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Remi Guillaume Bagré, Sayouba Traoré, Abdoulaye Compaoré

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Exploring the boundaries of fuzzy copulas: Construction, independence and survival analysis

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Abstract. In this article, we redefine fuzzy random variables using clear examples. We construct fuzzy distribution functions while graphically representing these fuzzy distribution functions. Then we clearly construct a fuzzy copula linking two fuzzy random variables, their marginals and the fuzzy joint distribution function. Finally, we construct new specific fuzzy copulas such as the minimum fuzzy copula, the maximum fuzzy copula and the survival fuzzy copula.

2020 AMS Classification: 03E72, 60A10, 62A86, 62H05

Keywords: Copula, Fuzzy copula, Fuzzy set, Fuzzy random variable, Fuzzy cumulative function.

Corresponding Author: Remi Guillaume Bagré (baremgui@yahoo.fr)

1. Introduction

The introduction of copulas in the modeling of multivariate stochastic dependence was motivated mainly by certain shortcomings of the traditional dependence measurement tool [1] such as the linear Bravais-Pearson coefficient. Indeed, it should be noted that this dependency tool has a few limitations in practice, the second - order moment must be finite for this coefficient to be defined, it only integrates linear dependency (rare in finance and the environment), and a zero correlation does not necessarily imply independence. The copula is an innovative tool for modeling the dependency structure of several random variables. The discovery of copulas has made it possible to understand and prevent risks in many fields, including finance, actuarial science and agriculture. The word copula was used in a mathematical sense by Sklar (1959) in multivariate theory. The copula existence theorem is generally addressed to Sklar [2],

$$H(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)). \tag{1.1}$$

Copulas make it possible to model dependence by means of a dependence function, which is more practical and more appropriate, and whose applications have made it possible to control risks in fields such as finance and agriculture.

Copulas were initially used in finance and insurance to assess portfolio risk and asset correlation. They have gained in popularity in these fields because they enable complex dependency structures to be modeled, independently of marginal distributions, which was not possible with traditional tools based on linear correlation coefficients. In addition to finance, copulas have found applications in the spatial domain, and authors such as Bagré and Loyara [3] have applied copulas to the environment, notably in the field of climate change. Also in medicine, copulas have been applied using survival copulas [2]

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v),$$

where $\hat{C}$ is the survival copula for the copula C.
However, classical copulas come up against certain limitations, particularly when it comes to modeling imprecise, uncertain or incomplete data, such as that often encountered in real-life contexts.

It is in this context that fuzzy sets are often used to deal with the imprecision affecting certain features. Following the introduction of fuzzy set theory [4], numerous attempts have been made to develop fuzzy statistical methods. Fuzzy random variables were introduced by Kwakernaak [5] as a natural generalization of random variables to represent relationships between the results of a random experiment and inexact non-statistical data. The use of fuzzy sets has made it possible to extend classical copulas by introducing fuzzy copulas. Fuzzy copulas offer a more flexible framework for dealing with uncertainties, by making it possible to model not only the dependence between variables, but also the imprecise nature of the data. This is particularly useful in fields such as environmental risk modeling or the management of complex systems, where data is often fuzzy by nature.

Some authors, such as Stylianos [6], have attempted to construct fuzzy copulas from fuzzy random variables, but have a number of shortcomings, namely the definition of a triangular fuzzy random variable, and the failure to use H-difference or gH-difference to guarantee existence before performing certain operations on fuzzy numbers.

To make up for these shortcomings, we have clearly defined the fuzzy random variables we use to construe and clearly represent fuzzy repartition functions of a fuzzy random variable and a fuzzy joint distribution function.

The paper is organized as follows: In Section 2, we define the basic concepts required to understand the paper. In Section 3, we present our results, i.e., the definition and representation of fuzzy distribution functions and fuzzy joint distribution functions, the construction of new fuzzy copulas such as independent fuzzy copulas, minimum fuzzy copulas, maximum fuzzy copulas and the survival fuzzy copula. In Section 4, we conclude with some perspectives.

2. Preliminaries

This section briefly reviews several concepts and terminology related to copula, fuzzy num bers, and fuzzy random variable.

2.1. **Copula.**

Definition 2.1 ([2])**.** Let $I = [0, 1]$. Then a function $C : I^2 \to I$ is called a *bivariate copula*, if it satisfies the following properties:

$$C(u, 0) = 0, \tag{2.1}$$

$$C(0, v) = 0, \tag{2.2}$$

$$C(u, 1) = u, \tag{2.3}$$

$$C(1, v) = v, \tag{2.4}$$

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \tag{2.5}$$

where $u_1 \leq u_2$ and $v_1 \leq v_2$.

Definition 2.2 ([1])**.** Let (X, Y) be a a random vector.

(i) The *distribution* of (X, Y) is characterized by:

$$H(x, y) = P(X \leq x, Y \leq y) \ \forall (x, y) \in \mathbb{R}^2,$$

where H is called the *bivariate distribution function.*

(ii) The *marginal distributions* F of X and G of Y in (X, Y) are defined respectively as follows: for each $x \in X$ and each $y \in Y$,

$$F(x) = P(X \leq x) = H(x, +\infty) \text{ and } G(y) = P(Y \leq y) = H(+\infty, y).$$

The main result on copula is given by the following theorem.

Theorem 2.3 ([2])**.** *Let H be a bivariate distribution with marginal distributions F and G. Then $H(u, v)$ can be written in terms of a unique function C such that*

$$H(u, v) = C(F(u), G(v)). \tag{2.6}$$

If F and G are continuous, then C is unique and otherwise, C is uniquely determined on $RanF \times RanG$.

2.2. **Fuzzy set.**

Fuzzy set theory was introduced by Zadeh [4] in 1965, and its initial intuition is the need to handle categories that are not dichotomous.

Definition 2.4 ([7])**.** Let X be a set called the universe. Then a *fuzzy set* $\tilde{A}$ of X is defined by

$$\tilde{A} = \{(x, u_{\tilde{A}}(x)), x \in X\}, \tag{2.7}$$

where $u_{\tilde{A}}$: $X \to I$ is called the *membership function* of $\tilde{A}$.

Let $\alpha \in I$. Then the *α-level subset* or *α-cut* of $\tilde{A}$, denoted $\tilde{A}^{[\alpha]}$, is a subset of X defined by:

$$\tilde{A}^{[\alpha]} = \{x \in X | u_{\tilde{A}}(x) \geq \alpha\}. \tag{2.8}$$

Definition 2.5 ([8])**.** A *fuzzy number* is a convex and normalized fuzzy set of the set of real numbers. The set of fuzzy numbers is denoted by $F(\mathbb{R})$. To each fuzzy set at level α, we associate a fuzzy interval denoted by:

$$\tilde{A}^{[\alpha]} = [\tilde{A}^L_\alpha, \tilde{A}^R_\alpha]. \tag{2.9}$$

Definition 2.6 ([6])**.** Let $\tilde{A}$ a fuzzy number. Then the function $Cr : F(\mathbb{R}) \times \mathbb{R} \to I$ defined by

$$Cr\{\tilde{A} \leq x\} = \frac{1}{2}[\sup_{y \leq x} u_{\tilde{A}}(y) + 1 - \sup_{y>x} u_{\tilde{A}}(y)]. \tag{2.10}$$

is called the *indicator*.

Definition 2.7 ([6])**.** The *α-pessimistic value* of $\tilde{A}$ is defined by

$$\tilde{A}_\alpha = \inf\{x \in \tilde{A}^{[0]} : Cr\{\tilde{A} \leq x\} \geq \alpha\}. \tag{2.11}$$

To each fuzzy number $\tilde{A}$, we associate $\tilde{A}_\alpha$ defined by

$$\tilde{A}_\alpha = \begin{cases} \tilde{A}^L_{2\alpha} & \text{if} \quad \alpha \in [0, \frac{1}{2}] \\ \tilde{A}^L_{2(1-\alpha)} & \text{if} \quad \alpha \in [\frac{1}{2}, 1] \end{cases} \tag{2.12}$$

and

$$\tilde{A}^{[\alpha]} = [\tilde{A}_{\frac{\alpha}{2}}, \tilde{A}_{1-\frac{\alpha}{2}}]. \tag{2.13}$$

Definition 2.8 ([6, 9])**.** Let $\tilde{A},\ \tilde{B} \in F(\mathbb{R})$ and $\tilde{A}^{[\alpha]} = [\tilde{A}^L_\alpha, \tilde{A}^R_\alpha],\ \tilde{B}^{[\alpha]} = [\tilde{B}^L_\alpha, \tilde{B}^R_\alpha]$. Then

(i) $(\tilde{A} \oplus \tilde{B})^{[\alpha]} = [\tilde{A}^L_\alpha + \tilde{B}^L_\alpha, \tilde{A}^R_\alpha + \tilde{B}^R_\alpha]$,

(ii) $(\tilde{A} \ominus_{gH} \tilde{B})^{[\alpha]} = [\min\{\tilde{A}^L_\alpha - \tilde{B}^L_\alpha, \tilde{A}^R_\alpha - \tilde{B}^R_\alpha\},\ \max\{\tilde{A}^L_\alpha - \tilde{B}^L_\alpha, \tilde{A}^R_\alpha - \tilde{B}^R_\alpha\}]$.

Definition 2.9 ([9])**.** Given $\tilde{u}, \tilde{v} \in F(\mathbb{R})$, if $\tilde{w} \in F(\mathbb{R})$ exists such that

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Longleftrightarrow \begin{cases} \tilde{u} = \tilde{v} + \tilde{w} \\ \quad \text{or} \quad \tilde{v} = \tilde{u} + (-1)\tilde{w}. \end{cases} \tag{2.14}$$

then $\tilde{w}$ is called the *gH-difference*.

Definition 2.10 ([7])**.** A fuzzy number $\tilde{m}$ is said to *be of type* $L - R$, if its membership function $u_{\tilde{m}}$ is written as follows

$$u_{\tilde{m}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{if} \quad x \leq m \\ 1 & \text{if} \quad x \in [m, n] \\ R(\frac{x-n}{\beta}) & \text{if} \quad x \geq n, \end{cases} \tag{2.15}$$

where α and β are positive real numbers and represent the left and right spreads of $\tilde{m}$. R and L are reference functions. When L and R are linear and $m = n$, then $\tilde{m}$ is called a *triangular fuzzy number*, denoted by $\tilde{m} = (m, \alpha, \beta)_{LR}$.

Remark 2.11. In practice, the most commonly used reference function in the literature is

$$L(t) = R(t) = 1 - t.$$

The membership function

$$u_{\tilde{a}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & \text{if } x \leq m \\ R\left(\frac{x-m}{\beta}\right) & \text{if } x \geq m. \end{cases}$$

By setting $a_1 = m-\alpha$, $a_2 = m$ and $a_3 = m+\beta$, the differences α and β are explicitly determined by:

$$\alpha = a_2 - a_1 \quad \text{and} \quad \beta = a_3 - a_2.$$

We then obtain a very simple triangular fuzzy number of type L-L or R-R, given by the following definition, which is easy to handle.

Definition 2.12 ([10])**.** A fuzzy number $\tilde{a}$ is said to be *LR-triangular*, if it is in the forma $\tilde{a} = (a_1, a_2, a_3)$, where a_1, a_2 and a_3 are real, and has as a membership function:

$$u_{\tilde{a}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & \text{if} \quad a_1 \leq x \leq a_2 \\ \dfrac{a_3-x}{a_3-a_2} & \text{if} \quad a_2 \leq x \leq a_3 \\ 0 \quad \text{otherelse}. \end{cases} \tag{2.16}$$

Definition 2.13 ([10])**.** Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ two fuzzy triangular numbers. We have:

(i) $\tilde{a} \approx \tilde{b} \Longleftrightarrow a_i = b_i \ \forall i = 1, 2, 3,$

(ii) $\tilde{a} \succeq \tilde{b} \Longleftrightarrow a_i \geq b_i \ \forall i = 1, 2, 3,$

(iii) $\tilde{a} \preceq \tilde{b} \Longleftrightarrow \begin{cases} a_1 \leq b_1 \\ a_1 - a_2 \leq b_1 - b_2 \\ a_1 + a_3 \leq b_1 + b_3. \end{cases}$

Definition 2.14 ([7])**.** Let $\tilde{m} = (m, \alpha, \beta)_{LR}$ and $\tilde{n} = (n, \gamma, \sigma)_{LR}$ two L-R fuzzy triangular number. We have:

(i) $\tilde{m} \approx \tilde{n} \Leftrightarrow m = n,\ \alpha = \lambda,\ \beta = \sigma,$

(ii) $\tilde{m} \preceq \tilde{n} \Leftrightarrow m \leq n,\ m - \alpha \leq n - \gamma$ and $m + \beta \leq n + \sigma.$

Definition 2.15 ([8])**.** Let $\tilde{A}$ and $\tilde{B}$ two fuzzy LR-fuzzy number. We have:

$$(\tilde{A} \otimes \tilde{B})_\alpha = \begin{cases} \tilde{A}_\alpha \times \tilde{B}_\alpha & \text{if} \quad \tilde{A},\ \tilde{B} \succeq \tilde{0} \\ \tilde{A}_{1-\alpha} \times \tilde{B}_{1-\alpha} & \text{if} \quad \tilde{A},\ \tilde{B} \preceq \tilde{0} \\ \tilde{A}_{1-\alpha} \times \tilde{B}_\alpha & \text{if} \quad \tilde{A} \succeq \tilde{0},\ \tilde{B} \preceq \tilde{0}. \end{cases} \tag{2.17}$$

Definition 2.16 ([11])**.** Let $\tilde{f} : U \subset \mathbb{R}^2 \to F(\mathbb{R})$ and $(x_0, y_0) \in U$. Then $\tilde{f}$ is said to be:

(i) *gH-differenciable with respect to* x, if it exists a fuzzy number $\frac{\partial \tilde{f}(x_0,y_0)}{\partial x} \in F(\mathbb{R})$ such that

$$\frac{\partial \tilde{f}(x_0, y_0)}{\partial x} = \lim_{h \to 0^-} \frac{\tilde{f}(x_0 + h, y) \ominus_{gH} \tilde{f}(x_0, y)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(x_0 + h, y) \ominus_{gH} \tilde{f}(x_0, y)}{h}. \tag{2.18}$$

(ii) *gH-differenciable with respect to y*, if it exists a fuzzy number $\frac{\partial \tilde{f}(x_0,y_0)}{\partial y} \in F(\mathbb{R})$ such that
(2.19)
$$\frac{\partial \tilde{f}(x_0,y_0)}{\partial y} = \lim_{h\to 0^-} \frac{\tilde{f}(x,y_0+h) \ominus_{gH} \tilde{f}(x,y_0)}{h} = \lim_{h\to 0^+} \frac{\tilde{f}(x,y_0+h) \ominus_{gH} \tilde{f}(x,y_0)}{h}.$$

2.3. **Fuzzy random variables.**

Definition 2.17 ([5])**.** Let $(\Omega, \mathbf{A}, P)$ be a probabilistic space. Then a *fuzzy random variabl* is a mapping from Ω to the set of fuzzy numbers $F(\mathbb{R})$.

Definition 2.18 ([12])**.** Let $(\Omega, \mathbf{A}, P)$ be a probabilistic space. Then a *fuzzy random variable* of *type L-R* $\tilde{X}$ is defined as follows: for each $w \in \Omega$,

$$\tilde{X}(w) = (\underline{x}(w), \overline{x}(w), a, b), \tag{2.20}$$

where $\underline{X}$ and $\overline{X}$ are real random variables defined by $\underline{x}(w) = \inf\tilde{X}(w)$ and $\overline{x}(w) = \sup\tilde{X}(w)$ respectively, and a and b are positive real numbers representing the left and right spreads of x respectively.

Definition 2.19 ([12])**.** Let $\alpha \in I$ and $\tilde{X}$ be a fuzzy random variable of type L-R. Then the *α-cut* of $\tilde{X}$, denoted by $\tilde{X}^{[\alpha]}$, is defined by

$$\tilde{X}^{[\alpha]} = [\underline{x}(w) - aL^{-1}(\alpha), \overline{x}(w) + bR^{-1}(\alpha)].$$

We define the *α-pessimistic value* of $\tilde{X}$, denoted by $\tilde{X}_\alpha$, is defined as follows:

$$\tilde{X}_\alpha = \begin{cases} \underline{x} - aL^{-1}(2\alpha) & \text{if} \quad 0 \le \alpha \le \dfrac{1}{2} \\ \overline{x} + bR^{-1}(2(1-\alpha)) & \text{if} \quad \dfrac{1}{2} \le \alpha \le 1, \end{cases} \tag{2.21}$$

where a and b are positive real numbers representing the left and right spreads of x, and $L : \mathbb{R}^+ \to I$, $R : \mathbb{R}^+ \to I$ are left and right shap function respectively and $L(0) = R(0) = 1$.
We can deduce that ([13])

$$\tilde{X}^{[\alpha]} = [\tilde{X}_{\frac{\alpha}{2}}, \tilde{X}_{1-\frac{\alpha}{2}}]. \tag{2.22}$$

3. Results

3.1. **Preminary results.**

In this subsection, we restate the classical minimum and maximum copulas in a different way.

Definition 3.1 ([1])**.** Let U be an open subset of $\mathbb{R}^n$. Then a function $f : U \to \mathbb{R}$ is said to be of class C^2, if it has continuous first and second partial derivatives at every point of U, equivalently,

(i) all first-order partial derivatives $\frac{\partial f}{\partial x_i}$ exist and are continuous on U,

(ii) all second-order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are continuous on U.

This means that f is twice continuously differentiable on U.

Example 3.2. Let U be an open subset of $\mathbb{R}^2$ and consider the function $f : U \to \mathbb{R}$ defined by: for each $(x, y) \in U$,

$$f(x, y) = x + y + 1.$$

Then clearly, $\frac{\partial f(x,y)}{\partial x} = 1$, $\frac{\partial f(x,y)}{\partial y} = 1$. Thus these derivatives exist everywhere and are continuous. On the other hand, we have

$$\frac{\partial^2 f(x, y)}{\partial x^2} = 0,\ \frac{\partial^2 f(x, y)}{\partial y^2} = 0,\ \frac{\partial^2 f(x, y)}{\partial x \partial y} = 0,\ \frac{\partial^2 f(x, y)}{\partial y \partial x} = 0.$$

So These derivatives are also continuous everywhere. Hence f is of class C^2.

A bivariate copula C is said to be *2-increasing* [1], if it is of the class C^2 and $\frac{\partial^2 C}{\partial u \partial v}(u, v) \geq 0$ or satisfies the property (2.5).

Proposition 3.3. *Let $C : I^2 \to I$ be the function defined by*

$$C(u, v) = \frac{u + v - 1 + |u + v - 1|}{2}.$$

Then C is a maximum bivariate copula.

Proof. Let $u,\ v \in [0, 1]$. Then we have

$$\begin{aligned} C(0, v) &= \frac{0 + v - 1 + |0 + v - 1|}{2} \\ &= \frac{v - 1 + |v - 1|}{2} \\ &= \frac{v - 1 - v + 1}{2} \\ &= 0. \end{aligned}$$

Similarly, we show that $C(u, 0) = 0$. On the other hand, we get

$$\begin{aligned} C(u, 1) &= \frac{u + 1 - 1 + |u + 1 - 1|}{2} \\ &= \frac{u + |u|}{2} \\ &= \frac{2u}{2} \\ &= u. \end{aligned}$$

Similarly, we show that $C(1, v) = v$. Thus properties (2.1)–(2.4) hold.

Now, we must prove that C is 2-increasing, i.e., $\frac{\partial^2 C}{\partial u \partial v}(u, v) \geq 0$.

If $u + v \geq 1$, then $|u + v - 1| = u + v - 1$. Thus $C(u, v) = u + v - 1$. So we have

$$\frac{\partial^2 C}{\partial u \partial v}(u, v) = \frac{\partial}{\partial u}(\frac{\partial C}{\partial v}(u + v - 1)) = \frac{\partial}{\partial u}(1) = 0.$$

If $u + v \leq 1$, then $|u + v - 1| = -u - v = 1$. Thus $C(u, v) = 0$. So we get

$$\frac{\partial^2 C}{\partial u \partial v}(u, v) = \frac{\partial^2 C}{\partial u \partial v}(0) = 0.$$

Hence C is 2-increasing. Therefore C is a maximum bivariate copula. □

Proposition 3.4. *Let $C : I^2 \to I$ be the function defined by*

$$C(u,v) = \frac{u+v-|u-v|}{2}.$$

Then C is a minimum bivariate copula.

Proof. Let $u,\ v \in I$. Then we have

$$\begin{aligned} C(0,v) &= \frac{0+v-|0-v|}{2} \\ &= \frac{v-|v|}{2} \\ &= \frac{v-v}{2} \\ &= 0, \end{aligned}$$

$$\begin{aligned} C(u,1) &= \frac{u+1-|u-1|}{2} \\ &= \frac{u+1+u-1}{2} \\ &= \frac{2u}{2} \\ &= u. \end{aligned}$$

Similarly, we get $C(u,0) = 0$ and $C(1,v) = v$. Thus properties (2.1)–(2.4) hold. Furthermore, the function C is of the class C^∞, i.e., that all partial derivatives of all orders of C are continuous. So we have $\frac{\partial^2 C}{\partial u \partial v}(u,v) \geq 0$. Hence C is 2-increasing. Therefore C is a minimum bivariate copula. □

The representation of the maximum and the minimum copula is given by Figure 1.

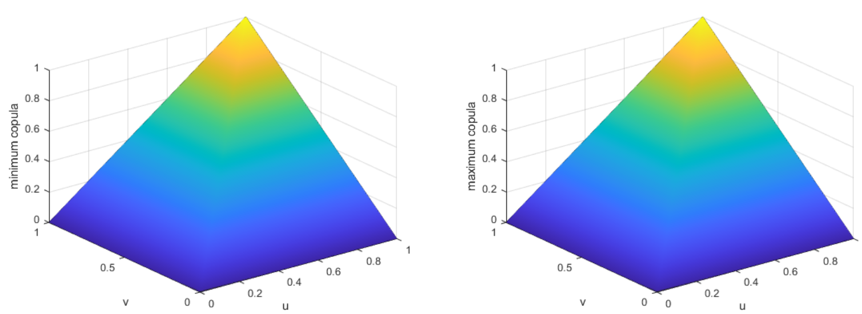

Figure 1. Minimum copula on the left and maximum copula on the right

3.2. **Fuzzy cumulative distribution functions.**

3.2.1. *Cumulative distribution function of a fuzzy random variable.*

This section extend a concept of cumulative joint distribution function to the fuzzy random variable and their represenation.

Definition 3.5. Let $\tilde{X}$ be a fuzzy random variable and $\tilde{X}_\alpha$ its α pessimistic value. We define the *cumulative distribution function* (CDF) of the fuzzy random variable $\tilde{X}$, denoted by $\tilde{F}$ or $\tilde{F}_{\tilde{X}}$, as

$$\tilde{F}(x) = \tilde{F}_{\tilde{X}}(x) = P(\tilde{X}_\alpha \leq x). \tag{3.1}$$

The α-*cut* of the cumulative distribution function $\tilde{F}_{\tilde{X}}$ of $\tilde{X}$ can be defined as

$$(\tilde{F}_{\tilde{X}})^{[\alpha]}(x) = [P(\tilde{X}_{\frac{\alpha}{2}} \leq x), P(\tilde{X}_{1-\frac{\alpha}{2}} \leq x)]. \tag{3.2}$$

Example 3.6. Let X be a real random variable following a normal distribution $\mathcal{N}(0,1)$, and $\tilde{X}$ a fuzzy normal random variable. Here we simulated 1,000 values following the standard normal distribution, and by taking $m = -3, 23$, $n = 3.58$ and $\alpha = \beta = 2$ and $L(t) = R(t) = 1 - t$, we have:

$$\begin{aligned} L(\frac{m-x}{\alpha}) &= 1 - \frac{-3.23 - x}{2} \\ &= \frac{5.23 + x}{2}, \end{aligned}$$

and

$$\begin{aligned} R(\frac{x-n}{\beta}) &= 1 - \frac{x - 3.58}{2} \\ &= \frac{5.58 - x}{2}. \end{aligned}$$

Based on (2.15), we deduce the membership function of $\tilde{X}$ defined by $u_{\tilde{X}}(x)$ is the membership function of $\tilde{X}$ defined by

$$u_{\tilde{X}}(x) = \begin{cases} \dfrac{x+5.23}{2} & \text{if} \quad x \leq -3.23 \\ 1 \ \text{if} & -3.23 \leq x \leq 3.58 \\ \dfrac{5.58-x}{2} & \text{if} \quad x \geq 3.58. \end{cases} \tag{3.3}$$

The α-pessimistic of $\tilde{X}$ is given by:

$$\tilde{X}_\alpha = \begin{cases} 4\alpha - 5.23 & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2} \\ 4\alpha + 1.58 & \text{if} \quad \frac{1}{2} \leq \alpha \leq 1. \end{cases} \tag{3.4}$$

By solving $\dfrac{x+5.23}{2} = \alpha$ and using the equation 2.21, $L^{-1}(2\alpha) = -4\alpha + 5.23$ and $a = 1$

Similarly, by solving $\dfrac{5.58 - x}{2} = \alpha$, $R^{-1}(2(1-\alpha)) = -4\alpha - 1.58$ and $b = 1$

Then we have

$$\tilde{X} \rightsquigarrow \mathcal{N}(4\alpha - 5.23, 1) \text{ if } 0 \leq \alpha \leq \frac{1}{2}$$

and

$$\tilde{X} \rightsquigarrow \mathcal{N}(4\alpha + 1.58, 1) \text{ if } \frac{1}{2} \leq \alpha \leq 1,$$

where $\tilde{X} \rightsquigarrow \mathcal{N}(4\alpha - 5.23, 1)$ means that $\tilde{X}$ follows the normal distribution of parameters $4\alpha - 5.23$ and 1.

For $0 \leq \alpha \leq \frac{1}{2}$,

$$\begin{aligned}\tilde{F}_{\tilde{X}}(x) &= P(\tilde{X}_\alpha \leq x)\\ &= P(\underline{x} - aL^{-1}(2\alpha) \leq x)\\ &= P(\underline{x} - 4\alpha + 5.23 \leq x)\\ &= P(\underline{x} \leq x + 4\alpha - 5.23)\\ &= \Phi(x + 4\alpha - 5.23),\end{aligned}$$

where Φ is the cumulative distribution function of the normal random variable.

For $\frac{1}{2} \leq \alpha \leq 1$,

$$\begin{aligned}\tilde{F}_{\tilde{X}}(x) &= P(\tilde{X}_\alpha \leq x)\\ &= P(\overline{x} + bR^{-1}(2(1-\alpha)) \leq x)\\ &= P(\overline{x} + 4\alpha + 1.58 \leq x)\\ &= P(\overline{x} \leq x - 4\alpha - 1.58).\\ &= \Phi(x - 4\alpha - 1.58).\end{aligned}$$

We discretize α with a step of 0.1. The graphical representation of the cumulative distribution function Φ of $\tilde{X}$ is given by Figure 2.

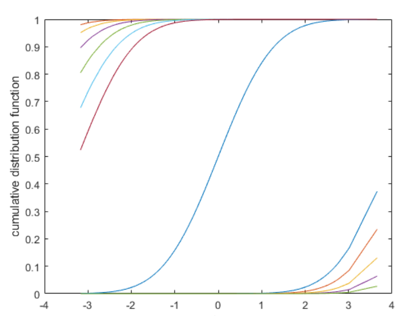

FIGURE 2. Fuzzy normal distribution function with α discretized in steps of 0.1

If α is discretized with a step of 0.01, the graphical representation is given by Figure 3.

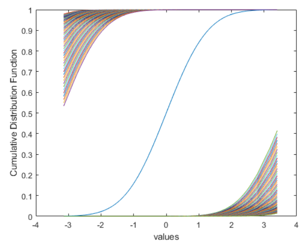

FIGURE 3. Fuzzy normal distribution function with α sampled at increments of 0.01

Example 3.7. Let X be a real random variable following a uniform distribution on the interval $[1,4]$ and $\tilde{X}$ a fuzzy uniform random variable. $u_{\tilde{X}}(x)$ is the membership function of $\tilde{X}$ defined by

$$\begin{aligned} L(\frac{m-x}{\alpha}) &= 1-\frac{1-x}{2} \\ &= \frac{1+x}{2} \end{aligned}$$

and

$$\begin{aligned} R(\frac{x-n}{\beta}) &= 1-\frac{x-4}{2} \\ &= \frac{6-x}{2}. \end{aligned}$$

We deduce the membership function of $\tilde{X}$ defined by

$$u_{\tilde{X}}(x) = \begin{cases} \dfrac{x+1}{2} & \text{if} \quad x \leq 1 \\ 1 \ \text{if} \ 1 \leq x \leq 4 \\ \dfrac{6-x}{2} & \text{if} \quad x \geq 4 \\ 0 \ \text{otherwise.} \end{cases} \tag{3.5}$$

The α-pessimiste of $\tilde{X}$ is given by

$$\tilde{X}_\alpha = \begin{cases} 4\alpha - 1 & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2}, \\ 4\alpha + 2 & \text{if} \quad \frac{1}{2} \leq \alpha \leq 1. \end{cases} \tag{3.6}$$

Then we have

$$\tilde{X} \rightsquigarrow \mathcal{U}(1+4\alpha-1, 4+4\alpha-1) \text{ if } 0 \leq \alpha \leq \frac{1}{2}$$

and

$$\tilde{X} \rightsquigarrow \mathcal{U}(1+4\alpha+2, 4+4\alpha+2) \text{ if } \frac{1}{2} \leq \alpha \leq 1,$$

where $\tilde{X} \rightsquigarrow \mathcal{U}(1+4\alpha+2, 4+4\alpha+2)$ means that $\tilde{X}$ follows the uniform distribution on the interval $[1+4\alpha+2, 4+4\alpha+2]$.

For $0 \leq \alpha \leq \frac{1}{2}$,

$$\begin{aligned}\tilde{F}_{\tilde{X}}(x) = P(\tilde{X}_\alpha \leq x) &= P(\underline{x} - aL^{-1}(2\alpha) \leq x)\\ &= P(4\alpha \leq \underline{x} \leq 4\alpha + 3)\\ &= \Phi(4\alpha + 3) - \Phi(4\alpha),\end{aligned}$$

where Φ is the cumulative distribution function of the uniform random variable.

For $\frac{1}{2} \leq \alpha \leq 1$,

$$\begin{aligned}\tilde{F}_{\tilde{X}}(x) = P(\tilde{X}_\alpha \leq x) &= P(\overline{x} + bR^{-1}(2(1-\alpha)) \leq x)\\ &= P(4\alpha + 3 \leq \overline{x} \leq 4\alpha + 6)\\ &= \Phi(4\alpha + 6) - \Phi(4\alpha + 3).\end{aligned}$$

We discretize α with a step size of 0.1 and the graphical representation of the cumulative distribution function Φ of $\tilde{X}$ is given by Figure 4.

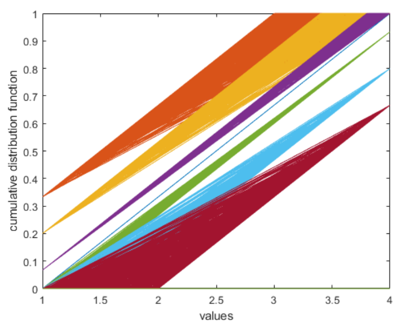

Figure 4. Fuzzy uniform cumulative distribution function with α with alpha discretized in steps of 0.1

If we discretize α with a step size of 0.01, we have Figure 5.

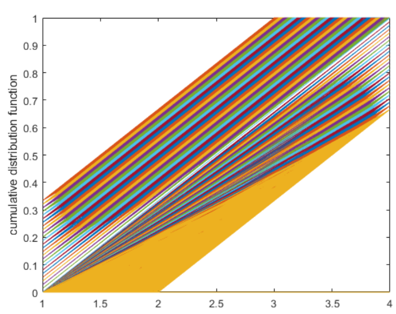

Figure 5. Fuzzy uniform distribution function with α sampled at increments of 0.01

3.2.2. *Two-variables fuzzy joint distribution function.*

Definition 3.8. Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy random variables and let $\alpha \in I$. Let $\tilde{X}_\alpha$ and $\tilde{Y}_\alpha$ be the α-pessimistic values of the respective fuzzy random variables $\tilde{X}$ and $\tilde{Y}$. Then we have

$$\tilde{H}(x,y) = \tilde{H}_{\tilde{X},\tilde{Y}}(x,y) = P(\tilde{X}_\alpha \leq x, \tilde{Y}_\alpha \leq y). \tag{3.7}$$

For $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\begin{aligned}\tilde{H}(x,y) &= \tilde{H}_{\tilde{X}_\alpha,\tilde{Y}_\alpha}(x,y) \\ &= P(\tilde{X}_{\frac{\alpha}{2}} \leq x, \tilde{Y}_{\frac{\alpha}{2}} \leq y).\end{aligned}$$

For $\frac{1}{2} \leq \alpha \leq 1$, we have

$$\begin{aligned}\tilde{H}(x,y) &= \tilde{H}_{\tilde{X}_\alpha,\tilde{Y}_\alpha}(x,y) \\ &= P(\tilde{X}_{1-\frac{\alpha}{2}} \leq x, \tilde{Y}_{1-\frac{\alpha}{2}} \leq y).\end{aligned}$$

Thus We can define the α-cut of the fuzzy distribution function $\tilde{H}$, denoted $\tilde{H}^{[\alpha]}$ by

$$\tilde{H}^{[\alpha]} = [H_{\tilde{X}_{\frac{\alpha}{2}},\tilde{Y}_{\frac{\alpha}{2}}}, H_{\tilde{X}_{1-\frac{\alpha}{2}},\tilde{Y}_{1-\frac{\alpha}{2}}}]. \tag{3.8}$$

Example 3.9. Let X and Y two random variables with a joint distribution

$$H(x,y) = \frac{(x+1)(1-e^{-y})}{(x-1)e^{-y}+2}, (x,y) \in [-1,1] \times [0,+\infty], \tag{3.9}$$

where X follows a uniform distribution on $[-1,1]$ and Y follows an exponential distribution with parameter 1. Let X be a real random variable following a uniform distribution on the interval $[-1,1]$ and $u_{\tilde{X}}$ be the membership function of $\tilde{X}$ defined by

$$u_{\tilde{X}}(x) = \begin{cases} \dfrac{x+3}{2} & \text{if} \quad -3 \leq x \leq -1 \\ 1 \ \text{if} \quad -1 \leq x \leq 1 \\ \dfrac{3-x}{2} & \text{if} \quad 1 \leq x \leq 3 \\ 0 \ \text{otherwise.} \end{cases} \tag{3.10}$$

The α-pessimistic is

$$\tilde{X}_\alpha = \begin{cases} 4\alpha - 3 & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2} \\ 4\alpha - 1 & \text{if} \quad \frac{1}{2} \leq \alpha \leq 1. \end{cases} \tag{3.11}$$

Then we have

$$\tilde{X} \rightsquigarrow \mathcal{U}(-1+4\alpha-3, 1+4\alpha-1) \text{ if } 0 \leq \alpha \leq \frac{1}{2}$$

and

$$\tilde{X} \rightsquigarrow \mathcal{U}(1+4\alpha+2, 4+4\alpha+2) \text{ if } \frac{1}{2} \leq \alpha \leq 1.$$

The membership function of $\tilde{Y}$ is given by

$$u_{\tilde{Y}}(x) = \begin{cases} x-1 & \text{if} \quad 1 \leq x \leq 2 \\ 3-x & \text{if} \quad 2 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

The α-pessimistic value of $\tilde{Y}$ is given by

$$\tilde{Y}_\alpha = \begin{cases} 2\alpha + 1 & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2} \\ 2\alpha + 1 & \text{if} \quad \frac{1}{2} \leq \alpha \leq 1. \end{cases} \tag{3.13}$$

For $0 \leq \alpha \leq \frac{1}{2}$,

$$\begin{aligned} \tilde{H}(x,y) &= \tilde{H}_{\tilde{X}_\alpha, \tilde{Y}_\alpha}(x,y) \\ &= P(\tilde{X}_{\frac{\alpha}{2}} \leq x, \tilde{Y}_{\frac{\alpha}{2}} \leq y) \\ &= H_{\tilde{X}_{\frac{\alpha}{2}}, \tilde{Y}_{\frac{\alpha}{2}}}(x,y) \\ &= P(\underline{x} - 2\alpha + 3 \leq x, \underline{y} - \alpha - 1 \leq y) \\ &= P(\underline{x} \leq x + 2\alpha - 3, \underline{y} \leq y + \alpha + 1) \\ &= \frac{(x+2\alpha-2)(1-e^{-y-\alpha-1})}{(x+2\alpha-4)e^{-y-\alpha-1}+2}. \end{aligned}$$

For $\frac{1}{2} \leq \alpha \leq 1$,

$$\begin{aligned} \tilde{H}(x,y) &= \tilde{H}_{\tilde{X}_\alpha, \tilde{Y}_\alpha}(x,y) \\ &= P(\tilde{X}_{1-\frac{\alpha}{2}} \leq x, \tilde{Y}_{1-\frac{\alpha}{2}} \leq y) \\ &= H_{\tilde{X}_{1-\frac{\alpha}{2}}, \tilde{Y}_{1-\frac{\alpha}{2}}}(x,y) \\ &= \frac{(x+2\alpha-2)(1-e^{-y-\alpha+3})}{(x+2\alpha-4)e^{-y-\alpha+3}+2}. \end{aligned}$$

3.3. **Main results.**

In this subsection, we construct fuzzy copulas.

Proposition 3.10. *Let $\tilde{X}$ and $\tilde{Y}$ two fuzzy random variables with respective fuzzy distribution functions $\tilde{F}$ and $\tilde{G}$ and $\tilde{H}$ a fuzzy joint distribution function. Then we have*

$$\tilde{H}(x,y)_\alpha = C((\tilde{F}(x))_\alpha, (\tilde{G}(y))_\alpha) \tag{3.14}$$

and

$$\tilde{C}(x,y)_\alpha = H((\tilde{F}^{-1}(x))_\alpha, (\tilde{G}^{-1}(y))_\alpha), \tag{3.15}$$

where H is a joint distribution function, C a bivariate copula and $\tilde{C}(x,y)_\alpha$, $\tilde{H}(x,y)_\alpha$, $\tilde{F}(x))_\alpha$, $\tilde{G}(y))_\alpha$ are respectvely the α-pessimistic of C, H, F, G.

Proof.

$$\begin{aligned}\tilde{H}(x,y) = \tilde{H}_{\tilde{X},\tilde{Y}}(x,y) &= P(\tilde{X}_\alpha \leq x, \tilde{Y}_\alpha \leq y)\\ &= P(\tilde{F}(\tilde{X}_\alpha) \leq \tilde{F}(x), \tilde{G}(\tilde{Y}_\alpha) \leq \tilde{G}(y))\\ &= C((\tilde{F}(x))_\alpha, (\tilde{G}(y))_\alpha).\end{aligned}$$

$$\begin{aligned}\tilde{C}(x,y)_\alpha &= P(\tilde{F}(\tilde{X}_\alpha) \leq x, \tilde{G}(\tilde{Y}_\alpha) \leq y)\\ &= P(\tilde{X}_\alpha \leq \tilde{F}^{-1}(x), \tilde{Y}_\alpha \leq \tilde{G}^{-1}(y))\\ &= H(\tilde{F}^{-1}(x)_\alpha, \tilde{G}^{-1}(y)_\alpha).\end{aligned}$$

Then $\tilde{H}$ and $\tilde{C}$ are well-defined.

Now, we will prove that $\tilde{C}$ is a fuzzy copula.

$$\begin{aligned}\tilde{C}(0,y)_\alpha &= H((\tilde{F}^{-1}(0))_\alpha, (\tilde{G}^{-1}(y))_\alpha)\\ &= H((\tilde{F}(0))_\alpha^{-1}, (\tilde{G}^{-1}(y))_\alpha)\\ &= H(-\infty, (\tilde{G}^{-1}(y))_\alpha)\\ &= \tilde{0}_\alpha,\end{aligned}$$

$$\begin{aligned}\tilde{C}(1,y)_\alpha &= H((\tilde{F}^{-1}(1))_\alpha, (\tilde{G}^{-1}(y))_\alpha)\\ &= H((\tilde{F}(1))_\alpha^{-1}, (\tilde{G}^{-1}(y))_\alpha)\\ &= H(+\infty, (\tilde{G}^{-1}(y))_\alpha)\\ &= G((\tilde{G}^{-1}(y))_\alpha)\\ &= \tilde{v}_\alpha.\end{aligned}$$

Similarly, $\tilde{C}(x,0)_\alpha = \tilde{0}_\alpha$ and $\tilde{C}(u,1)_\alpha = \tilde{u}_\alpha$. Let $x_1,\ y_1,\ x_2,\ y_2 \in I$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Then we have

$$\tilde{C}(x_1,y_1) \oplus \tilde{C}(x_2,y_2) = H((\tilde{F}^{-1}(x_1), \tilde{G}^{-1}(y_1)) \oplus H((\tilde{F}^{-1}(x_2), \tilde{G}^{-1}(y_2))$$

and

$$\tilde{C}(x_1,y_2) \oplus \tilde{C}(x_2,y_1) = H(\tilde{F}^{-1}(x_1), \tilde{G}^{-1}(y_2)) \oplus H(\tilde{F}^{-1}(x_2), \tilde{G}^{-1}(y_1)).$$

Since H is 2-increasing, we get

$H(\tilde{F}^{-1}(x_1), \tilde{G}^{-1}(y_1)) \oplus H(\tilde{F}^{-1}(x_2), \tilde{G}^{-1}(y_2))$

$\succeq H(\tilde{F}^{-1}(x_1), \tilde{G}^{-1}(y_2)) \oplus H(\tilde{F}^{-1}(x_2), \tilde{G}^{-1}(y_1))$.

Thus we have

$$\tilde{C}(x_1,y_1) \oplus \tilde{C}(x_2,y_2) \succeq \tilde{C}(x_1,y_2) \oplus \tilde{C}(x_2,y_1).$$

So $\tilde{C}$ is 2-increasing. □

The α-cut of $\tilde{C}$ is done by

$$\tilde{C}(x,y)^{[\alpha]} = [H(\tilde{F}^{-1}(x)_{\frac{\alpha}{2}}, \tilde{G}^{-1}_{\frac{\alpha}{2}}(y)), H(\tilde{F}^{-1}_{1-\frac{\alpha}{2}}(x), \tilde{G}^{-1}_{1-\frac{\alpha}{2}}(y))]. \tag{3.16}$$

Example 3.11 ([2])**.** Let X and Y two random variables with a joint distribution

$$H(x,y)=\frac{(x+1)(1-e^{-y})}{(x-1)e^{-y}+2},(x,y)\in[-1,1]\times[0,+\infty], \tag{3.17}$$

where X follows a uniform distribution on $[-1,1]$ and Y follows an exponential distribution with parameter 1. The copula associated with this distribution is the copula

$$C(u,v)=\frac{uv}{u+v-uv}. \tag{3.18}$$

The α-pssimistic of $\tilde{X}$ and $\tilde{Y}$ are done by:

$$\tilde{X}_\alpha=\begin{cases}4\alpha-3 & \text{if} \quad 0\le\alpha\le\frac{1}{2}\\ 4\alpha-1 & \text{if} \quad \frac{1}{2}\le\alpha\le 1,\end{cases}$$

$$\tilde{Y}_\alpha=\begin{cases}2\alpha+1 & \text{if} \quad 0\le\alpha\le\frac{1}{2}\\ 2\alpha+1 & \text{if} \quad \frac{1}{2}\le\alpha\le 1.\end{cases}$$

For $0\le\alpha\le\frac{1}{2}$, we have

$$\begin{aligned}\tilde{F}_{\tilde{X}_\alpha}&=P(\tilde{X}_{\frac{\alpha}{2}}\le x)\\&=P(\underline{x}-2\alpha+3\le x)\\&=P(\underline{x}\le x+2\alpha-3)\\&=\frac{x+2\alpha-3+1}{2}\\&=\frac{x+2\alpha-2}{2}.\end{aligned}$$

Then we get

$$(\tilde{F}^{-1}_{\tilde{X}}(u))_\alpha=2u+2-2\alpha,\ \tilde{G}_{\tilde{Y}_\alpha}(y)=1-e^{-y-\alpha-1},\ (\tilde{G}^{-1}_{\tilde{X}}(v))_\alpha=-ln(1-v)-\alpha-1.$$

Thus we have

$$\begin{aligned}C^L(u,v)_\alpha=\tilde{C}(u,v)_\alpha&=H(\tilde{F}^{-1}(u))_\alpha,(\tilde{G}^{-1}(v))_\alpha)\\&=\frac{(2u+2-2\alpha+1)(1-e^{-(-ln(1-v)-\alpha-1)})}{(2u+2-2\alpha-1)e^{-(-ln(1-v)-\alpha-1)}+2}\\&=\frac{(2u+3-2\alpha)(1-(1-v)e^{-\alpha-1)})}{(2u+1-2\alpha)(1-v)e^{-\alpha-1)}+2}.\end{aligned}$$

For $\frac{1}{2}\le\alpha\le 1$, similarly we have

$$C^R(u,v)_\alpha=\frac{(2u+1-2\alpha)(1-(1-v)e^{-\alpha-1)})}{(2u-1-2\alpha)(1-v)e^{-\alpha-1)}+2}. \tag{3.19}$$

The graphical representaton of this copula is done by Figure 6.

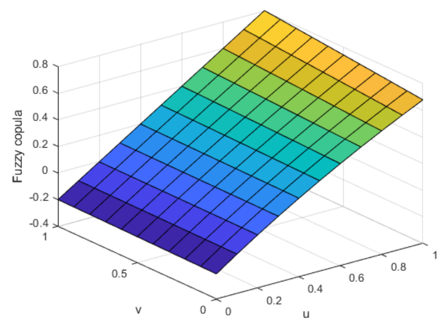

FIGURE 6. Fuzzy copula for $\alpha = 0.7$

3.3.1. *Construction of fuzzy copulas.*

In this subsecton, we constuct the novel copulas

3.3.2. *Fuzzy independent copula.*

Let $\tilde{X}$ and $\tilde{Y}$ two independent fuzzy random variables, we have

$$\begin{aligned}\tilde{C}_\alpha(x,y) &= H(\tilde{F}^{-1}(x))_\alpha, (\tilde{G}^{-1}(y))_\alpha) \\ &= \tilde{F}(\tilde{(}F^{-1}(x))_\alpha.\tilde{G}(\tilde{(}G^{-1}(y))_\alpha \\ &= \tilde{x}_\alpha.\tilde{y}_\alpha.\end{aligned}$$

The representation of α-cut of the independant fuzzy copula is Figure 7.

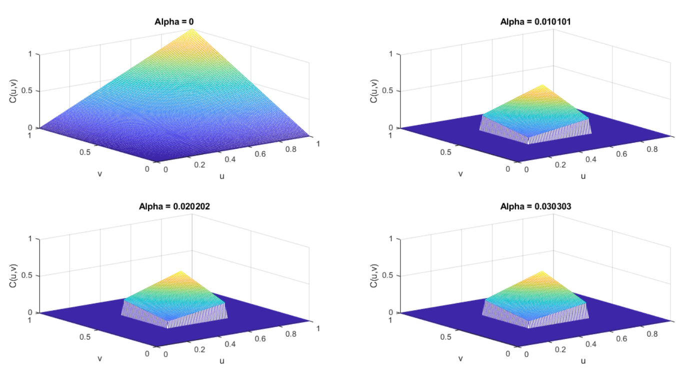

FIGURE 7. The α- cuts of fuzzy independant copulas

3.3.3. *Fuzzy minimum and maximum copulas.*

Proposition 3.12. *Let $\tilde{M} : I^2 \to I$ be the function defined as follows: for all $x,\ y,\ \alpha \in I$*

$$\tilde{M}(x,y)_\alpha = \frac{1}{2}(\tilde{x}_\alpha \oplus \tilde{y}_\alpha \ominus |\tilde{x}_\alpha \ominus \tilde{y}_\alpha|. \tag{3.20}$$

Then $\tilde{M}$ is a fuzzy minimum copula.

Proof. Let $x,\ y,\ \alpha \in I$. Then we have

$$\begin{aligned}\tilde{M}(0,y)_\alpha =& \frac{1}{2}(\tilde{0}_\alpha \oplus \tilde{y}_\alpha \ominus |\tilde{0}_\alpha \ominus \tilde{y}_\alpha| \\ &= \frac{1}{2}|\tilde{y}_\alpha \ominus \tilde{y}_\alpha| \\ &= \frac{1}{2}(\tilde{0}_\alpha) \\ &= \tilde{0}_\alpha,\end{aligned}$$

$$\begin{aligned}\tilde{M}(x,0)_\alpha =& \frac{1}{2}(\tilde{x}_\alpha \oplus \tilde{0}_\alpha \ominus |\tilde{x}_\alpha \ominus \tilde{0}_\alpha| \\ &= \frac{1}{2}(\tilde{x}_\alpha \ominus \tilde{x}_\alpha) \\ &= \frac{1}{2}(\tilde{0}_\alpha) \\ &= \tilde{0}_\alpha,\end{aligned}$$

$$\begin{aligned}\tilde{M}(x,1)_\alpha =& \frac{1}{2}(\tilde{x}_\alpha \oplus \tilde{1}_\alpha \ominus |\tilde{x}_\alpha \ominus \tilde{1}_\alpha| \\ &= \frac{1}{2}(\tilde{x}_\alpha \oplus \tilde{1}_\alpha \oplus \tilde{x}_\alpha \ominus \tilde{1}_\alpha) \\ &= \frac{1}{2}(\tilde{x}_\alpha \oplus \tilde{x}_\alpha) \\ &= \tilde{x}_\alpha,\end{aligned}$$

$$\begin{aligned}\tilde{M}(1,y)_\alpha =& \frac{1}{2}(\tilde{1}_\alpha \oplus \tilde{x}_\alpha \ominus |\tilde{1}_\alpha \ominus \tilde{y}_\alpha| \\ &= \frac{1}{2}(\tilde{1}_\alpha \oplus \tilde{y}_\alpha \ominus \tilde{1}_\alpha \oplus \tilde{y}_\alpha) \\ &= \frac{1}{2}(\tilde{y}_\alpha \oplus \tilde{y}_\alpha) \\ &= \tilde{y}_\alpha.\end{aligned}$$

Furthermore, we have

$$\tilde{M}(x_1,y_1)_\alpha \oplus \tilde{M}(x_2,y_2)_\alpha \ominus \tilde{M}(x_1,y_2)_\alpha \ominus \tilde{M}(y_1,x_2)_\alpha \succeq \tilde{0} \tag{3.21}$$

for all $x_1, x_2, y_1, x_1, y_1 \in I$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. □

The representation of α-cut of the minimum fuzzy copula is Figure 8.

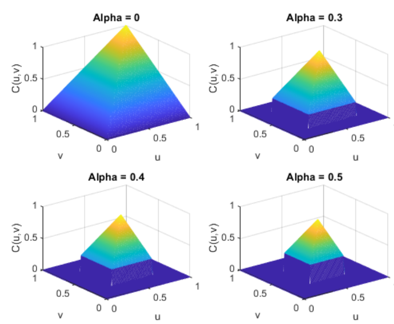

FIGURE 8. Fuzzy minimum copula

Proposition 3.13. *Let $\tilde{W} : I^2 \to I$ be the function defined as follows: for all $u,\ v,\ \alpha \in I$,*

$$\tilde{W}(u,v)_\alpha = \frac{\tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus \tilde{1} \oplus |\tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus \tilde{1}|}{2}. \tag{3.22}$$

Then $\tilde{W}$ is a fuzzy maximum copula.

Proof. The proof is the same way as Proposition 3.12.

The representation of α-cut of the maximum fuzzy copula is Figure 9.

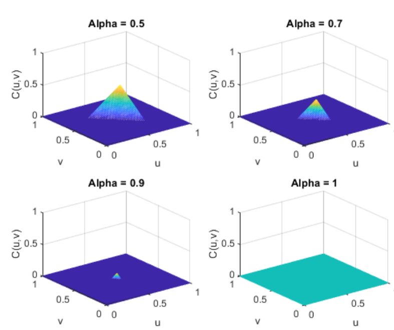

FIGURE 9. Fuzzy maximum copula

□

3.3.4. *Fuzzy survival copula.*

Fuzzy survival copulas enable modeling the dependence between fuzzy variables related to the survival of individuals in a population while managing data uncertainty.

Definition 3.14. The fuzzy random variables of interest represent the lifetimes of individuals or objects in some population. The *probability* of an individual living or surviving beyond time x is given by the survival function

$$\hat{F}(x) = P[\tilde{X}_\alpha > x] = 1 - \tilde{F}(x),$$

where $\tilde{F}$ is the cumulative distribution of $\tilde{X}$ and $\alpha \in I$.

Definition 3.15. Let be $(\tilde{X}, \tilde{Y})$ be a pair of fuzzy random variables with joint distribution function $\tilde{H}$, the *joint survival function* is given by

$$\hat{H}(x, y) = P[\tilde{X}_\alpha > x, \tilde{Y}_\alpha > y].$$

Proposition 3.16 (Fuzzy survival copula)**.** *Let $\tilde{X}$ and $\tilde{Y}$ two fuzzy random variables and C a copula. Let $\tilde{\xi} : I^2 \to I$ be the function defined as follows: for all , $u,\ v,\ \alpha \in I$,*

$$\tilde{\xi}(u, v)_\alpha = \tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{C}_\alpha(1 \ominus_{gH} u, 1 \ominus_{gH} v). \tag{3.23}$$

Then $\tilde{\xi}$ is a fuzzy maximum or minimum copula.

In this case, $\tilde{\xi}$ is called the *survival fuzzy copula* of C.

Proof. Let $u,\ V,\ \alpha \in I$. Then we have

$$\begin{aligned}
\tilde{\xi}(0, v)_\alpha &= \tilde{0}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{C}(1, 1 \ominus_{gH} v)_\alpha \\
&= \tilde{v}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{1}_\alpha \ominus_{gH} \tilde{v}_\alpha \\
&= \tilde{0}_\alpha,
\end{aligned}$$

$$\begin{aligned}
\tilde{\xi}(u, 0)_\alpha &= \tilde{u}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{C}(1 \ominus_{gH} u, 1)_\alpha \\
&= \tilde{u}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{1}_\alpha \ominus_{gH} \tilde{u}_\alpha \\
&= \tilde{0}_\alpha,
\end{aligned}$$

$$\begin{aligned}
\tilde{\xi}(1, v)_\alpha &= \tilde{1}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{1} \oplus \tilde{C}(0, 1 \ominus_{gH} v)_\alpha \\
&= \tilde{v}_\alpha,
\end{aligned}$$

$$\begin{aligned}
\tilde{\xi}(u, 1)_\alpha &= \tilde{u}_\alpha \oplus \tilde{1}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{C}(1 \ominus_{gH} u, 0)_\alpha \\
&= \tilde{u}_\alpha.
\end{aligned}$$

Moreover, we get

$$\frac{\partial^2 \tilde{\xi}}{\partial u \partial v}(u, v)_\alpha = \frac{\partial}{\partial u}(\frac{\partial \tilde{\xi}}{\partial v}(u, v)) = \frac{\partial}{\partial u}(1 + \frac{\partial \tilde{C}(u, v)}{\partial v}) = 0 + \frac{\partial^2 \tilde{C}(u, v)}{\partial u \partial v} \succeq 0. \tag{3.24}$$

Thus ξ is 2-increasing. □

Example 3.17. Let us determine the fuzzy survival copula associated with the independent copula.

$$\begin{aligned}
\xi(u, v)_\alpha &= \tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{C}_\alpha(1 - u, 1 - v) \\
&= \tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus (\tilde{1}_\alpha \ominus_{gH} \tilde{u}_\alpha)(\tilde{1}_\alpha \ominus_{gH} \tilde{v}_\alpha) \\
&= \tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{u}_\alpha \ominus_{gH} \tilde{v}_\alpha \oplus \tilde{u}_\alpha \tilde{u}_\alpha \\
&= \tilde{u}_\alpha \tilde{v}_\alpha.
\end{aligned}$$

This reflects the fact that the independence of random variables is preserved, even when considering their complementary events.

Example 3.18. Let X and Y be two random variables with a joint distribution

$$H(x,y) = \frac{(x+1)(1-e^{-y})}{(x-1)e^{-y}+2}, (x,y) \in [-1,1] \times [0,+\infty], \tag{3.25}$$

where X follows a uniform distribution on $[-1,1]$ and Y follows an exponential distribution with parameter 1. Then we have

$$C(u,v) = H(F^{-1}(u), G^{-1}(v)) \tag{3.26}$$

$$= \frac{uv}{u+v-uv}, \tag{3.27}$$

where $F(u) = \frac{u+1}{2}$ and $G(v) = 1 - e^{-v}$.

Now, we want to construct the fuzzy survival copula associate to this copula.

$$\begin{aligned}
\xi(u,v)_\alpha &= \tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \tilde{C}_\alpha(1-u, 1-v) \\
&= \tilde{u}_\alpha \oplus \tilde{v}_\alpha \ominus_{gH} \tilde{1}_\alpha \oplus \frac{(\tilde{1} \ominus_{gH} u)(\tilde{1} \ominus_{gH} v)}{\tilde{1} \ominus_{gH} u \oplus \tilde{1} \ominus_{gH} v \ominus_{gH} (\tilde{1} \ominus_{gH} u)(\tilde{1} \ominus_{gH} v)} \\
&= \frac{\tilde{u}_\alpha^2 \tilde{v}_\alpha \ominus_{gH} \tilde{v}_\alpha^2 \tilde{u}_\alpha \oplus 2\tilde{u}_\alpha \tilde{v}_\alpha}{\tilde{1} \ominus_{gH} \tilde{u}_\alpha \tilde{v}_\alpha}.
\end{aligned}$$

The fuzzy survival copula associated with this fuzzy survival copula is shown on Figure 10.

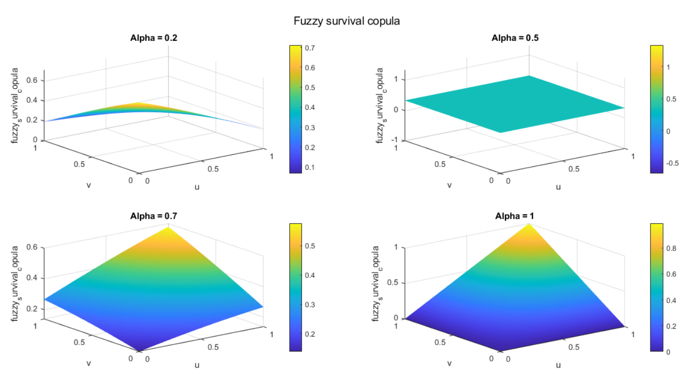

FIGURE 10. Fuzzy survival copula

3.3.5. *Survival copula analysis.* Fuzzy survival copulas model the dependence between fuzzy variables representing the survival of individuals in a population, accounting for data uncertainty. Fuzzy survival copulas model the dependence between fuzzy random variables by incorporating imprecise margins and parameters. Their construction involves fuzzifying the margins, selecting a classical copula, and applying an appropriate transformation. Useful in epidemiology, and risk management, they better handle data uncertainty.

The fuzzy survival copula captures the dependence between X and Y in fuzzy form, taking uncertainty into account through the parameter α.

Now by doing an analysis and simulation with (3.18) simulating, the simulation generates n random points for u and v and calculates the fuzzy margins and survival copula for each (u, v) pair. The parameter α controls the degree of uncertainty, and a sample of 1000 points is simulated for α=0.3 and $\alpha = 0.7$ The data generated is organized in a data frame comprising the values, fuzzy margins. The results are displayed as follows:

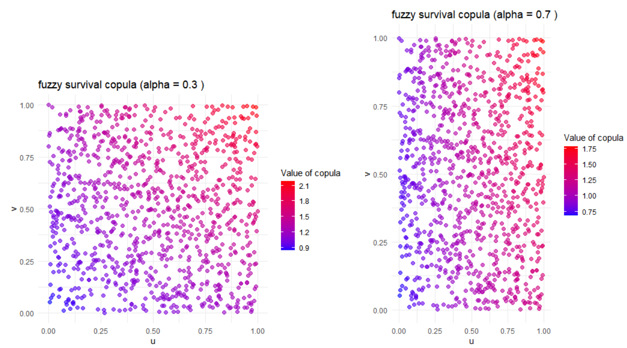

FIGURE 11. Fuzzy survival copula for $\alpha = 0.3$ and $\alpha = 0.7$

A scatter plot shows the points (u, v) colored according to copula value, ranging from blue (low dependence) to red (high dependence). For u and v close to 1, the value of the copula tends to be high (red), indicating strong dependency. Areas where u or v are low show reduced dependencies (blue).

The statistical summary of the fuzzy copula is calculated to obtain information on the distribution of values (minimum, maximum, mean, median). Correlations are calculated between the fuzzy margins and the copula. This allows us to assess the relationship between margins and the overall dependency modeled by the copula. We have

min value of the fuzzy copula	-3.23
maxvalue of the fuzzy copula	5.58
r	0.5
n	1000

TABLE 1. Statistical Analysis of the copula

The results allow visualization and analysis of the impact of α on the fuzzy survival copula and its margins. A high value of αincreases uncertainty in the margins and spreads the dependence curves in the copula. Correlation analysis helps understand how the fuzzy margins influence the dependence between variables.

4. Conclusion

In conclusion, the construction of fuzzy copulas offers several advantages. It allows us to take into account the uncertain nature of the data and to express the dependencies between variables in a more realistic way. We have clearly defined the concepts of fuzzy random variables, fuzzy cumulative distribution functions, fuzzy distribution functions and fuzzy copulas. In addition, we have constructed fuzzy copulas such as the minimum fuzzy copula, the maximum fuzzy copula and the fuzzy survival copula. This contribution provides researchers with tools to better represent the complexity of systems due to uncertainty in various domains. Looking ahead, we plan to build fuzzy copulas in 3 or 4 dimensions and apply them in medicine or the environment, where data is generally fuzzy, to better study dependence between variables and make predictions.

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Remi Guillaume Bagré (baremgui@yahoo.fr)
Département de Mathématiques, Laboratoire de Mathématiques, Informatique et Applications, Université Norbert Zongo, BP 376 Koudougou, Burkina Faso

Sayouba Traoré (tagawere2@gmail.com)
Département de Mathématiques, Laboratoire de Mathématiques, Informatique et Applications, Université Norbert Zongo, BP 376 Koudougou, Burkina Faso

Abdoulaye Compaoré (ab.compaore1@gmail.com)
Département de Mathématiques, Laboratoire de Mathématiques, Informatique et Applications, Université Norbert Zongo, BP 376 Koudougou, Burkina Faso