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Abstract. Using the imprecise set by Baruah, subalgebras and (closed) ideals of BCK/BCI-algebras are addressed. Concepts of imprecise subalgebras and (closed) imprecise ideals in BCK/BCI-algebras are introduced, and several properties are investigated. Imprecise subalgebras are made by using the BCK-part of a BCI-algebra and the initial section of a BCK-algebra. The membership t-cut and reference s-cut are used to form the characterization of imprecise subalgebras and imprecise ideals. By presenting examples, it is found that the two concepts imprecise subalgebras (resp. imprecise ideals) and intuitionistic fuzzy subalgebras (resp. intuitionistic fuzzy ideals) are independent of each other. A relationship is established between imprecise subalgebras and imprecise ideals. We explore imprecise subalgebras and (closed) imprecise ideals in relation to homomorphism. The imprecise subalgebras and (closed) imprecise ideals are explored with respect to homomorphisms.

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1. Introduction

The term inaccurate sets typically arises in situations where classical set theory (crisp sets) is extended to deal with ambiguity or uncertainty. Representative inaccurate sets include fuzzy sets, rough sets, or interval-value sets, etc. Fuzzy sets, introduced by Zadeh [1, 2], are a generalized theory of classical sets. As everyone knows, Zadeh's fuzzy set does not follow the universal laws of classical sets, e.g., Complement Laws. In this regard, Baruah [3, 4, 5] introduced a new concept, called an imprecise set, that satisfies the complement laws.

In this paper, we would like to deal with subalgebras and ideals in BCK/BCI-algebras using the imprecise set by Baruah. We introduce the concept of imprecise subalgebras and (closed) imprecise ideals in BCK/BCI-algebras, and investigate several properties. Using the BCK-part of a BCI-algebra $(X,*)_0$ and the initial section of a BCK-algebra $(X,*)_0$, we make an imprecise subalgebra of $(X,*)_0$. We deal with the characterization of imprecise subalgebras and imprecise ideals by membership t-cut and reference s-cut. We give examples showing that the two concepts imprecise subalgebras (resp. imprecise ideals) and intuitionistic fuzzy subalgebras (resp. intuitionistic fuzzy ideals) are independent of each other. We establish a relationship between imprecise subalgebras and imprecise ideals.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (See [6, 7, 8, 9]) and was extensively investigated by several researchers.

If a set X has a special element 0 and a binary operation "$*$" satisfying the conditions:

(I_1) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)$ $(((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0)$,
(I_2) $(\forall \mathfrak{a}, \mathfrak{b} \in X)$ $((\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0)$,
(I_3) $(\forall \mathfrak{a} \in X)$ $(\mathfrak{a} * \mathfrak{a} = 0)$,
(I_4) $(\forall \mathfrak{a}, \mathfrak{b} \in X)$ $(\mathfrak{a} * \mathfrak{b} = 0,\ \mathfrak{b} * \mathfrak{a} = 0 \ \Rightarrow\ \mathfrak{a} = \mathfrak{b})$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall \mathfrak{a} \in X)$ $(0 * \mathfrak{a} = 0)$,

then X is called a *BCK-algebra*. In what follows, BCK/BCI-algebra is expressed as $(X,*)_0$.

The order relation $\leq_X$ in a BCK/BCI-algebra $(X,*)_0$ is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \leq_X \mathfrak{b} \ \Leftrightarrow\ \mathfrak{a} * \mathfrak{b} = 0). \tag{2.1}$$

Every BCK/BCI-algebra $(X,*)_0$ satisfies the following conditions (See [6, 9]):

$$(\forall \mathfrak{a} \in X)\,(\mathfrak{a} * 0 = \mathfrak{a})\,, \tag{2.2}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,(\mathfrak{a} \leq_X \mathfrak{b} \ \Rightarrow\ \mathfrak{a} * \mathfrak{c} \leq_X \mathfrak{b} * \mathfrak{c},\ \mathfrak{c} * \mathfrak{b} \leq_X \mathfrak{c} * \mathfrak{a})\,, \tag{2.3}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = (\mathfrak{a} * \mathfrak{c}) * \mathfrak{b})\,. \tag{2.4}$$

A BCI-algebra $(X,*)_0$ is said to be *p-semisimple* (See [6]) if $0 * (0 * \mathfrak{a}) = \mathfrak{a}$ for all $\mathfrak{a} \in X$.

Every p-semisimple BCI-algebra $(X,*)_0$ satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)((\mathfrak{a} * \mathfrak{c}) * (\mathfrak{b} * \mathfrak{c}) = \mathfrak{a} * \mathfrak{b}). \tag{2.5}$$

A subset A of a BCK/BCI-algebra $(X,*)_0$ is called

- a *subalgebra* of X (See [6, 9]) if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b} \in A)(\mathfrak{a} * \mathfrak{b} \in A). \tag{2.6}$$

- an *ideal* of $(X,*)_0$ (See [6, 9]), if it satisfies:

$$0 \in A, \tag{2.7}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} * \mathfrak{b} \in A,\ \mathfrak{b} \in A \ \Rightarrow\ \mathfrak{a} \in A). \tag{2.8}$$

An ideal A of a BCI-algebra $(X,*)_0$ is said to be *closed* (See [6]), if $0 * \mathfrak{a} \in A$ for all $\mathfrak{a} \in A$.

Let X be a nonempty set and consider two mappings:

$$\mathcal{M} : X \to [0,1] \text{ and } \mathcal{R} : X \to [0,1]$$

which are called the *membership function* and *reference function*, respectively, on X. Then the *intuitionistic fuzzy set* on X (See [10, 11]) is defined to be of the form:

$$\mathcal{I} := \{\mathfrak{a}, \mathcal{M}(\mathfrak{a}), \mathcal{R}(\mathfrak{a}) \mid \mathfrak{a} \in X\} \tag{2.9}$$

satisfying $\mathcal{M}(\mathfrak{a}) + \mathcal{R}(\mathfrak{a}) \in [0,1]$. The *imprecise set* on X (See [3, 4, 5]) is defined to be of the form (2.9) satisfying $0 \leq \mathcal{R}(\mathfrak{a}) \leq \mathcal{M}(\mathfrak{a}) \leq 1$, and it is simply denoted by $(\mathcal{M}, \mathcal{R})$. So we can consider the imprecise set $(\mathcal{M}, \mathcal{R})$ on X as the following function:

$$(\mathcal{M}, \mathcal{R}) : X \to [0,1] \times [0,1], \ \mathfrak{a} \mapsto (\mathcal{M}(\mathfrak{a}), \mathcal{R}(\mathfrak{a}))$$

that satisfies $0 \leq \mathcal{R}(\mathfrak{a}) \leq \mathcal{M}(\mathfrak{a}) \leq 1$.

3. Imprecise subalgebras and ideals

In what follows, let $(X,*)_0$ and $(Y,*)_0$ denote BCK/BCI-algebras unless otherwise specified.

For every imprecise set $(\mathcal{M}, \mathcal{R})$ on X, we consider a relation $\succeq$ and $\barwedge$ as follows:

$$(\forall x, y \in X)\,((\mathcal{M}, \mathcal{R})(x) \succeq (\mathcal{M}, \mathcal{R})(y) \ \Leftrightarrow \ \mathcal{M}(x) \geq \mathcal{M}(y),\, \mathcal{R}(x) \leq \mathcal{R}(y))$$

and

$$(\forall x, y \in X)\,((\mathcal{M}, \mathcal{R})(x) \barwedge (\mathcal{M}, \mathcal{R})(y) = (\mathcal{M}(x) \wedge \mathcal{M}(y),\, \mathcal{R}(x) \vee \mathcal{R}(y)))$$

respectively, where $\mathcal{M}(x) \wedge \mathcal{M}(y) = \min\{\mathcal{M}(x), \mathcal{M}(y)\}$ and $\mathcal{R}(x) \vee \mathcal{R}(y) = \max\{\mathcal{R}(x), \mathcal{R}(y)\}$.

It is clear that if $(\mathcal{M}, \mathcal{R})(x) \succeq (\mathcal{M}, \mathcal{R})(y)$, then

$$(\mathcal{M}, \mathcal{R})(x) \barwedge (\mathcal{M}, \mathcal{R})(y) = (\mathcal{M}, \mathcal{R})(y).$$

Definition 3.1. An imprecise set $(\mathcal{M}, \mathcal{R})$ on X is called an *imprecise subalgebra* of $(X,*)_0$, if it satisfies:

$$(\forall x, y \in X)\,((\mathcal{M}, \mathcal{R})(x * y) \succeq (\mathcal{M}, \mathcal{R})(x) \barwedge (\mathcal{M}, \mathcal{R})(y))\,. \tag{3.1}$$

Example 3.2. Let $X = \{\ell_0, \ell_1, \ell_2, \ell_3\}$ be a set with the binary operation $*$ which is given in Table 1.

Table 1. Cayley table for the binary operation $*$

$*$	ℓ_0	ℓ_1	ℓ_2	ℓ_3
ℓ_0	ℓ_0	ℓ_0	ℓ_0	ℓ_0
ℓ_1	ℓ_1	ℓ_0	ℓ_0	ℓ_1
ℓ_2	ℓ_2	ℓ_1	ℓ_0	ℓ_2
ℓ_3	ℓ_3	ℓ_3	ℓ_3	ℓ_0

Then $(X,*)_{\ell_0}$ is a BCK-algebra (See [9]). Let $(\mathcal{M}, \mathcal{R})$ be an imprecise set on X defined by Table 2.

Table 2. Table for the imprecise set $(\mathcal{M}, \mathcal{R})$

$x \in X$	ℓ_0	ℓ_1	ℓ_2	ℓ_3
$(\mathcal{M}, \mathcal{R})(x)$	$(0.9, 0.3)$	$(0.8, 0.6)$	$(0.6, 0.4)$	$(0.7, 0.5)$

It is routine to confirm that $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of $(X, *)_{\ell_0}$.

Lemma 3.3. *Every imprecise subalgebra* $(\mathcal{M}, \mathcal{R})$ *of* $(X, *)_0$ *satisfies:*

$$(\forall x \in X)((\mathcal{M}, \mathcal{R})(0) \succeq (\mathcal{M}, \mathcal{R})(x)). \tag{3.2}$$

Proof. Using (I_3) and (3.1), we have

$$(\mathcal{M}, \mathcal{R})(0) = (\mathcal{M}, \mathcal{R})(x * x) \succeq (\mathcal{M}, \mathcal{R})(x) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(x) = (\mathcal{M}, \mathcal{R})(x)$$

for all $x \in X$. □

Using the BCK-part of a BCI-algebra $(X, *)_0$, we make an imprecise subalgebra of $(X, *)_0$.

Theorem 3.4. *Let* $(X, *)_0$ *be a BCI-algebra and define an imprecise set* $(\mathcal{M}, \mathcal{R})$ *on* X *as follows:*

$$(\mathcal{M}, \mathcal{R}) : X \to [0,1] \times [0,1], \ x \mapsto \begin{cases} (t_x, s_x) & \text{if } x \in A, \\ (t^*, s^*) & \text{otherwise,} \end{cases}$$

where A *is the BCK-part of* $(X, *)_0$*, i.e.,* $A := \{x \in X \mid 0 \leq_X x\}$ *and* $(t_x, s_x), (t^*, s^*) \in [0,1] \times [0,1]$ *with* $t_x \geq s_x$, $t^* \geq s^*$, $t_x > t^*$ *and* $s_x < s^*$. *If* $(t_{x*y}, s_{x*y}) = (t_x \wedge t_y, s_x \vee s_y)$ *for all* $x, y \in X$*, then* $(\mathcal{M}, \mathcal{R})$ *is an imprecise subalgebra of* $(X, *)_0$*.*

Proof. Let $x, y \in X$ be such that $(t_{x*y}, s_{x*y}) = (t_x \wedge t_y, s_x \vee s_y)$. If $x \in A$ and $y \in A$, then $(\mathcal{M}, \mathcal{R})(x) = (t_x, s_x)$ and $(\mathcal{M}, \mathcal{R})(y) = (t_y, s_y)$. Thus

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(x * y) &= (t_{x*y}, s_{x*y}) = (t_x \wedge t_y, s_x \vee s_y) \\ &= (t_x, s_x) \,\overline{\wedge}\, (t_y, s_y) = (\mathcal{M}, \mathcal{R})(x) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y)\end{aligned}$$

It is clear that if $x \notin A$ or $y \notin A$, then $(\mathcal{M}, \mathcal{R})(x) = (t^*, s^*)$ or $(\mathcal{M}, \mathcal{R})(y) = (t^*, s^*)$. So $(\mathcal{M}, \mathcal{R})(x*y) \succeq (t^*, s^*) = (\mathcal{M}, \mathcal{R})(x) \overline{\wedge} (\mathcal{M}, \mathcal{R})(y)$. Hence $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of $(X, *)_0$. □

Using the initial section of a BCK-algebra $(X, *)_0$, we make an imprecise subalgebra of $(X, *)_0$.

Theorem 3.5. *Let* $(X, *)_0$ *be a BCK-algebra and define an imprecise set* $(\mathcal{M}, \mathcal{R})$ *on* X *as follows:*

$$(\mathcal{M}, \mathcal{R}) : X \to [0,1] \times [0,1], \ x \mapsto \begin{cases} (t_x, s_x) & \text{if } x \in A(\mathfrak{a}), \\ (t^*, s^*) & \text{otherwise,} \end{cases}$$

where $A(\mathfrak{a}) := \{x \in X \mid \mathfrak{a} \leq_X x\}$ *and* $(t_x, s_x), (t^*, s^*) \in [0,1] \times [0,1]$ *with* $t_x \geq s_x$, $t^* \geq s^*$, $t_x > t^*$ *and* $s_x < s^*$. *If* $(t_{x*y}, s_{x*y}) = (t_x \wedge t_y, s_x \vee s_y)$ *for all* $x, y \in X$*, then* $(\mathcal{M}, \mathcal{R})$ *is an imprecise subalgebra of* $(X, *)_0$*.*

Proof. This is similar to the proof of Theorem 3.4. □

Given an imprecise set $(\mathcal{M},\mathcal{R})$ on X and $(t,s)\in[0,1]\times[0,1]$, we define

$$\mathcal{M}_t := \{x\in X \mid \mathcal{M}(x)\geq t\} \text{ and } \mathcal{R}^s := \{x\in X \mid \mathcal{R}(x)\leq s\}$$

which are called the *membership* t*-cut* and *reference* s*-cut* of $(\mathcal{M},\mathcal{R})$ respectively. The *imprecise* (t,s)*-cut* of $(\mathcal{M},\mathcal{R})$ is denoted by $(\mathcal{M},\mathcal{R})_t^s$ and is defined as follows:

$$(\mathcal{M},\mathcal{R})_t^s = \mathcal{M}_t\cap\mathcal{R}^s.$$

Theorem 3.6. *An imprecise set* $(\mathcal{M},\mathcal{R})$ *on* X *is an imprecise subalgebra of* $(X,*)_0$ *if and only if the nonempty membership* t*-cut and the nonempty reference* s*-cut of* $(\mathcal{M},\mathcal{R})$ *are subalgebras of* $(X,*)_0$ *for all* $(t,s)\in[0,1]\times[0,1]$.

Proof. Let $(\mathcal{M},\mathcal{R})$ be an imprecise subalgebra of $(X,*)_0$, and let $(t,s)\in[0,1]\times[0,1]$ be such that $\mathcal{M}_t\neq\varnothing$ and $\mathcal{R}^s\neq\varnothing$. If $x,y\in\mathcal{M}_t\cap\mathcal{R}^s$, then $\mathcal{M}(x)\geq t$, $\mathcal{M}(y)\geq t$, $\mathcal{R}(x)\leq s$, and $\mathcal{R}(y)\leq s$. It follows that $\mathcal{M}(x*y)\geq\mathcal{M}(x)\wedge\mathcal{M}(y)\geq t$ and $\mathcal{R}(x*y)\leq\mathcal{R}(x)\vee\mathcal{R}(y)\leq s$. Thus $x*y\in\mathcal{M}_t\cap\mathcal{R}^s$. So $\mathcal{M}_t$ and $\mathcal{R}^s$ are subalgebras of $(X,*)_0$.

Conversely, assume that the nonempty membership t-cut and the nonempty reference s-cut of $(\mathcal{M},\mathcal{R})$ are subalgebras of $(X,*)_0$ for all $(t,s)\in[0,1]\times[0,1]$. If (3.1) is not valid, then there exist $\mathfrak{a},\mathfrak{b}\in X$ such that

$$(\mathcal{M},\mathcal{R})(\mathfrak{a}*\mathfrak{b})\not\succeq(\mathcal{M},\mathcal{R})(\mathfrak{a})\,\bar{\curlywedge}\,(\mathcal{M},\mathcal{R})(\mathfrak{b}).$$

Thus $\mathcal{M}(\mathfrak{a}*\mathfrak{b})<t\leq\mathcal{M}(\mathfrak{a})\wedge\mathcal{M}(\mathfrak{b})$ or $\mathcal{R}(\mathfrak{a}*\mathfrak{b})>s\geq\mathcal{R}(\mathfrak{a})\vee\mathcal{R}(\mathfrak{b})$ for some $(t,s)\in(0,1]\times[0,1)$. It follows that $\mathfrak{a},\mathfrak{b}\in\mathcal{M}_t\cap\mathcal{R}^s$, but $\mathfrak{a}*\mathfrak{b}\notin\mathcal{M}_t$ or $\mathfrak{a}*\mathfrak{b}\notin\mathcal{R}^s$. This is a contradiction So (3.1) is valid, that is, $(\mathcal{M},\mathcal{R})$ is an imprecise subalgebra of $(X,*)_0$. □

Corollary 3.7. *If an imprecise set* $(\mathcal{M},\mathcal{R})$ *on* X *is an imprecise subalgebra of* $(X,*)_0$*, then its nonempty imprecise* (t,s)*-cut is a subalgebra of* $(X,*)_0$ *for all* $(t,s)\in[0,1]\times[0,1]$.

Proof. This is straightforward. □

The two concepts, intuitionistic fuzzy subalgebra (see [12]) and imprecise subalgebra, are independent of each other as shown in the following example.

Example 3.8. (i) In Example 3.2, the imprecise subalgebra $(\mathcal{M},\mathcal{R})$ is not an intuitionistic fuzzy subalgebra since $\mathcal{M}(\ell_0)+\mathcal{R}(\ell_0)=0.9+0.3=1.2\notin[0,1]$.

(ii) Consider the BCK-algebra $(X,*)_{\ell_0}$ in Example 3.2 and let $(\mathcal{M},\mathcal{R})$ be an intuitionistic fuzzy set on X defined by Table 3.

Table 3. Table for the intuitionistic fuzzy set $(\mathcal{M},\mathcal{R})$

$x\in X$	ℓ_0	ℓ_1	ℓ_2	ℓ_3
$(\mathcal{M},\mathcal{R})(x)$	$(0.4,0.4)$	$(0.4,0.5)$	$(0.3,0.5)$	$(0.2,0.7)$

It is routine to check that $(\mathcal{M},\mathcal{R})$ is an intuitionistic fuzzy subalgebra of $(X,*)_{\ell_0}$. But it is not an imprecise subalgebra of $(X,*)_{\ell_0}$ since $\mathcal{M}(x)\not\geq\mathcal{R}(x)$ for $x\in\{\ell_1,\ell_2,\ell_3\}$.

Theorem 3.9. *If an imprecise subalgebra $(\mathcal{M},\mathcal{R})$ of $(X,*)_0$ satisfies $\mathcal{M}(x)+\mathcal{R}(x)\in[0,1]$ for all $x\in X$, then it is an intuitionistic fuzzy subalgebra of $(X,*)_0$. Also, if an intuitionistic fuzzy subalgebra $(\mathcal{M},\mathcal{R})$ of $(X,*)_0$ satisfies $\mathcal{M}(x)\geq\mathcal{R}(x)$ for all $x\in X$, then it is an imprecise subalgebra of $(X,*)_0$.*

Proof. This is a straightforward. □

Definition 3.10. An imprecise set $(\mathcal{M},\mathcal{R})$ on X is called an *imprecise ideal* of $(X,*)_0$ if it satisfies (3.2) and

$$(\forall x,y\in X)\,((\mathcal{M},\mathcal{R})(x)\succeq(\mathcal{M},\mathcal{R})(x*y)\,\bar{\wedge}\,(\mathcal{M},\mathcal{R})(y)).\tag{3.3}$$

Example 3.11. Let $X=\{\ell_0,\ell_1,\ell_2,\ell_3,\ell_4\}$ be a set with the binary operation $*$ which is given in Table 4.

Table 4. Cayley table for the binary operation $*$

$*$	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4
ℓ_0	ℓ_0	ℓ_0	ℓ_0	ℓ_0	ℓ_0
ℓ_1	ℓ_1	ℓ_0	ℓ_1	ℓ_0	ℓ_0
ℓ_2	ℓ_2	ℓ_2	ℓ_0	ℓ_0	ℓ_0
ℓ_3	ℓ_3	ℓ_3	ℓ_3	ℓ_0	ℓ_0
ℓ_4	ℓ_4	ℓ_3	ℓ_4	ℓ_1	ℓ_0

Then $(X,*)_{\ell_0}$ is a BCK-algebra (see [9]). Let $(\mathcal{M},\mathcal{R})$ be an imprecise set on X defined by Table 5.

Table 5. Table for the imprecise set $(\mathcal{M},\mathcal{R})$

$x\in X$	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4
$(\mathcal{M},\mathcal{R})(x)$	$(0.9,0.3)$	$(0.8,0.3)$	$(0.9,0.4)$	$(0.6,0.5)$	$(0.6,0.5)$

It is routine to confirm that $(\mathcal{M},\mathcal{R})$ is an imprecise ideal of $(X,*)_{\ell_0}$.

Lemma 3.12. *Every imprecise ideal $(\mathcal{M},\mathcal{R})$ of $(X,*)_0$ satisfies:*

$$(\forall x,y\in X)\,(x\leq_X y\ \Rightarrow\ (\mathcal{M},\mathcal{R})(x)\succeq(\mathcal{M},\mathcal{R})(y)).\tag{3.4}$$

Proof. Let $x,y\in X$ be such that $x\leq_X y$. Then $x*y=0$. Thus

$$\begin{aligned}(\mathcal{M},\mathcal{R})(x)&\succeq(\mathcal{M},\mathcal{R})(x*y)\,\bar{\wedge}\,(\mathcal{M},\mathcal{R})(y)\\&=(\mathcal{M},\mathcal{R})(0)\,\bar{\wedge}\,(\mathcal{M},\mathcal{R})(y)\\&=(\mathcal{M},\mathcal{R})(y)\end{aligned}$$

by (3.3) and (3.2). □

Lemma 3.13. *Every imprecise ideal $(\mathcal{M},\mathcal{R})$ of $(X,*)_0$ satisfies:*

$$(\forall x,y,z\in X)\,(x*y\leq_X z\ \Rightarrow\ (\mathcal{M},\mathcal{R})(x)\succeq(\mathcal{M},\mathcal{R})(z)\,\bar{\wedge}\,(\mathcal{M},\mathcal{R})(y)).\tag{3.5}$$

Proof. Let $x, y, z \in X$ be such that $x * y \leq_X z$. Then $(x * y) * z = 0$. Thus

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(x) &\succeq (\mathcal{M}, \mathcal{R})(x * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) \\ &\succeq ((\mathcal{M}, \mathcal{R})((x * y) * z) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(z)) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) \\ &= ((\mathcal{M}, \mathcal{R})(0) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(z)) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) \\ &= (\mathcal{M}, \mathcal{R})(z) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y)\end{aligned}$$

by (3.3) and (3.2). □

Theorem 3.14. *If* $(\mathcal{M}, \mathcal{R})$ *is an imprecise ideal of* $(X, *)_0$*, then*

$$(\mathcal{M}, \mathcal{R})(x) \succeq (\mathcal{M}, \mathcal{R})(\mathfrak{a}_1) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(\mathfrak{a}_2) \,\overline{\wedge} \cdots \overline{\wedge}\, (\mathcal{M}, \mathcal{R})(\mathfrak{a}_n) \tag{3.6}$$

for all $x, \mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n \in X$ *with* $(\cdots((x * \mathfrak{a}_1) * \mathfrak{a}_2) * \cdots) * \mathfrak{a}_n = 0$.

Proof. This is straightforward by Lemma 3.12, Lemma 3.13 and the mathematical induction on n. □

Theorem 3.15. *In a BCK-algebra* $(X, *)_0$*, every imprecise ideal is an imprecise subalgebra.*

Proof. Let $(\mathcal{M}, \mathcal{R})$ be an imprecise ideal of a BCK-algebra $(X, *)_0$. Since $x * y \leq_X x$ for all $x, y \in X$, we have $(\mathcal{M}, \mathcal{R})(x * y) \succeq (\mathcal{M}, \mathcal{R})(x)$ by Lemma 3.12. Then

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(x * y) \succeq (\mathcal{M}, \mathcal{R})(x) &\succeq (\mathcal{M}, \mathcal{R})(x * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) \\ &\succeq (\mathcal{M}, \mathcal{R})(x) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y).\end{aligned}$$

This shows that $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of a BCK-algebra $(X, *)_0$. □

The converse of Theorem 3.15 may not be true. In fact, the imprecise subalgebra $(\mathcal{M}, \mathcal{R})$ of $(X, *)_{\ell_0}$ in Example 3.2 is not an imprecise ideal of $(X, *)_{\ell_0}$ since

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(\ell_2) = (0.6, 0.4) \not\succeq (0.8, 0.6) &= (\mathcal{M}, \mathcal{R})(\ell_1) \\ &= (\mathcal{M}, \mathcal{R})(\ell_2 * \ell_1) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(\ell_1).\end{aligned}$$

We explore conditions for an imprecise subalgebra to be an imprecise ideal in BCK-algebras.

Theorem 3.16. *If an imprecise subalgebra* $(\mathcal{M}, \mathcal{R})$ *of a BCK-algebra* $(X, *)_0$ *satisfies the condition* (3.5)*, then it is an imprecise ideal of* $(X, *)_0$.

Proof. Let $(\mathcal{M}, \mathcal{R})$ be an imprecise subalgebra of a BCK-algebra $(X, *)_0$ that satisfies the condition (3.5). Then it satisfies the condition (3.2) by Lemma 3.3. Since $x * (x * y) \leq_X y$ for all $x, y \in X$, we get

$$(\mathcal{M}, \mathcal{R})(x) \succeq (\mathcal{M}, \mathcal{R})(x * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y)$$

by the condition (3.5). Thus $(\mathcal{M}, \mathcal{R})$ is an imprecise ideal of a BCK-algebra $(X, *)_0$. □

The following example shows that any imprecise subalgebra of a BCI-algebra $(X, *)_0$ may not be an imprecise ideal of $(X, *)_0$.

Table 6. Cayley table for the binary operation $*$

$*$	ℓ_0	ℓ_1	ℓ_a	ℓ_b	ℓ_c
ℓ_0	ℓ_0	ℓ_0	ℓ_a	ℓ_a	ℓ_a
ℓ_1	ℓ_1	ℓ_0	ℓ_a	ℓ_a	ℓ_a
ℓ_a	ℓ_a	ℓ_a	ℓ_0	ℓ_0	ℓ_0
ℓ_b	ℓ_b	ℓ_a	ℓ_1	ℓ_0	ℓ_0
ℓ_c	ℓ_c	ℓ_a	ℓ_1	ℓ_1	ℓ_0

Example 3.17. Let $X = \{\ell_0, \ell_1, \ell_a, \ell_b, \ell_c\}$ be a set with the binary operation $*$ which is given in Table 6.
Then $(X, *)_{\ell_0}$ is a BCI-algebra (See [6, 9]). Let $(\mathcal{M}, \mathcal{R})$ be an imprecise set on X defined by Table 7.

Table 7. Table for the imprecise set $(\mathcal{M}, \mathcal{R})$

$x \in X$	ℓ_0	ℓ_1	ℓ_a	ℓ_b	ℓ_c
$(\mathcal{M}, \mathcal{R})(x)$	$(0.9, 0.15)$	$(0.8, 0.23)$	$(0.7, 0.45)$	$(0.6, 0.47)$	$(0.5, 0.49)$

It is routine to verify that $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of $(X, *)_{\ell_0}$. But it is not an imprecise ideal of $(X, *)_{\ell_0}$ since

$$(\mathcal{M}, \mathcal{R})(\ell_b) = (0.6, 0.47) \not\succeq (0.7, 0.45) = (\mathcal{M}, \mathcal{R})(\ell_b * \ell_a) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(\ell_a).$$

Theorem 3.18. *In a p-semisimple BCI-algebra $(X, *)_0$, every imprecise subalgebra of $(X, *)_0$ is an imprecise ideal of $(X, *)_0$.*

Proof. Let $(\mathcal{M}, \mathcal{R})$ be an imprecise subalgebra of a p-semisimple BCI-algebra $(X, *)_0$. Then $(\mathcal{M}, \mathcal{R})$ satisfies the condition (3.2) by Lemma 3.3. Thus

$$(\mathcal{M}, \mathcal{R})(0 * y) \succeq (\mathcal{M}, \mathcal{R})(0) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) = (\mathcal{M}, \mathcal{R})(y)$$

for all $y \in X$. Since $(X, *)_0$ is a p-semisimple BCI-algebra, it follows that

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(x) &= (\mathcal{M}, \mathcal{R})(x * 0) = (\mathcal{M}, \mathcal{R})((x * y) * (0 * y)) \\ &\succeq (\mathcal{M}, \mathcal{R})(x * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(0 * y) \\ &\succeq (\mathcal{M}, \mathcal{R})(x * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y)\end{aligned}$$

for all $x, y \in X$. So $(\mathcal{M}, \mathcal{R})$ is an imprecise ideal of $(X, *)_0$. □

Corollary 3.19. *If a BCI-algebra $(X, *)_0$ satisfies one of the following:*

(i) $X = \{0 * x \mid x \in X\}$,
(ii) *every element x in X is minimal,*
(iii) $(x, y \in X)$ $(x * (0 * y) = y * (0 * x))$,
(iv) $(x \in X)$ $(0 * x = 0 \;\Rightarrow\; x = 0)$,

then every imprecise subalgebra of $(X, *)_0$ is an imprecise ideal of $(X, *)_0$.

Theorem 3.20. *An imprecise set* $(\mathcal{M}, \mathcal{R})$ *on* X *is an imprecise ideal of* $(X, *)_0$ *if and only if the nonempty membership* t*-cut and the nonempty reference* s*-cut of* $(\mathcal{M}, \mathcal{R})$ *are ideals of* $(X, *)_0$ *for all* $(t, s) \in [0, 1] \times [0, 1]$.

Proof. Let $(\mathcal{M}, \mathcal{R})$ be an imprecise ideal of $(X, *)_0$, and let $(t, s) \in [0, 1] \times [0, 1]$ be such that $\mathcal{M}_t \neq \varnothing$ and $\mathcal{R}^s \neq \varnothing$. It is clear that $0 \in \mathcal{M}_t \cap \mathcal{R}^s$ by (3.2). Let $x, y \in X$ be such that $x * y \in \mathcal{M}_t \cap \mathcal{R}^s$ and $y \in \mathcal{M}_t \cap \mathcal{R}^s$. Then $\mathcal{M}(x * y) \geq t$, $\mathcal{M}(y) \geq t$, $\mathcal{R}(x * y) \leq s$, and $\mathcal{R}(y) \leq s$. It follows that

$$\mathcal{M}(x) \geq \mathcal{M}(x * y) \wedge \mathcal{M}(y) \geq t \text{ and } \mathcal{R}(x) \leq \mathcal{R}(x * y) \vee \mathcal{R}(y) \leq s.$$

Thus $x \in \mathcal{M}_t \cap \mathcal{R}^s$. So $\mathcal{M}_t$ and $\mathcal{R}^s$ are ideals of $(X, *)_0$.

Conversely, assume that the nonempty membership t-cut and the nonempty reference s-cut of $(\mathcal{M}, \mathcal{R})$ are ideals of $(X, *)_0$ for all $(t, s) \in [0, 1] \times [0, 1]$. For every $x \in X$, let $(\mathcal{M}, \mathcal{R})(x) = (t, s)$. Then $x \in \mathcal{M}_t \cap \mathcal{R}^s$. Thus $\mathcal{M}_t$ and $\mathcal{R}^s$ are nonempty. Since $\mathcal{M}_t$ and $\mathcal{R}^s$ are ideals of $(X, *)_0$, we have $0 \in \mathcal{M}_t \cap \mathcal{R}^s$. So $(\mathcal{M}, \mathcal{R})(0) \succeq (t, s) = (\mathcal{M}, \mathcal{R})(x)$ for all $x \in X$. If $(\mathcal{M}, \mathcal{R})$ does not satisfy (3.3), then there exist $\mathfrak{a}, \mathfrak{b} \in X$ such that

$$(\mathcal{M}, \mathcal{R})(\mathfrak{a}) \not\succeq (\mathcal{M}, \mathcal{R})(\mathfrak{a} * \mathfrak{b}) \,\bar{\wedge}\, (\mathcal{M}, \mathcal{R})(\mathfrak{b}),$$

that is, $\mathcal{M}(\mathfrak{a}) < \mathcal{M}(\mathfrak{a} * \mathfrak{b}) \wedge \mathcal{M}(\mathfrak{b})$ or $\mathcal{R}(\mathfrak{a}) > \mathcal{R}(\mathfrak{a} * \mathfrak{b}) \vee \mathcal{R}(\mathfrak{b})$. If we take $(t, s) = (\mathcal{M}(\mathfrak{a} * \mathfrak{b}) \wedge \mathcal{M}(\mathfrak{b}), \mathcal{R}(\mathfrak{a} * \mathfrak{b}) \vee \mathcal{R}(\mathfrak{b}))$, then $\mathfrak{a} * \mathfrak{b} \in \mathcal{M}_t \cap \mathcal{R}^s$ and $\mathfrak{b} \in \mathcal{M}_t \cap \mathcal{R}^s$. Since $\mathcal{M}_t$ and $\mathcal{R}^s$ are ideals of $(X, *)_0$, we get $\mathfrak{a} \in \mathcal{M}_t \cap \mathcal{R}^s$, and so $\mathcal{M}(\mathfrak{a}) \geq t$ and $\mathcal{R}(\mathfrak{a}) \leq s$. This is a contradiction. Thus $(\mathcal{M}, \mathcal{R})$ satisfies (3.3). So $(\mathcal{M}, \mathcal{R})$ is an imprecise ideal of $(X, *)_0$. □

In a BCI-algebra, any imprecise ideal may not be an imprecise subalgebra as shown in the following example.

Example 3.21. Let $(Y, *)_0$ be a BCI-algebra and $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Let $X := Y \times \mathbb{Z}$ and $\odot$ the binary operation on X defined by $(x, \mathfrak{a}) \odot (y, \mathfrak{b}) = (x * y, \mathfrak{a} - \mathfrak{b})$ for all $(x, \mathfrak{a}), (y, \mathfrak{b}) \in X$. Then $(X, \odot)_{(0,0)}$ is a BCI-algebra (See [6]). Define an imprecise set $(\mathcal{M}, \mathcal{R})$ on X as follows:

$$(\mathcal{M}, \mathcal{R}) : X \to [0, 1] \times [0, 1],\ (x, \mathfrak{a}) \mapsto \begin{cases} (t, s) & \text{if } (x, \mathfrak{a}) \in A, \\ (t^*, s^*) & \text{otherwise,} \end{cases}$$

where $A := Y \times \mathbb{N}_0$ for the set $\mathbb{N}_0$ of nonnegative integers and $(t, s), (t^*, s^*) \in [0, 1] \times [0, 1]$ with $t \geq s$, $t^* \geq s^*$, $t > t^*$ and $s < s^*$. It is common to check that $(\mathcal{M}, \mathcal{R})$ is an imprecise ideal of $(X, *)_0$. But it is not an imprecise subalgebra of $(X, *)_0$ since

$$\begin{aligned} (\mathcal{M}, \mathcal{R})((0, 0) \odot (0, 1)) &= (\mathcal{M}, \mathcal{R})(0, -1) = (t^*, s^*) \\ &\not\succeq (t, s) = (\mathcal{M}, \mathcal{R})(0, 0) \,\bar{\wedge}\, (\mathcal{M}, \mathcal{R})(0, 1). \end{aligned}$$

The following example confirms that the imprecise ideal is a concept independent of the intuitionistic fuzzy ideal (See [12]) in BCK/BCI-algebras.

Example 3.22. (i) In Example 3.11, the imprecise ideal $(\mathcal{M}, \mathcal{R})$ of $(X, *)_{\ell_0}$ is not an intuitionistic fuzzy ideal of $(X, *)_{\ell_0}$ since $(\mathcal{M}, \mathcal{R})$ does not satisfies the condition $\mathcal{M}(x) + \mathcal{R}(x) \in [0, 1]$ for some $x \in X$.

(ii) Consider the BCI-algebra $(X, \odot)_{(0,0)}$ in Example 3.21 and let $(\mathcal{M}, \mathcal{R})$ be an imprecise set on X defined as follows:

$$(\mathcal{M}, \mathcal{R}) : X \to [0,1] \times [0,1],\ (x, \mathfrak{a}) \mapsto \begin{cases} \left(\frac{1}{3n}, \frac{1}{2n}\right) & \text{if } (x, \mathfrak{a}) \in A, \\ (0,1) & \text{otherwise,} \end{cases}$$

where n is a natural number and $A := Y \times \mathbb{N}_0$ for the set $\mathbb{N}_0$ of nonnegative integers. Then $(\mathcal{M}, \mathcal{R})$ is an intuitionistic fuzzy ideal of $(X, \odot)_{(0,0)}$, but it does not satisfy the condition $\mathcal{M}(x, \mathfrak{a}) \geq \mathcal{R}(x, \mathfrak{a})$ for some $(x, \mathfrak{a}) \in X$. Thus $(\mathcal{M}, \mathcal{R})$ can not be an imprecise ideal of $(X, \odot)_{(0,0)}$.

Definition 3.23. An imprecise ideal $(\mathcal{M}, \mathcal{R})$ of a BCI-algebra $(X, *)_0$ is said to be *closed* if $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of $(X, *)_0$.

Example 3.24. Let $X = \{\ell_0, \ell_1, \ell_a, \ell_b, \ell_c\}$ be a set with the binary operation $*$ which is given in Table 8.

Table 8. Cayley table for the binary operation $*$

$*$	ℓ_0	ℓ_1	ℓ_a	ℓ_b	ℓ_c
ℓ_0	ℓ_0	ℓ_0	ℓ_a	ℓ_b	ℓ_c
ℓ_1	ℓ_1	ℓ_0	ℓ_a	ℓ_b	ℓ_c
ℓ_a	ℓ_a	ℓ_a	ℓ_0	ℓ_c	ℓ_b
ℓ_b	ℓ_b	ℓ_b	ℓ_c	ℓ_0	ℓ_a
ℓ_c	ℓ_c	ℓ_c	ℓ_b	ℓ_a	ℓ_0

Then $(X, *)_{\ell_0}$ is a BCI-algebra (see [6, 9]). Let $(\mathcal{M}, \mathcal{R})$ be an imprecise set on X defined by Table 9.

Table 9. Table for the imprecise set $(\mathcal{M}, \mathcal{R})$

$x \in X$	ℓ_0	ℓ_1	ℓ_a	ℓ_b	ℓ_c
$(\mathcal{M}, \mathcal{R})(x)$	$(0.9, 0.2)$	$(0.8, 0.3)$	$(0.6, 0.5)$	$(0.6, 0.4)$	$(0.7, 0.5)$

It is common to verify that $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of $(X, *)_0$.

Theorem 3.25. *Let $(X, *)_0$ be a BCI-algebra and define an imprecise set $(\mathcal{M}, \mathcal{R})$ on X as follows:*

$$(\mathcal{M}, \mathcal{R}) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (t_x, s_x) & \text{if } x \in A, \\ (t^*, s^*) & \text{otherwise,} \end{cases}$$

where $A := \{x \in X \mid 0 \leq_X x\}$ and $(t_x, s_x), (t^*, s^*) \in [0,1] \times [0,1]$ with $t_x \geq s_x$, $t^* \geq s^*$, $t_x > t^*$ and $s_x < s^*$. If $(t_{x*y}, s_{x*y}) = (t_x \wedge t_y, s_x \vee s_y)$ for all $x, y \in X$, then $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of $(X, *)_0$.

Proof. Using Theorem 3.4, we know that $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of $(X, *)_0$. It is sufficient to verify that $(\mathcal{M}, \mathcal{R})$ satisfies the condition (3.3). Let

$x, y \in X$. If $x * y \in A$ and $y \in A$, then $(\mathcal{M}, \mathcal{R})(x * y) = (t_{x*y}, s_{x*y})$ and $(\mathcal{M}, \mathcal{R})(y) = (t_y, s_y)$. Thus

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(x) = (t_x, s_x) &\succeq (t_x \wedge t_y, s_x \vee s_y) = ((t_x \wedge t_y) \wedge t_y, (s_x \vee s_y) \vee s_y) \\ &= (t_{x*y}, s_{x*y}) \,\overline{\wedge}\, (t_y, s_y) = (\mathcal{M}, \mathcal{R})(x * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y).\end{aligned}$$

It is clear that if $x * y \notin A$ or $y \notin A$, then $(\mathcal{M}, \mathcal{R})(x * y) = (t^*, s^*)$ or $(\mathcal{M}, \mathcal{R})(y) = (t^*, s^*)$. So $(\mathcal{M}, \mathcal{R})(x) \succeq (t^*, s^*) = (\mathcal{M}, \mathcal{R})(x * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y)$. Hence $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of $(X, *)_0$. □

In Example 3.21, we saw that an imprecise ideal is not an imprecise subalgebra in BCI-algebras, so the question arises as to what happens in p-semisimple BCI-algebras where the condition of BCI-algebras is strengthened. However, the next example shows that an imprecise ideal is not an imprecise subalgebra even in p-semisimple BCI-algebras.

Example 3.26. Let $\mathbb{Z}$ be the set of integers. Then $(\mathbb{Z}, -)_0$ is a BCI-algebra (See [6]). Define an imprecise set $(\mathcal{M}, \mathcal{R})$ on $\mathbb{Z}$ as follows:

$$(\mathcal{M}, \mathcal{R}) : \mathbb{Z} \to [0,1] \times [0,1],\ x \mapsto \begin{cases} \left(\frac{0.98}{n}, \frac{0.98}{2n}\right) & \text{if } x \in \mathbb{N}, \\ \left(\frac{0.66}{n}, \frac{0.66}{2n}\right) & \text{otherwise,} \end{cases}$$

where $\mathbb{N}$ is the set of natural numbers and $n \in \mathbb{N}$. Then $(\mathcal{M}, \mathcal{R})$ is an imprecise ideal of $(\mathbb{Z}, -)_0$, and we have

$$\mathcal{M}_t = \begin{cases} \varnothing & \text{if } t \in \left(\frac{0.98}{n}, 1\right], \\ \mathbb{N} & \text{if } t \in \left(\frac{0.66}{n}, \frac{0.98}{n}\right], \\ \mathbb{Z} & \text{if } t \in \left[0, \frac{0.66}{n}\right], \end{cases} \quad \text{and } \mathcal{R}_s = \begin{cases} \mathbb{Z} & \text{if } t \in \left[\frac{0.98}{2n}, 1\right], \\ \mathbb{N} & \text{if } t \in \left[\frac{0.66}{2n}, \frac{0.98}{2n}\right), \\ \varnothing & \text{if } t \in \left[0, \frac{0.66}{2n}\right). \end{cases}$$

Note that $\mathbb{N}$ is not a subalgebra of $(\mathbb{Z}, -)_0$ since $2 - 5 = -3 \notin \mathbb{N}$. Thus $(\mathcal{M}, \mathcal{R})$ is not an imprecise subalgebra of $(\mathbb{Z}, -)_0$ by Theorem 3.6.

We provide a way to know whether an imprecise ideal is closed.

Theorem 3.27. *An imprecise ideal $(\mathcal{M}, \mathcal{R})$ of a BCI-algebra $(X, *)_0$ is closed if and only it it satisfies:*

$$(\forall x \in X)((\mathcal{M}, \mathcal{R})(0 * x) \succeq (\mathcal{M}, \mathcal{R})(x)). \tag{3.7}$$

Proof. Assume that $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of a BCI-algebra $(X, *)_0$. Then it is an imprecise subalgebra of $(X, *)_0$. Thus

$$(\mathcal{M}, \mathcal{R})(0 * x) \succeq (\mathcal{M}, \mathcal{R})(0) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(x) = (\mathcal{M}, \mathcal{R})(x)$$

for all $x \in X$.

Conversely, let $(\mathcal{M}, \mathcal{R})$ be an imprecise ideal of a BCI-algebra $(X, *)_0$ that satisfies (3.7). Using (I_3), (2.4), (3.3) and (3.7) induces

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(y * x) &\succeq (\mathcal{M}, \mathcal{R})((y * x) * y) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) \\ &= (\mathcal{M}, \mathcal{R})((y * y) * x) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) \\ &= (\mathcal{M}, \mathcal{R})(0 * x) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y) \\ &\succeq (\mathcal{M}, \mathcal{R})(x) \,\overline{\wedge}\, (\mathcal{M}, \mathcal{R})(y)\end{aligned}$$

for all $x, y \in X$. Then $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of $(X, *)_0$. Thus $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of $(X, *)_0$. □

Theorem 3.28. *If $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of a BCI-algebra $(X, *)_0$, then the set $C := \{x \in X \mid (\mathcal{M}, \mathcal{R})(x) = (\mathcal{M}, \mathcal{R})(0)\}$ is a closed ideal of $(X, *)_0$.*

Proof. It is obvious that $0 \in C$. Let $x, y \in X$ be such that $x * y \in C$ and $y \in C$. Then $(\mathcal{M}, \mathcal{R})(x * y) = (\mathcal{M}, \mathcal{R})(0) = (\mathcal{M}, \mathcal{R})(y)$. It follows from (3.3) that

$$(\mathcal{M}, \mathcal{R})(x) \succeq (\mathcal{M}, \mathcal{R})(x * y) \barwedge (\mathcal{M}, \mathcal{R})(y) = (\mathcal{M}, \mathcal{R})(0).$$

The combination of this and (3.2) induces $(\mathcal{M}, \mathcal{R})(x) = (\mathcal{M}, \mathcal{R})(0)$. Thus $x \in C$. So C is an ideal of $(X, *)_0$. If $x \in C$, then $(\mathcal{M}, \mathcal{R})(0 * x) \succeq (\mathcal{M}, \mathcal{R})(x) = (\mathcal{M}, \mathcal{R})(0)$ by Theorem 3.27. Thus $(\mathcal{M}, \mathcal{R})(0 * x) = (\mathcal{M}, \mathcal{R})(0)$. So $0 * x \in C$. Hence C is a closed ideal of $(X, *)_0$. □

Let $f : X \to Y$ be a homomorphism from $(X, *)_0$ to $(Y, *)_0$. Given an imprecise set $(\mathcal{M}, \mathcal{R})$ on Y, we define a new imprecise set $(\mathcal{M}, \mathcal{R})^f = (\mathcal{M}^f, \mathcal{R}^f)$ on X as follows:

$$\mathcal{M}^f(x) = \mathcal{M}(f(x)) \text{ and } \mathcal{R}^f(x) = \mathcal{R}(f(x))$$

for all $x \in X$. It is clear that $(\mathcal{M}, \mathcal{R})^f$ is an imprecise set on X, and it can be considered as the following function.

$$(\mathcal{M}, \mathcal{R})^f : X \to [0, 1] \times [0, 1], x \mapsto (\mathcal{M}, \mathcal{R})(f(x)). \tag{3.8}$$

Theorem 3.29. *Let $f : X \to Y$ be a homomorphism from $(X, *)_0$ to $(Y, *)_0$. If $(\mathcal{M}, \mathcal{R})$ is an imprecise ideal (resp., imprecise subalgebra) of $(Y, *)_0$, then $(\mathcal{M}, \mathcal{R})^f$ is an imprecise ideal (resp., imprecise subalgebra) of $(X, *)_0$.*

Proof. Let $(\mathcal{M}, \mathcal{R})$ be an imprecise ideal of $(Y, *)_0$. For every $x, \mathfrak{a} \in X$, we have

$$(\mathcal{M}, \mathcal{R})^f(0) = (\mathcal{M}, \mathcal{R})(f(0)) = (\mathcal{M}, \mathcal{R})(0) \succeq (\mathcal{M}, \mathcal{R})(f(x)) = (\mathcal{M}, \mathcal{R})^f(x)$$

and

$$\begin{aligned} (\mathcal{M}, \mathcal{R})^f(x) = (\mathcal{M}, \mathcal{R})(f(x)) &\succeq (\mathcal{M}, \mathcal{R})(f(x) * f(\mathfrak{a})) \barwedge (\mathcal{M}, \mathcal{R})(f(\mathfrak{a})) \\ &= (\mathcal{M}, \mathcal{R})(f(x * \mathfrak{a})) \barwedge (\mathcal{M}, \mathcal{R})(f(\mathfrak{a})) \\ &= (\mathcal{M}, \mathcal{R})^f(x * \mathfrak{a}) \barwedge (\mathcal{M}, \mathcal{R})^f(\mathfrak{a}). \end{aligned}$$

Then $(\mathcal{M}, \mathcal{R})^f$ is an imprecise ideal of $(X, *)_0$. If $(\mathcal{M}, \mathcal{R})$ is an imprecise subalgebra of $(Y, *)_0$, then

$$\begin{aligned} (\mathcal{M}, \mathcal{R})^f(x * \mathfrak{a}) = (\mathcal{M}, \mathcal{R})(f(x * \mathfrak{a})) &= (\mathcal{M}, \mathcal{R})(f(x) * f(\mathfrak{a})) \\ &\succeq (\mathcal{M}, \mathcal{R})(f(x)) \barwedge (\mathcal{M}, \mathcal{R})(f(\mathfrak{a})) \\ &= (\mathcal{M}, \mathcal{R})^f(x) \barwedge (\mathcal{M}, \mathcal{R})^f(\mathfrak{a}) \end{aligned}$$

for all $x, \mathfrak{a} \in X$. Thus $(\mathcal{M}, \mathcal{R})^f$ is an imprecise subalgebra of $(X, *)_0$. □

Theorem 3.30. *Let $f : X \to Y$ be a homomorphism from a BCI-algebra $(X, *)_0$ to a BCI-algebra $(Y, *)_0$. If $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of $(Y, *)_0$, then the imprecise set $(\mathcal{M}, \mathcal{R})^f$ in (3.8) is a closed imprecise ideal of $(X, *)_0$.*

Proof. Assume that $(\mathcal{M},\mathcal{R})$ is a closed imprecise ideal of $(Y,*)_0$. Then $(\mathcal{M},\mathcal{R})^f$ is an imprecise ideal of $(X,*)_0$ (See Theorem 3.29), and

$$\begin{aligned}(\mathcal{M},\mathcal{R})^f(0*x) &= (\mathcal{M},\mathcal{R})(f(0*x)) = (\mathcal{M},\mathcal{R})(f(0)*f(x))\\ &= (\mathcal{M},\mathcal{R})(0*f(x)) \succeq (\mathcal{M},\mathcal{R})(f(x))\\ &= (\mathcal{M},\mathcal{R})^f(x)\end{aligned}$$

for all $x \in X$. It follows from Theorem 3.27 that $(\mathcal{M},\mathcal{R})^f$ is a closed imprecise ideal of $(X,*)_0$. □

By tightening the condition of f, the converse of Theorems 3.29 and 3.30 can be constructed as follows.

Theorem 3.31. *Let $f : X \to Y$ be an onto homomorphism from $(X,*)_0$ to $(Y,*)_0$. Given an imprecise set $(\mathcal{M},\mathcal{R})$ on Y, if the imprecise set $(\mathcal{M},\mathcal{R})^f$ in (3.8) is an imprecise ideal (resp., imprecise subalgebra) of $(X,*)_0$, then $(\mathcal{M},\mathcal{R})$ is an imprecise ideal (resp., imprecise subalgebra) of $(Y,*)_0$.*

Proof. Let $y, \mathfrak{b} \in Y$. Then $f(x) = y$ and $f(\mathfrak{a}) = \mathfrak{b}$ for some $x, \mathfrak{a} \in X$ since f is onto. Assume that $(\mathcal{M},\mathcal{R})^f$ is an imprecise subalgebra of $(X,*)_0$. Then

$$\begin{aligned}(\mathcal{M},\mathcal{R})(y*\mathfrak{b}) &= (\mathcal{M},\mathcal{R})(f(x)*f(\mathfrak{a})) = (\mathcal{M},\mathcal{R})(f(x*\mathfrak{a}))\\ &= (\mathcal{M},\mathcal{R})^f(x*\mathfrak{a}) \succeq (\mathcal{M},\mathcal{R})^f(x) \barwedge (\mathcal{M},\mathcal{R})^f(\mathfrak{a})\\ &= (\mathcal{M},\mathcal{R})(f(x)) \barwedge (\mathcal{M},\mathcal{R})(f(\mathfrak{a}))\\ &= (\mathcal{M},\mathcal{R})(y) \barwedge (\mathcal{M},\mathcal{R})(\mathfrak{b}).\end{aligned}$$

Thus $(\mathcal{M},\mathcal{R})$ is an imprecise subalgebra of $(X,*)_0$. Suppose that $(\mathcal{M},\mathcal{R})^f$ is an imprecise ideal of $(X,*)_0$. Then

$$\begin{aligned}(\mathcal{M},\mathcal{R})(0) &= (\mathcal{M},\mathcal{R})(f(0)) = (\mathcal{M},\mathcal{R})^f(0)\\ &\succeq (\mathcal{M},\mathcal{R})^f(x) = (\mathcal{M},\mathcal{R})(f(x))\\ &= (\mathcal{M},\mathcal{R})(y)\end{aligned}$$

and

$$\begin{aligned}(\mathcal{M},\mathcal{R})(y) &= (\mathcal{M},\mathcal{R})(f(x)) = (\mathcal{M},\mathcal{R})^f(x)\\ &\succeq (\mathcal{M},\mathcal{R})^f(x*\mathfrak{a}) \barwedge (\mathcal{M},\mathcal{R})^f(\mathfrak{a})\\ &= (\mathcal{M},\mathcal{R})(f(x*\mathfrak{a})) \barwedge (\mathcal{M},\mathcal{R})(f(\mathfrak{a}))\\ &= (\mathcal{M},\mathcal{R})(f(x)*f(\mathfrak{a})) \barwedge (\mathcal{M},\mathcal{R})(f(\mathfrak{a}))\\ &= (\mathcal{M},\mathcal{R})(y*\mathfrak{b}) \barwedge (\mathcal{M},\mathcal{R})(\mathfrak{b}).\end{aligned}$$

Thus $(\mathcal{M},\mathcal{R})$ is an imprecise ideal of $(X,*)_0$. □

Theorem 3.32. *Let $f : X \to Y$ be an onto homomorphism from a BCI-algebra $(X,*)_0$ to a BCI-algebra $(Y,*)_0$. Given an imprecise set $(\mathcal{M},\mathcal{R})$ on Y, if the imprecise set $(\mathcal{M},\mathcal{R})^f$ in (3.8) is a closed imprecise ideal of $(X,*)_0$, then $(\mathcal{M},\mathcal{R})$ is a closed imprecise ideal of $(Y,*)_0$.*

Proof. Let $y \in Y$. Then $f(x) = y$ for some $x \in X$ since f is onto. Assume that $(\mathcal{M}, \mathcal{R})^f$ is a closed imprecise ideal of $(X, *)_0$. Then $(\mathcal{M}, \mathcal{R})$ is an imprecise ideal of $(X, *)_0$ (See Theorem 3.31) and

$$\begin{aligned}(\mathcal{M}, \mathcal{R})(0 * y) &= (\mathcal{M}, \mathcal{R})(f(0) * f(x)) = (\mathcal{M}, \mathcal{R})(f(0 * x)) \\ &= (\mathcal{M}, \mathcal{R})^f(0 * x) \succeq (\mathcal{M}, \mathcal{R})^f(x) \\ &= (\mathcal{M}, \mathcal{R})(f(x)) = (\mathcal{M}, \mathcal{R})(y).\end{aligned}$$

Thus $(\mathcal{M}, \mathcal{R})$ is a closed imprecise ideal of $(Y, *)_0$ by Theorem 3.27. □

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