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Abstract. In this paper, we study a numerical approximation of a parabolic system with three components. Under some assumptions, we prove that the solution of a semidiscrete form of above problem blows up at the center point in a finite time and estimate its semidiscrete blow-up time. We show that the semidiscrete simultaneous blow-up occurs. We also establish the convergence of the semidiscrete simultaneous blow-up time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments to illustrate our analysis.

2020 AMS Classification: 35B50, 35K40, 35K45, 65M06

Keywords: Parabolic system, Semidiscretization, Simultaneous blow-up, Numerical blow-up time, Convergence.

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1. Introduction

Consider the following nonlinear parabolic system:

$$u_t(x,t) = u_{xx}(x,t),\ (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

$$v_t(x,t) = v_{xx}(x,t),\ (x,t) \in (0,1) \times (0,T), \tag{1.2}$$

$$w_t(x,t) = w_{xx}(x,t),\ (x,t) \in (0,1) \times (0,T), \tag{1.3}$$

$$-u_x(0,t) = u^{p_{11}}(0,t) + v^{p_{12}}(0,t),\ t \in (0,T), \tag{1.4}$$

$$-v_x(0,t) = v^{p_{22}}(0,t) + w^{p_{23}}(0,t),\ t \in (0,T), \tag{1.5}$$

$$-w_x(0,t) = w^{p_{33}}(0,t) + u^{p_{31}}(0,t),\ t \in (0,T), \tag{1.6}$$

$$u_x(1,t) = u^{p_{11}}(1,t) + v^{p_{12}}(1,t),\ t \in (0,T), \tag{1.7}$$

$$v_x(1,t) = v^{p_{22}}(1,t) + w^{p_{23}}(1,t),\ t \in (0,T), \tag{1.8}$$

$$w_x(1,t) = w^{p_{33}}(1,t) + u^{p_{31}}(1,t),\ t \in (0,T), \tag{1.9}$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) \text{ for each } \ x \in [0,1], \tag{1.10}$$

where p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, $p_{31} \geq 0$; $u_0(x)$, $v_0(x)$ and $w_0(x)$ are positive smooth functions satisfying the compatibility conditions. T represents the maximal existence time of the solutions. The existence and uniqueness of local classical solutions to (1.1)–(1.10) are well known in [1].

Nonlinear parabolic system (1.1)–(1.10) comes from chemical reactions, heat transfer, where $u_0(x)$, $v_0(x)$ and $w_0(x)$ represent the thickness of three kinds of chemical reactants, the temperatures of three different materials during a propagation. Certain specific forms of (1.1)–(1.10) have been widely analysed for critical exponents, blow-up rates, blow-up sets, and even blow-up profiles (See [2, 3, 4, 5, 6, 7]). The study of blow-up and simultaneous blow-up has been the subject of research by several authors such that (See [8, 9, 10, 11, 12]).

In [9], the authors have proved that if ($p_{33} > 1$, $p_{11} \leq 1$, $p_{22} \leq 1$, $p_{33} < p_{23} + 1$, $p_{12} > \dfrac{p_{33}-1}{p_{23}+1-p_{33}}$ and $p_{31} < \dfrac{p_{33}-1}{p_{12}\,(p_{23}+1-p_{33})+1-p_{33}}$) or ($p_{22} > 1$, $p_{11} \leq 1$, $p_{33} \leq 1$, $p_{22} < p_{12}+1$, $p_{31} > \dfrac{p_{22}-1}{p_{12}+1-p_{22}}$ and $p_{23} < \dfrac{p_{22}-1}{p_{31}\,(p_{12}+1-p_{22})+1-p_{22}}$) or ($p_{11} > 1$, $p_{22} \leq 1$, $p_{33} \leq 1$, $p_{11} < p_{31}+1$, $p_{23} > \dfrac{p_{11}-1}{p_{31}+1-p_{11}}$ and $p_{12} < \dfrac{p_{11}-1}{p_{23}\,(p_{31}+1-p_{11})+1-p_{11}}$). Then, for every positive initial data, simultaneous blow-up occurs.

Simultaneous blow-up is defined as $\|u(\cdot,t)\|_\infty < +\infty$, $\|v(\cdot,t)\|_\infty < +\infty$, $\|w(\cdot,t)\|_\infty < +\infty$ and $\lim_{t\to T} \sup \min\{\|u(\cdot,t)\|_\infty, \|v(\cdot,t)\|_\infty, \|w(\cdot,t)\|_\infty\} = +\infty$, where $\|u(\cdot,t)\|_\infty = \max_{0\leq x\leq 1} |u(x,t)|$, $\|v(\cdot,t)\|_\infty = \max_{0\leq x\leq 1} |v(x,t)|$, $\|w(\cdot,t)\|_\infty = \max_{0\leq x\leq 1} |w(x,t)|$.

But these authors do not provide us with exact value of the simultaneous blow-up time of the solution of problem (1.1)–(1.10) although necessary for the work of some engineers. This is why we opt for a numerical study of this problem to determine good approximate values of the simultaneous blow-up time of its solution. In this paper, we are interesting in the numerical study of the above problem. A similar study has been undertaken in [13, 14, 15, 16, 17].

This paper is organized as follows. In the section 2, using the finite difference method we construct a semidiscrete scheme of the continuous problem and we give some properties concerning the semidiscrete scheme. In section 3, under some assumptions, we prove that the solution of a semidiscrete form of the continuous problem blows up in a finte time and estimate its semidiscrete blow-up time. In section 4, we propose the criteria of the simultaneous blow-up of the semidiscrete scheme. In section 5, we show the convergence of the semidiscrete simultaneous blow-up time to the theoretical one when the mesh size goes to zero. Finally, in last section, we give some numerical experiments for a best illustration of our analysis.

2. Properties of the semidiscrete scheme

In this section, we construct a semidiscrete scheme of the continuous problem (1.1)–(1.10). After, we give some properties of this scheme.

Let $I \geqslant 3$ be a positive integer and let $h = \dfrac{1}{I-1}$. Define the grid $x_i = (i-1)h$ with $i = 1, \ldots, I$. Approximate the solution (u, v, w) of (1.1)–(1.10) by the solution $\big(U_h(t) = (U_1(t), \ldots, U_I(t))^T, V_h(t) = (V_1(t), \ldots, V_I(t))^T, W_h(t) = (W_1(t), \ldots, W_I(t))^T\big)$ and approximate the initial data (u_0, v_0, w_0) of the same problem by $\big(\phi_{1,h} = (\phi_{1,1}, \ldots, \phi_{1,I})^T, \phi_{2,h} = (\phi_{2,1}, \ldots, \phi_{2,I})^T, \phi_{3,h} = (\phi_{3,1}, \ldots, \phi_{3,I})^T\big)$ of the following system of ODEs whose is obtained using the finite difference method

$$U_i'(t) = \delta^2 U_i(t) + d_i(U_i^{p_{11}}(t) + V_i^{p_{12}}(t)), \quad i = 1, \ldots, I, \ t \in (0, T_h^b), \tag{2.1}$$

$$V_i'(t) = \delta^2 V_i(t) + d_i(V_i^{p_{22}}(t) + W_i^{p_{23}}(t)), \quad i = 1, \ldots, I, \ t \in (0, T_h^b), \tag{2.2}$$

$$W_i'(t) = \delta^2 W_i(t) + d_i(W_i^{p_{33}}(t) + U_i^{p_{31}}(t)), \quad i = 1, \ldots, I, \ t \in (0, T_h^b), \tag{2.3}$$

$$U_i(0) = \phi_{1,i}, \ V_i(0) = \phi_{2,i}, \ W_i(0) = \phi_{3,i} \quad i = 1, \ldots, I, \tag{2.4}$$

where

$$\delta^2 U_i(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2}, \quad i = 2, \ldots, I-1,$$

$$\delta^2 U_1(t) = \frac{2U_2(t) - 2U_1(t)}{h^2},$$

$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$

$$d_1 = \frac{2}{h}, d_I = \frac{2}{h}, d_i = 0, \quad i = 2, \ldots, I-1.$$

Definition 2.1. We say that $(\underline{U_h}, \underline{V_h}, \underline{W_h}) \in \big(C^1([0, T_h^b), \mathbb{R}^I)\big)^3$ is a *lower solution* of (2.1)–(2.4), if

$$\begin{aligned}
\underline{U_i}'(t) &\leq \delta^2 \underline{U_i}(t) + d_i(\underline{U_i}^{p_{11}}(t) + \underline{V_i}^{p_{12}}(t)), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),\\
\underline{V_i}'(t) &\leq \delta^2 \underline{V_i}(t) + d_i(\underline{V_i}^{p_{22}}(t) + \underline{W_i}^{p_{23}}(t)), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),\\
\underline{W_i}'(t) &\leq \delta^2 \underline{W_i}(t) + d_i(\underline{W_i}^{p_{33}}(t) + \underline{U_i}^{p_{31}}(t)), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),\\
0 < \underline{U_i}(0) \leq \phi_{1,i}, \ 0 < \underline{V_i}(0) &\leq \phi_{2,i}, \ 0 < \underline{W_i}(0) \leq \phi_{3,i} \qquad i = 1, \ldots, I,
\end{aligned}$$

where (U_h, V_h, W_h) is the solution of (2.1)–(2.4). On the other hand, we say that $(\overline{U_h}, \overline{V_h}, \overline{W_h}) \in \big(C^1([0, T_h^b), \mathbb{R}^I)\big)^3$ is an *upper solution* of (2.1)–(2.4), if these inequalities are reversed.

The following lemma is the discrete form of the maximum principle.

Lemma 2.2. *Let* $e_h, c_h, \alpha_h, \beta_h, \lambda_h, \gamma_h \in C^0([0, T_h^b), \mathbb{R}^I)$ *and* $U_h, V_h, W_h \in C^1([0, T_h^b), \mathbb{R}^I)$ *such that*

$$\begin{aligned} U_i'(t) - \delta^2 U_i(t) - e_i(t)U_i(t) - c_i(t)V_i(t) &\geq 0, \quad i = 1, \ldots, I, \ t \in (0, T_h^b), \\ V_i'(t) - \delta^2 V_i(t) - \alpha_i(t)V_i(t) - \beta_i(t)W_i(t) &\geq 0, \quad i = 1, \ldots, I, \ t \in (0, T_h^b), \\ W_i'(t) - \delta^2 W_i(t) - \lambda_i(t)W_i(t) - \gamma_i(t)U_i(t) &\geq 0, \quad i = 1, \ldots, I, \ t \in (0, T_h^b), \\ U_i(0) \geq 0, \ V_i(0) \geq 0, \ W_i(0) &\geq 0, \qquad\quad i = 1, \ldots, I. \end{aligned}$$

Then we have

$$U_i(t) \geq 0, \ V_i(t) \geq 0, \ W_i(t) \geq 0, \quad i = 1, \ldots, I, \ t \in [0, T_h^b).$$

Proof. Let $T_0 < T_h^b$ and let $(N_h(t), M_h(t), K_h(t)) = (e^{\nu t}U_h(t), e^{\nu t}V_h(t), e^{\nu t}W_h(t))$, where ν is a real. We find that $(N_h(t), M_h(t), K_h(t))$ satisfies the following inequalities: for $i = 1, \ldots, I, t \in (0, T_h^b)$,

$$N_i'(t)(t) - \delta^2 N_i(t) - (e_i(t) + \nu)N_i(t) - c_i(t)M_i(t) \geq 0, \tag{2.5}$$

$$M_i'(t) - \delta^2 M_i(t) - (\alpha_i(t) + \nu)M_i(t) - \beta_i(t)K_i(t) \geq 0, \tag{2.6}$$

$$K_i'(t) - \delta^2 K_i(t) - (\lambda_i(t) + \nu)K_i(t) - \gamma_i(t)N_i(t) \geq 0, \tag{2.7}$$

$$N_i(0) \geq 0, \ M_i(0) \geq 0, \ K_i(0) \geq 0. \tag{2.8}$$

Set $m = \min\left\{\min_{1\leq i\leq I, t\in[0,T_0]} N_i(t), \min_{1\leq i\leq I, t\in[0,T_0]} M_i(t), \min_{1\leq i\leq I, t\in[0,T_0]} K_i(t)\right\}$. Since for $i \in \{1, \ldots, I\}$, N_i, M_i and K_i are continuous functions on the compact $[0, T_0]$, we write $m = N_{i_0}(t_{i_0})$ for a certain $i_0 \in \{1, \ldots, I\}$.

Assume $m < 0$ and $\nu < 0$ such that

$$(e_{i_0}(t_{i_0}) + \nu) < 0, \quad (\alpha_{i_0}(t_{i_0}) + \nu) < 0 \quad \textit{and} \quad (\lambda_{i_0}(t_{i_0}) + \nu) < 0.$$

If $t_{i_0} = 0$, then $N_{i_0}(0) < 0$, which contradicts (2.8). Rhus $t_{i_0} \neq 0$.

If $1 \leq i_0 \leq I$, then we have

$$N_{i_0}'(t_{i_0}) = \lim_{k\to 0} \frac{N_{i_0}(t_{i_0}) - N_{i_0}(t_{i_0} - k)}{k} \leq 0 \quad \textit{and} \quad \delta^2 N_{i_0}(t_{i_0}) \geq 0.$$

Furthermore by a straightforward computation, we have

$$N_{i_0}'(t_{i_0}) - \delta^2 N_{i_0}(t_{i_0}) - (e_{i_0}(t_{i_0}) + \nu)N_{i_0}(t_{i_0}) - c_{i_0}(t_{i_0})M_{i_0}(t_{i_0}) < 0,$$

but this inequality contradicts (2.5) and the proof is completed. □

Lemma 2.3. *Let* $(\underline{U_h}, \underline{V_h}, \underline{W_h}), (\overline{U_h}, \overline{V_h}, \overline{W_h}) \in \left(C^1([0, T_h^b), \mathbb{R}^I)\right)^3$ *be lower and upper solutions of* (2.1)–(2.4) *respectively such that*

$$(\underline{U_h}(0), \underline{V_h}(0), \underline{W_h}(0)) \leq (\overline{U_h}(0), \overline{V_h}(0), \overline{W_h}(0)).$$

Then the following inequality holds:

$$(\underline{U_h}(t), \underline{V_h}(t), \underline{W_h}(t)) \leq (\overline{U_h}(t), \overline{V_h}(t), \overline{W_h}(t)).$$

Proof. Let us define $(N_h(t), M_h(t), K_h(t)) = (\overline{U_h}(t), \overline{V_h}(t), \overline{W_h}(t)) - (\underline{U_h}(t), \underline{V_h}(t), \underline{W_h}(t))$. By a straightforward computation and using the Mean value theorem, we obtain for $i = 1, \dots, I$,

$$N_i'(t) - \delta^2 N_i(t) - p_{11} d_i(\alpha_i'(t))^{p_{11}-1} N_i(t) - p_{12} d_i(\lambda_i'(t))^{p_{12}-1} M_i(t) \geq 0, \tag{2.9}$$

$$M_i'(t) - \delta^2 M_i(t) - p_{22} d_i(\lambda_i'(t))^{p_{22}-1} M_i(t) - p_{23} d_i(\gamma_i'(t))^{p_{23}-1} K_i(t) \geq 0, \tag{2.10}$$

$$K_i'(t) - \delta^2 K_i(t) - p_{33} d_i(\gamma_i'(t))^{p_{33}-1} K_i(t) - p_{31} d_i(\alpha_i'(t))^{p_{31}-1} N_i(t) \geq 0, \tag{2.11}$$

$$N_i(0) \geq 0, \ M_i(0) \geq 0, \ K_i(0) \geq 0, \tag{2.12}$$

where $\alpha_i'(t)$, $\lambda_i'(t)$ and $\gamma_i'(t)$ lie, respectively, between $\overline{U_i}(t)$ and $\underline{U_i}(t)$, between $\overline{V_i}(t)$ and $\underline{V_i}(t)$ and between $\overline{W_i}(t)$ and $\underline{W_i}(t)$ for $i \in \{1, \dots, I\}$.
We can rewrite (2.9)–(2.11) as

$$\begin{aligned} N_i'(t) - \delta^2 N_i(t) - e_i(t) N_i(t) - c_i(t) M_i(t) \geq 0, \quad i = 1, \dots, I, \ t \in (0, T_h^b), \\ M_i'(t) - \delta^2 M_i(t) - \alpha_i(t) M_i(t) - \beta_i(t) K_i(t) \geq 0, \quad i = 1, \dots, I, \ t \in (0, T_h^b), \\ K_i'(t) - \delta^2 K_i(t) - \lambda_i(t) K_i(t) - \gamma_i(t) N_i(t) \geq 0, \quad i = 1, \dots, I, \ t \in (0, T_h^b), \end{aligned}$$

where $e_i(t) = p_{11} d_i(\alpha_i'(t))^{p_{11}-1}$, $c_i(t) = p_{12} d_i(\lambda_i'(t))^{p_{12}-1}$, $\alpha_i(t) = p_{22} d_i(\lambda_i'(t))^{p_{22}-1}$, $\beta_i(t) = p_{23} d_i(\gamma_i'(t))^{p_{23}-1}$, $\lambda_i(t) = p_{33} d_i(\gamma_i'(t))^{p_{33}-1}$ and $\gamma_i(t) = p_{31} d_i(\alpha_i'(t))^{p_{31}-1}$, for $i = 1, \dots, I$, $\forall t \in (0, T_h^b)$. According to Lemma 2.2, $N_i(t) \geq 0$, $M_i(t) \geq 0$, $K_i(t) \geq 0$, for $i = 1, \dots, I$, $\forall t \in (0, T_h^b)$ and the proof is completed. □

The next lemma gives the properties of the semidiscrete solution.

Lemma 2.4. *Let $(U_h, V_h, W_h) \in \left(C^1([0, T_h^b), \mathbb{R}^I)\right)^3$ be the solution of (2.1)–(2.4) with an initial data $(\phi_{1,h}, \phi_{2,h}, \phi_{3,h})$ lower solution such that $0 < \phi_{1,i} < \phi_{1,i+1}$, $0 < \phi_{2,i} < \phi_{2,i+1}$ and $0 < \phi_{3,i} < \phi_{3,i+1}$ for $i = 1, \dots, I-1$. Then we have*
(1) $(U_i(t), V_i(t), W_i(t)) \geq (\phi_{1,i}, \phi_{2,i}, \phi_{3,i}) > 0$, $i = 1, \dots, I$, $t \in [0, T_h^b)$,
(2) $(U_{i+1}(t), V_{i+1}(t), W_{i+1}(t)) > (U_i(t), V_i(t), W_i(t))$, $i = 1, \dots, I-1$, $t \in [0, T_h^b)$,
(3) $(U_i'(t), V_i'(t), W_i'(t)) > 0$, $i = 1, \dots, I$, $t \in [0, T_h^b)$.

Proof. (1) Since $(\phi_{1,h}, \phi_{2,h}, \phi_{3,h})$ is a lower solution of (2.1)–(2.4), by the Lemma 2.3, we have

$$(U_i(t), V_i(t), W_i(t)) \geq (\phi_{1,i}, \phi_{2,i}, \phi_{3,i}) > 0, \ i = 1, \dots, I, \ t \in [0, T_h^b).$$

(2) We argue by contradiction. Assume that t_0 is the first $t > 0$ such that $(R_i, F_i, S_i)(t) = (U_{i+1} - U_i, V_{i+1} - V_i, W_{i+1} - W_i)(t) > 0$ for $1 \leqslant i \leqslant I-1$, $t \in [0, t_0)$ but $R_{i_0}(t_0) = U_{i_0+1}(t_0) - U_{i_0}(t_0) = 0$ or $F_{i_0}(t_0) = V_{i_0+1}(t_0) - V_{i_0}(t_0) = 0$ or $S_{i_0}(t_0) = W_{i_0+1}(t_0) - W_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, \dots, I-1\}$. Assume that $R_{i_0}(t_0) = U_{i_0+1}(t_0) - U_{i_0}(t_0) = 0$. Without loss of generality, we suppose that i_0 is

the smallest integer which satisfies the above equality. Then we obtain

$$R_1'(t)=\frac{U_1(t)-2U_2(t)+U_3(t)}{h^2}-\left(\frac{2U_2(t)-2U_1(t)}{h^2}-\frac{2}{h}\left(U_1^{p_{11}}(t)+V_1^{p_{12}}(t)\right)\right),$$

$$R_i'(t)=\frac{U_i(t)-2U_{i+1}(t)+U_{i+2}(t)}{h^2}-\frac{U_{i-1}(t)-2U_i(t)+U_{i+1}(t)}{h^2},\ 2\leq i\leq I-2,$$

$$R_{I-1}'(t)=\frac{2U_{I-1}(t)-2U_I(t)}{h^2}-\frac{U_{I-2}(t)-2U_{I-1}(t)+U_I(t)}{h^2}+\frac{2}{h}\left(U_I^{p_{11}}(t)+V_I^{p_{12}}(t)\right),$$

$$R_1'(t)=\frac{R_2(t)-3R_1(t)}{h^2}+\frac{2}{h}\left(U_1^{p_{11}}(t)+V_1^{p_{12}}(t)\right), \tag{2.13}$$

$$R_i'(t)=\frac{R_{i-1}(t)-2R_i(t)+R_{i+1}(t)}{h^2},\ 2\leq i\leq I-2, \tag{2.14}$$

$$R_{I-1}'(t)=\frac{R_{I-2}(t)-3R_{I-1}(t)}{h^2}+\frac{2}{h}\left(U_I^{p_{11}}(t)+V_I^{p_{12}}(t)\right). \tag{2.15}$$

According to the hypotheses on t_0, we have the following inequalities

$$R_{i_0}'(t_0)=\lim_{\epsilon\to 0}\frac{R_{i_0}(t_0)-R_{i_0}(t_0-\epsilon)}{\epsilon}\leq 0,$$

$$\delta^2R_{i_0}(t_0)=\frac{R_{i_0-1}(t_0)-2R_{i_0}(t_0)+R_{i_0+1}(t_0)}{h^2}>0,\ \text{if}\quad 2\leq i_0\leq I-2,$$

$$\frac{R_{i_0+1}(t_0)-3R_{i_0}(t_0)}{h^2}>0 \text{ if } i_0=1,$$

$$\frac{-3R_{i_0}(t_0)+R_{i_0-1}(t_0)}{h^2}>0 \text{ if } i_0=I-1,$$

which implies

$$R_{i_0}'(t_0)-\frac{R_{i_0-1}(t_0)-2R_{i_0}(t_0)+R_{i_0+1}(t_0)}{h^2}<0,\ \text{if}\quad 1\leq i_0\leq I-1,$$

$$R_{i_0}'(t_0)-\frac{R_{i_0+1}(t_0)-3R_{i_0}(t_0)}{h^2}-\frac{2}{h}\left(U_{i_0}^{p_{11}}(t)+V_{i_0}^{p_{12}}(t)\right)<0,\quad\text{if}\quad i_0=1,$$

$$R_{i_0}'(t_0)-\frac{-3R_{i_0}(t_0)+R_{i_0-1}(t_0)}{h^2}-\frac{2}{h}\left(U_{i_0}^{p_{11}}(t)+V_{i_0}^{p_{12}}(t)\right)<0,\quad\text{if}\quad i_0=I-1.$$

Thus we have a contradiction with (2.13)–(2.15), which leads to the desired result.

(3) Denote $J_i(t)=U_i(t+\zeta)-U_i(t)$, $L_i(t)=V_i(t+\zeta)-V_i(t)$ and $G_i(t)=W_i(t+\zeta)-W_i(t)$ for $i=1,\ldots,I$ and $t\in[0,T_h^b)$. Using (1) and (2.4), we show that $J_i(0)\geq 0$, $L_i(0)\geq 0$ and $G_i(0)\geq 0$ for $i=1,\ldots,I$. It is not hard to see thanks to proof of lemma 2.3 that

$$J_i'(t)-\delta^2J_i(t)-e_i'(t)J_i(t)-c_i'(t)L_i(t)\geq 0,\quad i=1,\ldots,I,\quad t\in[0,T_h^b),$$

$$L_i'(t)-\delta^2L_i(t)-\alpha_i'(t)L_i(t)-\beta_i'(t)G_i(t)\geq 0,\quad i=1,\ldots,I,\quad t\in[0,T_h^b),$$

$$G_i'(t)-\delta^2G_i(t)-\lambda_i'(t)G_i(t)-\gamma_i'(t)J_i(t)\geq 0,\quad i=1,\ldots,I,\quad t\in[0,T_h^b),$$

where $e_i'(t) = p_{11}d_i(\alpha_i^{''}(t))^{p_{11}-1}$, $c_i'(t) = p_{12}d_i(\lambda_i^{''}(t))^{p_{12}-1}$, $\alpha_i^{'}(t) = p_{22}d_i(\lambda_i^{''}(t))^{p_{22}-1}$, $\beta_i^{'}(t) = p_{23}d_i(\gamma_i^{''}(t))^{p_{23}-1}$, $\lambda_i^{'}(t) = p_{33}d_i(\gamma_i^{''}(t))^{p_{33}-1}$ and $\gamma_i^{'}(t) = p_{31}d_i(\alpha_i^{''}(t))^{p_{31}-1}$. We have $\alpha_i^{''}(t)$, $\lambda_i^{''}(t)$ and $\gamma_i^{''}(t)$ lie, respectively, between $U_i(t+\zeta)$ and $U_i(t)$, between $V_i(t+\zeta)$ and $V_i(t)$, between $W_i(t+\zeta)$ and $W_i(t)$, for $i = 1, \dots, I$, $\forall t \in [0, T_h^b)$. It follows from lemma 2.2 that $J_i(t) \geq 0$, $L_i(t) \geq 0$ and $G_i(t) \geq 0$ for $i = 1, \dots, I$, $t \in [0, T_h^b)$. This implies that U_i, V_i and W_i increase in time. This ends the proof. □

3. Semidiscrete blow-up solution

In this section under some assumptions, we show that the solution (U_h, V_h, W_h) of (2.1)–(2.4) blows up in a finite time.

Definition 3.1. We say that the solution (U_h, V_h, W_h) of (2.1)–(2.4) *blows up in a finite time*, if there exist a strictly positive finite time T_h^b such that for $t \in [0, T_h^b)$,

$$\max\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty, \|W_h(t)\|_\infty\} < +\infty$$

and

$$\lim_{t\to T_h^b} (\|U_h(t)\|_\infty + \|V_h(t)\|_\infty + \|W_h(t)\|_\infty) = +\infty,$$

where $\|U_h(t)\|_\infty = \max_{1\leq i\leq I} |U_i(t)|$, $\|V_h(t)\|_\infty = \max_{1\leq i\leq I} |V_i(t)|$, $\|W_h(t)\|_\infty = \max_{1\leq i\leq I} |W_i(t)|$. In this case, T_h^b is called the *blow-up time* of the solution (U_h, V_h, W_h).

In the continuation, we have the conditions of global existence of the solution of the semi-dicrete problem.

Theorem 3.2. *The solution* (U_h, V_h, W_h) *of the system* (2.1)–(2.4) *exists globally, if*

$$\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} \leq 1.$$

Proof. Suppose $\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} \leq 1$. Let $\overline{U}_h$, $\overline{V}_h$ and $\overline{W}_h$ defined by: for $i = 1, \dots, I,\ t \in (0, T_h^b)$,

$$\overline{U}_i(t) = A\mathrm{e}^{m_1 t+(i-1)n_1 h}, \overline{V}_i(t) = B\mathrm{e}^{m_2 t+(i-1)n_2 h}, \overline{W}_i(t) = L\mathrm{e}^{m_3 t+(i-1)n_3 h},$$

where A, B, L, m_j, n_j $(j = 1, 2, 3)$ are positive constants satisfying:

$$A \geq \max_{1\leq i\leq I} |U_i(0)|,\ B \geq \max_{1\leq i\leq I} |V_i(0)|,\ L \geq \max_{1\leq i\leq I} |W_i(0)|,$$

$$\begin{cases} m_1 = m_2 p_{12} = m_3 p_{23} p_{12} \geq \max\{Y, J, X\} \\ n_1 = n_2 p_{12} = n_3 p_{23} p_{12} \end{cases}$$

and

$$\begin{cases} m_2 = m_3 p_{23} \\ n_2 = n_3 p_{23}, \end{cases}$$

where
$Y = \frac{2}{h^2}\left(\mathrm{e}^{n_1 h} - 1 + h\left(A^{p_{11}} + B^{p_{12}}\right)/A\right)$, $J = \frac{2}{h^2}\left(\mathrm{e}^{n_2 h} - 1 + h\left(B^{p_{22}} + F^{p_{23}}\right)/B\right)$,

$X = \frac{2}{h^2}\left(\mathrm{e}^{n_3 h} - 1 + h\left(L^{p_{33}} + A^{p_{31}}\right)/L\right).$

Since $p_{23}p_{12}p_{31} \leq 1$, it is not hard to see that for $i = 1, \ldots, I$,

$$\mathrm{e}^{m_1 t+(i-1)n_1 h} = \mathrm{e}^{m_2 p_{12} t+(i-1)n_2 p_{12} h} = \mathrm{e}^{m_3 p_{23} p_{12} t+(i-1)n_3 p_{23} p_{12} h},$$

$$\mathrm{e}^{m_2 t+(i-1)n_2 h} = \mathrm{e}^{m_3 p_{23} t+(i-1)n_3 p_{23} h},$$

$$\mathrm{e}^{m_3 t+(i-1)n_3 h} \geq \mathrm{e}^{m_3 p_{23} p_{12} p_{31} t+(i-1)n_3 p_{23} p_{12} p_{31} h} = \mathrm{e}^{m_1 p_{31} t+(i-1)n_1 p_{31} h}.$$

Since $p_{11} \leq 1$, we have

$$A^{p_{11}} + B^{p_{12}} \geq A^{p_{11}}\mathrm{e}^{m_1(p_{11}-1)t+(i-1)n_1(p_{11}-1)h} + B^{p_{12}},$$

$$2\frac{\left(A^{p_{11}} + B^{p_{12}}\right)/A}{h}\overline{U}_i(t) \geq 2\frac{\left(A^{p_{11}}\mathrm{e}^{m_1(p_{11}-1)t+(i-1)n_1(p_{11}-1)h} + B^{p_{12}}\right)/A}{h}\overline{U}_i(t),$$

$$2\frac{\left(A^{p_{11}} + B^{p_{12}}\right)/A}{h}\overline{U}_i(t)$$
$$\geq \frac{\left(2A^{p_{11}}\mathrm{e}^{m_1 p_{11} t+(i-1)n_1 p_{11} h} + 2B^{p_{12}}\mathrm{e}^{m_1 t+(i-1)n_1 h}\right)/A}{h}\mathrm{e}^{-m_1 t-(i-1)n_1 h}\overline{U}_i(t),$$

$$2\frac{\left(A^{p_{11}} + B^{p_{12}}\right)/A}{h}\overline{U}_i(t) \geq \frac{2A^{p_{11}}\mathrm{e}^{m_1 p_{11} t+(i-1)n_1 p_{11} h} + 2B^{p_{12}}\mathrm{e}^{m_1 t+(i-1)n_1 h}}{h},$$

$$2\frac{\left(A^{p_{11}} + B^{p_{12}}\right)/A}{h}\overline{U}_i(t) \geq \frac{2A^{p_{11}}\mathrm{e}^{m_1 p_{11} t+(i-1)n_1 p_{11} h} + 2B^{p_{12}}\mathrm{e}^{m_2 p_{12} t+(i-1)n_2 p_{12} h}}{h}.$$

Then we have

$$2\frac{\left(A^{p_{11}} + B^{p_{12}}\right)/A}{h}\overline{U}_i(t) \geq \frac{2\overline{U}_i^{p_{11}}(t) + 2\overline{V}_i^{p_{12}}(t)}{h}. \tag{3.1}$$

By the same way, we prove that for $i = 1, \ldots, I$,

$$2\frac{\left(B^{p_{22}} + L^{p_{23}}\right)/B}{h}\overline{V}_i(t) \geq \frac{2\overline{V}_i^{p_{22}}(t) + 2\overline{W}_i^{p_{23}}(t)}{h} \tag{3.2}$$

and

$$2\frac{\left(L^{p_{33}} + A^{p_{31}}\right)/L}{h}\overline{W}_i(t) \geq \frac{2\overline{W}_i^{p_{33}}(t) + 2\overline{U}_i^{p_{31}}(t)}{h}. \tag{3.3}$$

Thus we have

$$\delta^2\overline{U}_1(t) = \frac{2\overline{U}_2(t) - 2\overline{U}_1(t)}{h^2},\ t \in (0, T_h^b),$$

$$\delta^2\overline{U}_i(t) = \frac{\overline{U}_{i-1}(t) - 2\overline{U}_i(t) + \overline{U}_{i+1}(t)}{h^2},\ 2 \leq i \leq I-1,\ t \in (0, T_h^b),$$

$$\delta^2\overline{U}_I(t) = \frac{2\overline{U}_{I-1}(t) - 2\overline{U}_I(t)}{h^2},\ t \in (0, T_h^b),$$

$$\delta^2\overline{U}_1(t) = \frac{2A\mathrm{e}^{m_1 t+(1+1-1)n_1 h} - 2A\mathrm{e}^{m_1 t+(1-1)n_1 h}}{h^2},\ t \in (0, T_h^b),$$

$$\delta^2\overline{U}_i(t) = \frac{A\mathrm{e}^{m_1 t+(i-1-1)n_1 h} - 2A\mathrm{e}^{m_1 t+(i-1)n_1 h} + A\mathrm{e}^{m_1 t+(i+1-1)n_1 h}}{h^2},$$

$2 \le i \le I-1,\ t \in (0, T_h^b),$

$$\delta^2 \overline{U}_I(t) = \frac{2A\mathrm{e}^{m_1 t+(I-1-1)n_1 h} - 2A\mathrm{e}^{m_1 t+(I-1)n_1 h}}{h^2},\ t \in (0, T_h^b).$$

$$\begin{aligned}
\delta^2 \overline{U}_1(t) &= 2A\mathrm{e}^{m_1 t+(1-1)n_1 h}\frac{\mathrm{e}^{n_1 h}-1}{h^2},\ t \in (0, T_h^b),\\
\delta^2 \overline{U}_i(t) &= 2A\mathrm{e}^{m_1 t+(i-1)n_1 h}\frac{\mathrm{e}^{-n_1 h}-2+\mathrm{e}^{n_1 h}}{h^2},\ 2 \le i \le I-1,\ t \in (0, T_h^b),\\
\delta^2 \overline{U}_I(t) &= 2A\mathrm{e}^{m_1 t+(I-1)n_1 h}\frac{\mathrm{e}^{-n_1 h}-1}{h^2},\ t \in (0, T_h^b),
\end{aligned}$$

$$\begin{aligned}
\delta^2 \overline{U}_1(t) &= 2\frac{\mathrm{e}^{n_1 h}-1}{h^2}\overline{U}_1(t),\ t \in (0, T_h^b),\\
\delta^2 \overline{U}_i(t) &= 2\frac{\cosh(n_1 h)-1}{h^2}\overline{U}_i(t),\ 2 \le i \le I-1,\ t \in (0, T_h^b),\\
\delta^2 \overline{U}_I(t) &= 2\frac{\mathrm{e}^{-n_1 h}-1}{h^2}\overline{U}_I(t),\ t \in (0, T_h^b).
\end{aligned}$$

By the same way, we also prove that

$$\begin{aligned}
\delta^2 \overline{V}_1(t) &= 2\frac{\mathrm{e}^{n_2 h}-1}{h^2}\overline{V}_1(t),\ t \in (0, T_h^b),\\
\delta^2 \overline{V}_i(t) &= 2\frac{\cosh(n_2 h)-1}{h^2}\overline{V}_i(t),\ 2 \le i \le I-1,\ t \in (0, T_h^b),\\
\delta^2 \overline{V}_I(t) &= 2\frac{\mathrm{e}^{-n_2 h}-1}{h^2}\overline{V}_I(t),\ t \in (0, T_h^b).
\end{aligned}$$

And

$$\begin{aligned}
\delta^2 \overline{W}_1(t) &= 2\frac{\mathrm{e}^{n_3 h}-1}{h^2}\overline{W}_1(t),\ t \in (0, T_h^b),\\
\delta^2 \overline{W}_i(t) &= 2\frac{\cosh(n_3 h)-1}{h^2}\overline{W}_i(t),\ 2 \le i \le I-1,\ t \in (0, T_h^b),\\
\delta^2 \overline{W}_I(t) &= 2\frac{\mathrm{e}^{-n_3 h}-1}{h^2}\overline{W}_I(t),\ t \in (0, T_h^b).
\end{aligned}$$

For $i = 1, \ldots, I$,

$$\begin{cases}
\overline{U_i}'(t) = m_1 A\mathrm{e}^{m_1 t+(i-1)n_1 h},\ t \in (0, T_h^b),\\
\overline{V_i}'(t) = m_2 B\mathrm{e}^{m_2 t+(i-1)n_2 h},\ t \in (0, T_h^b),\\
\overline{W_i}'(t) = m_3 L\mathrm{e}^{m_3 t+(i-1)n_3 h},\ t \in (0, T_h^b),
\end{cases}$$

which implies that for $i = 1, \ldots, I$,

$$\overline{U_i}'(t) = m_1 \overline{U}_i(t), \quad \overline{V_i}'(t) = m_2 \overline{V}_i(t), \quad \overline{W_i}'(t) = m_3 \overline{W}_i(t), \quad t \in (0, T_h^b).$$

So we have, for $i = 1, \ldots, I$,

$$\overline{U_i}'(t) \ge Y\overline{U}_i(t), \quad \overline{V_i}'(t) \ge J\overline{V}_i(t), \quad \overline{W_i}'(t) \ge X\overline{W}_i(t), \quad t \in (0, T_h^b),$$

which implies that for $i = 1, \ldots, I$,

$$\overline{U_i}'(t) \geq \frac{2}{h^2}\left(e^{n_1 h} - 1 + h\left(A^{p_{11}} + B^{p_{12}}\right)/A\right)\overline{U}_i(t), \quad t \in (0, T_h^b),$$
$$\overline{V_i}'(t) \geq \frac{2}{h^2}\left(e^{n_2 h} - 1 + h\left(B^{p_{22}} + L^{p_{23}}\right)/B\right)\overline{V}_i(t), \quad t \in (0, T_h^b),$$
$$\overline{W_i}'(t) \geq \frac{2}{h^2}\left(e^{n_3 h} - 1 + h\left(L^{p_{33}} + A^{p_{31}}\right)/L\right)\overline{W}_i(t), \quad t \in (0, T_h^b),$$

which implies that for $i = 1, \ldots, I$,

$$\overline{U_i}'(t) \geq \frac{2}{h^2}\left(e^{n_1 h} - 1\right)\overline{U}_i(t) + \frac{2}{h}\left(\left(A^{p_{11}} + B^{p_{12}}\right)/A\right)\overline{U}_i(t), \quad t \in (0, T_h^b),$$
$$\overline{V_i}'(t) \geq \frac{2}{h^2}\left(e^{n_2 h} - 1\right)\overline{V}_i(t) + \frac{2}{h}\left(\left(B^{p_{22}} + L^{p_{23}}\right)/B\right)\overline{V}_i(t), \quad t \in (0, T_h^b),$$
$$\overline{W_i}'(t) \geq \frac{2}{h^2}\left(e^{n_3 h} - 1\right)\overline{W}_i(t) + \frac{2}{h}\left(\left(L^{p_{33}} + A^{p_{31}}\right)/L\right)\overline{W}_i(t), \quad t \in (0, T_h^b).$$

As

$$\begin{cases} e^{n_1 h} - 1 \geq \cosh(n_1 h) - 1 \geq e^{-n_1 h} - 1, \\ e^{n_2 h} - 1 \geq \cosh(n_2 h) - 1 \geq e^{-n_2 h} - 1, \\ e^{n_3 h} - 1 \geq \cosh(n_3 h) - 1 \geq e^{-n_3 h} - 1, \end{cases}$$

from (3.1), (3.2) and (3.3), we obtain

$$\overline{U_i}'(t) \geq \delta^2 \overline{U}_i(t) + \frac{2}{h}\left(\overline{U}_i^{p_{11}}(t) + \overline{V}_i^{p_{12}}(t)\right), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),$$
$$\overline{V_i}'(t) \geq \delta^2 \overline{V}_i(t) + \frac{2}{h}\left(\overline{V}_i^{p_{22}}(t) + \overline{W}_i^{p_{23}}(t)\right), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),$$
$$\overline{W_i}'(t) \geq \delta^2 \overline{W}_i(t) + \frac{2}{h}\left(\overline{W}_i^{p_{33}}(t) + \overline{U}_i^{p_{31}}(t)\right), \quad i = 1, \ldots, I, \ t \in (0, T_h^b).$$

Hence we have the following inequalities:

$$\overline{U_i}'(t) \geq \delta^2 \overline{U}_i(t) + d_i\left(\overline{U}_i^{p_{11}}(t) + \overline{V}_i^{p_{12}}(t)\right), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),$$
$$\overline{V_i}'(t) \geq \delta^2 \overline{V}_i(t) + d_i\left(\overline{V}_i^{p_{22}}(t) + \overline{W}_i^{p_{23}}(t)\right), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),$$
$$\overline{W_i}'(t) \geq \delta^2 \overline{W}_i(t) + d_i\left(\overline{W}_i^{p_{33}}(t) + \overline{U}_i^{p_{31}}(t)\right), \quad i = 1, \ldots, I, \ t \in (0, T_h^b),$$
$$\overline{U}_i(0) \geq U_i(0), \quad \overline{V}_i(0) \geq V_i(0), \quad \overline{W}_i(0) \geq W_i(0), \qquad i = 1, \ldots, I.$$

From lemma 2.3, we can conclude that $(\overline{U_h}, \overline{V_h}, \overline{W_h})$ is an upper solution of (2.1)–(2.4). Therefore the global existence of the solution of (2.1)–(2.4) and the proof is complete. □

Theorem 3.3. *Let (U_h, V_h, W_h) be the solution of the semidiscrete problem (2.1)–(2.4). If $p_{11},\ p_{22},\ p_{33} > 1$ and there exists a real $\eta \in (0, 1)$ such that*

(3.4) $$\delta^2 \phi_{1,i} + d_i \phi_{1,i}^{p_{11}} + d_i \phi_{2,i}^{p_{12}} \geq \eta(d_i \phi_{1,i}^{p_{11}} + d_i \phi_{2,i}^{p_{12}}), \quad i = 1, \ldots, I,$$

(3.5) $$\delta^2 \phi_{2,i} + d_i \phi_{2,i}^{p_{22}} + d_i \phi_{3,i}^{p_{23}} \geq \eta(d_i \phi_{2,i}^{p_{22}} + d_i \phi_{3,i}^{p_{23}}), \quad i = 1, \ldots, I,$$

(3.6) $$\delta^2 \phi_{3,i} + d_i \phi_{3,i}^{p_{33}} + d_i \phi_{1,i}^{p_{31}} \geq \eta(d_i \phi_{3,i}^{p_{33}} + d_i \phi_{1,i}^{p_{31}}), \quad i = 1, \ldots, I,$$

then (U_h, V_h, W_h) *blows up in a finite time* T_h^b.

Proof. We only demonstrate case where $p_{11} > 1$.
Define the functions $q_h(t)$, $s_h(t)$ and $z_h(t)$ such that

$$\begin{aligned}
&q_h(t) = U_h'(t) - \eta\left((U_h(t))^{p_{11}} + (V_h(t))^{p_{12}}\right),\ t \in (0, T_h^b),\\
&s_h(t) = V_h'(t) - \eta\left((V_h(t))^{p_{22}} + (W_h(t))^{p_{23}}\right),\ t \in (0, T_h^b),\\
&z_h(t) = W_h'(t) - \eta\left((W_h(t))^{p_{33}} + (U_h(t))^{p_{31}}\right),\ t \in (0, T_h^b).
\end{aligned}$$

With a straightforward calculation, we get: $i = 1, \ldots, I,\ t \in (0, T_h^b)$,

$$\begin{aligned}
&q_i'(t) - \delta^2 q_i(t) - p_{11} d_i (U_i(t))^{p_{11}-1} q_i(t) - p_{12} d_i (V_i(t))^{p_{12}-1} s_i(t) \geq 0,\\
&s_i'(t) - \delta^2 s_i(t) - p_{22} d_i (V_i(t))^{p_{22}-1} s_i(t) - p_{23} d_i (W_i(t))^{p_{23}-1} z_i(t) \geq 0,\\
&z_i'(t) - \delta^2 z_i(t) - p_{33} d_i (W_i(t))^{p_{33}-1} z_i(t) - p_{31} d_i (U_i(t))^{p_{31}-1} q_i(t) \geq 0.
\end{aligned}$$

Thanks to lemma 2.2, we get $q_h(t) \geq 0$, $s_h(t) \geq 0$ and $z_h(t) \geq 0$.
Assume $\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} = p_{11} > 1$, since $q_h(t) \geq 0$, we obtain the following inequality

$$\frac{d\|U_h(t)\|_\infty}{dt} \geq \eta \|U_h(t)\|_\infty^{p_{11}}. \tag{3.7}$$

Then U_h blows up in a finite time T_h^b and integrating (3.7) from 0 to T_h^b, we prove that $T_h^b \leq \dfrac{\|\phi_{1,h}\|_\infty^{1-p_{11}}}{\eta(p_{11}-1)}$.
Analogously, we show that V_h and W_h blow up in a finite time T_h^b if $p_{22} > 1$ and $p_{33} > 1$. Thus we get $T_h^b \leq \dfrac{\|\phi_{2,h}\|_\infty^{1-p_{22}}}{\eta(p_{22}-1)}$ and $T_h^b \leq \dfrac{\|\phi_{3,h}\|_\infty^{1-p_{33}}}{\eta(p_{33}-1)}$.
So if $p_{11} > 1$, $p_{22} > 1$ and $p_{33} > 1$, then (U_h, V_h, W_h) blows up in a finite time T_h^b and we have

$$T_h^b \leq \min\left\{\frac{\|\phi_{1,h}\|_\infty^{1-p_{11}}}{\eta(p_{11}-1)}, \frac{\|\phi_{2,h}\|_\infty^{1-p_{22}}}{\eta(p_{22}-1)}, \frac{\|\phi_{3,h}\|_\infty^{1-p_{33}}}{\eta(p_{33}-1)}\right\}.$$

Now, we prove that (U_h, V_h, W_h) blows up in a finite time when $p_{12}\ p_{23}\ p_{31} > 1$.
Assume that $\max\{p_{11},\ p_{22},\ p_{33},\ p_{12}p_{23}p_{31}\} = p_{12}p_{23}p_{31} > 1$.
We may assume without loss of generality that $\inf_{1 \leq i \leq I} \phi_{j,i} \geq c > 0$ $(j = 1, 2, 3)$.
Let $(\theta_1, \theta_2, \theta_3)^T$ be the solution of

$$\begin{pmatrix} 1 & -p_{12} & 0 \\ 0 & 1 & -p_{23} \\ -p_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and denote $T_0 = R\rho^{-1} - 1$, $\underline{U_i}(t) = (R - \rho - \rho t)^{-\theta_1}$, $\underline{V_i}(t) = (R - \rho - \rho t)^{-\theta_2}$, $\underline{W_i}(t) = (R - \rho - \rho t)^{-\theta_3}$ for $i = 1, \ldots, I$, $t \in [0, T_0]$ (R, ρ are positive constants to be determined). A simple computation shows for $i = 1, \ldots, I,\ t \in [0, T_0]$. Denote $-\theta_2 p_{12} = -\theta_1 - 1$, $-\theta_3 p_{23} = -\theta_2 - 1$ and $-\theta_1 p_{31} = -\theta_3 - 1$. Then we have

$\underline{U_i}'(t) - \delta^2 \underline{U_i}(t) - d_i(\underline{U_i}^{p_{11}}(t) + \underline{V_i}^{p_{12}}(t))$

$$= (\theta_1\rho - \frac{2}{h})(R-\rho-\rho t)^{-\theta_2 p_{12}} - \frac{2}{h}(R-\rho-\rho t)^{-\theta_1 p_{11}},$$
$$\underline{V_i}'(t) - \delta^2\underline{V_i}(t) - d_i(\underline{V_i}^{p_{22}}(t) + \underline{W_i}^{p_{23}}(t))$$
$$= (\theta_2\rho - \frac{2}{h})(R-\rho-\rho t)^{-\theta_3 p_{23}} - \frac{2}{h}(R-\rho-\rho t)^{-\theta_2 p_{22}},$$
$$\underline{W_i}'(t) - \delta^2\underline{W_i}(t) - d_i(\underline{W_i}_i^{p_{33}}(t) + \underline{U_i}^{p_{31}}(t))$$
$$= (\theta_3\rho - \frac{2}{h})(R-\rho-\rho t)^{-\theta_3 p_{31}} - \frac{2}{h}(R-\rho-\rho t)^{-\theta_3 p_{33}},$$

$$\underline{U_i}(0) = (R-\rho)^{-\theta_1},\ \underline{V_i}(0) = (R-\rho)^{-\theta_2},\ \underline{W_i}(0) = (R-\rho)^{-\theta_3}.$$

Take ρ to be so small $\theta_j\rho \le \frac{2}{h}$ and R to be so large such that $R \ge \rho + c^{-\frac{1}{\theta_j}}$, $(j = 1,2,3)$, we have

$$\underline{U_i}'(t) - \delta^2\underline{U_i}(t) - d_i(\underline{U_i}^{p_{11}}(t) + \underline{V_i}^{p_{12}}(t)) \le 0,$$
$$\underline{V_i}'(t) - \delta^2\underline{V_i}(t) - d_i(\underline{V_i}^{p_{22}}(t) + \underline{W_i}^{p_{23}}(t)) \le 0,$$
$$\underline{W_i}'(t) - \delta^2\underline{W_i}(t) - d_i(\underline{W_i}_i^{p_{33}}(t) + \underline{U_i}^{p_{31}}(t)) \le 0,$$

$$0 < \underline{U_i}(0) \le (R-\rho)^{-\theta_1},\ 0 < \underline{V_i}(0) \le (R-\rho)^{-\theta_2},\ 0 < \underline{W_i}(0) \le (R-\rho)^{-\theta_3}.$$

Thus $\left(\underline{U_h}, \underline{V_h}, \underline{W_h}\right)$ is a lower solution of (2.1)–(2.4). Since $\max_{t\to T_0}(\underline{U_i}(t), \underline{V_i}(t), \underline{W_i}(t)) \to +\infty$. So $\left(\underline{U_h}, \underline{V_h}, \underline{W_h}\right)$ blows up. This ends the proof. □

Remark 3.4. With an integration of the inequality (3.7) over (t, T_h^b), we have

$$\frac{1}{p_{11}-1}\frac{1}{\|U_h(t)\|_\infty^{p_{11}-1}} \ge \eta\left(T_h^b - t\right)$$

and there exists a constant $\kappa_{11} > 0$ such that

$$\|U_h(t)\|_\infty \le \kappa_{11}\left(T_h^b - t\right)^{-\frac{1}{p_{11}-1}}, t \in (0, T_h^b),$$

where $p_{11} > 1$.
For $p_{22} > 1$ and $p_{33} > 1$, there exists constants $\kappa_{22} > 0$ and $\kappa_{33} > 0$ such that

$$\|V_h(t)\|_\infty \le \kappa_{22}\left(T_h^b - t\right)^{-\frac{1}{p_{22}-1}}, t \in (0, T_h^b),$$

where $p_{22} > 1$.

$$\|W_h(t)\|_\infty \le \kappa_{33}\left(T_h^b - t\right)^{-\frac{1}{p_{33}-1}}, t \in (0, T_h^b),$$

where $p_{33} > 1$.

4. Simultaneous blow-up

We identify simultaneous blow-up in this section. We consider (U_h, V_h, W_h) the solution of (2.1)–(2.4).

Definition 4.1. We say that the solution (U_h, V_h, W_h) of (2.1)–(2.4) *blows up simultaneously in a finite time*, if there exist a finite time $T_h^b > 0$ such that for $t \in [0, T_h^b)$,

$$\max\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty, \|W_h(t)\|_\infty\} < +\infty$$

and

$$\lim\sup_{t \to T_h^b} \min\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty, \|W_h(t)\|_\infty\} = +\infty,$$

where $\|U_h(t)\|_\infty = \max_{1 \le i \le I} |U_i(t)|$, $\|V_h(t)\|_\infty = \max_{1 \le i \le I} |V_i(t)|$, $\|W_h(t)\|_\infty = \max_{1 \le i \le I} |W_i(t)|$, $t \in [0, T_h^b)$.
In this case, T_h^b is called the *simultaneous blow-up time* of the solution (U_h, V_h, W_h).

In this subsection, we give sufficient conditions for the existence simultaneous blow-up.

Theorem 4.2. *Assume* $p_{33} > 1$, $p_{11} \le 1$, $p_{22} \le 1$, $p_{33} < p_{23}+1$, $p_{12} > \dfrac{p_{33} - 1}{p_{23} + 1 - p_{33}}$ *and*

$$p_{31} < \frac{p_{33} - 1}{p_{12}\left(p_{23} + 1 - p_{33}\right) + 1 - p_{33}}.$$

Then for every positive initial data, simultaneous blow-up occurs.

Proof. This proof consists of two steps.
Step 1. V_h and W_h blow up simultaneously at T_h^b.
First, assume that W_h remains bounded up to time T_h^b. Since $p_{11}, p_{22} \le 1$, U_h and V_h also remain bounded. This is a contradiction to the blow-up property of solution (U_h, V_h, W_h) for $p_{33} > 1$.
Next, assume that V_h remains bounded up to time T_h^b. Since $p_{11} \le 1$, there exists $M > 0$ such that $U_i(t) \le M$ for $i = 1, \dots, I$, $\forall t \in (0, T_h^b)$. Then W_h satisfies

$$W_I'(t) \le \frac{2}{h}\left(W_I^{p_{33}}(t) + M^{p_{31}}\right) \ \forall t \in (0, T_h^b),$$

because $\forall t \in (0, T_h^b)$, $W_i(t) \le W_{i+1}(t)$ for $i = 1, \dots, I$ (Lemma 2.4),
which implies that

$$\frac{W_I'(t)}{W_I^{p_{33}}(t)} \le \frac{2}{h}\left(1 + \frac{M^{p_{31}}}{W_I^{p_{33}}(t)}\right) \ \forall t \in (0, T_h^b),$$

which implies that

$$\frac{W_I'(t)}{W_I^{p_{33}}(t)} \le \frac{2}{h}\left(1 + M^{p_{31}}\right), \ \forall t \in (0, T_h^b), \ \textit{because}\left(W_I(t) \ge 1\right).$$

Integrating this inequality from t to T_h^b, we obtain

$$W_I(t) \ge c\left(T_h^b - t\right)^{-\dfrac{1}{p_{33} - 1}}, \ t \in [0, T_h^b), \tag{4.1}$$

where $c = \left(\frac{2}{h}\left(1+M^{p_{31}}\right)(p_{33}-1)\right)^{-\frac{1}{p_{33}-1}}$.
From (2.2) and (4.1), for $t \in (0, T_h^b)$, we have

$$V_I'(t) \geq \frac{2V_{I-1}(t) - 2V_I(t)}{h^2} + \frac{2}{h}\left(V_I^{p_{22}}(t) + c^{p_{23}}\left(T_b^h - t\right)^{-\frac{p_{23}}{p_{33}-1}}\right), \tag{4.2}$$

as V_h is bounded. Thus there exists a constant N_1 dependent of h such that

$$\begin{aligned} V_I'(t) &\geq N_1 + \frac{2c^{p_{23}}}{h}\left(T_h^b - t\right)^{-\frac{p_{23}}{p_{33}-1}},\ t \in (0, T_h^b), \\ V_I'(t) &\geq N_1 + N_2\left(T_h^b - t\right)^{-\frac{p_{23}}{p_{33}-1}},\ t \in (0, T_h^b), \end{aligned}$$

with $N_2 = \frac{2c^{p_{23}}}{h}$.
Integrating this inequality from t_0 to T_h^b, we have

$$V_I(T_h^b) \geq V_I(t_0) + N_1\left(T_h^b - t_0\right) + N_2\int_{t_0}^{T_h^b}\left(T_h^b - t\right)^{-\frac{p_{23}}{p_{33}-1}} dt.$$

The boundedness of V_h requires $p_{23}+1 < p_{33}$, which contradicts $p_{23}+1 > p_{33}$. So V_h must blow up.

Step 2. U_h blows up at T_h^b.
Assume that U_h remains bounded up to time T_h^b.
From (4.2), there exists a constant b dependent of h such that

$$V_I'(t) \geq b\frac{2c^{p_{23}}}{h}\left(T_h^b - t\right)^{-\frac{p_{23}}{p_{33}-1}},\ t \in (0, T_h^b).$$

Integrating this inequality from t_0 to t, we obtain for $t \in (t_0, T_h^b)$,

$$V_I(t) \geq V_I(t_0) + \frac{r(p_{33}-1)}{p_{33}-p_{23}-1}\left(T_h^b - t_0\right)^{\frac{p_{33}-p_{23}-1}{p_{33}-1}} + \frac{r(p_{33}-1)}{p_{23}+1-p_{33}}\left(T_h^b - t\right)^{-\frac{p_{23}+1-p_{33}}{p_{33}-1}},$$

where $r = b\frac{2c^{p_{23}}}{h}$.
Then there exists a constant q such that

$$V_I(t) \geq q\left(T_h^b - t\right)^{-\frac{p_{23}+1-p_{33}}{p_{33}-1}},\ t \in (0, T_h^b). \tag{4.3}$$

From (4.3) and (2.1), we have

$U_I'(t)$

$$\geq \frac{2U_{I-1}(t)-2U_I(t)}{h^2}+\frac{2}{h}\left(U_I^{p_{11}}(t)+q^{p_{12}}\left(T_h^b-t\right)^{-\frac{p_{12}\left(p_{23}+1-p_{33}\right)}{p_{33}-1}}\right),\ t\in(0,T_h^b),$$

as U_h is bounded. Thus there exists a constant K_1 dependent of h such that

$$\begin{aligned} U_I'(t) &\geq K_1+\frac{2q^{p_{12}}}{h}\left(T_h^b-t\right)^{-\frac{p_{12}\left(p_{23}+1-p_{33}\right)}{p_{33}-1}},\ t\in(t_0,T_h^b),\\ U_I'(t) &\geq K_1+K_2\left(T_h^b-t\right)^{-\frac{p_{12}\left(p_{23}+1-p_{33}\right)}{p_{33}-1}},\ t\in(t_0,T_h^b), \end{aligned}$$

with $K_2=\dfrac{2q^{p_{12}}}{h}$.

Integrating this inequality from t_0 to T_h^b, we have

$$U_I(T_h^b) \geq U_I(t_0)+K_1\left(T_h^b-t_0\right)+K_2\int_{t_0}^{T_h^b}\left(T_h^b-t\right)^{-\frac{p_{12}\left(p_{23}+1-p_{33}\right)}{p_{33}-1}}dt.$$

The boundedness of U_h requires $p_{12}<\dfrac{p_{33}-1}{p_{23}+1-p_{33}}$, which contradicts $p_{12}>\dfrac{p_{33}-1}{p_{23}+1-p_{33}}$. So U_h must blow up. That means U_h, V_h, W_h must blow up simultaneously and the proof is completed. □

Theorem 4.3. *Assume* $p_{22}>1$, $p_{11}\leq 1$, $p_{33}\leq 1$, $p_{22}<p_{12}+1$, $p_{31}>\dfrac{p_{22}-1}{p_{12}+1-p_{22}}$ *and*

$$p_{23}<\frac{p_{22}-1}{p_{31}\left(p_{12}+1-p_{22}\right)+1-p_{22}}.$$

Then for every positive initial data, simultaneous blow-up occurs.

Proof. The proof is similar to Theorem 4.2. □

Theorem 4.4. *Assume* $p_{11}>1$, $p_{22}\leq 1$, $p_{33}\leq 1$, $p_{11}<p_{31}+1$, $p_{23}>\dfrac{p_{11}-1}{p_{31}+1-p_{11}}$ *and*

$$p_{12}<\frac{p_{11}-1}{p_{23}\left(p_{31}+1-p_{11}\right)+1-p_{11}}.$$

Then for every positive initial data, simultaneous blow-up occurs.

Proof. The proof is similar to Theorem 4.2. □

5. Convergence of the semidiscrete blow-up time

In this section, first under some assumptions, we show that the solution of the semidiscrete problem converges to the solution of the continuous problem when the mesh size goes to zero. Then, we prove that the semediscrete simultaneous blow-up time converges to the theoretical one when the mesh size tends to zero. Before, we denote

$$\begin{aligned} u_h(t) &= \left(u(x_1,t),\ldots,u(x_I,t)\right)^T,\quad v_h(t)=\left(v(x_1,t),\ldots,v(x_I,t)\right)^T,\\ w_h(t) &= \left(w(x_1,t),\ldots,w(x_I,t)\right)^T. \end{aligned}$$

Theorem 5.1. *Assume that the problem (1.1)–(1.10) has a solution* $(u,v,w) \in \left(C^{4,1}\left([0,1]\times[0,T^*]\right)\right)^3$ *and the initial data* $(\phi_{1,h},\phi_{2,h},\phi_{3,h})$ *at (2.4) satisfies*

$$\|\phi_{1,h}-u_h(0)\|_\infty = o(1), h\to 0, \tag{5.1}$$

$$\|\phi_{2,h}-v_h(0)\|_\infty = o(1), h\to 0, \tag{5.2}$$

$$\|\phi_{3,h}-w_h(0)\|_\infty = o(1), h\to 0. \tag{5.3}$$

Then, for h sufficiently small, the problem (2.1)–(2.4) has a unique solution $(U_h,V_h,W_h) \in \left(C^1\left([0,T^*],\mathbb{R}^I\right)\right)^3$ *such that, as* $h\to 0$

$$\max_{t\in[0,T^*]}\|U_h(t)-u_h(t)\|_\infty = O(\|\phi_{1,h}-u_h(0)\|_\infty+\|\phi_{2,h}-v_h(0)\|_\infty+\|\phi_{3,h}-w_h(0)\|_\infty+h),$$
$$\max_{t\in[0,T^*]}\|V_h(t)-v_h(t)\|_\infty = O(\|\phi_{1,h}-u_h(0)\|_\infty+\|\phi_{2,h}-v_h(0)\|_\infty+\|\phi_{3,h}-w_h(0)\|_\infty+h),$$
$$\max_{t\in[0,T^*]}\|W_h(t)-w_h(t)\|_\infty = O(\|\phi_{1,h}-u_h(0)\|_\infty+\|\phi_{2,h}-v_h(0)\|_\infty+\|\phi_{3,h}-w_h(0)\|_\infty+h).$$

Proof. Let $\xi>0$ be such that

$$(\|u\|_\infty,\|v\|_\infty,\|w\|_\infty) < \xi,\ t\in[0,T^*]. \tag{5.4}$$

Let $t(h)\leqslant T^*$ be the greatest value of $t>0$. Then for $t\in(0,t(h))$, we have

$$\max\{\|U_h(t)-u_h(t)\|_\infty,\|V_h(t)-v_h(t)\|_\infty,\|W_h(t)-w_h(t)\|_\infty\}<1. \tag{5.5}$$

The relation (5.1)–(5.3) implies that $t(h)>0$ for h small enough. Using the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \le 1+\xi,\ \text{for } t\in(0,t(h)), \tag{5.6}$$

$$\|V_h(t)\|_\infty \le 1+\xi,\ \text{for } t\in(0,t(h)), \tag{5.7}$$

$$\|W_h(t)\|_\infty \le 1+\xi,\ \text{for } t\in(0,t(h)). \tag{5.8}$$

Let $(e_{1,i},e_{2,i},e_{3,i})(t)=(U_i-u_i,V_i-v_i,W_i-w_i)(t)$ for $i=1,\dots,I$, $\forall t\in[0,T^*]$ be the discretization error. These error functions verify

$$e'_{1,i}(t)=\delta^2e_{1,i}(t)+p_{11}d_i(\alpha_i(t))^{p_{11}-1}e_{1,i}(t)+p_{12}d_i(\beta_i(t))^{p_{12}-1}e_{2,i}(t)+O(h),$$
$$e'_{2,i}(t)=\delta^2e_{2,i}(t)+p_{22}d_i(\beta_i(t))^{p_{22}-1}e_{2,i}(t)+p_{23}d_i(\lambda_i(t))^{p_{23}-1}e_{3,i}(t)+O(h),$$
$$e'_{3,i}(t)=\delta^2e_{3,i}(t)+p_{33}d_i(\lambda_i(t))^{p_{33}-1}e_{3,i}(t)+p_{31}d_i(\alpha_i(t))^{p_{31}-1}e_{1,i}(t)+O(h),$$

where $\alpha_i(t)$, $\beta_i(t)$ and $\lambda_i(t)$ lie, respectively, between $U_i(t)$ and $u(x_i,t)$, between $V_i(t)$ and $v(x_i,t)$ and between $W_i(t)$ and $w(x_i,t)$, for $i=1,\dots,I$. Using (5.4) and (5.6)–(5.8), there exist P and Q positive constants such that

$$e'_{1,i}(t)\le\delta^2e_{1,i}(t)+d_iP|e_{1,i}(t)|+d_iP|e_{2,i}(t)|+Qh,\quad i=1,\dots,I,\quad t\in[0,T^*],$$
$$e'_{2,i}(t)\le\delta^2e_{2,i}(t)+d_iP|e_{2,i}(t)|+d_iP|e_{3,i}(t)|+Qh,\quad i=1,\dots,I,\quad t\in[0,T^*],$$
$$e'_{3,i}(t)\le\delta^2e_{3,i}(t)+d_iP|e_{3,i}(t)|+d_iP|e_{1,i}(t)|+Qh,\quad i=1,\dots,I,\quad t\in[0,T^*].$$

Let $(g,f,l)\in\left(C^{4,1}([0,1],[0,T^*])\right)^3$ be such that

$$g(x,t)=(\|\phi_{1,h}-u_h(0)\|_\infty+\|\phi_{2,h}-v_h(0)\|_\infty+\|\phi_{3,h}-w_h(0)\|_\infty+Qh)\,\mathrm{e}^{(H+4)t+2x^2-3x}$$

and $g = f = l$, $\forall (x,t) \in [0,1]\times[0,T^*]$, with H, Q positive constants. Then by Lemma 2.3, we prove that

$$(|e_{1,i}(t)|, |e_{2,i}(t)|, |e_{3,i}(t)|) < (g(x_i,t), f(x_i,t), l(x_i,t)) \text{ with } 1 \leq i \leq I \text{ for } t \in (0, t(h)).$$

Thus we get

$$\|U_h(t) - u_h(t)\|_\infty$$
$$\leqslant (\|\phi_{1,h} - u_h(0)\|_\infty + \|\phi_{2,h} - v_h(0)\|_\infty + \|\phi_{3,h} - w_h(0)\|_\infty + Qh)\, \mathrm{e}^{(H+4)t},$$
$$\|V_h(t) - v_h(t)\|_\infty$$
$$\leq (\|\phi_{1,h} - u_h(0)\|_\infty + \|\phi_{2,h} - v_h(0)\|_\infty + \|\phi_{3,h} - w_h(0)\|_\infty + Qh)\, \mathrm{e}^{(H+4)t},$$
$$\|W_h(t) - w_h(t)\|_\infty$$
$$\leq (\|\phi_{1,h} - u_h(0)\|_\infty + \|\phi_{2,h} - v_h(0)\|_\infty + \|\phi_{3,h} - w_h(0)\|_\infty + Qh)\, \mathrm{e}^{(H+4)t},$$

where $t \in (0, t(h))$.

Assume that $t(h) < T^*$. Then from (5.5), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty$$
$$\leq (\|\phi_{1,h} - u_h(0)\|_\infty + \|\phi_{2,h} - v_h(0)\|_\infty + \|\phi_{3,h} - w_h(0)\|_\infty + Qh)\, \mathrm{e}^{(H+4)t(h)}.$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $1 \leqslant 0$, which is impossible. Consequently $t(h) = T^*$ and we conclude the proof. □

Theorem 5.2. *Suppose that the problem (1.1)–(1.10) has a solution (u,v,w) which blows up in a finite time T such that $(u,v,w) \in \left(C^{4,1}([0,1]\times[0,T))\right)^3$ and the initial data at (2.4) satisfies (5.1)–(5.3). Under the assumption of Theorem 3.3, the problem (2.1)–(2.4) has a solution (U_h, V_h, W_h) which blows up in a finite time T_h^b and we have*

$$\lim_{h\to 0} T_h^b = T.$$

Proof. Set $\sigma > 0$, there exists $\varsigma > 0$ such that

$$\frac{y^{1-p_{11}}}{\eta(p_{11}-1)} \leq \frac{\sigma}{2}, \quad \varsigma \leq y. \tag{5.9}$$

Since u blows up in a finite time T, there exists a time $T_1 \in (T - \sigma/2; T)$ such that $\|u(.,t)\|_\infty \geq 2\varsigma$ for $t \in [T_1, T)$. Denote $T_2 = \dfrac{T_1 + T}{2}$, we see easily that $\sup_{t\in[0,T_2]} \|u(.,t)\|_\infty < \infty$. It follows from Theorem 5.1 that for h sufficiently small

$$\sup_{t\in[0,T_2]} \|U_h(t) - u_h(t)\|_\infty \leq \varsigma.$$

Applying the triangle inequality, we get

$$\|U_h(T_2)\|_\infty \geq \|u_h(T_2)\|_\infty - \|U_h(T_2) - u_h(T_2)\|_\infty \geq \varsigma.$$

From Theorem 3.3, U_h blows up at the time T_h^b. We deduce from Remark 3.4 and (5.9) that

$$|T_h^b - T| \leq |T_h^b - T_2| + |T_2 - T| \leq \frac{\|U_h(T_2)\|_\infty^{1-p_{11}}}{\eta(p_{11}-1)} + \frac{\sigma}{2} \leqslant \sigma.$$

Cases where $p_{22}, p_{33} > 1$ or $p_{12}p_{23}p_{31} > 1$ are resolved analogously. □

6. Numerical experiments

In this section, we present some numerical approximations to the simultaneous blow-up time of (1.1)–(1.10). We consider the following explicit scheme:

$$U_1^{(n+1)} = \left(1 - \frac{2\Delta t_n}{h^2}\right) U_1^{(n)} + \frac{2\Delta t_n}{h^2} U_2^{(n)} + \frac{2\Delta t_n}{h} \left((U_1^{(n)})^{p_{11}} + (V_1^{(n)})^{p_{12}}\right),$$

$$U_i^{(n+1)} = \frac{\Delta t_n}{h^2} U_{i-1}^{(n+1)} + \left(1 - \frac{2\Delta t_n}{h^2}\right) U_i^{(n)} + \frac{\Delta t_n}{h^2} U_{i+1}^{(n+1)}, \quad 2 \leq i \leq I-1,$$

$$U_I^{(n+1)} = \frac{2\Delta t_n}{h^2} U_{I-1}^{(n)} + \left(1 - \frac{2\Delta t_n}{h^2}\right) U_I^{(n)} + \frac{2\Delta t_n}{h} \left((U_I^{(n)})^{p_{11}} + (V_I^{(n)})^{p_{12}}\right),$$

$$V_1^{(n+1)} = \left(1 - \frac{2\Delta t_n}{h^2}\right) V_1^{(n)} + \frac{2\Delta t_n}{h^2} V_2^{(n)} + \frac{2\Delta t_n}{h} \left((V_1^{(n)})^{p_{22}} + (W_1^{(n)})^{p_{23}}\right),$$

$$V_i^{(n+1)} = \frac{\Delta t_n}{h^2} V_{i-1}^{(n+1)} + \left(1 - \frac{2\Delta t_n}{h^2}\right) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i+1}^{(n+1)}, \quad 2 \leq i \leq I-1,$$

$$V_I^{(n+1)} = \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} + \left(1 - \frac{2\Delta t_n}{h^2}\right) V_I^{(n)} + \frac{2\Delta t_n}{h} \left((V_I^{(n)})^{p_{22}} + (W_I^{(n)})^{p_{23}}\right),$$

$$W_1^{(n+1)} = \left(1 - \frac{2\Delta t_n}{h^2}\right) W_1^{(n)} + \frac{2\Delta t_n}{h^2} W_2^{(n)} + \frac{2\Delta t_n}{h} \left((W_1^{(n)})^{p_{33}} + (U_1^{(n)})^{p_{31}}\right),$$

$$W_i^{(n+1)} = \frac{\Delta t_n}{h^2} W_{i-1}^{(n+1)} + \left(1 - \frac{2\Delta t_n}{h^2}\right) W_i^{(n)} + \frac{\Delta t_n}{h^2} W_{i+1}^{(n+1)}, \quad 2 \leq i \leq I-1,$$

$$W_I^{(n+1)} = \frac{2\Delta t_n}{h^2} W_{I-1}^{(n)} + \left(1 - \frac{2\Delta t_n}{h^2}\right) W_I^{(n)} + \frac{2\Delta t_n}{h} \left((W_I^{(n)})^{p_{33}} + (U_I^{(n)})^{p_{31}}\right),$$

$$U_i^{(0)} = \phi_{1,i}, \quad V_i^{(0)} = \phi_{2,i}, W_i^{(0)} = \phi_{3,i}, \quad 1 \leq i \leq I,$$

where $n \geq 0$, $p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, p_{31} \geq 0$,

$\Delta t_{n,0} = \frac{\tau h}{2} \min\{\|U_h^{(n)}\|_\infty^{1-p_{11}}, \|V_h^{(n)}\|_\infty^{1-p_{12}}, \|V_h^{(n)}\|_\infty^{1-p_{22}},$
$\|W_h^{(n)}\|_\infty^{1-p_{23}}, \|W_h^{(n)}\|_\infty^{1-p_{33}}, \|U_h^{(n)}\|_\infty^{1-p_{31}}\},$

$\Delta t_n = \min\left\{\frac{h^2}{2}, \Delta t_{n,0}\right\}$ for $\tau \in]0,1[$.

We also consider the implicit scheme:

$$\left(1 + \frac{2\Delta t_n}{h^2}\right) U_1^{(n+1)} - \frac{2\Delta t_n}{h^2} U_2^{(n+1)} = U_1^{(n)} + \frac{2\Delta t_n}{h} \left((U_1^{(n)})^{p_{11}} + (V_1^{(n)})^{p_{12}}\right),$$

$$(-\frac{\Delta t_n}{h^2}) U_{i-1}^{(n+1)} + \left(1 + \frac{2\Delta t_n}{h^2}\right) U_i^{(n+1)} - (\frac{\Delta t_n}{h}) U_{i+1}^{(n+1)} = U_i^{(n)}, \quad 2 \leq i \leq I-1,$$

$$(-\frac{\Delta t_n}{h^2}) U_{I-1}^{(n+1)} + \left(1 + \frac{2\Delta t_n}{h^2}\right) U_I^{(n+1)} = U_I^{(n)} + \frac{2\Delta t_n}{h} \left((U_I^{(n)})^{p_{11}} + (V_I^{(n)})^{p_{12}}\right),$$

$$\left(1 + \frac{2\Delta t_n}{h^2}\right) V_1^{(n+1)} - \frac{2\Delta t_n}{h^2} V_2^{(n+1)} = V_1^{(n)} + \frac{2\Delta t_n}{h} \left((V_1^{(n)})^{p_{22}} + (W_1^{(n)})^{p_{23}}\right),$$

$$(-\frac{\Delta t_n}{h^2}) V_{i-1}^{(n+1)} + \left(1 + \frac{2\Delta t_n}{h^2}\right) V_i^{(n+1)} - (\frac{\Delta t_n}{h}) V_{i+1}^{(n+1)} = V_i^{(n)}, \quad 2 \leq i \leq I-1,$$

$$(-\frac{\Delta t_n}{h^2})V_{I-1}^{(n+1)} + \left(1+\frac{2\Delta t_n}{h^2}\right)V_I^{(n+1)} = V_I^{(n)} + \frac{2\Delta t_n}{h}\left((V_I^{(n)})^{p_{22}} + (W_I^{(n)})^{p_{23}}\right),$$

$$\left(1+\frac{2\Delta t_n}{h^2}\right)W_2^{(n+1)} - \frac{2\Delta t_n}{h^2}W_1^{(n+1)} = W_1^{(n)} + \frac{2\Delta t_n}{h}\left((W_1^{(n)})^{p_{33}} + (U_1^{(n)})^{p_{31}}\right),$$

$$(-\frac{\Delta t_n}{h^2})W_{i-1}^{(n+1)} + \left(1+\frac{2\Delta t_n}{h^2}\right)W_i^{(n+1)} - (\frac{\Delta t_n}{h})W_{i+1}^{(n+1)} = W_i^{(n)}, \quad 2 \le i \le I-1,$$

$$(-\frac{\Delta t_n}{h^2})W_{I-1}^{(n+1)} + \left(1+\frac{2\Delta t_n}{h^2}\right)W_I^{(n+1)} = W_I^{(n)} + \frac{2\Delta t_n}{h}\left((W_I^{(n)})^{p_{33}} + (U_I^{(n)})^{p_{31}}\right),$$

$$U_i^{(0)} = \phi_{1,i}, \quad V_i^{(0)} = \phi_{2,i}, W_i^{(0)} = \phi_{3,i}, \quad 1 \le i \le I,$$

where $n \ge 0,\ p_{11}, p_{22}, p_{33}, p_{12}, p_{23}, p_{31} \ge 0$,
$\Delta t_n = \frac{\tau h}{2}\min\{\|U_h^{(n)}\|_\infty^{1-p_{11}}, \|V_h^{(n)}\|_\infty^{1-p_{12}}, \|V_h^{(n)}\|_\infty^{1-p_{22}},$
$\|W_h^{(n)}\|_\infty^{1-p_{23}}, \|W_h^{(n)}\|_\infty^{1-p_{33}}, \|U_h^{(n)}\|_\infty^{1-p31}\}$
for $0 < \tau < 1$. In both cases we use : $\tau = h$, $\phi_{1,i} = \phi_{2,i} = \phi_{3,i} = ((i-1)h)^2$, $i = 1, \ldots, I$. In Tables 1-6, in rows, we present the numerical simultaneous blow-up times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512. The numerical simultaneous blow-up time $T^n = \sum_{j=0}^{n-1}\Delta t_j$ is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

<u>**First case:**</u> $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (1/4; 2; 1/4; 2; 2; 1/4)$.

Table 1. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.225104579	718	0.34	-
32	0.218606750	1851	0.38	-
64	0.216650020	5537	1.59	1.73
128	0.216077530	18724	10.73	1.77
256	0.215913572	68622	76.75	1.80
512	0.215867375	263043	604.25	1.83

Table 2. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.227827774	723	0.11	-
32	0.219371864	1857	0.36	-
64	0.216852808	5545	15.20	1.74
128	0.216129818	18733	129.44	1.80
256	0.215926878	68633	1390.11	1.83
512	0.215870739	263055	29936.25	1.85

Second case: $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (1/4; 4; 4; 1/2; 1/2; 4)$.

Table 3. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.058637168	216	0.11	-
32	0.054339712	513	0.20	-
64	0.053017052	1411	0.42	1.70
128	0.052624347	4461	2.64	1.75
256	0.052510692	15677	17.33	1.79
512	0.052478413	58741	143.78	1.82

Table 4. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.060905443	219	0.06	-
32	0.055030180	517	0.14	-
64	0.053206110	1415	4.44	1.69
128	0.052673671	4467	28.97	1.78
256	0.052523277	15684	316.70	1.82
512	0.052481590	58748	7427.47	1.85

Third case: $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (4; 2; 1/4; 4; 1/2; 4)$.

Table 5. Explicit Euler method

I	T^n	n	$CPUt$	s
16	0.049450109	206	0.06	-
32	0.045105246	474	0.13	-
64	0.043768015	1257	0.48	1.70
128	0.043371418	3854	3.78	1.75
256	0.043256776	13257	15.08	1.79
512	0.043224252	49072	115.27	1.82

Table 6. Implicit Euler method

I	T^n	n	$CPUt$	s
16	0.051797292	208	0.06	-
32	0.045832940	478	0.16	-
64	0.043968386	1262	3.80	1.68
128	0.043423753	3860	26.28	1.78
256	0.043270130	13264	271.16	1.83
512	0.043227623	49080	5997.19	1.85

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes.

First case: $I = 32$ and $(p_{11}, p_{12}, p_{22}, p_{23}, p_{33}, p_{31}) = (1/4; 2; 1/4; 2; 2; 1/4)$

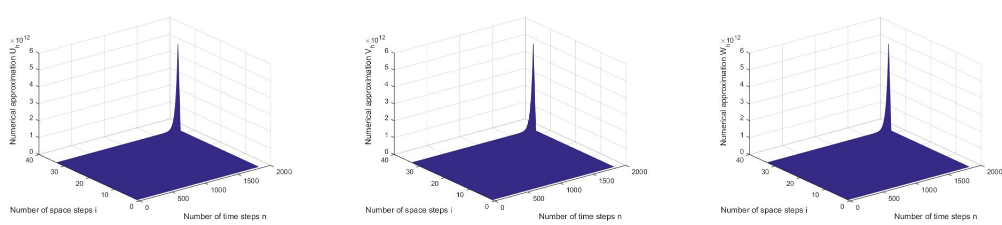

Figure 1. (Explicit scheme): Evolution of components U_h, V_h and W_h according to space and time.

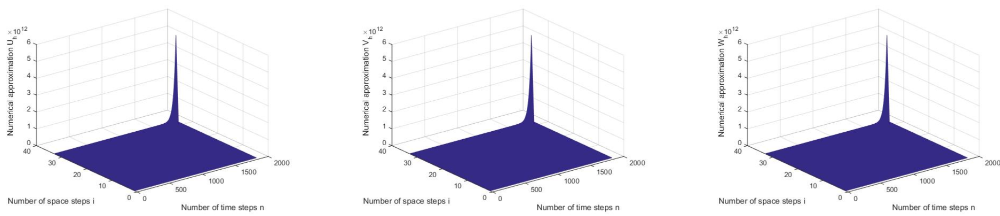

FIGURE 2. (Implicit scheme): Evolution of components U_h, V_h and W_h according to space and time.

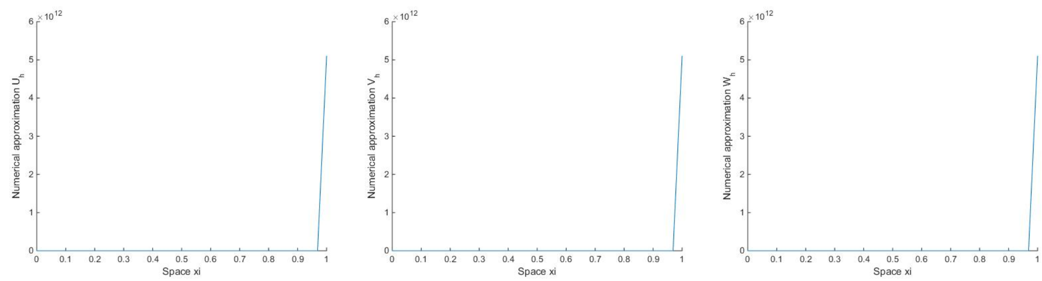

FIGURE 3. (Explicit scheme): Evolution of components U_h, V_h and W_h according to the node.

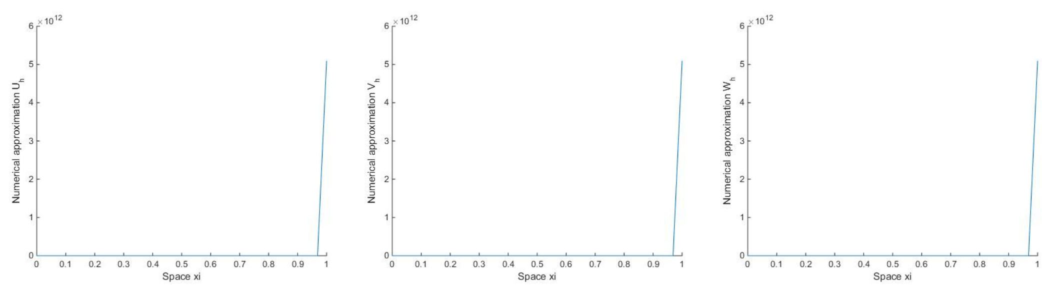

FIGURE 4. (Implicit scheme): Evolution of components U_h, V_h and W_h according to the node.

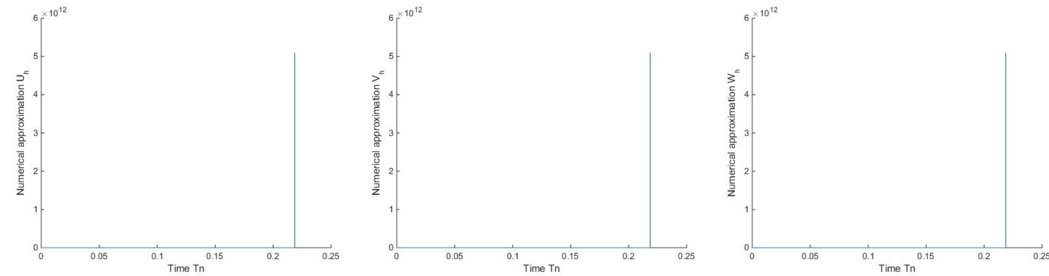

FIGURE 5. (Explicit scheme): Evolution of the norm of components U_h, V_h and W_h according to the time.

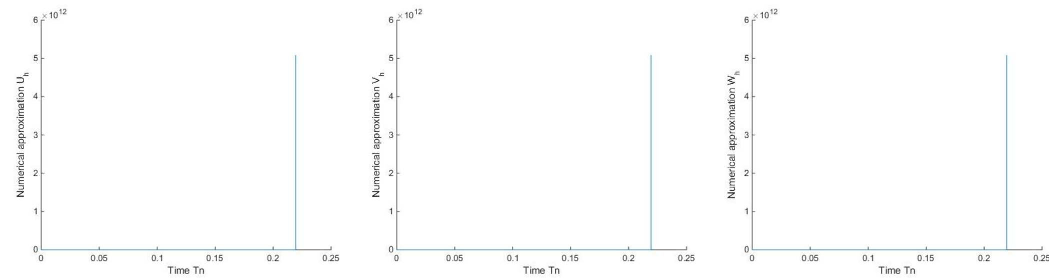

FIGURE 6. (Implicit scheme): Evolution of the norm of components U_h, V_h and W_h according to the time.

<u>Second case</u>: $I = 32$ and $(p_{11}, p_{12}, p_{22}, p_{23}, p_{33}, p_{31}) = (1/4; 4; 4; 1/2; 1/2; 4)$

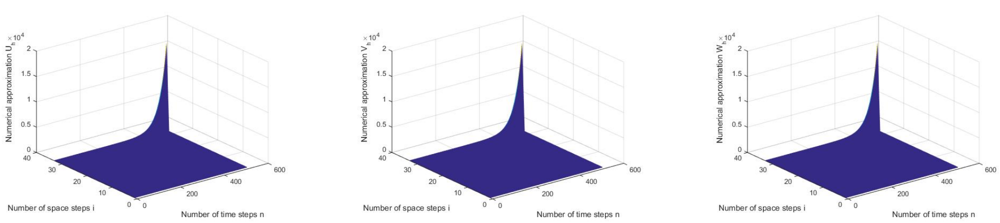

FIGURE 7. (Explicit scheme): Evolution of components U_h, V_h and W_h according to space and time.

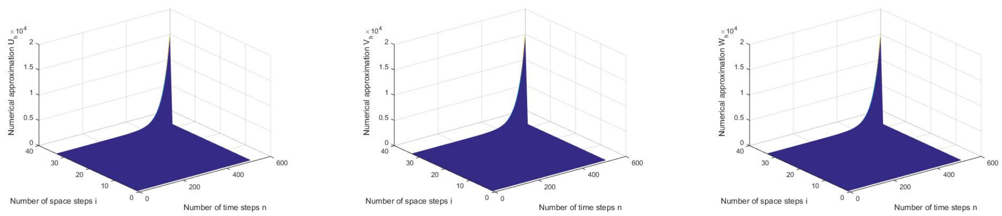

FIGURE 8. (Implicit scheme): Evolution of components U_h, V_h and W_h according to space and time.

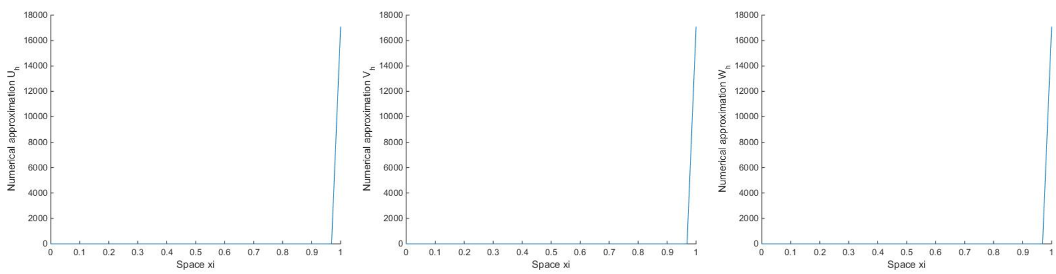

FIGURE 9. (Explicit scheme): Evolution of components U_h, V_h and W_h according to the node.

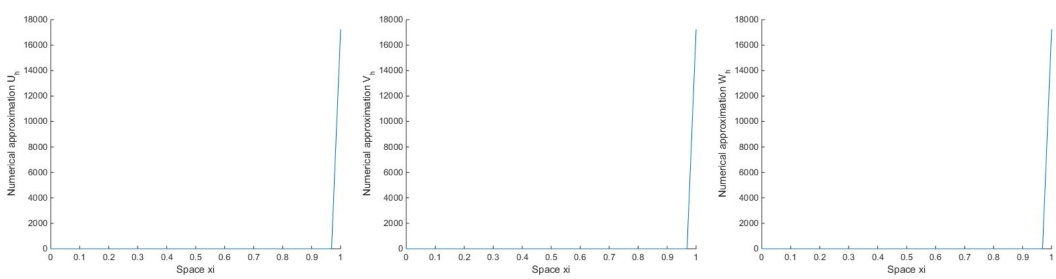

FIGURE 10. (Implicit scheme): Evolution of components U_h, V_h and W_h according to the node.

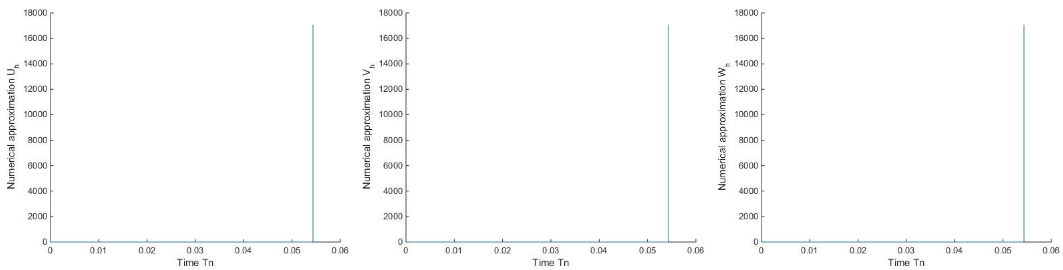

FIGURE 11. (Explicit scheme): Evolution of the norm of components U_h, V_h and W_h according to the time.

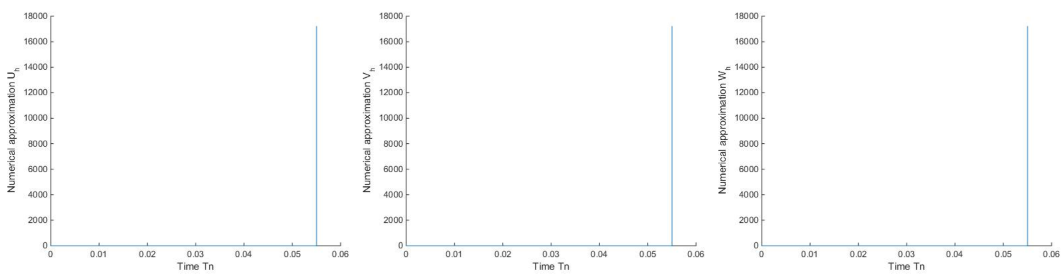

FIGURE 12. (Implicit scheme): Evolution of the norm of components U_h, V_h and W_h according to the time.

<u>**Third case**</u>: $I = 32$ and $(p_{11}, p_{12}, p_{22}, p_{23}, p_{33}, p_{31}) = (4; 2; 1/4; 4; 1/2; 4)$

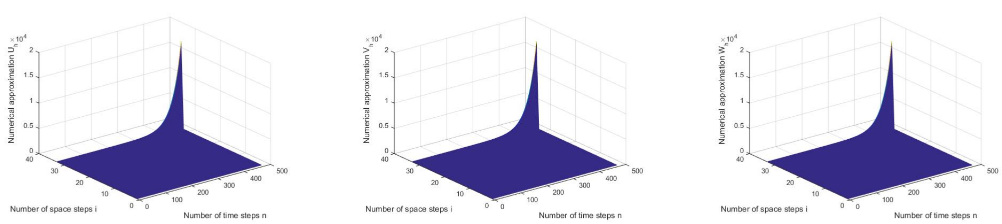

FIGURE 13. (Explicit scheme): Evolution of components U_h, V_h and W_h according to space and time.

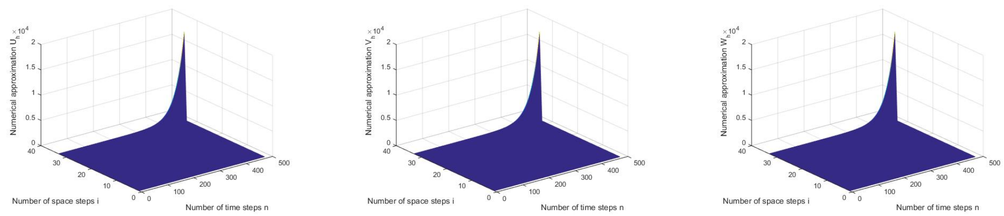

FIGURE 14. (Implicit scheme): Evolution of components U_h, V_h and W_h according to space and time.

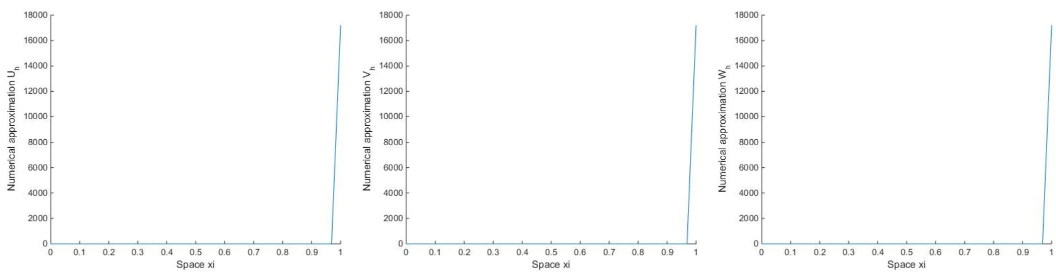

FIGURE 15. (Explicit scheme): Evolution of components U_h, V_h and W_h according to the node.

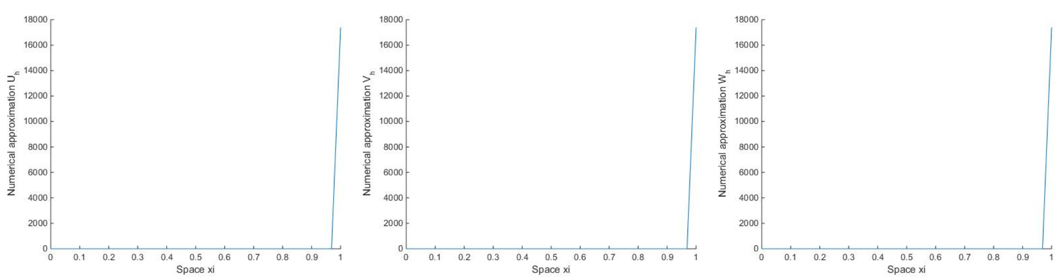

FIGURE 16. (Implicit scheme): Evolution of components U_h, V_h and W_h according to the node.

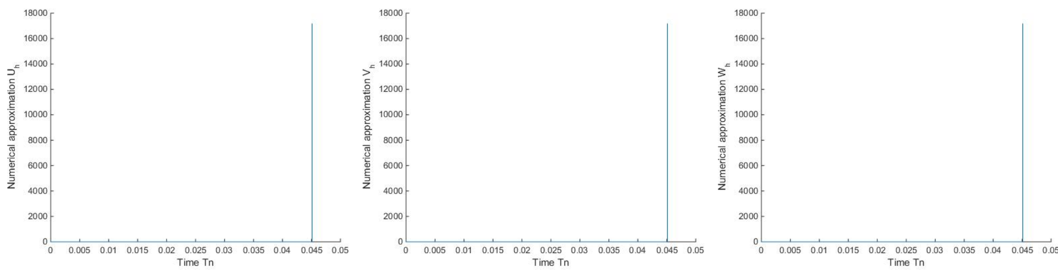

FIGURE 17. (Explicit scheme): Evolution of the norm of components U_h, V_h and W_h according to the time.

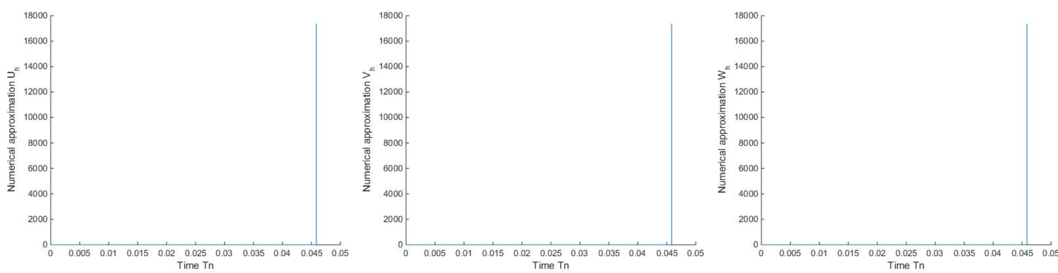

FIGURE 18. (Implicit scheme): Evolution of the norm of components U_h, V_h and W_h according to the time.

7. Main results

In fact, when ($p_{33} > 1$, $p_{11} \leq 1$, $p_{22} \leq 1$, $p_{33} < p_{23} + 1$, $p_{12} > \dfrac{p_{33}-1}{p_{23}+1-p_{33}}$ and $p_{31} < \dfrac{p_{33}-1}{p_{12}\left(p_{23}+1-p_{33}\right)+1-p_{33}}$) or ($p_{22} > 1$, $p_{11} \leq 1$, $p_{33} \leq 1$, $p_{22} < p_{12} + 1$, $p_{31} > \dfrac{p_{22}-1}{p_{12}+1-p_{22}}$ and $p_{23} < \dfrac{p_{22}-1}{p_{31}\left(p_{12}+1-p_{22}\right)+1-p_{22}}$) or ($p_{11} > 1$, $p_{22} \leq 1$, $p_{33} \leq 1$, $p_{11} < p_{31} + 1$, $p_{23} > \dfrac{p_{11}-1}{p_{31}+1-p_{11}}$ and $p_{12} < \dfrac{p_{11}-1}{p_{23}\left(p_{31}+1-p_{11}\right)+1-p_{11}}$) simultaneous blow-up cannot be produce only in one point, which confirms the results known theoretical of the simultaneous blow-up of the problem (1.1)–(1.10) (See [9]).

For Tables 1 and 2 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (1/4; 2; 1/4; 2; 2; 1/4)$, an approximate value of the simultaneous blow-up time is 0.2.

For Tables 3 and 4 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (1/4; 4; 4; 1/2; 1/2; 4)$ an approximate value of the simultaneous blow-up time is 0.05.

For Tables 5 and 6 with $(p_{11}; p_{12}; p_{22}; p_{23}; p_{33}; p_{31}) = (4; 2; 1/4; 4; 1/2; 4)$ an approximate value of the simultaneous blow-up time is 0.04.

We remark that when we refine the spatial step, simultaneous blow-up times decrease for explicit Euler method and implicit Euler method. Ultimately, we see that the CPU times are increasing, as we refine the spatial step, or if we compare CPUT of explicit method with that of implicit method.

8. Conclusion

In this paper using the finite difference method, we have constructed a semidiscrete scheme in space of the continuous problem. We have proved that the solution of a semidiscrete form of the continuous problem blows up in a finite time at the center point and we have estimated its semidiscrete blow-up time. Under some assumptions, we have showed that the semidiscrete simultaneous blow-up occurs. We also have established the convergence of the semidiscrete simultaneous blow-up time to the theoretical one when the mesh size tends to zero. Finally, we have given some numerical experiments to illustrate our analysis.

References

[1] [4] O. A. Ladyženskaja, V. A. Sol'onnikov and N. N. Uralceva, Linear and quasi-linear equations of parabolic type, Amer. Math. Soc. Transl. (2) 23 (1968) 571–629.

[2] [1] K. Deng, K. Fila and H. A Levine, Blow-up rates for parabolic systems, Z. Angew. Math. Phys. 47 (1996) 132–143.

[3] [2] K. Deng, K. Fila and H. A Levine, On critical exponents for a system of the heat equation coupled in the boundary conditions, Acta Math. Univ. comenian. (N.S.) 63(1994) 169–192.

[4] [6] Z. G. Lin and C. H. Xie, The blow-up rate for a system of heat equations with nonlinear boundary conditions, Nonlinear Anal. 34 (1998) 767–778.

[5] [12] M. Pedersen and Z. G. Lin, Blow-up estimates of the positive solution of the parabolic system, J. Math. Appl. Anal. 255 (2001) 551–563.

[6] [16] M. X. Wang, Blow-up estimates for semilinear parabolic systems with nonlinear boundary conditions, Appl. Math. Lett. 16 (2003) 543–549.

[7] [17] M. X. Wang, Blow-up estimates for semilinear parabolic systems, J. Differential Equations. 170 (2001) 317-324.

[8] [5] F. J. Li, B. C. Liu and S. N. Zheng, Simultaneous and non-simultaneous blow-up for heat equations with coupled nonlinear boundary flux, Z. angew. Math. Phys. 58(2007) 717–735.

[9] [7] B. Liu and F. Li, Non-simultaneous blow-up in a parabolic system with three components, Nonlinear Analysis. 70 (2009) 1813–1829.

[10] [11] M. Pedersen and Z. G. Lin, Blow-up Analysis for a System of Heat Equations Coupled Through a Nonlinear Boundary condition, Appl. Math. Lett. 14 (2001) 171–176.

[11] [13] J. P. Pinasco and J. D. Rossi, Simultaneous versus non-simultaneous blow-up, New Zealand, J. Math. 29 (2000) 55-59.

[12] [14] J. D. Rossi and P. Souplet, Coexistence of simultaneous blow-up in a semilinear parabolic system, Differential Integral Equations. 18 (2005) 405–418.

[13] [3] K. B. Edja, K. N'guessan, B. J. -C. Koua and K. A. Touré, Numerical blow-up for a Heat Equation with coupled nonlinear boundary flues, Far East J. Math. Sci. 117 (2019) 119–138.

[14] [8] M. A. Rasheed, R. A. Hameed, A. N. Khalaf and H. A. Kareem, Estimating the blow-up time for chipot-weissler equation using crank-nicolson method, Journal of applied mathematics and informatic, 43 (1) (2025) 123–137.

[15] [9] M. I. Khalil, I. Hashim, M. A. Rasheed and E. S. Ismail, Numerical finite-difference approximation of a coupled parabolic system with blow-up, Journal of mathematics and computer science 32 (4) (2023) 387–407.

[16] [10] M. I. Khalil, I. Hashim, M. A. Rasheed, F. Samat and S. Momani, Numerical finite-difference approximation of a coupled reaction-diffusion system with grandient terms, European journal of pure and applied mathematics 17 (3) (2024) 1516–1538.

[17] [15] M. Senol, S. Atpinar, Z. Zararsiz, S. Salahshour and A. Ahmadian, Approximation solution of time-fractional fuzzy partial differential equations. Comp. Appl. Math. 38 18 (2019) https://doi.org/10.1007/s40314-019-0796-6.

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