1 \cap V_2 \neq \varnothing$ and the edge set $E_1 \cup E_2 \cup E^{'}$, where $E^{'}$ is the set of all edges between vertices of G_1^* and vertices of G_2^*. In fact, $G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E^{'})$.

Definition 2.7 ([31]). Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs. Then we say that G_1^* and G_2^* are isomorphic, denoted by $G_1^* \simeq G_2^*$, if there is a bijection $f : V_1 \to V_2$ such that for any $x,\ y \in V_1$, $xy \in E_1$ if and only if $f(x)f(y) \in E_2$.

Definition 2.8 ([19]). Let X be a nonempty set. Then a pair $A = (A^P, A^N)$ is called a bipolar-valued fuzzy set (or, bipolar fuzzy set) in X, if $A^P : X \to I$ and $A^N : X \to -I$ are mappings.

In particular, the bipolar fuzzy empty set [resp. the bipolar fuzzy whole set] (See [32]), denoted by $0_{bp} = (0^P_{bp}, 0^N_{bp})$ [resp. $1_{bp} = (1^P_{bp}, 1^N_{bp})$], is a bipolar fuzzy set in X

defined by: for each $x \in X$,

$$0^P_{bp}(x) = 0 = 0^N_{bp}(x) \text{ [resp. } 1^P_{bp}(x) = 1 \text{ and } 1^N_{bp}(x) = -1].$$

We will denote the set of all bipolar fuzzy sets in X as $BPF(X)$.

For each $x \in X$, we use the positive membership degree $A^P(x)$ to denote the satisfaction degree of the element x to the property corresponding to the bipolar fuzzy set A and the negative membership degree $A^N(x)$ to denote the satisfaction degree of the element x to some implicit counter-property corresponding to the bipolar fuzzy set A.

If $A^P(x) \neq 0$ and $A^N(x) = 0$, then it is the situation that x is regarded as having only positive satisfaction for A. If $A^P(x) = 0$ and $A^N(x) \neq 0$, then it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter-property of A. It is possible for some $x \in X$ to be such that $A^P(x) \neq 0$ and $A^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of X.

It is obvious that for each $A \in BPF(X)$ and $x \in X$, if $0 \leq A^P(x) - A^N(x) \leq 1$, then A is an intutionistic fuzzy set introduced by Atanassov [33]. In fact, $A^P(x)$ [resp. $-A^N(x)$] denotes the membership degree [resp. non-membership degree] of x to A.

Definition 2.9 ([19])**.** Let X be a nonempty set and let $A,\ B \in BPF(X)$.

(i) We say that A is subset of B, denoted by $A \subset B$, if for each $x \in X$,

$$A^P(x) \leq B^P(x) \text{ and } A^N(x) \geq B^N(x).$$

(ii) The complement of A, denoted by $A^c = ((A^c)^P, (A^c)^N)$, is a bipolar fuzzy set in X defined as: for each $x \in X$, $A^c(x) = (1 - A^P(x), -1 - A^P(x))$, i.e.,

$$(A^c)^P(x) = 1 - A^P(x),\ (A^c)^N(x) = -1 - A^N(x).$$

(iii) The intersection of A and B, denoted by $A \cap B$, is a bipolar fuzzy set in X defined as:for each $x \in X$,

$$(A \cap B)(x) = (A^P(x) \wedge B^P(x), A^N(x) \vee B^N(x)).$$

(iv) The union of A and B, denoted by $A \cup B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cup B)(x) = (A^P(x) \vee B^P(x), A^N(x) \wedge B^N(x)).$$

Definition 2.10 ([34])**.** Let X be a nonempty set and let $(A_j)_{j \in J} \subset BPF(X)$.

(i) The intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$, is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$(\bigcap_{j \in J} A_j)(x) = (\bigwedge_{j \in J} A^P_j(x), \bigvee_{j \in J} A^N_j(x)).$$

(ii) The union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$, is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$(\bigcup_{j \in J} A_j)(x) = (\bigvee_{j \in J} A^P_j(x), \bigwedge_{j \in J} A^N_j(x)).$$

Result 2.11 ([34], Proposition 3.8)**.** *Let $A \in BPF(X)$ and let $(A_j)_{j\in J} \subset BPF(X)$. Then*

(1) (Generalized distributive laws): $A \cup (\bigcap_{j\in J} A_j) = \bigcap_{j\in J}(A \cup A_j)$,
$A \cap (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}(A \cap A_j)$,

(2) (Generalized DeMorgan's laws): $(\bigcup_{j\in J} A_j)^c = \bigcap_{j\in J} A_j^c$, $(\bigcap_{j\in J} A_j)^c = \bigcup_{j\in J} A_j^c$.

Definition 2.12 ([35, 36])**.** Let X be a nonempty set. Then a mapping $A = (A^P, A^N) : X \to [I] \times [-I]$ is called an interval-valued bipolar fuzzy set (in short, IVB set) in X.

In fact, $A^P(x) = [A^{P,-}(x), A^{P,+}(x)]$ and $A^N(x) = [A^{N,-}(x), A^{N,+}(x)]$, for each $x \in X$.

In particular, the interval-valued bipolar fuzzy empty set [resp. the interval-valued bipolar fuzzy whole set], denoted by $\mathbf{0}_{IVB} = (\mathbf{0}^P_{IVB}, \mathbf{0}^N_{IVB})$ [resp. $\mathbf{1}_{IVB} = (\mathbf{1}^P_{IVB}, \mathbf{1}^N_{IVB})$], is an IVB set in X defined by: for each $x \in X$,

$$\mathbf{0}^P_{IVB}(x) = [0,0] = \mathbf{0}^N_{IVB}(x) \text{ [resp. } \mathbf{1}^P_{IVB}(x) = [1,1],\ \mathbf{1}^N_{IVB}(x) = [-1,-1]].$$

We will denote the set of all IVB sets in X as $IVB(X)$.

For each $x \in X$, we use the positive membership degree $A^P(x)$ to denote the satisfaction degree of the element x to the property corresponding to the IVB set A and the negative membership degree $A^N(x)$ to denote the satisfaction degree of the element x to some implicit counter-property corresponding to the IVB set A.

If $A^P(x) \neq [0,0]$ and $A^N(x) = [0,0]$, then it is the situation that x is regarded as having only positive satisfaction for A. If $A^P(x) = [0,0]$ and $A^N(x) \neq [0,0]$, then it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter-property of A. It is possible for some $x \in X$ to be such that $A^P(x) \neq [0,0]$ and $A^N(x) \neq [0,0]$ when the membership function of the property overlaps that of its counter-property over some portion of X.

Refer to [3, 2] for the definition of an interval-valued fuzzy set and the order, the equality, the union, the intersection of interval-valued fuzzy sets and the complement of an interval-valued fuzzy set.

Definition 2.13 ([36])**.** Let X be a nonempty set and let $A,\ B \in IVB(X)$, $(A_j)_{j\in J}$ be a subfamily of $IVB(X)$.

(i) We say that A is a subset of B, denoted by $A \subset B$, if for each $x \in X$,

$$A^P(x) \leq B^P(x) \text{ and } A^N(x) \geq B^N(x), \text{ i.e.,}$$

$$A^{P,-}(x) \leq B^{P,-}(x),\ A^{P,+}(x) \leq B^{P,+}(x),\ A^{N,-}(x) \geq B^{N,-}(x),\ A^{N,+}(x) \geq B^{N,+}(x).$$

(ii) We say that A is equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

(iii) The union of A and B, denoted by $A \cup B$, is an IVB set in X defined by: for each $x \in X$,

$$(A \cup B)(x) = (A^P(x) \vee B^P(x), A^N(x) \wedge B^N(x)),$$

where $A^P(x) \vee B^P(x) = [A^{P,-}(x) \vee B^{P,-}(x), A^{P,+}(x) \vee B^{P,+}(x)]$
and

$$A^N(x) \wedge B^N(x) = [A^{N,-}(x) \wedge B^{N,-}(x), A^{N,+}(x) \wedge B^{N,+}(x)].$$

(iv) The union of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} A_j$, is an IVB set in X defined by: for each $x \in X$,

$$(\bigcup_{j\in J} A_j)(x) = (\bigvee_{j\in J} A_j^P(x), \bigwedge_{j\in J} A_j^N(x)),$$

where $\bigvee_{j\in J} A_j^P(x) = [\bigvee_{j\in J} A_j^{P,-}(x), \bigvee_{j\in J} A_j^{P,+}(x)]$
and
$\bigwedge_{j\in J} A_j^N(x) = [\bigwedge_{j\in J} A_j^{N,-}(x), \bigwedge_{j\in J} A_j^{N,+}(x)]$.

(v) The intersection of A and B, denoted by $A \cap B$, is an IVB set in X defined by: for each $x \in X$,

$$(A \cap B)(x) = (A^P(x) \wedge B^P(x), A^N(x) \vee B^N(x)),$$

where $A^P(x) \wedge B^P(x) = [A^{P,-}(x) \wedge B^{P,-}(x), A^{P,+}(x) \wedge B^{P,+}(x)]$
and
$A^N(x) \vee B^N(x) = [A^{N,-}(x) \vee B^{N,-}(x), A^{N,+}(x) \vee B^{N,+}(x)]$.

(vi) The intersection of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$, is an IVB set in X defined by: for each $x \in X$,

$$(\bigcap_{j\in J} A_j)(x) = (\bigwedge_{j\in J} A_j^P(x), \bigvee_{j\in J} A_j^N(x)),$$

where $\bigwedge_{j\in J} A_j^P(x) = [\bigwedge_{j\in J} A_j^{P,-}(x), \bigwedge_{j\in J} A_j^{P,+}(x)]$
and
$\bigvee_{j\in J} A_j^N(x) = [\bigvee_{j\in J} A_j^{N,-}(x), \bigvee_{j\in J} A_j^{N,+}(x)]$.

(Vii) the complement of A, denoted by $A^c = ((A^c)^P, (A^c)^N)$, is an IVB set in X defined as follows: for each $x, \in X$,

$$(A^c)^P(x) = 1 - A^P(x) = [1 - A^{P,+}(x), 1 - A^{P,-}(x)],$$
$$(A^c)^N(x) = -1 - A^N(x) = [-1 - A^{N,+}(x), -1 - A^{N,-}(x)].$$

Definition 2.14 ([18])**.** Let X be a nonempty set. Then a mapping $R = (R^P, R^N) : X \times X \to I \times -I$ is called a bipolar fuzzy relation on X.

Definition 2.15 ([18])**.** Let X be a nonempty set, let A be a bipolar fuzzy set in X and let R be a bipolar fuzzy relation on X. Then

(i) R is called a bipolar fuzzy relation on A, if for any $x,\ y \in X$,

$$R^P(x, y) \leq A^P(x) \wedge A^P(y) \text{ and } R^N(x, y) \geq A^N(x) \vee A^N(y),$$

(ii) R is said to be symmetric, if $R^P(x, y) = R^P(y, x)$ and $R^N(x, y) = R^N(y, x)$, for each $(x, y) \in X \times X$.

Definition 2.16 ([28])**.** Let $G^* = (V, E)$ be a graph. Then a pair $G = (A, B)$ is called a bipolar fuzzy graph of G^*, if it satisfies the following conditions:

(i) A is a bipolar fuzzy set in V,

(ii) B is a bipolar fuzzy set in $E \subset V \times V$ such that for each $\{x, y\} \in E$,

$$B^P(\{x, y\}) \leq A^P(x) \wedge A^P(y) \text{ and } B^N(\{x, y\}) \geq A^N(x) \vee A^N(y).$$

In this case, we will call A the bipolar fuzzy vertex set of V and B the bipolar fuzzy edge set of E, respectively.

It is obvious that B is a bipolar fuzzy symmetric relation on A. Thus by using the notation xy for an element of E, $G=(A,B)$ is a bipolar fuzzy graph of $G^*=(V,E)$, if for all $xy \in E$,

$$B^P(xy) \leq A^P(x) \wedge A^P(y) \text{ and } B^N(xy) \geq A^N(x) \vee A^N(y).$$

3. Basic properties of interval-valued bipolar fuzzy graphs

In this section, we introduce the notion of interval-valued bipolar fuzzy graphs and some operations between them, we find some of their basic properties and give some examples.

Definition 3.1. Let X be a nonempty set. Then a mapping $R=(R^P,R^N): X \times X \to [I] \times [-I]$ is called an interval-valued bipolar fuzzy relation (in short, IVBR) on X.

In fact, $R^P(x)=[R^{P,-}(x), R^{P,+}(x)]$ and $R^N(x)=[R^{N,-}(x), R^{N,+}(x)]$, for each $x \in X$.

In particular, the interval-valued bipolar fuzzy empty relation [resp. the interval-valued bipolar fuzzy whole relation], denoted by $\mathbf{0}_{IVBR}=(\mathbf{0}^P_{IVBR}, \mathbf{0}^N_{IVBR})$ [resp. $\mathbf{1}_{IVBR}=(\mathbf{1}^P_{IVB}, \mathbf{1}^N_{IVB})$], is an IVB set in X defined by: for each $x \in X$,

$$\mathbf{0}^P_{IVBR}(x)=[0,0]=\mathbf{0}^N_{IVBR}(x) \text{ [resp. } \mathbf{1}^P_{IVBR}(x)=[1,1],\ \mathbf{1}^N_{IVBR}(x)=[-1,-1]].$$

We will denote the set of all IVBRs on X as $IVBR(X)$.

Example 3.2. Let $R: I \times I \to [I] \times [-I]$ be the mapping defined as follows: for any $x,\ y \in I$,

$$R(x,y)=([\frac{x+y}{5}, \frac{x+y}{2}], [-\frac{x+y}{3}, -\frac{x+y}{6}]).$$

Then clearly R is an IVBR on I.

Definition 3.3. Let X be a nonempty set, let $A \in IVB(X)$ and $R \in IVBR(X)$. Then

(i) R is called an interval-valued bipolar fuzzy relation on A, if for any $x,\ y \in X$,

$$R^P(x,y) \leq A^P(x) \wedge A^P(y) \text{ and } R^N(x,y) \geq A^N(x) \vee A^N(y),$$

(ii) R is said to be symmetric, if $R^P(x,y)=R^P(y,x)$ and $R^N(x,y)=R^N(y,x)$, for each $(x,y) \in X \times X$.

Example 3.4. Let R be the IVBR on I given in Example 3.2 and let A be the IVB set in I given by: for each $x \in I$,

$$A(x)=([\frac{1+x}{5}, \frac{1+x}{2}], [-\frac{1+x}{3}, -\frac{1+x}{6}]).$$

Then we can easily see that R is an IVBR on A. Moreover, R is symmetric.

Definition 3.5. Let $G^*=(V,E)$ be a graph. Then a pair $G=(A,B)$ is called an interval-valued bipolar fuzzy graph (in short, IVBG) of G^*, if it satisfies the following conditions:

(i) $A \in IVB(V)$,

(ii) $B \in IVB(E)$ such that for each $xy \in E \subset V \times V$,

$$B^P(xy) \leq A^P(x) \wedge A^P(y) \text{ and } B^N(xy) \geq A^N(x) \vee A^N(y),$$

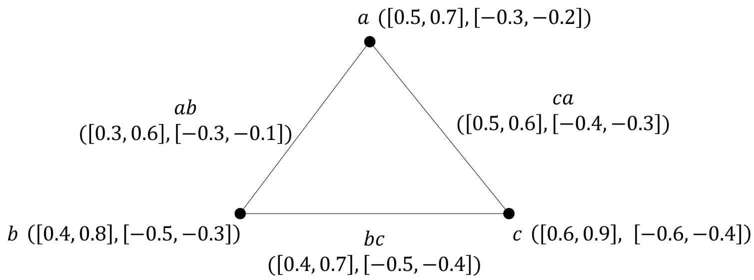

FIGURE 1. An IVFB of G^*.

where $B^P(xy) = [B^{P,-}(x,y), B^{P,+}(x,y)] \leq [A^{P,-}(x) \wedge A^{P,-}(y), A^{P,+}(x) \wedge A^{P,+}(y)]$ and

$B^N(xy) = [B^{N,-}(x,y), B^{N,+}(x,y)] \geq [A^{N,-}(x) \vee A^{N,-}(y), A^{N,+}(x) \vee A^{N,+}(y)]$.
In this case, A be called the interval-valued bipolar fuzzy vertex set of V and B the interval-valued bipolar fuzzy edge set of E, respectively.

It is obvious that B is an interval-valued bipolar fuzzy symmetric relation on A. Thus $G = (A, B)$ is an IVBG of $G^* = (V, E)$, if for all $xy \in E$,

$$B^P(xy) \leq A^P(x) \wedge A^P(y) \text{ and } B^N(xy) \geq A^N(x) \vee A^N(y).$$

Remark 3.6. Suppose $G = (A, B)$ is an IVBG of a graph $G^* = (V, E)$. Then from Definition 3.5, we have

(i) $A^- = (A^{P,-}, A^{N,-})$ and $A^+ = (A^{P,+}, A^{N,+})$ are bipolar fuzzy sets in V,

(ii) $B^- = (B^{P,-}, B^{N,-})$ and $B^+ = (B^{P,+}, B^{N,+})$ are bipolar fuzzy sets in V such that for all $xy \in E \subset V \times V$,

$$B^{P,-}(xy) \leq A^{P,-}(x) \wedge A^{P,-}(y) \text{ and } B^{N,-}(xy) \geq A^{N,-}(x) \vee A^{N,-}(y),$$

$$B^{P,+}(xy) \leq A^{P,+}(x) \wedge A^{P,+}(y) \text{ and } B^{N,+}(xy) \geq A^{N,+}(x) \vee A^{N,+}(y).$$

Thus G^- and G^+ are bipolar fuzzy graphs of G^*, where

$$G^- = (A^-, B^-) \text{ and } G^+ = (A^+), B^+).$$

Furthermore, we can easily see that if $A^N(x) = [0, 0]$ for each $x \in V$, then G^- and G^+ are fuzzy graphs of G^*.

Example 3.7. Let $G^* = (V, E)$ be a graph such that $V = \{a, b, c\}$ and $E = \{ab, bc, ca\}$. Let A be an IVB set in V and let B be an IVB set in $E \subset V \times V$ defined as follows:

V	$A(t)$
a	$([0.5, 0.7], [-0.3, -0.2])$
b	$([0.4, 0.8], [-0.5, -0.3])$
c	$([0.6, 0.9], [-0.6, -0.4])$

E	$B(x)$
ab	$([0.3, 0.6], [-0.3, -0.1])$
bc	$([0.4, 0.7], [-0.5, -0.4])$
ca	$([0.5, 0.6], [-0.4, -0.3])$

Then we can easily see that G is an IVBG of G^* (see Figure 1).

Remark 3.8. Let $G = (A, B)$ be an IVBG of a graph $G^* = (V, E)$. Then it is obvious that if $A^P(x) = A^N(x) = [0, 0]$ and $B^P(x) = B^N(x) = [0, 0]$ for some

$x,\ y \in V$, then there is no edge between x and y, and otherwise, there exists an edge between x and y.

Definition 3.9. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then the Cartesian product of G_1 and G_2, denoted by $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$, is defined as follows:

(i) $(A_1^P \times A_2^P)(x_1, x_2) = [A_1^{P,-}(x_1) \wedge A_2^{P,-}(x_2), A_1^{P,+}(x_1) \wedge A_2^{P,+}(x_2)],$
$= A_1^P(x_1) \wedge A_2^P(x_2),$
$(A_1^N \times A_2^N)(x_1, x_2) = [A_1^{N,-}(x_1) \vee A_2^{N,-}(x_2), A_1^{N,+}(x_1) \vee A_2^{N,+}(x_2)]$
$= A_1^N(x_1) \vee A_2^N(x_2),$
for each $(x_1, x_2) \in V_1 \times V_2$,

(ii) $(B_1^P \times B_2^P)((x, x_2)(x, y_2)) = [A_1^{P,-}(x) \wedge B_2^{P,-}(x_2y_2), A_1^{P,+}(x) \wedge B_2^{P,+}(x_2y_2)]$
$= A_1^P(x) \wedge B_2^P(x_2y_2),$
$(B_1^N \times B_2^N)((x, x_2)(x, y_2)) = [A_1^{N,-}(x) \vee B_2^{N,-}(x_2y_2), A_1^{N,+}(x) \vee B_2^{N,+}(x_2y_2)]$
$= A_1^N(x) \wedge B_2^N(x_2y_2),$
for each $x \in V_1$ and each $x_2y_2 \in E_2$,

(iii) $(B_1^P \times B_2^P)((x_1, y)(y_1, y)) = [B_1^{P,-}(x_1y_1) \wedge A_2^{P,-}(y), B_1^{P,+}(x_1y_1) \wedge A_2^{P,+}(y)]$
$= B_1^P(x_1y_1) \wedge A_2^P(y),$
$(B_1^N \times B_2^N)((x_1, y)(y_1, y)) = [B_1^{N,-}(x_1y_1) \vee A_2^{N,-}(y), B_1^{N,+}(x_1y_1) \vee A_2^{N,+}(y)]$
$= B_1^N(x_1y_1) \wedge A_2^N(y),$
for each $y \in V_2$ and each $x_1y_1 \in E_1$.

Proposition 3.10. *The Cartesian product of two IVBGs is an IVBG.*

Proof. Suppose $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then by Definition 2.4, $G^* = G_1^* \times G_2^* = (V, E)$ is a graph, where $V = V_1 \times V_2$ and $E = \{(x, x_2)(x, y_2) : x \in V_1,\ x_2y_2 \in E_2\} \cup \{(x_1, y)(x_2, y) : y \in V_2,\ x_1y_1 \in E_1\}$. From Definition Definition 3.9, it is clear that $A_1 \times A_2 \in IVB(V_1 \times V_2)$. Let us prove that $B_1 \times B_2$ satisfies the condition (ii) in Definition 3.5.

Let $x \in V_1$ and let $x_2y_2 \in E_2$. Then
$(B_1^P \times B_2^P)((x, x_2)(x, y_2)) = A_1^P(x) \wedge B_2^P(x_2y_2)$
$\leq A_1^P(x) \wedge (A_2^P(x_2) \wedge A_2^P(y_2))$
$= (A_1^P(x) \wedge A_2^P(x_2)) \wedge (A_1^P(x) \wedge A_2^P(y_2))$
$= (A_1^P \times A_2^P)(x, x_2) \wedge (A_1^P \times A_2^P)(x, y_2).$
Similarly, we can show that

$$(B_1^N \times B_2^N)((x, x_2)(x, y_2)) \geq (A_1^N(x) \vee A_2^N(x_2)) \vee (A_1^N(x) \wedge A_2^N(y_2)).$$

Now let $y \in V_2$ and let $x_1y_1 \in E_1$. Then
$(B_1^P \times B_2^P)((x_1, y)(y_1, y)) = B_1^P(x_1y_1) \wedge A_2^P(y)$
$\leq (A_1^P(x_1) \wedge (A_2^P(y_1)) \wedge A_2^P(y)$
$= (A_1^P(x_1) \wedge A_2^P(y)) \wedge (A_1^P(y_1) \wedge A_2^P(y))$
$= (A_1^P \times A_2^P)(x_1, y) \wedge (A_1^P \times A_2^P)(y_1, y).$
Similarly, we can prove that

$$B_1^N \times B_2^N)((x_1, y)(y_1, y)) \geq (A_1^N \times A_2^N)(x_1, y) \vee (A_1^N \times A_2^N)(y_1, y).$$

Thus $G_1 \times G_2$ is an IVBG of $G_1^* \times G_2^*$. □

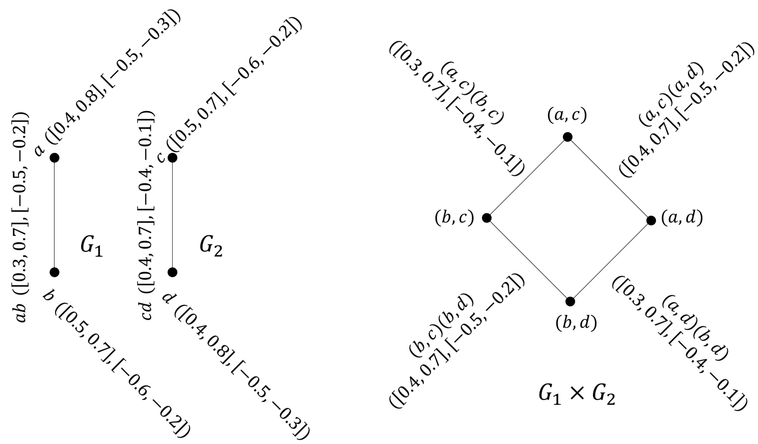

FIGURE 2. G_1, G_2, and $G_1 \times G_2$.

Example 3.11. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$ and $E_2 = \{cd\}$. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be the IVBGs of G_1^* and G_2^*, respectively defined as follows:

$$A_1(a) = ([0.4, 0.8], [-0.5, -0.3]),\ A_1(b) = ([0.5, 0.7], [-0.6, -0.2]),$$
$$B_1(ab) = ([0.3, 0.7], [-0.5, -0.2]),$$
$$A_2(c) = ([0.5, 0.7], [-0.6, -0.2]),\ A_2(d) = ([0.4, 0.8], [-0.5, -0.3]),$$
$$B_2(cd) = ([0.4, 0.7], [-0.4, -0.1]).$$

Then we can easily calculate that:

$$(B_1 \times B_2)((a, c)(a, d)) = ([0.4, 0.7], [-0.5, -0.2]),$$
$$(B_1 \times B_2)((b, c)(b, d)) = ([0.4, 0.7], [-0.5, -0.2]),$$
$$(B_1 \times B_2)((a, c)(b, c)) = ([0.3, 0.7], [-0.4, -0.1]),$$
$$(B_1 \times B_2)((a, d)(b, d)) = ([0.3, 0.7], [-0.4, -0.1]).$$

Thus we can easily see that $G_1 \times G_2$ is an IVBG of $G_1^* \times G_2^*$ (see Figure 2).

Definition 3.12. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then the composition of G_1 and G_2, denoted by $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$, is defined as follows:

(i) $(A_1^P \circ A_2^P)(x_1, x_2) = [A_1^{P,-}(x_1) \wedge A_2^{P,-}(x_2), A_1^{P,+}(x_1) \wedge A_2^{P,+}(x_2)]$
$= A_1^P(x_1) \wedge A_2^P(x_2)$,
$(A_1^N \circ A_2^N)(x_1, x_2) = [A_1^{N,-}(x_1) \vee A_2^{N,-}(x_2), A_1^{N,+}(x_1) \vee A_2^{N,+}(x_2)]$
$= A_1^N(x_1) \vee A_2^N(x_2)$,

for each $(x_1, x_2) \in V_1 \times V_2$,

(ii) $(B_1^P \circ B_2^P)((x, x_2)(x, y_2)) = [A_1^{P,-}(x) \wedge B_2^{P,-}(x_2y_2), A_1^{P,+}(x) \wedge B_2^{P,+}(x_2y_2)]$
$= A_1^P(x) \wedge B_2^P(x_2y_2)$,
$(B_1^N \circ B_2^N)((x, x_2)(x, y_2)) = [A_1^{N,-}(x) \vee B_2^{N,-}(x_2y_2), A_1^{N,+}(x) \vee B_2^{N,+}(x_2y_2)]$

$= A_1^N(x) \wedge B_2^N(x_2y_2)$,
for each $x \in V_1$ and each $x_2y_2 \in E_2$,
(iii) $(B_1^P \circ B_2^P)((x_1,y)(y_1,y)) = [B_1^{P,-}(x_1y_1) \wedge A_2^{P,-}(y), B_1^{P,+}(x_1y_1) \wedge A_2^{P,+}(y)]$
$= B_1^P(x_1y_1) \wedge A_2^P(y)$,
$(B_1^N \circ B_2^N)((x_1,y)(y_1,y)) = [B_1^{N,-}(x_1y_1) \vee A_2^{N,-}(y), B_1^{N,+}(x_1y_1) \vee A_2^{N,+}(y)]$
$= B_1^N(x_1y_1) \wedge A_2^N(y)$,
for each $y \in V_2$ and each $x_1y_1 \in E_1$,
(iv) $(B_1^P \circ B_2^P)((x_1,x_2)(y_1,y_2)) = A_2^P(x_2) \wedge A_2^P(y_2) \wedge B_1^P(x_1y_1)$,
$(B_1^N \circ B_2^N)((x_1,x_2)(y_1,y_2)) = A_2^N(x_2) \vee A_2^N(y_2) \vee B_1^P(x_1y_1)$,
for any $(x_1,x_2)(y_1,y_2) \in E^0 - E$.

Proposition 3.13. *The composition of two IVBGs is an IVBG.*

Proof. Suppose $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then by Definition 3.12 and the proof of Proposition 3.12, we have

$$(B_1^P \circ B_2^P)((x,x_2)(x,y_2)) \leq (A_1^P \circ A_2^P)(x,x_2) \wedge (A_1^P \circ A_2^P)(x,x_2), \tag{3.1}$$

for each $x \in V_1$ and each $x_2y_2 \in E_2$,

$$(B_1^N \circ B_2^N)((x,x_2)(x,y_2)) \geq (A_1^N \circ A_2^N)(x,x_2) \vee (A_1^N \circ A_2^N)(x,x_2), \tag{3.2}$$

for each $x \in V_1$ and each $x_2y_2 \in E_2$,

$$(B_1^P \circ B_2^P)((x_1,y)(y_1,y)) \leq (A_1^P \circ A_2^P)(x_1,y) \wedge (A_1^P \circ A_2^P)(y_1,y), \tag{3.3}$$

for each $y \in V_2$ and each $x_1y_1 \in E_1$,

$$(B_1^N \circ B_2^N)((x_1,y)(y_1,y)) \geq (B_1^N \circ B_2^N)(x_1,y) \vee (B_1^N \circ B_2^N)(y_1,y), \tag{3.4}$$

for each $y \in V_2$ and each $x_1y_1 \in E_1$.

Now let $(x_1,x_2)(y_1,y_2) \in E^0 - E$. Then clearly, $x_1y_1 \in E_1$ and $x_2 \neq y_2$. Thus
$(B_1^P \circ B_2^P)(x_1,x_2)(y_1,y_2) = A_2^P(x_2) \wedge A_2^P(y_2) \wedge B_1^P(x_1y_1)$
$\leq (A_2^P(x_2) \wedge A_2^P(y_2)) \wedge (A_1^P(x_1) \wedge A_1^P(y_1))$
$= (A_1^P(x_1) \wedge A_2^P(x_2)) \wedge (A_1^P(y_1) \wedge A_2^P(y_2))$
$= (A_1^P \circ A_2^P)(x_1,x_2) \wedge (A_1^P \circ A_2^P)(y_1,y_2)$.
Similarly, we have $(B_1^N \circ B_2^N)(x_1,x_2)(y_1,y_2) \geq (A_1^N \circ A_2^N)(x_1,x_2) \vee (A_1^N \circ A_2^N)(y_1,y_2)$.
So $G_1[G_2]$ is an IVBG of $G_1^*[G_2^*]$. □

Example 3.14. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be as in Example 3.11. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be the IVBGs of G_1^* and G_2^*, respectively defined as follows:

$$A_1(a) = ([0.4, 0.5], [-0.6, -0.4]),\ A_1(b) = ([0.3, 0.6], [-0.7, -0.3]),$$
$$B_1(ab) = ([0.2, 0.4], [-0.5, -0.3]),$$
$$A_2(c) = ([0.5, 0.8], [-0.7, -0.5]),\ A_2(d) = ([0.6, 0.7], [-0.5, -0.4]),$$
$$B_2(cd) = ([0.4, 0.6], [-0.4, -0.2]).$$

Then we have:

$$(B_1 \circ B_2)((a,c)(a,d)) = ([0.4, 0.5], [-0.4, -0.2]),$$
$$(B_1 \circ B_2)((b,c)(b,d)) = ([0.3, 0.6], [-0.4, -0.2]),$$

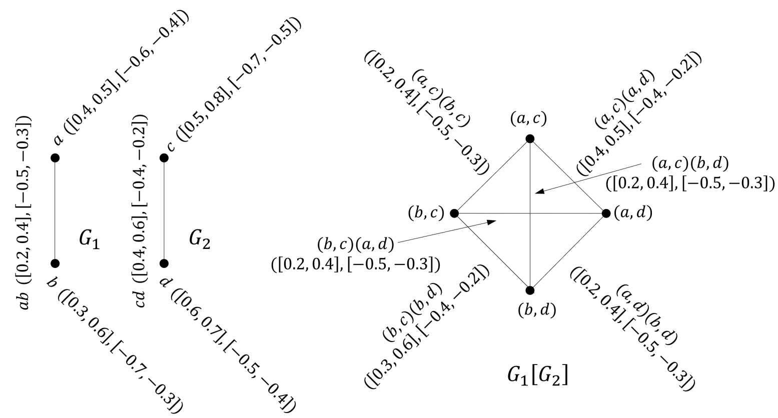

FIGURE 3. G_1, G_2, and $G_1[G_2]$.

$$(B_1 \circ B_2)((a,c)(b,c)) = ([0.2, 0.4], [-0.5, -0.3]),$$

$$(B_1 \circ B_2)((a,d)(b,d)) = ([0.2, 0.4], [-0.5, -0.3]),$$

$$(B_1 \circ B_2)((a,c)(b,d)) = ([0.2, 0.4], [-0.5, -0.3]),$$

$$(B_1 \circ B_2)((b,c)(a,d)) = ([0.2, 0.4], [-0.5, -0.3]).$$

Thus we can easily see that $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ is an IVBG of $G_1^*[G_2^*]$ (see Figure 3).

Definition 3.15. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then the union of G_1 and G_2, denoted by $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$, is defined as follows:

(i)

$$(A_1^P \cup A_2^P)(x) = \begin{cases} A_1^P(x) & \text{if } x \in V_1 \cap V_2^c \\ A_2^P(x) & \text{if } x \in V_2 \cap V_1^c \\ A_1^P(x) \vee A_2^P(x) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

(ii)

$$(A_1^N \cup A_2^N)(x) = \begin{cases} A_1^N(x) & \text{if } x \in V_1 \cap V_2^c \\ A_2^N(x) & \text{if } x \in V_2 \cap V_1^c \\ A_1^N(x) \wedge A_2^N(x) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

(iii)

$$(B_1^P \cup B_2^P)(xy) = \begin{cases} B_1^P(xy) & \text{if } xy \in E_1 \cap E_2^c \\ B_2^P(xy) & \text{if } xy \in E_2 \cap E_1^c \\ B_1^P(xy) \vee B_2^P(xy) & \text{if } xy \in E_1 \cap E_2, \end{cases}$$

(iv)

$$(B_1^N \cup B_2^N)(xy) = \begin{cases} B_1^N(xy) & \text{if } xy \in E_1 \cap E_2^c \\ B_2^N(xy) & \text{if } xy \in E_2 \cap E_1^c \\ B_1^N(xy) \wedge B_2^N(xy) & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

Proposition 3.16. *The union of two IVBGs is an IVBG.*

Proof. Suppose $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. We prove that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an IVBG of $G_1^* \cup G_2^*$. Since the conditions (i) and (ii) in the above definition for $A_1 \cup A_2$ are automatically satisfied, we show only the conditions for $B_1 \cup B_2$.

Suppose $xy \in E_1 \cap E_2$. Then
$(B_1^P \cup B_2^P)(xy) = B_1^P(xy) \vee B_2^P(xy)$
$\leq (A_1^P(x) \wedge A_1^P(y)) \vee (A_2^P(x) \wedge A_2^P(y))$
$= (A_1^P(x) \vee A_2^P(x)) \wedge (A_1^P(y) \vee A_2^P(y))$
$= (A_1^P \cup A_2^P)(x) \wedge (A_1^P \cup A_2^P)(y)$,
$(B_1^N \cup B_2^N)(xy) = B_1^N(xy) \wedge B_2^N(xy)$
$\geq (A_1^N(x) \vee A_1^N(y)) \wedge (A_2^N(x) \vee A_2^N(y))$
$= (A_1^N(x) \wedge A_2^N(x)) \vee (A_1^N(y) \wedge A_2^N(y))$
$= (A_1^N \cup A_2^N)(x) \vee (A_1^N \cup A_2^N)(y)$.

Suppose $xy \in E_1 \cap E_2^c$ or $xy \in E_2 \cap E_1^c$. Then similarly, we can prove:

$$(B_1^P \cup B_2^P)(xy) \leq (A_1^P \cup A_2^P)(x) \wedge (A_1^P \cup A_2^P)(y),$$
$$(B_1^N \cup B_2^N)(xy) \geq (A_1^N \cup A_2^N)(x) \vee (A_1^N \cup A_2^N)(y).$$

Thus $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an IVBG of $G_1^* \cup G_2^*$. □

Example 3.17. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs such $V_1 = \{a, b, c, d, e\}$, $E_1 = \{ab, bc, be, ce, ad, ed\}$, $V_2 = \{a, b, c, d, f\}$ and $E_1 = \{ab, bc, cf, bf, bd\}$. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be the IVBGs of G_1^* and G_2^*, respectively defined as follows:

V_1	$A_1(t)$
a	$([0.2, 0.4], [-0.5, -0.2])$
b	$([0.4, 0.5], [-0.6, -0.3])$
c	$([0.3, 0.6], [-0.7, -0.5])$
d	$([0.3, 0.7], [-0.8, -0.6])$
e	$([0.2, 0.6], [-0.4, -0.3])$

E_1	$B_1(x)$
ab	$([0.1, 0.3], [-0.4, -0.1])$
bc	$([0.2, 0.4], [-0.5, -0.2])$
ce	$([0.5, 0.6], [-0.4, -0.3])$
be	$([0.1, 0.5], [-0.3, -0.2])$
ad	$([0.1, 0.3], [-0.4, -0.1])$
de	$([0.1, 0.6], [-0.3, -0.2])$

V_2	$A_2(t)$
a	$([0.2, 0.4], [-0.5, -0.2])$
b	$([0.2, 0.5], [-0.6, -0.4])$
c	$([0.3, 0.6], [-0.7, -0.5])$
d	$([0.2, 0.6], [-0.8, -0.7])$
f	$([0.4, 0.6], [-0.4, -0.2])$

E_2	$B_2(x)$
ab	$([0.1, 0.3], [-0.3, -0.1])$
bc	$([0.2, 0.4], [-0.5, -0.3])$
cf	$([0.1, 0.5], [-0.3, -0.1])$
bf	$([0.1, 0.4], [-0.3, -0.2])$
bd	$([0.2, 0.5], [-0.3, -0.1])$

Then from the above definition, we have:

$$(A_1 \cup A_2)(a) = ([0.2, 0.4], [-0.5, -0.2]), \ (A_1 \cup A_2)(b) = ([0.4, 0.5], [-0.5, -0.4]),$$
$$(A_1 \cup A_2)(c) = ([0.3, 0.6], [-0.7, -0.5]), \ (A_1 \cup A_2)(d) = ([0.3, 0.7], [-0.8, -0.7]),$$

$$(A_1 \cup A_2)(e) = ([0.2, 0.6], [-0.4, -0.3]),\ (A_1 \cup A_2)(f) = ([0.4, 0.6], [-0.4, -0.2]),$$
$$(B_1 \cup B_2)(ab) = ([0.1, 0.3], [-0.4, -0.1]),\ (B_1 \cup B_2)(bc) = ([0.2, 0.4], [-0.5, -0.3]),$$
$$(B_1 \cup B_2)(ce) = ([0.5, 0.6], [-0.4, -0.3]),\ (B_1 \cup B_2)(be) = ([0.1, 0.5], [-0.3, -0.2]),$$
$$(B_1 \cup B_2)(ad) = ([0.1, 0.3], [-0.4, -0.1]),\ (B_1 \cup B_2)(de) = ([0.1, 0.6], [-0.3, -0.2]),$$
$$(B_1 \cup B_2)(cf) = ([0.1, 0.5], [-0.3, -0.1]),\ (B_1 \cup B_2)(bf) = ([0.1, 0.4], [-0.3, -0.2]),$$
$$(B_1 \cup B_2)(bd) = ([0.2, 0.5], [-0.3, -0.1]).$$

Thus we can easily see that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an IVBG of $G_1^* \cup G_2^*$ (see Figure 4).

Theorem 3.18. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \varnothing$. Let $A_1 \in IVB(V_1)$, $A_2 \in IVB(V_2)$, $B_1 \in IVB(E_1)$ and $B_2 \in IVB(E_2)$. Then $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an IVBG of $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ if and only if $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are IVBGs of G_1^* and G_2^*, respectively.*

Proof. Suppose $G_1 \cup G_2$ is an IVBG of G^* and let $xy \in E_1$. Then clearly, $xy \notin E_2$ and $x,\ y \in V_1 - V_2$. Thus

$$B_1^P(xy) = (B_1^P \cup B_2^P)(xy) \leq (A_1^P \cup A_2^P)(x) \wedge (A_1^P \cup A_2^P)(y) = A_1^P(x) \wedge A_1^P(y),$$
$$B_1^N(xy) = (B_1^N \cup B_2^N)(xy) \geq (A_1^N \cup A_2^N)(x) \vee (A_1^N \cup A_2^N)(y) = A_1^N(x) \vee A_1^N(y).$$

So $G_1 = (A_1, B_1)$ is an IVBG of G_1^*. Similarly, we can prove that $G_2 = (A_2, B_2)$ is an IVBG of G_2^*.

The converse is given by Proposition 3.16. □

Definition 3.19. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then the intersection of G_1 and G_2, denoted by $G_1 \cap G_2 = (A_1 \cap A_2, B_1 \cap B_2)$, is defined as follows:

(i) $(A_1^P \cap A_2^P)(x) = A_1^P(x) \wedge A_2^P(x)$, $(A_1^N \cap A_2^N)(x) = A_1^N(x) \vee A_2^N(x)$, for each $x \in V_1 \cap V_2$,

(ii) $(B_1^P \cap B_2^P)(xy) = B_1^P(xy) \wedge B_2^P(xy)$, $(B_1^N \cap B_2^N)(xy) = B_1^N(xy) \vee B_2^N(xy)$, for each $xy \in E_1 \cap E_2$.

Proposition 3.20. *The intersection of two IVBGs is an IVBG.*

Proof. The proof is straightforward. □

Example 3.21. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs given in Example 3.17. Consider two IVBGs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of G_1^* and G_2^*, respectively defined in Example 3.17. Then we have:

$$(A_1 \cap A_2)(a) = ([0.2, 0.4], [-0.5, -0.2]),\ (A_1 \cap A_2)(b) = ([0.2, 0.5], [-0.6, -0.3]),$$
$$(A_1 \cap A_2)(c) = ([0.3, 0.6], [-0.7, -0.5]),\ (A_1 \cap A_2)(d) = ([0.2, 0.6], [-0.8, -0.6]),$$
$$(B_1 \cap B_2)(ab) = ([0.1, 0.3], [-0.3, -0.1]),\ (B_1 \cap B_2)(bc) = ([0.2, 0.4], [-0.5, -0.2]).$$

Thus we can easily see that $G_1 \cap G_2 = (A_1 \cap A_2, B_1 \cap B_2)$ is an IVBG of $G_1^* \cap G_2^*$ (see Figure 5).

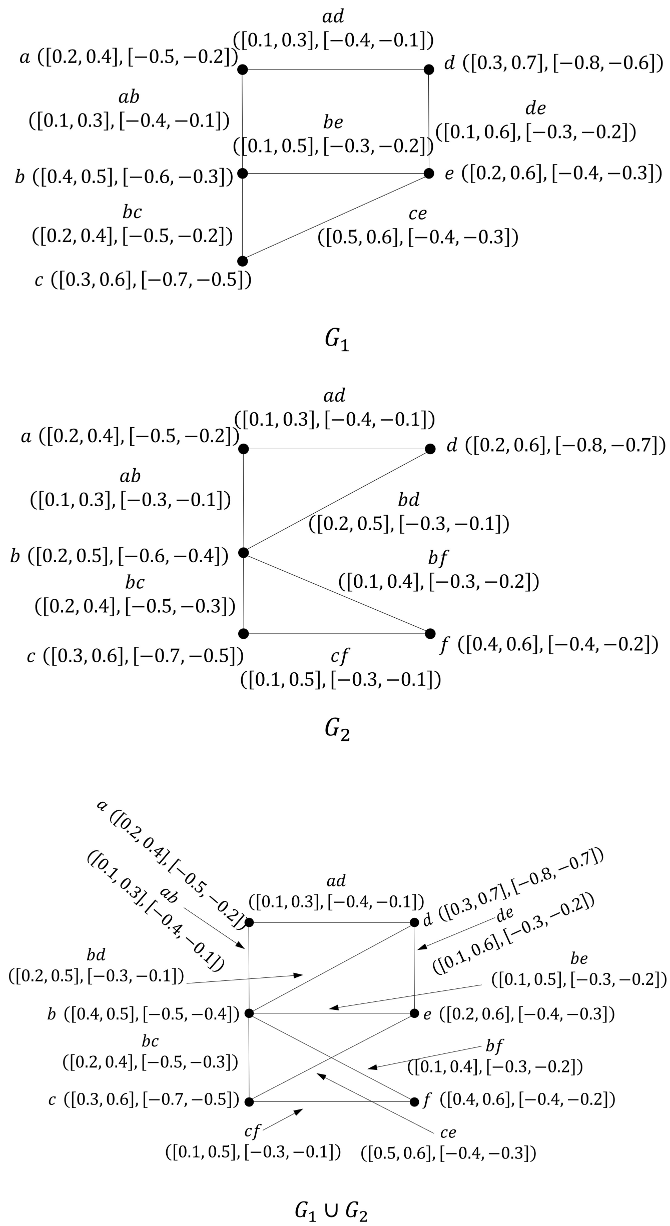

FIGURE 4. G_1, G_2, and $G_1 \cup G_2$.

Definition 3.22. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then the join of G_1 and G_2, denoted by $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$, is defined as follows:

(i) $(A_1^P + A_2^P)(x) = (A_1^P \cup A_2^P)(x),\ (A_1^N + A_2^N)(x) = (A_1^N \cap A_2^N)(x)$,
for each $x \in V_1 \cup V_2$,
(ii) $(B_1^P + B_2^P)(xy) = (B_1^P \cup B_2^P)(xy),\ (B_1^N + B_2^N)(xy) = (B_1^N \cap B_2^N)(xy)$,
for each $xy \in E_1 \cap E_2$,
(ii) $(B_1^P + B_2^P)(xy) = A_1^P(x) \vee A_2^P(y),\ (B_1^N + B_2^N)(xy) = B_1^N(x) \wedge B_2^N(y)$,
for each $xy \in E^{'}$.

Proposition 3.23. *The join of two IVBGs is an IVBG.*

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. We show that $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ is an IVBG of $G_1^* + G_2^* = (V_1 + V_2, E_1 + E_2)$. From Proposition 3.16, it is sufficient to prove the case when $xy \in E^{'}$.

Suppose $xy \in E^{'}$. Then

$(B_1^P + B_2^P)(xy) = A_1^P(x) \vee A_2^P(y)$
$\leq (A_1^P \cup A_2^P)(x) \vee (A_1^P \cup A_2^P)(y)$
$= (A_1^P + A_2^P)(x) \vee (A_1^P + A_2^P)(y)$,

$(B_1^N + B_2^N)(xy) = A_1^N(x) \wedge A_2^N(y)$
$\geq (A_1^N \cup A_2^N)(x) \wedge (A_1^N \cup A_2^N)(y)$
$= (A_1^N + A_2^N)(x) \wedge (A_1^N + A_2^N)(y)$.

Thus $G_1 + G_2$ is an IVBG of $G_1^* + G_2^*$. □

The following is an immediate result of Theorem 3.18 and Proposition 3.23.

Theorem 3.24. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \varnothing$. Let $A_1 \in IVB(V_1)$, $A_2 \in IVB(V_2)$, $B_1 \in IVB(E_1)$ and $B_2 \in IVB(E_2)$. Then $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ is an IVBG of $G^* = G_1^* + G_2^* = (V_1 + V_2, E_1 + E_2)$ if and only if $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are IVBGs of G_1^* and G_2^*, respectively.*

Definition 3.25. Let $G = (A, B)$ be an IVBG of graph $G^* = (V, E)$. Then $G = (A, B)$ is said to be complete, if for any $x,\ y \in V$,

$$B(xy) = (A^P(x) \wedge A^P(y), A^N(x) \vee A^N(y)).$$

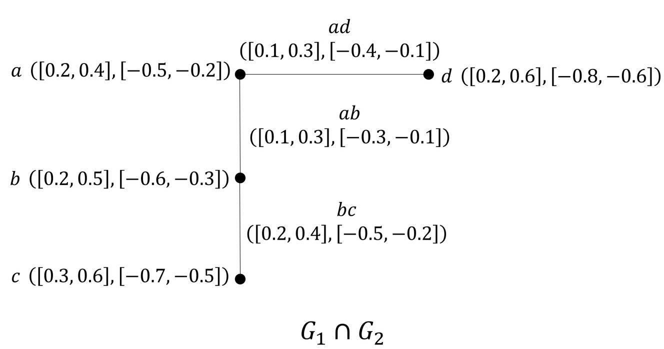

FIGURE 5. $G_1 \cap G_2$.

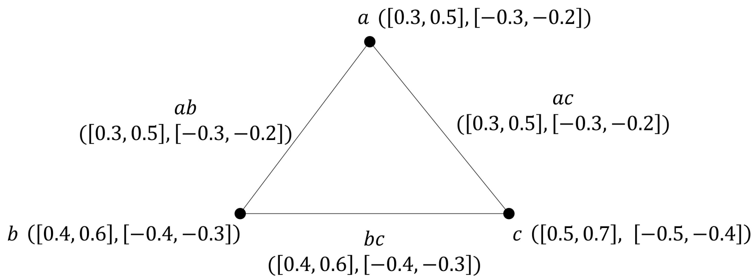

Figure 6. A complete IVBG G.

It is obvious that if G is a complete IVBG of a graph G^*, then G^- and G^+ are bipolar fuzzy complete graphs of G^*.

Example 3.26. Let $G^* = (V, E)$ such that $V = \{a, b, c\}$. Let $A \in IVB(V)$ and let $B \in IVB(V \times V)$ be defined as follows:

V	$A(t)$
a	$([0.3, 0.5], [-0.3, -0.2])$
b	$([0.4, 0.6], [-0.4, -0.3])$
c	$([0.5, 0.7], [-0.5, -0.4])$

$V \times V$	$B(t)$
ab	$([0.3, 0.5], [-0.3, -0.2])$
bc	$([0.4, 0.6], [-0.4, -0.3])$
ac	$([0.3, 0.5], [-0.3, -0.2])$

Then by routine computations, it is easy to see that G is a complete IVBG of G^* (see Figure 6).

Remark 3.27. The union of two complete IVBGs is not necessarily a complete IVBG.

Example 3.28. Let $G^* = (V, E)$ be a graph such that $V = \{a, b, c\}$. Consider two complete IVBGs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined as follows, respectively:

$$A_1(a) = ([0.7, 0.9], [-0.4, -0.2]),\ A_1(b) = ([0.5, 0.8], [-0.6, -0.3]),$$
$$A_1(c) = ([0.6, 0.7], [-0.5, -0.1]),$$
$$B_1(ab) = ([0.5, 0.8], [-0.4, -0.2]),\ B_1(bc) = ([0.5, 0.7], [-0.5, -0.1]),$$
$$B_1(ac) = ([0.6, 0.8], [-0.4, -0.1]),$$
$$A_2(a) = ([0.3, 0.6], [-0.4, -0.2]),\ A_2(b) = ([0.4, 0.7], [-0.3, -0.1]),$$
$$A_2(c) = ([0.8, 0.9], [-0.5, -0.3]),$$
$$B_2(ab) = ([0.3, 0.6], [-0.3, -0.1]),\ B_2(bc) = ([0.4, 0.7], [-0.3, -0.1]),$$
$$B_2(ac) = ([0.3, 0.6], [-0.4, -0.2]).$$

Then we have:

$$(A_1 \cup A_2)(a) = ([0.7, 0.9], [-0.4, -0.2]),\ (A_1 \cup A_2)(b) = ([0.5, 0.8], [-0.6, -0.3]),$$
$$(A_1 \cup A_2)(c) = ([0.8, 0.9], [-0.5, -0.3]),$$
$$(B_1 \cup B_2)(ab) = ([0.5, 0.8], [-0.4, -0.2]),\ (B_1 \cup B_2)(bc) = ([0.5, 0.7], [-0.5, -0.1]),$$
$$(B_1 \cup B_2)(ac) = ([0.6, 0.8], [-0.4, -0.2]).$$

But $(A_1 \cup A_2)(a) \wedge (A_1 \cup A_2)(b) = ([0.5, 0.8], [-0.6, -0.3])$. Thus $(B_1 \cup B_2)(ab) \neq (A_1 \cup A_2)(a) \wedge (A_1 \cup A_2)(b)$. So $G_1 \cup G_2$ is not a complete IVBG.

Proposition 3.29. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two complete IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G_1 \cap G_2$ is a complete IVBG.*

Proof. Let $x,\ y \in V_1 \cap V_2$. Then
$(B_1^P \cap B_2^P)(xy) = B_1^P(xy) \wedge B_2^P(xy)$
$= (A_1^P(x) \wedge A_1^P(y)) \wedge (A_2^P(x) \wedge A_2^P(y))$
$= (A_1^P(x) \wedge A_2^P(x)) \wedge (A_1^P(y) \wedge A_2^P(y))$
$= (A_1^P \cap A_2^P)(x) \wedge (A_1^P \cap A_2^P)(y)$,
$(B_1^N \cap B_2^N)(xy) = B_1^N(xy) \vee B_2^N(xy)$
$= (A_1^N(x) \vee A_1^N(y)) \vee (A_2^N(x) \vee A_2^N(y))$
$= (A_1^N(x) \vee A_2^N(x)) \vee (A_1^N(y) \vee A_2^N(y))$
$= (A_1^N \cap A_2^N)(x) \wedge (A_1^N \cap A_2^N)(y)$.
Thus $G_1 \cap G_2$ is a complete IVBG. □

Proposition 3.30. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two complete IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, where $V_1 \cap V_2 = \varnothing$. Then $G_1 + G_2$ is a complete IVBG.*

Proof. suppose $xy \in E^{'}$. Then
$(B_1^P + B_2^P)(xy) = A_1^P(x) \vee A_2^P(y)$
$= (A_1^P \cup A_2^P)(x) \vee (A_1^P \cup A_2^P)(y)$
$= (A_1^P + A_2^P)(x) \vee (A_1^P + A_2^P)(y)$,
$(B_1^N + B_2^N)(xy) = A_1^N(x) \wedge A_2^N(y)$
$= (A_1^N \cup A_2^P)(x) \wedge (A_1^N \cup A_2^N)(y)$
$= (A_1^N + A_2^N)(x) \wedge (A_1^N + A_2^N)(y)$. Suppose $xy \in E_1 \cup E_2$. Then we can easily see that

$$(B_1^P + B_2^P)(xy) = (A_1^P + A_2^P)(x) \vee (A_1^P + A_2^P)(y),$$
$$(B_1^N + B_2^N)(xy) = (A_1^N + A_2^N)(x) \wedge (A_1^N + A_2^N)(y).$$

Thus $G_1 + G_2$ is a complete IVBG. □

The following is an immediate result of Proposition 3.13 and Definition 3.25.

Proposition 3.31. *If $G = (A, B)$ is a complete IVBG, then so is $G[G]$.*

4. Isomorphisms of IVBGs and isometric IVBGs

In this section, we define an isomorphism and an isometric between interval-valued bipolar fuzzy graphs and investigate various properties.

Definition 4.1. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then a homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ which satisfies the following conditions:
(i) $A_1^P(x) \leq A_2^P(f(x)),\ A_1^N(x) \geq A_2^N(f(x))$ for each $x \in V_1$,
(ii) $B_1^P(xy) \leq B_2^P(f(x)f(y)),\ B_1^N(xy) \geq A_2^N(f(x)f(y))$ for each $xy \in E_1$.
In particular, if $G_1 = G_2 = G$, then the homomorphism f over G is called an endomorphism.

Definition 4.2. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then an isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:
(i) $A_1(x) = A_2(f(x))$ for each $x \in V_1$,
(ii) $B_1(xy) = B_2(f(x)f(y))$ for each $xy \in E_1$.
In this case, G_1 and G_2 are said to be isomorphic and is denoted by $G_1 \simeq G_2$.

In particular, if $G_1 = G_2 = G$, then the isomorphism f over G is called an automorphism. We will denote the set of all interval-valued bipolar automorphisms of G as $Aut(G)$.

Remark 4.3. Let $G = (A, B)$ be an IVBG of graph $G^* = (V, E)$ and let $e : V \to V$ be the mapping defined by $e(x) = x$ for each $x \in V$. Then clearly, $e \in Aut(G)$.

Definition 4.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then a weak isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:
(i) f is a homomorphism,
(ii) $A_1(x) = A_2(f(x))$ for each $x \in V_1$.

It is obvious that a weak isomorphism preserves the weights of nodes but not necessarily the weights of the arcs.

Example 4.5. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graph such that $V_1 = \{a_1, b_1\}$ and $V_2 = \{a_2, b_2\}$. Consider two IVBGs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined as follows, respectively:

$$A_1(a_1) = ([0.5, 0.8], [-0.5, -0.3]),\ A_1(b_1) = ([0.6, 0.7], [-0.6, -0.2]),$$
$$B_1(a_1b_1) = ([0.4, 0.6], [-0.4, -0.1]),$$
$$A_2(a_2) = ([0.6, 0.7], [-0.6, -0.2]),\ A_2(b_2) = ([0.5, 0.8], [-0.5, -0.3]),$$
$$B_2(a_2b_2) = ([0.2, 0.5], [-0.4, -0.1]).$$

Let $f : V_1 \to V_2$ be the mapping defined by $f(a_1) = a_2$ and $f(b_1) = b_2$. Then clearly, f is bijective and a homomorphism. Moreover, we can easily see that:

$$A_1(a_1) = A_2(b_2),\ A_1(b_1) = A_2(a_2),\ B_1^N(a_1b_1) = B_2^N(a_2b_2).$$

But $B_1^P(a_1b_1) = [0.4, 0.6] \neq [0.2, 0.5] = B_2^P(a_2b_2) = B_2^P(f(a_1)f(b_1))$. Thus f is a weak isomorphism but not an isomorphism.

Definition 4.6. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then a co-weak isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:
(i) f is a homomorphism,
(ii) $B_1(xy) = B_2(f(x)f(y))$ for each $xy \in E_1$.

It is obvious that a co-weak isomorphism preserves the weights of arcs but not necessarily the weights of the nodes.

Example 4.7. Let $G^* = (V, E)$ be a graph such that $V = \{a, b, c\}$ and $E = \{ab, bc\}$. Consider the IVBG $G = (A, B)$ defined as follows:

$$A_1(a_1) = ([0.5, 0.8], [-0.5, -0.3]),\ A_1(b_1) = ([0.6, 0.7], [-0.6, -0.2]),$$

$$B_1(a_1b_1) = ([0.2, 0.5], [-0.4, -0.1]),$$
$$A_2(a_2) = ([0.6, 0.7], [-0.6, -0.2]),\ A_2(b_2) = ([0.5, 0.8], [-0.5, -0.3]),$$
$$B_2(a_2b_2) = ([0.2, 0.5], [-0.4, -0.1]).$$

Let $f : V_1 \to V_2$ be the mapping defined by $f(a_1) = a_2$ and $f(b_1) = b_2$. Then clearly, f is bijective and a homomorphism. Moreover, we can easily see that:

$$B_1(a_1b_1) = ([0.2, 0.5], [-0.4, -0.1]) = B_2(a_2b_2) = B_2(f(a_1)f(b_1)).$$

But $A_1(a_1) = ([0.5, 0.8], [-0.5, -0.3]) \neq ([0.6, 0.7], [-0.6, -0.2]) = A_2(a_2) = A_2(f(a_1))$. Thus f is a co-weak isomorphism but not an isomorphism.

Remark 4.8. In Definitions 4.4 and 4.6, we can easily see that if $G_1 = G_2$, then the weak and co-weak isomorphisms actually become isomorphic.

Proposition 4.9. *The isomorphism between interval-valued bipolar fuzzy graphs is an equivalence relation.*

Proof. Let $\mathcal{G}$ be the set of all IVBGs. Let R_I be the relation on $\mathcal{G}$ defined as follows: for any $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2) \in \mathcal{G}$,

$$(G_1, G_2) \in R_I \text{ if and only if there is an isomorphism } f : G_1 \to G_2,$$

where G_1 and G_2 are IVBGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively.

(i) R_I is reflexive: The proof is obvious.

(ii) R_I is symmetric: Let $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2) \in \mathcal{G}$ and suppose $(G_1, G_2) \in R_I$. Then there is an bijection $f : V_1 \to V_2$ defined by

$$f(x_1) = x_2 \text{ for each } x_1 \in V_1 \tag{4.1}$$

satisfying the following conditions:

$A_1(x_1) = A_2(f(x_1))$ for each $x_1 \in V_1$,

$B_1(x_1y_1) = B_2(f(x_1)f(y_1))$ for each $x_1y_1 \in E_1$.

Since f is bijective, by (4.1), it follows that $f^{-1}(x_2) = x_1$ for each $x_2 \in V_2$. Thus we have

$A_2(x_2) = A_1(f^{-1}(x_2))$ for each $x_2 \in V_2$,

$B_2(x_2y_2) = B_1(f^{-1}(x_2)f^{-1}(y_2))$ for each $x_2y_2 \in E_2$.

So $f^{-1} : G_2 \to G_1$ is an isomorphism. Hence $(G_2, G_1) \in R_I$.

(iii) R_I is transitive: Let $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2)$ and $G_3 = (A_3, B_3)$ be IVBGs of $G_1^* = (V_1, E_1),\ G_2^* = (V_2, E_2)$ and $G_3^* = (V_3, E_3)$, respectively. Suppose $(G_1, G_2),\ (G_2, G_3) \in R_I$. Then clearly there are isomorphisms $f : G_1 \to G_2$ and $g : G_2 \to G_3$. Thus there are bijective mappings $f : V_1 \to V_2$ and $g : V_2 \to V_3$ defined by $f(x_1) = x_2$ and $g(x_2) = x_3$ for each $x_1 \in V_1$ and each $x_2 \in V_2$, respectively such that

$$A_1(x_1) = A_2(f(x_1) = A_2(x_2) \text{ for each } x_1 \in V_1, \tag{4.2}$$

$$B_1(x_1y_1) = B_2(f(x_1)f(y_1)) = B_2(x_2y_2) \text{ for each } x_1y_1 \in E_1, \tag{4.3}$$

$$A_2(x_2) = A_3(f(x_2) = A_3(x_3) \text{ for each } x_2 \in V_2, \tag{4.4}$$

$$B_2(x_2y_2) = B_3(f(x_2)f(y_2)) = B_3(x_3y_3) \text{ for each } x_2y_2 \in E_2. \tag{4.5}$$

Since $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are bijective such that $f(x_1) = x_2$ and $g(x_2) = x_3$ for each $x_1 \in V_1$ and each $x_2 \in V_3$, $g \circ f : V_1 \to V_3$ is bijective such that $(g \circ f)(x_1) = g(f(x_1)) = x_3$ for each $x_1 \in V_1$. From (4.2) and (4.4), we have: for each $x_1 \in V_1$,

$$A_1(x_1) = A_2(f(x_1)) = A_2(x_2) = A_3(g(x_2)) = A_3(g \circ f)(x_1)).$$

Furthermore, from (4.3) and (4.5), we have: for each $x_1y_1 \in E_1$,

$$B_1(x_1y_1) = B_3((g \circ f)(x_1)(g \circ f)(y_1)).$$

So $g \circ f : G_1 \to G_3$ is an isomorphism, i.e., $(G_1, G_3) \in R_I$. Hence R_I is transitive. Therefore R_I is an equivalence relation on $\mathcal{G}$. This completes the proof. □

Proposition 4.10. *The weak isomorphism between interval-valued bipolar fuzzy graphs is a partial order relation.*

Proof. Let $\mathcal{G}$ be the set of all IVBGs. Let R_{WI} be the relation on $\mathcal{G}$ defined as follows: for any $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2) \in \mathcal{G}$,

$$(G_1, G_2) \in R_{WI} \text{ if and only if there is a weak isomorphism } f : G_1 \to G_2,$$

where G_1 and G_2 are IVBGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively.

(i) R_{WI} is reflexive: The proof is obvious.

(ii) R_{WI} is antisymmetric: Let $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2) \in \mathcal{G}$ and suppose $(G_1, G_2) \in R_{WI}$ and $(G_2, G_1) \in R_{WI}$. Let $f : G_1 \to G_2$ and $g : G_2 \to G_1$ be weak isomorphisms. Then clearly, $f : V_1 \to V_2$ and $g : V_2 \to V_1$ are bijective mappings, respectively defined by

$$f(x_1) = x_2 \text{ for each } x_1 \in V_1 \text{ and } g(x_2) = x_1 \text{ for each } x_2 \in V_2$$

satisfying the following conditions, respectively:

$$A_1(x_1) = A_2(f(x_1)) \text{ for each } x_1 \in V_1,$$

(4.6)
$$B_1^P(x_1y_1) \leq B_2^P(f(x_1)f(y_1)),\ B_1^N(x_1y_1) \geq B_2^N(f(x_1)f(y_1)) \text{ for each } x_1y_1 \in E_1,$$

$$A_2(x_2) = A_1(g(x_2)) \text{ for each } x_2 \in V_2,$$

(4.7)
$$B_2^P(x_2y_2) \leq B_1^P(g(x_2)g(y_2)),\ B_2^N(x_2y_2) \geq B_1^N(g(x_2)g(y_2)) \text{ for each } x_2y_2 \in E_2.$$

The inequalities (4.6) and (4.7) hold on the finite sets V_1 and V_2 only when G_1 and G_2 have the same number of edges and the corresponding edges have the same weight. Thus $G_1 = G_2$. So R_{WI} is symmetric.

(iii) R_{WI} is transitive: Let $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2)$ and $G_3 = (A_3, B_3)$ be IVBGs of $G_1^* = (V_1, E_1),\ G_2^* = (V_2, E_2)$ and $G_3^* = (V_3, E_3)$, respectively. Suppose $(G_1, G_2),\ (G_2, G_3) \in R_{WI}$. Let $f : G_1 \to G_2$ and $g : G_2 \to G_3$ be weak isomorphisms. Then clearly, $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are bijective mappings, respectively defined by

$$f(x_1) = x_2 \text{ for each } x_1 \in V_1 \text{ and } g(x_2) = x_3 \text{ for each } x_2 \in V_2$$

satisfying the following conditions, respectively:

(4.8)
$$A_1(x_1) = A_2(f(x_1)) = A_2(x_2) \text{ for each } x_1 \in V_1,$$

(4.9)
$$B_1^P(x_1y_1) \leq B_2^P(f(x_1)f(y_1)) = B_2^P(x_2y_2),\ B_1^N(x_1y_1) \geq B_2^N(f(x_1)f(y_1)) = B_2^N(x_2y_2)$$

for each $x_1y_1 \in E_1$,

(4.10)
$$A_2(x_2) = A_3(g(x_2)) = A_3(x_3) \text{ for each } x_2 \in V_2,$$

(4.11)
$$B_2^P(x_2y_2) \leq B_3^P(g(x_2)g(y_2)) = B_3^P(x_3y_3),\ B_2^N(x_2y_2) \geq B_3^N(g(x_2)g(y_2)) = B_3^N(x_3y_3)$$

for each $x_2y_2 \in E_2$. Since $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are bijective such that $f(x_1) = x_2$ for each $x_1 \in V_1$ and $g(x_2) = x_3$ for each $x_2 \in V_2$, $g \circ f : V_1 \to V_3$ is bijective such that $(g \circ f)(x_1) = g(f(_1)) = g(x_2) = x_3$ for each $x_1 \in V_1$. Thus from (4.8) and (4.10), we have: for each $x_1 \in V_1$,

$$A_1(x_1) = A_3((g \circ f)(x_1)).$$

Furthermore, from (4.9) and (4.11), we have: for each $x_1y_1 \in E_1$,

$$B_1^P(x_1y_1) \leq B_3^P(x_1y_1)((g \circ f)(x_1)(g \circ f)(x_2)),$$
$$B_1^N(x_1y_1) \geq B_3^N(x_1y_1)((g \circ f)(x_1)(g \circ f)(x_2)).$$

So $g \circ f : G_1 \to G_3$ is a weak isomorphism. Hence R_{WI} is transitive. Therefore R_{WI} is a partial order relation. This completes the proof. □

Definition 4.11. Let S be a semigroup and let $A \in IVB(S)$. Then A is called an interval-valued bipolar fuzzy subsemigroup of S, if it satisfies the following conditions:

$$A_P(xy) \geq A_P(x) \wedge A_P(y),\ A_N(xy) \leq A_N(x) \vee A_N(y)$$

for any $x,\ y \in S$.

Definition 4.12. Let G be a group and let $A \in IVB(G)$. Then A is called an interval-valued bipolar fuzzy subgroup of G, if it satisfies the following conditions:
(i) A is an interval-valued bipolar fuzzy subsemigroup of G,
(ii) $A(x^{-1}) = A(x)$ for each $x \in G$.

Now let us show how to associate an interval-valued bipolar fuzzy subgroup with an IVBG in a natural way.

Proposition 4.13. *Let $G = (A, B)$ be an IVBG of a graph $G^* = (V, E)$ and let $Aut(G)$ be the set of all automorphisms of G. We define the binary operation $\circ$ on $Aut(G)$ as follows: for any $f,\ g \in Aut(G)$ and any $x,\ y \in V$,*

$$B^P((g \circ f)(x)(g \circ f)(y)) = B^P(g(f(x))g(f(y))) \geq B^P(f(x)f(y)) \geq B^P(xy),$$
$$B^N((g \circ f)(x)(g \circ f)(y)) = B^N(g(f(x))g(f(y))) \leq B^N(f(x)f(y)) \geq B^N(xy),$$
$$B^P((g \circ f)(x)) = B^P(g(f(x))) \geq B^P(f(x)) \geq B^P(x),$$
$$B^N((g \circ f)(x)) = B^N(g(f(x))) \geq B^N(f(x)) \geq B^N(x).$$

Then $(Aut(G), \circ)$ forms a group.

Proof. It is obvious that $\circ$ is well-defined and satisfies associativity. Moreover, from Remark 4.3, $e \in Aut(G)$, $e \circ f = f = f \circ e$ and $A(f^{-1}) = A(f)$ for each $f \in Aut(G)$. Thus $(Aut(G), \circ)$ forms a group. □

Proposition 4.14. *Let $G = (A, B)$ be an IVBG of a graph $G^* = (V, E)$ and let $Aut(G)$ be the set of all automorphisms of G. Let*

$$g = (g^P, g^N) = ([g^{P,-}, g^{P,+}], [g^{N,-}, g^{N,+}])$$

be an interval-valued bipolar fuzzy set in $Aut(G)$ defined as follows: for each $f \in Aut(G)$,
$g^P(f) = sup\{B^P(f(x), f(y)) : (x, y) \in V \times V\}$
$= [\bigvee_{(x,y)\in V\times V} B^{P,-}(f(x), f(y)), \bigvee_{(x,y)\in V\times V} B^{P,+}(f(x), f(y))]$,
$g^N(f) = inf\{B^N(f(x), f(y)) : (x, y) \in V \times V\}$
$= [\bigwedge_{(x,y)\in V\times V} B^{N,-}(f(x), f(y)), \bigwedge_{(x,y)\in V\times V} B^{N,+}(f(x), f(y))]$.
Then g is an interval-valued bipolar fuzzy subgroup of $Aut(G)$.

Proof. The proof is straightforward. □

Proposition 4.15. *Every interval-valued bipolar fuzzy subgroup has an embedding into the interval-valued bipolar fuzzy subgroup of the group of automorphisms of some IVBG.*

Proof. The proof is straightforward. □

Definition 4.16. Let $G = (A, B)$ be an IVBG of graphs $G^* = (V, E)$. Then the complement of G, denoted by $\overline{G} = (\overline{A}, \overline{B})$, is an IVBG of $\overline{G^*} = (\overline{V}, \overline{E})$ defined as follows:
(i) $\overline{V} = V$, $\overline{E} = V \times V - E$,
(ii) $\overline{A} = (\overline{A^P}, \overline{A^N}) \in IVB(V)$ such that $\overline{A}(x) = A(x)$ for each $x \in V$,
(iii) $\overline{B} = (\overline{B^P}, \overline{B^N}) \in IVB(V \times V)$ such that

$$\overline{B}(xy) = (A^P(x) \wedge A^P(y) - B^P(xy), A^N(x) \vee A^N(y) - B^N(xy))$$

for each $xy \in V \times V$.

$$\overline{B^P}(xy) = \begin{cases} [0, 0] & \text{if } B^{P,-}(xy) > 0 \\ [A^{P,-}(x) \wedge A^{P,-}(y), A^{P,+}(x) \wedge A^{P,+}(y)] & \text{if } B^{P,+}(xy) = 0, \end{cases}$$

$$\overline{B^N}(xy) = \begin{cases} [0, 0] & \text{if } B^{N,+}(xy) < 0 \\ [A^{N,-}(x) \vee A^{N,-}(y), A^{N,+}(x) \wedge A^{N,+}(y)] & \text{if } B^{N,-}(xy) = 0. \end{cases}$$

Definition 4.17. Let $G = (A, B)$ be an IVBG of graphs $G^* = (V, E)$. Then $G = (A, B)$ is said to be self-complementary, if $G = \overline{\overline{G}}$.

Example 4.18. Let $G^* = (V, E)$ be a graph such that $V = \{a, b, c\}$ and $E = \{ab, bc\}$. Consider the IVBG $G = (A, B)$ defined as follows:

$$A(a) = ([0.1, 0.3], [-0.3, -0.1]),\ A(b) = ([0.2, 0.4], [-0.4, -0.2]),$$

$$A(c) = ([0.3, 0.5], [-0.5, -0.3]),$$

$$B(ab) = ([0.1, 0.3], [-0.3, -0.1]),\ B(bc) = ([0.2, 0.4], [-0.3, -0.2]).$$

Then clearly, $\overline{\overline{A}} = A$, $\overline{\overline{B}}(ab) = B(ab)$ and $\overline{\overline{B}}(bc) = B(bc)$. Thus $\overline{\overline{G}} = G$. So $G = (A, B)$ is self-complementary.

The following is the immediate result of Definitions 3.25 and 4.16.

Proposition 4.19. *Let $G = (A, B)$ be a complete IVBG of a graph $G^* = (V, E)$. Then G is self complementary.*

Proposition 4.20. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of a graph $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. If $G_1 \simeq G_2$, then $\overline{G_1} \simeq \overline{G_2}$.*

Proof. Suppose $G_1 \simeq G_2$. Then clearly, there is a bijective mapping $f : V_1 \to V_2$ satisfying the following conditions:

$$A_1(x) = A_2(f(x)) \text{ for each } x \in V_1,$$
$$B_1(x) = B_2(f(x)f(y)) \text{ for each } xy \in E_1.$$

Let $xy \in V_1 \times V_1$.

Suppose $B_1^{P,-}(xy) > 0$. Then clearly, $B_2^{P,-}(f(x)f(y)) > 0$. Thus $\overline{B_1^P}(xy) = \overline{B_2^P}(f(x)f(y)) = [0, 0]$.

Suppose $B_1^{P,+}(xy) = 0$. Then clearly, $B_2^{P,+}(f(x)f(y)) = 0$. Thus we have
$\overline{B_1^P}(xy) = [A_1^{P,-}(x) \wedge A_1^{P,-}(y), A_1^{P,+}(x) \wedge A_1^{P,+}(y)]$
$= [A_2^{P,-}(f(x)) \wedge A_2^{P,-}(f(y)), A_2^{P,+}(f(x)) \wedge A_2^{P,+}(f(y))]$
$= \overline{B_2^P}(f(x)f(y))$.

Suppose $B_1^{N,+}(xy) < 0$. Then clearly, $B_2^{N,+}(f(x)f(y)) < 0$. Thus $\overline{B_1^N}(xy) = \overline{B_2^N}(f(x)f(y)) = [0, 0]$.

Suppose $B_1^{N,-}(xy) = 0$. Then clearly, $B_2^{N,-}(f(x)f(y)) = 0$. Thus we have
$\overline{B_1^N}(xy) = [A_1^{N,-}(x) \vee A_1^{N,-}(y), A_1^{N,+}(x) \vee A_1^{N,+}(y)]$
$= [A_2^{N,-}(f(x)) \vee A_2^{N,-}(f(y)), A_2^{N,+}(f(x)) \vee A_2^{N,+}(f(y))]$
$= \overline{B_2^N}(f(x)f(y))$.

So in either cases, $\overline{B_1}(xy) = \overline{B_2}(f(x)f(y))$. Hence $\overline{G_1} \simeq \overline{G_2}$. □

Definition 4.21. Let $G = (A, B)$ be an IVBG of a graph $G^* = (V, E)$. Let $u = u_0, u_1, \cdots, u_i, \cdots, u_n = v$ be a path from u to v in G^*. Then the B^P distance and B^N distance from u to v, denoted by $\delta^P(u, v)$ and $\delta^N(u, v)$, defined by, respectively:

$$\delta^P(u, v) = \bigwedge\{\Sigma_{i=1}^n[\frac{1}{B^{P,-}(u_{i-1}u_i)}, \frac{1}{B^{P,+}(u_{i-1}u_i)}]\},$$
$$\delta^N(u, v) = \bigvee\{\Sigma_{i=1}^n[\frac{1}{B^{N,-}(u_{i-1}u_i)}, \frac{1}{B^{N,+}(u_{i-1}u_i)}]\}.$$

Definition 4.22. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of a graph $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then G_2 is said to be isometric from G_1 or G_1 is said to be isometric to G_2, if for each $v \in V_1$, there is a bijective mapping $g_v : V_1 \to V_2$ such that for each $u \in V_1$,

$$\delta_1^P(u, v) = \delta_2^P(g_v(u), g_v(v)),\ \delta_1^N(u, v) = \delta_2^N(g_v(u), g_v(v)).$$

Proposition 4.23. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVBGs of a graph $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. If $G_1 \simeq G_2$, then G_1 is isometric to G_2.*

Proof. Suppose $G_1 \simeq G_2$. Then clearly, there is a bijective mapping $g : V_1 \to V_2$ satisfying the following conditions:

$$A_1(x) = A_2(g(x)) \text{ for each } x \in V_1,\ B_1(xy) = B_2(g(x)g(y)) \text{ for each } xy \in E_1.$$

For any $u,\ v \in V_1$, let $u = u_0,\ u_1,\ \cdots,\ u_i,\ \cdots,\ u_n = v$ be a path from u to v in G^*. Then
$\delta_1^P(u,v) = \bigwedge\{\Sigma_{i=1}^n[\frac{1}{B_1^{P,-}(u_{i-1}u_i)}, \frac{1}{B_1^{P,+}(u_{i-1}u_i)}]\}$
$= \bigwedge\{\Sigma_{i=1}^n[\frac{1}{B_2^{P,-}(g(u_{i-1})g(u_i))}, \frac{1}{B_2^{P,+}(g(u_{i-1})g(u_i))}]\}$
$= \delta_2^P(g(u), g(v))$,
$\delta_1^N(u,v) = \bigvee\{\Sigma_{i=1}^n[\frac{1}{B_1^{N,-}(u_{i-1}u_i)}, \frac{1}{B_1^{N,+}(u_{i-1}u_i)}]\}$
$= \bigvee\{\Sigma_{i=1}^n[\frac{1}{B_2^{N,-}(g(u_{i-1})g(u_i))}, \frac{1}{B_2^{N,+}(g(u_{i-1})g(u_i))}]\}$
$= \delta_2^N(g(u), g(v))$.
Thus G_1 is isometric to G_2. □

Remark 4.24. (1) Even if G_1 is co-weak isomorphic to G_2, Proposition 4.23 holds.

(2) From Proposition 4.20, we know that if $G_1 \simeq G_2$, the $\overline{G_1} \simeq \overline{G_2}$. But there is not so in the case of isometry, i.e., G_2 is isometric from G_1 but $\overline{G_2}$ is not isometric from $\overline{G_1}$.

Example 4.25. Consider two IVBGs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 = \{a, b, c, d\}$, $E_1 = \{ab, bc, ad, bd\}$, $V_2 = \{x, u, v, w\}$ and $E_2 = \{xu, xv, xw, uv, vw\}$, respectively given by:

$$A_1(a) = ([\frac{1}{2}, \frac{1}{2}], [-1, -1]),\ A_1(b) = ([1, 1], [-1, -1]),$$

$$A_1(c) = ([\frac{1}{2}, \frac{1}{2}], [-1, -1]),\ A_1(d) = ([\frac{1}{5}, \frac{1}{5}], [-1, -1]),$$

$$B_1(ab) = ([\frac{1}{8}, \frac{1}{8}], [-\frac{1}{3}, -\frac{1}{3}]),\ B_1(bc) = ([\frac{1}{4}, \frac{1}{4}], [-\frac{1}{3}, -\frac{1}{3}]),$$

$$A_1(ad) = ([\frac{1}{9}, \frac{1}{9}], [-\frac{1}{2}, -\frac{1}{2}]),\ A_1(ad) = ([\frac{1}{9}, \frac{1}{9}], [-\frac{1}{2}, -\frac{1}{2}]),$$

$$A_2(x) = A_2(u) = A_2(v) = A_2(w) = ([1, 1], [-1, -1]),$$

$$B_2(xu) = ([\frac{1}{9}, \frac{1}{9}], [-\frac{1}{2}, -\frac{1}{2}]),\ B_2(xv) = ([\frac{1}{9}, \frac{1}{9}], [-\frac{1}{2}, -\frac{1}{2}]),$$

$$B_2(xw) = ([\frac{1}{13}, \frac{1}{13}], [-\frac{1}{5}, -\frac{1}{5}]),\ B_2(uv) = ([\frac{1}{8}, \frac{1}{8}], [-\frac{1}{3}, -\frac{1}{3}]),$$

$$B_2(vw) = ([\frac{1}{4}, \frac{1}{4}], [-\frac{1}{3}, -\frac{1}{3}]).$$

Then we can easily calculate that

$$\delta_1^P(a,b) = [8, 8],\ \delta_1^N(a,b) = [-3, -3],\ \delta_1^P(a,c) = [12, 12],\ \delta_1^N(a,c) = [-6, -6],$$

$$\delta_1^P(a,d) = [9, 9],\ \delta_1^N(a,d) = [-2, -2],\ \delta_1^P(b,d) = [9, 9],\ \delta_1^N(b,d) = [-2, -2],$$

$$\delta_1^P(c,d) = [13, 13],\ \delta_1^N(c,d) = [-5, -5],$$

$$\delta_2^P(u,v) = [8, 8],\ \delta_2^N(u,v) = [-3, -3],\ \delta_2^P(u,w) = [12, 12],\ \delta_2^N(u,w) = [-6, -6],$$

$$\delta_2^P(u,x) = [9, 9],\ \delta_2^N(u,x) = [-2, -2],\ \delta_2^P(u,w) = [4, 4],\ \delta_2^N(u,w) = [-3, -3],$$

$$\delta_2^P(v,x) = [9, 9],\ \delta_1^N(c,d) = [-5, -5].$$

Let $g : V_1 \to V_2$ be the mapping defined as follows:

$$g(a) = u,\ g(b) = v,\ g(c) = w,\ g(d) = x.$$

Then clearly, g is a bijective mapping preserving the distance between every pair of vertices in G_1 and G_2. Thus G_2 is isometric from G_1.

On the other hand, we have $\overline{G_1} = (\overline{A_1}, \overline{B_1})$ and $\overline{G_2} = (\overline{A_2}, \overline{B_2})$, such that

$$\overline{B_1}(a,d) = ([\frac{1}{2}, \frac{1}{2}], [-1,-1]),\ \overline{B_1}(c,d) = ([\frac{1}{5}, \frac{1}{5}], [-1,-1]),$$

$$\overline{B_2}(u,w) = ([1,1], [-1,-1]).$$

Moreover, $\delta_1(a,d) = ([2,2],[-1,-1])$, $\delta_1(c,d) = [5,5],[-1,-1])$ and $\delta_2(u,v) = ([1,1],[-1,-1])$. So there is not a bijective between $\overline{G_1}$ and $\overline{G_2}$ preserving distance. Hence $\overline{G_2}$ is not isometric from $\overline{G_1}$.

Proposition 4.26. *Isometry on IVBGs is an equivalence relation.*

Proof. Let $\mathcal{G}$ be the set of all IVBGs. Let R_{Iso} be the relation on $\mathcal{G}$ defined as follows: for any $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2) \in \mathcal{G}$,

$$(G_1, G_2) \in R_{Iso} \text{ if and only if } G_2 \text{ is isometric from } G_1,$$

where G_1 and G_2 are IVBGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively.

(i) R_{Iso} is reflexive: The proof is obvious.

(ii) R_{Iso} is symmetric: Let $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2) \in \mathcal{G}$ and suppose $(G_1, G_2) \in R_{Iso}$. Then clearly, G_2 is isometric from G_1. Thus there is an bijection $g : V_1 \to V_2$ defined by

$$f(v_1) = v_2 \text{ for each } v_1 \in V_1 \tag{4.12}$$

such that $\delta_1(u_1, v_1) = \delta_2(g(u_1), g(v_1)) = \delta_2(u_2, v_2)$ for any $u_1,\ v_1 \in V_1$.
Since g is bijective, $g^{-1} : V_2 \to V_1$ is bijective. By (4.12), $g^{-1}(u_2) = u_1$ and $g^{-1}(v_2) = v_1$ for any $u_2,\ v_2 \in V_2$. Moreover, we have

$$\delta_2(u_2, v_2) = \delta_1(u_1, v_1) = \delta_1(g^{-1}(u_2), g^{-1}(v_2)).$$

So G_1 is isometric from G_2, i.e., $(G_2, G_1) \in R_{Iso}$. Hence R_{Iso} is symmetric.

(iii) R_{Iso} is transitive: Let $G_1 = (A_1, B_1),\ G_2 = (A_2, B_2)$ and $G_3 = (A_3, B_3)$ be IVBGs of $G_1^* = (V_1, E_1),\ G_2^* = (V_2, E_2)$ and $G_3^* = (V_3, E_3)$, respectively. Suppose $(G_1, G_2),\ (G_2, G_3) \in R_I$. Then clearly, G_2 is isometric from G_1 and G_3 is isometric from G_2. Thus there are bijective mappings $f : V_1 \to V_2$ and $g : V_2 \to V_3$ defined by $f(v_1) = v_2$ and $g(v_2) = v_3$ for each $v_1 \in V_1$ and each $v_2 \in V_2$, respectively such that

$$\delta_1(u_1, v_1) = \delta_2(f(u_1), f(v_1)) = \delta_2(u_2, v_2) \text{ for any } u_1,\ v_1 \in V_1, \tag{4.13}$$

$$\delta_2(u_2, v_2) = \delta_3(g(u_2), g(v_2)) = \delta_3(u_3, v_3) \text{ for any } u_2,\ v_1 \in V_2. \tag{4.14}$$

Since $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are bijective such that $f(u_1) = u_2$ and $g(u_2) = u_3$ for each $u_1 \in V_1$ and each $u_2 \in V_3$, $g \circ f : V_1 \to V_3$ is bijective such that $(g \circ f)(u_1) = g(f(u_1)) = u_3$ for each $u_1 \in V_1$. From (4.13) and (4.14), we have: for any $u_1,\ v_1 \in V_1$,

$$\delta_1(u_1, v_1) = \delta_3((g \circ f)(u_1), (g \circ f)(v_1)).$$

So G_3 is isometric from G_1, i.e., $(G_1, G_3) \in R_{Iso}$. Hence R_{Iso} is transitive. Therefore R_{Iso} is an equivalence relation on $\mathcal{G}$. This completes the proof. □

5. Discussion

Compared to previous works on interval-valued fuzzy graphs [13] and bipolar fuzzy graphs [28], the proposed IVBFG model provides a unified structure that accommodates both bipolarity and interval uncertainty simultaneously. Unlike regular bipolar fuzzy graphs [29], which consider fixed membership values, our approach introduces flexibility in modeling systems where the degree of membership is inherently uncertain. The comparison with these existing models highlights the enhanced representational capability of IVBFGs.

6. Conclusions

In this paper, we introduced the concept of interval-valued bipolar fuzz graphs and give some basic operations between them, and we obtained some of their properties. Also we defined an isomorphism and an isometric between interval-valued bipolar fuzz graphs and studied some of their properties. In the future, we expect that one apply interval-valued bipolar fuzz graphs to database theory, an expert system, neural networks, the method for finding the shortest paths in networks and decision making problems, etc.

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