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Abstract. In this work, β-openness and β-compactness in graded ditopological texture spaces are defined and the properties of these concepts are examined. Furthermore, several relationships between the structures ditopological texture spaces and graded ditopological texture spaces in the context of β-openness are studied.

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1. Introduction

Abd El Monsef *et al.* introduced β-open sets in [1] and several studies have been developed this area such as [2, 3, 4].

In [5] Brown and Šostak introduced the theory of graded ditopology as a more comprehensive structure than ditopology presented in [6, 7] and fuzzy topology given independently by Šostak in [8] and Kubiak in [9]. In the structure of graded ditopology, openness and closedness are defined by independent grading functions instead of elements of a texture as in ditopological case. The theory of graded ditopological texture spaces continues to be developed by various recent studies such as [10, 11].

The purpose of this study is to generalize the concepts of β-openness and β-compactness in ditopological texture spaces defined in [12] to the structure of graded ditopological texture spaces and investigate the properties of these concepts in this structure. For this generalization, the spectral approach as in [13, 14, 15] is used and several relationships between the structures ditopological texture spaces and graded ditopological texture spaces in the context of β-openness are examined.

Our basic motivation is to define β-openness and β-compactness in g.d.t.s. and so develop the theory of graded ditopologies by investigating the properties of β-openness and β-compactness in g.d.t.s.

2. Preliminaries

Ditopological Texture Spaces ([6, 16, 17]) Let S be a set and $\mathcal{S} \subseteq \mathcal{P}(S)$ with $S, \varnothing \in \mathcal{S}$. If $\mathcal{S}$ is a point separating, complete, completely distributive lattice with respect to inclusion and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ coincide with unions $\bigcup$ then the pair $(S, \mathcal{S})$ is called a *texture* or a *texture space*.

In general, a texturing of S may not be closed under set complementation. However, if there is a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$ then σ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is called a *complemented texture* .

The $p-sets$ given by $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and the $q-sets$ given by $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}$ are essential to define several concepts in a texture space $(S, \mathcal{S})$.

Product of textures ([7, 16, 17]) Let $(S_j, \mathcal{S}_j)$, $j \in J$ be textures, $S = \prod_{j \in J} S_j$ and $A_k \in \mathcal{S}_k$ for some $k \in J$. If we write

$$E(k, A_k) = \prod_{j \in J} Y_j \text{ where } Y_j = \begin{cases} A_j & \text{if } j = k \\ S_j & \text{otherwise} \end{cases}$$

then the product texturing $\mathcal{S} = \bigotimes_{j \in J} \mathcal{S}_j$ of S consists of arbitrary intersections of elements of the set

$$\varepsilon = \{ \bigcup_{j \in J_1} E(j, A_j) \mid J_1 \subseteq J,\ A_j \in \mathcal{S}_j \text{ for } j \in J_1 \}.$$

Consider two textures $(S, \mathcal{S})$ and $(V, \mathcal{V})$. The p-sets and q-sets of the product texture $(S \times V, \mathcal{P}(S) \otimes \mathcal{V})$ will be denoted by $\overline{P}_{(s,v)}$, $\overline{Q}_{(s,v)}$ respectively.

If $\mathcal{P}(X)$ is the power set of a set X, then $(X, \mathcal{P}(X))$ is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$. The mapping $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is a complementation on the texture $(X, \mathcal{P}(X))$.

Definition 2.1 ([16]). Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be textures. Then

(i) $r \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a *relation* on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, if it satisfies
  R1 $r \not\subseteq \overline{Q}(s, v)$, $P_{s'} \not\subseteq Q_s \Rightarrow r \not\subseteq \overline{Q}(s', v)$,
  R2 $r \not\subseteq \overline{Q}(s, v) \Rightarrow \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}(s', v)$,

(ii) $R \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a $co-relation$ on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, if it satisfies
  CR1 $\overline{P}(s, v) \not\subseteq R$, $P_s \not\subseteq Q_{s'} \Rightarrow \overline{P}(s', v) \not\subseteq R$,
  CR2 $\overline{P}(s, v) \not\subseteq R \Rightarrow \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}(s', v) \not\subseteq R$,

(iii) A pair (r, R), where r is a relation and R a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ is called a *direlation* on $(S, \mathcal{S})$ to $(V, \mathcal{V})$.

For a texture $(S, \mathcal{S})$ the identity direlation $(i_{(S,\mathcal{S})}, I_{(S,\mathcal{S})})$ is defined by

$$i_{(S,\mathcal{S})} = \bigvee\{\overline{P}(s, s) \mid s \in S\} \text{ and } I_{(S,\mathcal{S})} = \bigcap\{\overline{Q}(s, s) \mid s \in S^\flat\}.$$

For $A \subseteq S$, $r^{\to}A = \bigcap\{Q_v \mid \forall s, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow A \subseteq Q_s\}$ is called the *A-section* of r and $R^{\to}A = \bigvee\{P_v \mid \forall s, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow P_s \subseteq A\}$ is called the *A-section* of R.

For $B \subseteq V$, $r^{\leftarrow}B = \bigvee\{P_s \mid \forall v, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow P_v \subseteq B\}$ is called the *B-presection* of r and $R^{\leftarrow}B = \bigcap\{Q_s \mid \forall v, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$ is called the *B-presection* of R.

Proposition 2.2 ([16])**.** *If (r,R) is a direlation on $(S,\mathcal{S})$ to $(V,\mathcal{V})$, then $r^{\to}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} r^{\to}A_i$, $R^{\to}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} R^{\to}A_i$, $r^{\leftarrow}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} r^{\leftarrow}B_j$ and $R^{\leftarrow}(\bigvee_{j\in J} B_j) = \bigvee_{j\in J} R^{\leftarrow}B_j$ for any $A_i \in \mathcal{S}$, $B_j \in \mathcal{V}$, $i \in I$, $j \in J$.*

Definition 2.3 ([16])**.** A direlation (f,F) from $(S,\mathcal{S})$ to $(V,\mathcal{V})$ is called a *difunction* from $(S,\mathcal{S})$ to $(V,\mathcal{V})$, if it satisfies the following two conditions:

(DF1) For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists v \in V$ with $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s',v)} \not\subseteq F$.

(DF2) For $v, v' \in V$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v')} \not\subseteq F \Rightarrow P_{v'} \not\subseteq Q_v$.

(f,F) is called *surjective*, if $\forall v, v' \in V$ $P_v \not\subseteq Q_{v'} \Rightarrow \exists s \in S$ with $f \not\subseteq \overline{Q}_{(s,v')}$ and $\overline{P}_{(s,v)} \not\subseteq F$.

In particular, the identity direlation (i_S, I_S) is a difunction on $(S,\mathcal{S})$.

Proposition 2.4 ([16])**.** (1) *$f^{\leftarrow}B = F^{\leftarrow}B$ for each $B \in \mathcal{V}$.*
(2) *$f^{\leftarrow}\varnothing = F^{\leftarrow}\varnothing = \varnothing$ and $f^{\leftarrow}V = F^{\leftarrow}V = S$.*
(3) *$f^{\leftarrow}(F^{\to}A) \subseteq A \subseteq F^{\leftarrow}(f^{\to}A)$ and $f^{\to}(F^{\leftarrow}B) \subseteq B \subseteq F^{\to}(f^{\leftarrow}B)$ for all $A \in \mathcal{S}$, $B \in \mathcal{V}$.*
(4) *If (f,F) is surjective, then $F^{\to}(f^{\leftarrow}B) = B = f^{\to}(F^{\leftarrow}B)$ for all $B \in \mathcal{V}$.*

A ditopology on a texture $(S,\mathcal{S})$ is a pair (τ,κ), where $\tau,\kappa \subseteq \mathcal{S}$ and the set of open sets τ satisfies

(T_1) $S, \varnothing \in \tau$,
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$,
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set of closed sets κ satisfies

(CT_1) $S, \varnothing \in \kappa$,
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$,
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

In this case, $(S,\mathcal{S},\tau,\kappa)$ is called a *ditopological texture space* (or "d.t.s." for short). So a ditopology can be considered as a "topology" in which there is no need to exist a relation between the open and closed sets [6].

Let $(S,\mathcal{S},\tau,\kappa)$ be a d.t.s. For a subset $A \in \mathcal{S}$, the closure (interior) of A is defined by $[A] = \bigcap\{B \in \kappa \mid A \subseteq B\}$ ($]A[= \bigvee\{B \in \tau \mid B \subseteq A\}$) respectively [17]. $A \in \mathcal{S}$ is called *semi open* (*semi closed*), if $A \subseteq [\,]A[\,]$ ($]\,[A]\,[\subseteq A$) respectively [18]. $A \in \mathcal{S}$ is called *β-open* (*β-closed*), if $A \subseteq [\,]\,[A]\,[\,]$ ($]\,[\,]A[\,]\,[\subseteq A$) respectively [12].

Definition 2.5 ([19])**.** Let $(S,\mathcal{S},\tau,\kappa)$ be a d.t.s. and $A \in \mathcal{S}$.

(i) A is called *compact*, if whenever $\{G_i \mid i \in I\}$ is an open cover of A (i.e. $\forall i \in I$ $G_i \in \tau$ and $A \subseteq \bigvee_{i\in I} G_i$) then there is a finite subset J of I with $A \subseteq \bigvee_{i\in J} G_i$. In particular, $(S,\mathcal{S},\tau,\kappa)$ is called *compact*, if S is compact.
(ii) A is called *cocompact*, if whenever $\{K_i \mid i \in I\}$ is a closed cocover of A (i.e. $\forall i \in I$ $K_i \in \kappa$ and $\bigcap_{i\in I} K_i \subseteq A$) then there is a finite subset J of I with $\bigcap_{i\in J} K_i \subseteq A$. In particular, $(S,\mathcal{S},\tau,\kappa)$ is called *cocompact*, if $\varnothing$ is compact.
(iii) $(S,\mathcal{S},\tau,\kappa)$ is called *stable*, if every $K \in \kappa$ with $K \neq S$ is compact.
(iv) $(S,\mathcal{S},\tau,\kappa)$ is called *costable*, if every $G \in \kappa$ with $G \neq \varnothing$ is cocompact.
(v) $(S,\mathcal{S},\tau,\kappa)$ is called *dicompact*, if it is compact, cocompact, stable and costable.

$A \in \mathcal{S}$ is called *β-compact* (*β-cocompact*). if every cover (cocover) of A by β-open (β-closed) sets has a finite subcover (subcocover) respectively and $(S, \mathcal{S}, \tau, \kappa)$ is called *β-compact* (*β-cocompact*), if S is β-compact (if $\varnothing$ is β-cocompact) respectively [12].

Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$ be d.t.s. and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction. (f, F) is called *β-continuous* (*$M\beta$-continuous*), if $F^{\leftarrow}(G)$ is β-open in $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ for every open (β-open) set G in $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. (f, F) is called *β-cocontinuous* (*$M\beta$-cocontinuous*), if $f^{\leftarrow}(K)$ is β-closed in $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ for every closed (β-closed) set K in $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. (f, F) is called *β-bicontinuous* (*$M\beta$-bicontinuous*), if it is both β-continuous and β-cocontinuous (both $M\beta$-continuous and $M\beta$-cocontinuous) respectively [12].

Graded Ditopological Texture Spaces [5] Consider two textures $(S, \mathcal{S})$ and $(V, \mathcal{V})$. A graded ditopological texture space (or "g.d.t.s." for short) is a tuple $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ where the mappings $\mathcal{T}, \mathcal{K} : \mathcal{S} \to \mathcal{V}$ satisfy following conditions:

(GT_1) $\mathcal{T}(S) = \mathcal{T}(\varnothing) = V$,
(GT_2) $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)$ $\forall A_1, A_2 \in \mathcal{S}$,
(GT_3) $\bigcap_{j \in J} \mathcal{T}(A_j) \subseteq \mathcal{T}(\bigvee_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$,
(GCT_1) $\mathcal{K}(S) = \mathcal{K}(\varnothing) = V$,
(GCT_2) $\mathcal{K}(A_1) \cap \mathcal{K}(A_2) \subseteq \mathcal{K}(A_1 \cup A_2)$ $\forall A_1, A_2 \in \mathcal{S}$,
(GCT_3) $\bigcap_{j \in J} \mathcal{K}(A_j) \subseteq \mathcal{K}(\bigcap_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$.

In this case, $\mathcal{T}$ is called a *$(V, \mathcal{V})$-graded topology* and $\mathcal{K}$ a *$(V, \mathcal{V})$-graded cotopology* on $(S, \mathcal{S})$. For $v \in V$ it is defined that

$$\mathcal{T}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{T}(A)\},\ \mathcal{K}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{K}(A)\}.$$

So $(\mathcal{T}^v, \mathcal{K}^v)$ is a ditopology on $(S, \mathcal{S})$ for each $v \in V$. Namely, if $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is a g.d.t.s., then there exists a d.t.s. $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ for each $v \in V$.

$[A]^v$ and $]A[^v$ stand for the closure and the interior of a set $A \in \mathcal{S}$ in the d.t.s. $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ respectively, so we have $[A]^v = \bigcap\{B \in \mathcal{S} \mid A \subseteq B,\ B \in \mathcal{K}^v\}$, $]A[^v = \bigvee\{B \in \mathcal{S} \mid B \subseteq A,\ B \in \mathcal{T}^v\}$.

Let $(S, \mathcal{S}, \sigma)$ be a complemented texture. If $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is a g.d.t.s. then $(S, \mathcal{S}, \mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma, V, \mathcal{V})$ is also a g.d.t.s. Additionally, $(\mathcal{T}, \mathcal{K})$ is called *complemented*, if $(\mathcal{T}, \mathcal{K}) = (\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$ and in this case, we say that $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ is a *complemented g.d.t.s.*

Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be g.d.t.s., $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. For the pair $((f, F), (h, H))$, (f, F) is called *continuous with respect to* (h, H), if $H^{\leftarrow}\mathcal{T}_2(A) \subseteq \mathcal{T}_1(F^{\leftarrow}A)$ $\forall A \in \mathcal{S}_2$, and *cocontinuous with respect to* (h, H), if $h^{\leftarrow}\mathcal{K}_2(A) \subseteq \mathcal{K}_1(f^{\leftarrow}A)$ $\forall A \in \mathcal{S}_2$. If (f, F) is continuous and cocontinuous with respect to (h, H), then it is said to be a *bicontinuous difunction with respect to* (h, H).

Example 2.6 ([5]). Consider the discrete texture $(V, \mathcal{V}) = (1, \mathcal{P}(1))$ (The notation 1 denotes the set $\{0\}$) and take a d.t.s. $(S, \mathcal{S}, \tau, \kappa)$. Then the mappings $\tau^g, \kappa^g : \mathcal{S} \to \mathcal{P}(1)$ defined by $\tau^g(A) = 1 \Leftrightarrow A \in \tau$ and $\kappa^g(A) = 1 \Leftrightarrow A \in \kappa$ form a g.d.t.s. $(S, \mathcal{S}, \tau^g, \kappa^g, V, \mathcal{V})$. In this case, (τ^g, κ^g) is called a *graded ditopology* on $(S, \mathcal{S})$ corresponding to ditopology (τ, κ). Thus g.d.t.s. are more general than d.t.s.

Definition 2.7 ([13])**.** Let $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ be a g.d.t.s. and $A \in \mathcal{S}$. The families defined by

$$\mathcal{C}(A) = \{P_v \in \mathcal{V} \mid [\mathcal{U} \subseteq \mathcal{T}^v,\ A \subseteq \bigvee \mathcal{U}] \Rightarrow \exists \mathcal{U}_0 \subseteq \mathcal{U}\ : A \subseteq \bigvee \mathcal{U}_0\}$$

$$\mathcal{C}^*(A) = \{P_v \in \mathcal{V} \mid [\mathcal{U} \subseteq \mathcal{K}^v,\ \bigwedge \mathcal{U} \subseteq A] \Rightarrow \exists \mathcal{U}_0 \subseteq \mathcal{U}\ : \bigwedge \mathcal{U}_0 \subseteq A\}$$

where $\mathcal{U}_0$ denotes a finite subfamily of $\mathcal{U}$, are called *compactness* and *co−compactness spectrums* of $A \in \mathcal{S}$ respectively. In particular, the compactness spectrum and the co-compactness spectrum of $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ are $\mathcal{C}(S)$ and $\mathcal{C}^*(\varnothing)$ respectively.

Definition 2.8 ([20])**.** Let $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ be a graded ditopological texture space and the mappings $p\mathcal{T}, p\mathcal{K}, s\mathcal{T}, s\mathcal{K} : \mathcal{S} \to \mathcal{V}$ be defined by

$$p\mathcal{T}(A) = \bigvee\{P_v \mid A \subseteq][A]^v[^v\},\ p\mathcal{K}(A) = \bigvee\{P_v \mid [\,]A[^v]^v \subseteq A\} \tag{2.1}$$

$$s\mathcal{T}(A) = \bigvee\{P_v \mid A \subseteq [\,]A[^v]^v\},\ s\mathcal{K}(A) = \bigvee\{P_v \mid][A]^v[^v \subseteq A\} \tag{2.2}$$

for all $A \in \mathcal{S}$. Then $p\mathcal{T}$ ($p\mathcal{K}$) is called pre-openness (pre-closedness) function; $s\mathcal{T}$ ($s\mathcal{K}$) is called *semi − openness* (*semi − closedness*) *function* of $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ and $p\mathcal{T}(A)$ ($p\mathcal{K}(A)$) is called pre-openness (pre-closedness) grade; $s\mathcal{T}(A)$ ($s\mathcal{K}(A)$) is called *semi − openness* (*semi − closedness*) *grade* of A respectively.

3. Main Results

Definition 3.1. Let $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ be a graded ditopological texture space and the mappings $\beta\mathcal{T}, \beta\mathcal{K} : \mathcal{S} \to \mathcal{V}$ be defined by

$$\beta\mathcal{T}(A) = \bigvee\{P_v \mid A \subseteq [\,][A]^v[^v]^v\},\ \beta\mathcal{K}(A) = \bigvee\{P_v \mid][\,]A[^v]^v[^v \subseteq A\} \tag{3.1}$$

for all $A \in \mathcal{S}$. Then $\beta\mathcal{T}$ ($\beta\mathcal{K}$) is called *β-openness* (*β-closedness*) *function* of $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ and $\beta\mathcal{T}(A)$ ($\beta\mathcal{K}(A)$) is called *β-openness* (*β-closedness*) *grade* of A respectively.

Proposition 3.2. *For a graded ditopological texture space $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ following statements are hold:*

(1) (a) $\mathcal{T} \subseteq p\mathcal{T} \subseteq \beta\mathcal{T}$ *and* $\mathcal{K} \subseteq p\mathcal{K} \subseteq \beta\mathcal{K}$,
 (b) $\mathcal{T} \subseteq s\mathcal{T} \subseteq \beta\mathcal{T}$ *and* $\mathcal{K} \subseteq s\mathcal{K} \subseteq \beta\mathcal{K}$,

(2) *The functions $\beta\mathcal{T}$ and $\beta\mathcal{K}$ satisfy the property (GT_3) and (GCT_3) respectively where the texture $(V,\mathcal{V})$ is discrete.*

Proof. (1) (a) For any $A \in \mathcal{S}$ and $v \in V$, we have

$$P_v \subseteq \mathcal{T}(A) \Rightarrow]A[^v = A \Rightarrow A =]A[^v \subseteq][A]^v[^v \Rightarrow P_v \subseteq p\mathcal{T}(A)$$

and

$$A \subseteq][A]^v[^v \Rightarrow A \subseteq][A]^v[^v \subseteq [\,][A]^v[^v]^v.$$

Then this implies

$$p\mathcal{T}(A) = \bigvee\{P_v \mid A \subseteq][A]^v[^v\} \subseteq \bigvee\{P_v \mid A \subseteq [\,][A]^v[^v]^v\} = \beta\mathcal{T}(A).$$

Thus we get $\mathcal{T} \subseteq p\mathcal{T} \subseteq \beta\mathcal{T}$. Similarly, it can be shown that $\mathcal{K} \subseteq p\mathcal{K} \subseteq \beta\mathcal{K}$.

(b) For any $A \in \mathcal{S}$ and $v \in V$, we have

$$P_v \subseteq \mathcal{K}(A) \Rightarrow [A]^v = A \Rightarrow][A]^v[^v =]A[^v \subseteq A \Rightarrow P_v \subseteq s\mathcal{K}(A)$$

and

$$][A]^v[^v \subseteq A \Rightarrow][\,]A[^v]^v[^v \subseteq][A]^v[^v \subseteq A.$$

Then this implies

$$s\mathcal{K}(A) = \bigvee\{P_v \mid][A]^v[^v \subseteq A\} \subseteq \bigvee\{P_v \mid][\,]A[^v]^v[^v \subseteq A\} = \beta\mathcal{K}(A).$$

Thus we get $\mathcal{K} \subseteq s\mathcal{K} \subseteq \beta\mathcal{K}$. Similarly, it can be shown that $\mathcal{T} \subseteq s\mathcal{T} \subseteq \beta\mathcal{T}$.

(2) Let $A_j \in \mathcal{S}$, $j \in J$ where J is an index set.

(a) Assume that $\bigcap_{j\in J} \beta\mathcal{T}(A_j) \nsubseteq \beta\mathcal{T}(\bigvee_{j\in J} A_j)$. Then $\bigcap_{j\in J} \beta\mathcal{T}(A_j) \nsubseteq Q_v$ and $P_v \nsubseteq \beta\mathcal{T}(\bigvee_{j\in J} A_j)$ for an element $v \in V$. Thus we get $P_v \subseteq \bigcap_{j\in J} \beta\mathcal{T}(A_j)$ by $\bigcap_{j\in J} \beta\mathcal{T}(A_j) \nsubseteq Q_v$. So considering $(V, \mathcal{V})$ is discrete, we have

$$\forall j \in J\ P_v = \{v\} \subseteq \beta\mathcal{T}(A_j) \Rightarrow \forall j \in J\ A_j \subseteq [\ ][A_j]^v[^v]^v \subseteq [\ ]\ [\bigvee_{j\in J} A_j]^v\ [^v\ ]^v$$

$$\Rightarrow \bigvee_{j\in J} A_j \subseteq [\ ]\ [\bigvee_{j\in J} A_j]^v\ [^v\ ]^v \Rightarrow P_v \subseteq \beta\mathcal{T}(\bigvee_{j\in J} A_j)$$

and this contradicts with $P_v \nsubseteq \beta\mathcal{T}(\bigvee_{j\in J} A_j)$. Hence $\bigcap_{j\in J} \beta\mathcal{T}(A_j) \subseteq \beta\mathcal{T}(\bigvee_{j\in J} A_j)$.

(b) Assume that $\bigcap_{j\in J} \beta\mathcal{K}(A_j) \nsubseteq \beta\mathcal{K}(\bigcap_{j\in J} A_j)$. Then $\bigcap_{j\in J} \beta\mathcal{K}(A_j) \nsubseteq Q_v$ and $P_v \nsubseteq \beta\mathcal{K}(\bigcap_{j\in J} A_j)$ for an element $v \in V$. Thus we get $P_v \subseteq \bigcap_{j\in J} \beta\mathcal{K}(A_j)$ by $\bigcap_{j\in J} \beta\mathcal{K}(A_j) \nsubseteq Q_v$. So considering $(V, \mathcal{V})$ is discrete, we have

$$\forall j \in J\ P_v = \{v\} \subseteq \beta\mathcal{K}(A_j) \Rightarrow \forall j \in J\ ][\ ]\bigcap_{j\in J} A_j[^v]^v[^v \subseteq][\ ]A_j[^v]^v[^v \subseteq A_j$$

$$\Rightarrow][\ ]\bigcap_{j\in J} A_j[^v]^v[^v \subseteq \bigcap_{j\in J} A_j \Rightarrow P_v \subseteq \beta\mathcal{K}(\bigcap_{j\in J} A_j)$$

and this contradicts with $P_v \nsubseteq \beta\mathcal{K}(\bigcap_{j\in J} A_j)$. Hence $\bigcap_{j\in J} \beta\mathcal{K}(A_j) \subseteq \beta\mathcal{K}(\bigcap_{j\in J} A_j)$. □

Example 3.3. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then a set $D \in \mathcal{S}$ is $\beta-$*open* (not $\beta-$*open*) in $(S, \mathcal{S}, \tau, \kappa)$ if and only if β-openness grade of D, $\beta\tau^g(D) = 1$ ($\beta\tau^g(D) = 0$) in the graded ditopological texture space $(S, \mathcal{S}, \tau^g, \kappa^g, 1, \mathcal{P}(1))$ respectively. Similarly, a set $D \in \mathcal{S}$ is $\beta-$ *closed* (not $\beta-$ *closed*) in $(S, \mathcal{S}, \tau, \kappa)$ if and only if β-closedness grade of D, $\beta\kappa^g(D) = 1$ ($\beta\kappa^g(D) = 0$) in $(S, \mathcal{S}, \tau^g, \kappa^g, 1, \mathcal{P}(1))$ respectively.

Proposition 3.4. *If* $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ *is a complemented g.d.t.s., then* $\beta\mathcal{K} \circ \sigma = \beta\mathcal{T}$ *and* $\beta\mathcal{T} \circ \sigma = \beta\mathcal{K}$.

Proof. Since $(S,\mathcal{S},\mathcal{T},\mathcal{K},\sigma,V,\mathcal{V})$ is complemented, we get $\sigma([A]^v)=]\sigma(A)[^v$ and $\sigma(]A[^v)=[\sigma(A)]^v$ for all $A\in\mathcal{S}$ and $v\in V$ by [20]. Then we have

$$(\beta\mathcal{K}\circ\sigma)(A)=\beta\mathcal{K}(\sigma(A))=\bigvee\{P_v \mid]\;[\;]\sigma(A)[^v\;]^v\;[^v\subseteq\sigma(A)\}$$
$$=\bigvee\{P_v \mid A\subseteq\sigma(]\;[\;]\sigma(A)[^v\;]^v\;[^v)\}=\bigvee\{P_v \mid A\subseteq[\sigma([\;]\sigma(A)[^v\;]^v)]^v\}$$
$$=\bigvee\{P_v \mid A\subseteq[\;]\sigma(]\sigma(A)[^v)[^v\;]^v\}=\bigvee\{P_v \mid A\subseteq[\;][\sigma(\sigma(A))]^v\;[^v\;]^v\}$$
$$=\bigvee\{P_v \mid A\subseteq[\;][A]^v\;[^v\;]^v\}=\beta\mathcal{T}(A)$$

and

$$(\beta\mathcal{T}\circ\sigma)(A)=\beta\mathcal{T}(\sigma(A))=\bigvee\{P_v \mid \sigma(A)\subseteq[\;]\;[\sigma(A)]^v\;[^v\;]^v\}$$
$$=\bigvee\{P_v \mid \sigma([\;]\;[\sigma(A)]^v\;[^v\;]^v)\subseteq A\}=\bigvee\{P_v \mid]\sigma(]\;[\sigma(A)]^v\;[^v)[^v\subseteq A\}$$
$$=\bigvee\{P_v \mid]\;[\sigma([\sigma(A)]^v)]^v\;[^v\subseteq A\}=\bigvee\{P_v \mid]\;[\;]\sigma(\sigma(A))[^v\;]^v\;[^v\subseteq A\}$$
$$=\bigvee\{P_v \mid]\;[\;]A[^v\;]^v\;[^v\subseteq A\}=\beta\mathcal{K}(A)$$

for all $A\in\mathcal{S}$. □

Definition 3.5. Let $(S_k,\mathcal{S}_k,\mathcal{T}_k,\mathcal{K}_k,V_k,\mathcal{V}_k)$, $k=1,2$ be g.d.t.s. and $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$, $(h,H):(V_1,\mathcal{V}_1)\to(V_2,\mathcal{V}_2)$ be difunctions. (f,F) is said to be:

(i) *β-continuous w.r.t. (h,H)*, if $H^{\leftarrow}\mathcal{T}_2(A)\subseteq\beta\mathcal{T}_1(F^{\leftarrow}A)$ for all $A\in\mathcal{S}_2$,
(ii) *$M\beta$-continuous w.r.t. (h,H)*, if $H^{\leftarrow}\beta\mathcal{T}_2(A)\subseteq\beta\mathcal{T}_1(F^{\leftarrow}A)$ for all $A\in\mathcal{S}_2$,
(iii) *β-cocontinuous w.r.t. (h,H)*, if $h^{\leftarrow}\mathcal{K}_2(A)\subseteq\beta\mathcal{K}_1(f^{\leftarrow}A)$ for all $A\in\mathcal{S}_2$,
(iv) *$M\beta$-cocontinuous w.r.t. (h,H)*, if $h^{\leftarrow}\beta\mathcal{K}_2(A)\subseteq\beta\mathcal{K}_1(f^{\leftarrow}A)$ for all $A\in\mathcal{S}_2$,
(v) *β-bicontinuous w.r.t. (h,H)*, if it is both β-continuous and β-cocontinuous w.r.t. (h,H),
(vi) *$M\beta$-bicontinuous w.r.t. (h,H)*, if it is both $M\beta$-continuous and $M\beta$-cocontinuous w.r.t. (h,H).

We say that $((f,F),(h,H))$ is a *relatively $(M)\beta$-bicontinuous difunction pair*, if (f,F) is $(M)\beta$-bicontinuous w.r.t. (h,H).

Example 3.6. Let $(S_k,\mathcal{S}_k,\tau_k,\kappa_k)$, $k=1,2$ be d.t.s. and $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ be a difunction. Consider graded ditopological texture spaces $(S_k,\mathcal{S}_k,\tau_k^g,\kappa_k^g,V,\mathcal{V})$, $k=1,2$ corresponding to ditopological texture spaces $(S_k,\mathcal{S}_k,\tau_k,\kappa_k)$, $k=1,2$.

(1) If (f,F) is $M\beta$-continuous, then we have $I_V^{\leftarrow}\beta\tau_2^g(A)=\beta\tau_2^g(A)\subseteq\beta\tau_1^g(F^{\leftarrow}A)$ for each $A\in\mathcal{S}_2$ and so (f,F) is $M\beta$-continuous w.r.t. the identity difunction (i_V,I_V). If (f,F) is $M\beta$-cocontinuous, then we have $i_V^{\leftarrow}\beta\kappa_2^g(A)=\beta\kappa_2^g(A)\subseteq\beta\kappa_1^g(f^{\leftarrow}A)$ for each $A\in\mathcal{S}_2$ and so (f,F) is $M\beta$-cocontinuous w.r.t. the identity difunction (i_V,I_V).

(2) If (f,F) is β-continuous, then we have $I_V^{\leftarrow}\tau_2^g(A)=\tau_2^g(A)\subseteq\beta\tau_1^g(F^{\leftarrow}A)$ for each $A\in\mathcal{S}_2$ and so (f,F) is β-continuous w.r.t. the identity difunction (i_V,I_V). If (f,F) is β-cocontinuous, then we have $i_V^{\leftarrow}\kappa_2^g(A)=\kappa_2^g(A)\subseteq\beta\kappa_1^g(f^{\leftarrow}A)$ for each $A\in\mathcal{S}_2$ and so (f,F) is β-cocontinuous w.r.t. the identity difunction (i_V,I_V).

In this sense, every β-bicontinuous ($M\beta$-bicontinuous) difunction between two d.t.s. is considered as β-bicontinuous ($M\beta$-bicontinuous) difunction w.r.t. identity difunction on the discrete texture on a singleton.

(3) Clearly, each relatively bicontinuous difunction pair is relatively β-bicontinuous by Proposition 3.2 (1).

Definition 3.7. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. and $A \in \mathcal{S}$. Then we say that the families

$$\beta\mathcal{C}(A) = \{P_v \mid [\mathcal{U} \subseteq \beta\mathcal{T}^v,\ A \subseteq \bigvee \mathcal{U}] \Rightarrow \exists \mathcal{U}_0 \subseteq \mathcal{U}\ : A \subseteq \bigvee \mathcal{U}_0\}$$

$$\beta\mathcal{C}^*(A) = \{P_v \mid [\mathcal{U} \subseteq \beta\mathcal{K}^v,\ \bigwedge \mathcal{U} \subseteq A] \Rightarrow \exists \mathcal{U}_0 \subseteq \mathcal{U}\ : \bigwedge \mathcal{U}_0 \subseteq A\}$$

where $\mathcal{U}_0$ denotes a finite subfamily of $\mathcal{U}$, are β-compactness and β-cocompactness spectrums of $A \in \mathcal{S}$ respectively. We also say that $\beta\mathcal{C}(S)$ and $\beta\mathcal{C}^*(\varnothing)$ are β-compactness spectrum and β-cocompactness spectrum of the g.d.t.s. $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ respectively.

Corollary 3.8. *Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. and $A \in \mathcal{S}$. Then $\beta\mathcal{C}(A) \subseteq \mathcal{C}(A)$ and $\beta\mathcal{C}^*(A) \subseteq \mathcal{C}^*(A)$.*

Proof. It is clear by Proposition 3.2. □

Proposition 3.9. *If $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ is a complemented g.d.t.s. and $A \in \mathcal{S}$ then $\beta\mathcal{C}(A) = \beta\mathcal{C}^*(\sigma(A))$. Namely, β-compactness and β-cocompactness spectrums of a complemented g.d.t.s. are equal.*

Proof. Let $P_v \in \beta\mathcal{C}^*(\sigma(A))$, $\mathcal{U} \subseteq \beta\mathcal{T}^v$ and $A \subseteq \bigvee \mathcal{U}$. Since $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ is complemented, we have $P_v \subseteq \beta\mathcal{T}(U) = (\beta\mathcal{K} \circ \sigma)(U) = \beta\mathcal{K}(\sigma(U))$ for all $U \in \mathcal{U}$ by Proposition 3.4. This implies $\sigma(\mathcal{U}) = \{\sigma(U) \mid U \in \mathcal{U}\} \subseteq \beta\mathcal{K}^v$. On the other hand, we have also $A \subseteq \bigvee \mathcal{U} \Rightarrow \sigma(\bigvee \mathcal{U}) \subseteq \sigma(A) \Rightarrow \bigwedge \sigma(\mathcal{U}) \subseteq \sigma(A)$. Then we get $\bigwedge \sigma(\mathcal{U}_0) \subseteq \sigma(A)$ for a finite subfamily $\mathcal{U}_0 \subseteq \mathcal{U}$ by $P_v \in \beta\mathcal{C}^*(\sigma(A))$. Also $\bigwedge \sigma(\mathcal{U}_0) \subseteq \sigma(A)$ implies $\sigma(\sigma(A)) \subseteq \sigma(\bigwedge \sigma(\mathcal{U}_0))$ and thus $A \subseteq \bigvee \mathcal{U}_0$. So we obtain that $P_v \in \beta\mathcal{C}(A)$, i.e. $\beta\mathcal{C}^*(\sigma(A)) \subseteq \beta\mathcal{C}(A)$. The other inclusion $\beta\mathcal{C}(A) \subseteq \beta\mathcal{C}^*(\sigma(A))$ can be shown similarly.

Since $\sigma(S) = \varnothing$, we get $\beta\mathcal{C}(S) = \beta\mathcal{C}^*(\sigma(S)) = \beta\mathcal{C}^*(\varnothing)$. Then β-compactness and β-cocompactness spectrums of $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ are equal. □

Theorem 3.10. *Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be g.d.t.s. and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ be difunctions.*

(1) *If (f, F) is $M\beta$-continuous w.r.t. (h, H) and $A \in \mathcal{S}_1$, then*

$$P_v \in \beta\mathcal{C}_1(A) \Rightarrow P_t \in \beta\mathcal{C}_2(f^{\to}A),$$

(2) *If (f, F) is $M\beta$-cocontinuous w.r.t. (h, H) and $A \in \mathcal{S}_1$, then*

$$P_v \in \beta\mathcal{C}_1^*(A) \Rightarrow P_t \in \beta\mathcal{C}_2^*(F^{\to}A)$$

where $P_v \in \mathcal{V}_1$, $P_t \in \mathcal{V}_2$ with $P_v \subseteq h^{\leftarrow}P_t$.

Proof. We omit the proof of (1) and prove (2) since the proofs of (1) and (2) are similar. Let $P_v \in \beta\mathcal{C}_1^*(A)$, $P_v \in \mathcal{V}_1$, $P_t \in \mathcal{V}_2$ and $P_v \subseteq h^{\leftarrow}P_t$. Considering the definition of $\beta\mathcal{C}_2^*(F^{\to}A)$, if we suppose that $(\mathcal{U} \subseteq \beta\mathcal{K}_2^t,\ \bigwedge \mathcal{U} \subseteq F^{\to}A)$ then we have

$$\bigwedge_{U \in \mathcal{U}} f^{\leftarrow}U = \bigwedge f^{\leftarrow}\mathcal{U} = f^{\leftarrow}(\bigwedge \mathcal{U}) \subseteq f^{\leftarrow}(F^{\to}A) \subseteq A$$

by Proposition 2.2 and Proposition 2.4 (3). Besides, since $\mathcal{U} \subseteq \beta\mathcal{K}_2^t \Rightarrow (\forall U \in \mathcal{U}\ P_t \subseteq \beta\mathcal{K}_2(U)) \Rightarrow P_t \subseteq \beta\mathcal{K}_2(\mathcal{U})$ and (f, F) is $M\beta$-cocontinuous w.r.t. (h, H) we get

$$P_v \subseteq h^{\leftarrow} P_t \subseteq h^{\leftarrow}(\beta\mathcal{K}_2(\mathcal{U})) \subseteq \beta\mathcal{K}_1(f^{\leftarrow}\mathcal{U}).$$

Since $P_v \in \beta\mathcal{C}_1^*(A)$, we have $\bigwedge_{U\in\mathcal{U}_0} f^{\leftarrow}U \subseteq A$ for a finite subfamily $\mathcal{U}_0 \subseteq \mathcal{U}$. Then we get

$$F^{\rightarrow}A \supseteq F^{\rightarrow}(\bigwedge_{U\in\mathcal{U}_0} f^{\leftarrow}U) = \bigwedge_{U\in\mathcal{U}_0} F^{\rightarrow}(f^{\leftarrow}U) \supseteq \bigwedge_{U\in\mathcal{U}_0} U = \bigwedge \mathcal{U}_0$$

by Proposition 2.2 and Proposition 2.4 (3). Thus we obtain that $P_t \in \beta\mathcal{C}_2^*(F^{\rightarrow}A)$. □

Corollary 3.11. *In addition to the conditions in the above theorem, if (f, F) is surjective, then the following results hold:*

(1) *if (f, F) is $M\beta$-continuous w.r.t. (h, H), then*

$$P_v \in \beta\mathcal{C}_1(S_1) \Rightarrow P_t \in \beta\mathcal{C}_2(S_2),$$

(2) *if (f, F) is $M\beta$-cocontinuous w.r.t. (h, H), then*

$$P_v \in \beta\mathcal{C}_1^*(\varnothing) \Rightarrow P_t \in \beta\mathcal{C}_2^*(\varnothing)$$

where $P_v \in \mathcal{V}_1$, $P_t \in \mathcal{V}_2$ with $P_v \subseteq h^{\leftarrow}P_t$.

Proof. We get $f^{\leftarrow}\varnothing = F^{\leftarrow}\varnothing = \varnothing$ and $f^{\leftarrow}S_2 = F^{\leftarrow}S_2 = S_1$ by using Proposition 2.4 (2). Besides, surjectivity of (f, F) implies $F^{\rightarrow}(f^{\leftarrow}S_2) = S_2 = f^{\rightarrow}(F^{\leftarrow}S_2)$ and $F^{\rightarrow}(f^{\leftarrow}\varnothing) = \varnothing = f^{\rightarrow}(F^{\leftarrow}\varnothing)$ by using Proposition 2.4 (4). Then we have $f^{\rightarrow}S_1 = S_2$ and $F^{\rightarrow}\varnothing = \varnothing$. Thus the result is obtained by Theorem 3.10. □

Definition 3.12. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. Then the families

$$\beta\Omega = \{P_v \mid [A \in \mathcal{S},\ A \neq S] \Rightarrow [P_v \subseteq \beta\mathcal{K}(A) \Rightarrow P_v \in \beta\mathcal{C}(A)]\}$$

$$\beta\Omega^* = \{P_v \mid [A \in \mathcal{S},\ A \neq \varnothing] \Rightarrow [P_v \subseteq \beta\mathcal{T}(A) \Rightarrow P_v \in \beta\mathcal{C}^*(A)]\}$$

are called *β-stableness spectrum* and *β-costableness spectrum* of the g.d.t.s. $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ respectively.

Proposition 3.13. *If $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ is a complemented g.d.t.s. then the β-stableness spectrum and the β-costableness spectrum are equal, i.e., $\beta\Omega = \beta\Omega^*$.*

Proof. Let $P_v \in \beta\Omega$, $A \in \mathcal{S}$, $A \neq \varnothing$ and $P_v \subseteq \beta\mathcal{T}(A)$. Since $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, \sigma, V, \mathcal{V})$ is complemented, we have $P_v \subseteq \beta\mathcal{T}(A) = (\beta\mathcal{K} \circ \sigma)(A) = \beta\mathcal{K}(\sigma(A))$. Moreover, $A \in \mathcal{S}$, $A \neq \varnothing$ implies $\sigma(A) \in \mathcal{S}$, $\sigma(A) \neq S$. Then using $P_v \in \beta\Omega$ and Proposition 3.9, we get $P_v \in \beta\mathcal{C}(\sigma(A)) = \beta\mathcal{C}^*(A)$. Thus we obtain that $\beta\Omega \subseteq \beta\Omega^*$. Similarly, it can be shown that $\beta\Omega^* \subseteq \beta\Omega$. □

Theorem 3.14. *Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be g.d.t.s. and $(f, F) : (S_1, \mathcal{S}_1) \rightarrow (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \rightarrow (V_2, \mathcal{V}_2)$ be difunctions. If (f, F) is surjective and $M\beta$-bicontinuous w.r.t. (h, H), then*

(1) $P_v \in \beta\Omega_1 \Rightarrow P_t \in \beta\Omega_2$,
(2) $P_v \in \beta\Omega_1^* \Rightarrow P_t \in \beta\Omega_2^*$

where $P_v \in \mathcal{V}_1$, $P_t \in \mathcal{V}_2$ with $P_v \subseteq h^{\leftarrow}P_t$.

Proof. (1) Let $P_v \in \beta\Omega_1$. Let $P_t \subseteq \beta\mathcal{K}_2(B)$ is given where $B \in \mathcal{S}_2$, $B \neq S_2$. Then we get $P_v \subseteq h^{\leftarrow}P_t \subseteq h^{\leftarrow}(\beta\mathcal{K}_2(B)) \subseteq \beta\mathcal{K}_1(f^{\leftarrow}B)$ by using that $P_v \subseteq h^{\leftarrow}P_t$ and (f, F) is $M\beta$-bicontinuous w.r.t. (h, H). Thus we have $P_v \subseteq \beta\mathcal{K}_1(f^{\leftarrow}B)$. On the other hand, since (f, F) is surjective and $B \neq S_2$, we get $f^{\leftarrow}B \neq S_1$. So we obtain that $P_v \in \beta\mathcal{C}_1(f^{\leftarrow}B)$ by using $P_v \in \beta\Omega_1$. Hence we have $P_t \in \beta\mathcal{C}_2(f^{\rightarrow}(f^{\leftarrow}B)) = \beta\mathcal{C}_2(B)$ by Theorem 3.10 and Proposition 2.4 (1) and (4). Therefore we get $P_t \in \beta\Omega_2$.

(2) Let $P_v \in \beta\Omega_1^*$. Let $P_t \subseteq \beta\mathcal{T}_2(B)$ is given where $B \in \mathcal{S}_2$, $B \neq \varnothing$. Then we get $P_v \subseteq h^{\leftarrow}P_t \subseteq h^{\leftarrow}(\beta\mathcal{T}_2(B)) = H^{\leftarrow}(\beta\mathcal{T}_2(B)) \subseteq \beta\mathcal{T}_1(F^{\leftarrow}B)$ by using that $P_v \subseteq h^{\leftarrow}P_t$ and (f, F) is $M\beta$-bicontinuous w.r.t. (h, H). Thus we have $P_v \subseteq \beta\mathcal{T}_1(F^{\leftarrow}B)$. On the other hand, since (f, F) is surjective and $B \neq \varnothing$, we get $F^{\leftarrow}B \neq \varnothing$. So we obtain that $P_v \in \beta\mathcal{C}_1^*(F^{\leftarrow}B)$ by using $P_v \in \beta\Omega_1^*$. Hence we have $P_t \in \beta\mathcal{C}_2^*(F^{\rightarrow}(F^{\leftarrow}B)) = \beta\mathcal{C}_2^*(B)$ by Theorem 3.10 and Proposition 2.4 (1) and (4). Therefore we get $P_t \in \beta\Omega_2^*$. □

Definition 3.15. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. Then the family defined by

$$\beta\mathcal{DC} = \beta\mathcal{C}(S) \cap \beta\mathcal{C}^*(\varnothing) \cap \beta\Omega \cap \beta\Omega^*$$

is called *β-dicompactness spectrum* of the g.d.t.s. $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$.

Corollary 3.16. *Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be g.d.t.s. and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ be difunctions. If (f, F) is surjective and $M\beta$-bicontinuous w.r.t. (h, H), then*

$$P_v \in \beta\mathcal{DC}_1 \Rightarrow P_t \in \beta\mathcal{DC}_2$$

where $P_v \in \mathcal{V}_1$, $P_t \in \mathcal{V}_2$ with $P_v \subseteq h^{\leftarrow}P_t$.

Proof. It is an immediate result of Corollary 3.11 and Theorem 3.14. □

Example 3.17. Let $(S, \mathcal{S}, \tau, \kappa)$ be a d.t.s. If $(S, \mathcal{S}, \tau, \kappa)$ is β-compact (β-cocompact, β-dicompact), then for the g.d.t.s. $(S, \mathcal{S}, \tau^g, \kappa^g, 1, \mathcal{P}(1))$, $P_v \in \beta\mathcal{C}(S)$ ($P_v \in \beta\mathcal{C}^*(\varnothing)$, $P_v \in \beta\mathcal{DC}$) respectively for all $v \in 1 = \{0\}$, i.e. $v = 0$.

Following proposition shows β-compactness relationships between ditopological texture spaces and graded ditopological texture spaces:

Proposition 3.18. *Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a g.d.t.s. Then the following hold for each $v \in V$:*

(1) *$P_v \in \beta\mathcal{C}(S) \Leftrightarrow$ The d.t.s. $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is β-compact,*
(2) *$P_v \in \beta\mathcal{C}^*(\varnothing) \Leftrightarrow$ The d.t.s. $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is β-cocompact,*
(3) *$P_v \in \beta\Omega \Leftrightarrow$ The d.t.s. $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is β-stable,*
(4) *$P_v \in \beta\Omega^* \Leftrightarrow$ The d.t.s. $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is β-costable,*
(5) *$P_v \in \beta\mathcal{DC} \Leftrightarrow$ The d.t.s. $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$ is β-dicompact.*

Proof. It is straightforward. □

Example 3.19. Take discrete textures $(S, \mathcal{S} = \mathcal{P}(S))$ and $(V, \mathcal{V} = \mathcal{P}(V))$ where $S \neq \varnothing$ and $V = \{x, y, z, r\}$. If we define the mappings $\mathcal{T}, \mathcal{K} : \mathcal{S} \to \mathcal{V}$ as

$$\mathcal{T}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ P_z = \{z\}, & \text{otherwise} \end{cases}$$

$$\mathcal{K}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ P_r = \{r\}, & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{S}$, then we get a g.d.t.s. $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$. Thus we obtain $\mathcal{T}^z = \mathcal{S} = \mathcal{P}(S)$, $\mathcal{T}^x = \mathcal{T}^y = \mathcal{T}^r = \{S, \varnothing\}$, $\mathcal{K}^r = \mathcal{S} = \mathcal{P}(S)$, $\mathcal{K}^x = \mathcal{K}^y = \mathcal{K}^z = \{S, \varnothing\}$. So we have $]A[^z = A$ and $[A]^r = A$ for all $A \in \mathcal{S}$. Also $A \neq S$, $A \in \mathcal{S}$ implies $]A[^x =]A[^y =]A[^r = \varnothing$ and $A \neq \varnothing$, $A \in \mathcal{S}$ implies $[A]^x = [A]^y = [A]^z = S$. Hence we get $\beta\mathcal{T}^x = \beta\mathcal{T}^y = \beta\mathcal{T}^z = \mathcal{S} = \mathcal{P}(S)$, $\beta\mathcal{T}^r = \{\varnothing, S\}$, $\beta\mathcal{K}^x = \beta\mathcal{K}^y = \beta\mathcal{K}^r = \mathcal{S} = \mathcal{P}(S)$, $\beta\mathcal{K}^z = \{\varnothing, S\}$. Therefore we obtain that

$$\beta\mathcal{C}(S) = \beta\mathcal{C}^*(\varnothing) = \beta\Omega = \beta\Omega^* = \beta\mathcal{DC} = \{\{x\}, \{y\}, \{z\}, \{r\}\}$$

if S is finite.

If a subset $A \in \mathcal{S}$ is infinite, then $A \subseteq \bigvee \mathcal{U} = \bigvee_{s \in A} \{s\}$ where $\mathcal{U} = \{P_s \mid s \in A\}$. Yet there is no finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \bigvee \mathcal{U}_0$. Thus we get that $\beta\mathcal{C}(S) = \beta\Omega = \{P_r\} = \{r\}$ if S is infinite.

On the other hand, for a subset $A \in \mathcal{S}$, if $S \setminus A$ is infinite, then $\bigwedge \mathcal{U} = \bigwedge_{x \in (S \setminus A)} ((S \setminus A) \setminus P_x) = \varnothing \subseteq A$ where $\mathcal{U} = \{(S \setminus A) \setminus P_x \mid x \in (S \setminus A)\}$. Yet there is no finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\bigwedge \mathcal{U}_0 \subseteq A$. So we have $\beta\mathcal{C}^*(\varnothing) = \beta\Omega^* = \{P_z\} = \{z\}$ and $\beta\mathcal{DC} = \varnothing$ if S is infinite.

Moreover, we get

$$p\mathcal{T}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ \{x, y, z\}, & \text{otherwise} \end{cases}$$

$$p\mathcal{K}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ \{x, y, r\}, & \text{otherwise} \end{cases}$$

$$s\mathcal{T}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ P_z = \{z\}, & \text{otherwise} \end{cases}$$

$$s\mathcal{K}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ P_r = \{r\}, & \text{otherwise} \end{cases}$$

$$\beta\mathcal{T}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ \{x, y, z\}, & \text{otherwise} \end{cases}$$

$$\beta\mathcal{K}(A) = \begin{cases} V, & A = \varnothing \text{ or } A = S \\ \{x, y, r\}, & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{S}$. Hence we have $\mathcal{T} \subseteq p\mathcal{T} = \beta\mathcal{T}$, $\mathcal{K} \subseteq p\mathcal{K} = \beta\mathcal{K}$, $\mathcal{T} = s\mathcal{T} \subseteq \beta\mathcal{T}$, $\mathcal{K} = s\mathcal{K} \subseteq \beta\mathcal{K}$ and this result is also an example for Proposition 3.2 (1).

4. Conclusion

This study focuses on generalizing the concepts of β-openness and β-compactness in d.t.s. to the g.d.t.s. To enhance the theory of g.d.t.s., it is important to examine the properties of these new concepts. The interrelations of openness, semi-openness, pre-openness and β-openness (closedness, semi-closedness, pre-closedness and β-closedness) grade of an element of a texture are investigated in a g.d.t.s.: $\mathcal{T} \subseteq p\mathcal{T} \subseteq \beta\mathcal{T}$, $\mathcal{K} \subseteq p\mathcal{K} \subseteq \beta\mathcal{K}$, $\mathcal{T} \subseteq s\mathcal{T} \subseteq \beta\mathcal{T}$, $\mathcal{K} \subseteq s\mathcal{K} \subseteq \beta\mathcal{K}$ (Proposition 3.2). In a complemented g.d.t.s. with complementation σ; $\beta\mathcal{K} \circ \sigma = \beta\mathcal{T}$ and $\beta\mathcal{T} \circ \sigma = \beta\mathcal{K}$

(Proposition 3.4). We also show that β-compactness and β-cocompactness spectrums of a complemented g.d.t.s. are equal in Proposition 3.9.

The relationships between the concepts of $M\beta$-continuity (β-continuity) in d.t.s. and relatively $M\beta$-continuity (relatively β-continuity) in g.d.t.s. are studied respectively. Besides, the properties of β-(di)compactness spectrum in g.d.t.s., the relationship between β-(di)compactness in d.t.s. and β-(di)compactness spectrum in g.d.t.s. are examined.

Thus, this work develops the theory of graded ditopology by these findings and it allows a more comprehensive approach to the theory.

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