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Applications of fixed point theorems in fuzzy Fréchet space to dynamic problems

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Abstract. This paper aims to present new results for fixed point theorems in fuzzy Fréchet spaces, focusing on fuzzy contraction mappings under certain conditions. We apply our results to solve dynamic problems using the derived fixed-point theorems.

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1. Introduction

Fixed point theory is a renowned and classical branch of mathematics with numerous applications. The Banach contraction mapping theorem is one of the most significant results in analysis. It is a widely used tool for addressing existing problems across various fields of mathematics. Many generalizations of the Banach mapping theorem are discussed in the literature (See [1, 2, 3, 4, 5, 6] among others). The concept of a fuzzy set, along with some fundamental operations on fuzzy sets, was introduced by Zadeh [7]. Katsaras [8] defined the concept of fuzzy seminorm in 1984. Later, in 2007, Sadeqi and Solaty [9] further developed this concept. In 2021, Jasim and Al-Nafie [10, 11] developed the concept of fuzzy Fréchet space using two equivalent approaches. In 2024, researchers in [1] established a common fixed point for mappings in fuzzy Fréchet space using the concepts of commuting and compatibility. The goal of this paper is to establish new fixed-point theorems for contraction mappings in fuzzy Fréchet space. Additionally, we apply our results to demonstrate the existence of solutions for dynamic problems.

2. Preliminaries

Definition 2.1 ([10])**.** A complete fuzzy topological vector space W is said to be a *fuzzy Fréchet space* (for short, FFS), if whose fuzzy topology τ_D induced by a countable separating family of fuzzy seminorms $D = \{\rho_j\}_{j\in J}$.

The construct of FFS, along with the concepts of fuzzy continuity, convergence and Cauchy sequence in FFS, are discussed in [10, 11].

Definition 2.2 ([12])**.** Let W be an FFS with family of fuzzy seminorms $D = \{\rho_j\}_{j\in J}$. A mapping $g : W \longrightarrow W$ is said to be a *fuzzy c-contraction*, if $\exists c \in (0, 1)$ such that $\rho_j(g(v) - g(u), h) \geq \rho_j(v - u, \frac{h}{c})\ \forall h > 0,\ \forall u,\ v \in W$ and $\forall \rho_j \in D$.

Theorem 2.3 ([12])**.** *If W be FFS, and $g : W \longrightarrow W$ be fuzzy c-contraction then g has a unique fixed point $u \in W$ and $\lim_{n\to\infty} g^n(v) = u\ \forall v \in W$.*

3. Mainresults

Let W be FFS with family of fuzzy seminorms $D = \{\rho_j\}_{j\in J}$.

Theorem 3.1. *Let $g : W \longrightarrow W$ be fuzzy continuous mapping and $\rho_j(., h)$ be a fuzzy continuous mapping $\forall h > 0$. Then g satisfies the following conditions:*

- $\rho_j(g(v_1) - g(v_2), h) \geq \rho_j(v_1 - v_2, h)\ \forall h > 0,\ \forall v_1,\ v_2 \in W,\ v_1 \neq v_2$ *and* $\forall \rho_j \in D$,
- *if the closure of $\overline{(g-I)(W)}$ is invariant to g and compact, then g has a unique fixed point $x \in \overline{(g - I)(W)}$. In addition, $\forall v_0 \in \overline{(g - I)(W)}$, the sequence $g^n(v_0)$ converge to v_0.*

Proof. Let $h > 0$. Based on the hypothesis, the continuous mapping $W \mapsto \rho_j(g(v) - v, h)$ reaches its maximum at a certain point $x \in \overline{(g - I)(W)}$. If $g(x) \neq x$, then we get: $\forall \rho_j \in D$,

$\rho_j(g(x) - x, h) = \max_{v\in(h-I)(W)} \rho_j(g(v) - v, h)$
$\geq \rho_j(g(g(x)) - g(x), h)$
$> \rho_j(g(x) - x, h).$

This is not possible. Thus x is a fixed point for g.

Let y be another fixed point of g such that $y \neq x$. Then we have

$$\rho_j(y - x, h) = \rho_j(g(y) - g(x), h) > \rho_j(y - x, h)\ \forall \rho_j \in D \text{ and } \forall h > 0.$$

This is not possible. Thus $y = x$. So x is the sole fixed point of g and also serves as a fixed point for g^n.

Consider $v_0 \in \overline{(g - I)(W)}$ and $\lim_{n\to\infty} g^n(v_0) = s$. Since g is a fuzzy continuous, $g(s) = \lim_{n\to\infty} g^{n+1}(v_0) = s$. Then s is a fixed point for g. Since x is the unique fixed point of g, it follows that $s = x$. □

The following example demonstrates the existence of fuzzy continuous mapping $FFS\ W$ that satisfies in theorem 2.3 this continuous mapping has the property that $g - I$ is not onto, which contrasts with scenario where g is a fuzzy c-contraction.

Example 3.2. In the FFS W, we consider $\rho_j(w,h) = \begin{cases} \frac{h}{h+|w|} & h > 0,\ w \in R \\ 0 & h = 0,\ w \in R. \end{cases}$

Let $P : R \to R$ be given by $P(w) = \begin{cases} w - \frac{w^2}{w^2+1} & w \geq 0 \\ w + \frac{w^2}{w^2+1} & w < 0. \end{cases}$

Then P is fuzzy continuous, the $\overline{(P-I)R}$ is invriant to P and a compact set, and $\rho_j(P(w_1) - P(w_2), h) > \rho_j(w_1 - w_2, h)\ \forall h > 0, \forall w_1, w_2 \in R, w_1 \neq w_2$ and $\forall \rho_j \in D$. Actually, it is straight forward to very that P is fuzzy continuous, the $\overline{(P-I)R}$ is $[-1,1]$ and $P[-1,1] = [\frac{-1}{2}, \frac{1}{2}] \subset [-1,1]$. Let $h > 0, w_1, w_2 > 0, w_1 \neq w_2$ and $\rho_j \in D$. Then we get

$$\begin{aligned}\rho_j(P(w_1) - P(w_2), h) &= \frac{h}{h + |P(w_1) - P(w_2)|} \\ &= \frac{h}{h + |w_1 - \frac{w_1^2}{w_1^2+1} - w_2 + \frac{w_2^2}{w_2^2+1}|} \\ &= \frac{h}{h + |(w_1 - w_2)(1 - \frac{w_1+w_2}{(w_1^2+1)(w_2^2+1)}|}.\end{aligned}$$

Since $|(1 - \frac{w_1+w_2}{(w_1^2+1)(w_2^2+1)}| < 1,\ \rho_j(P(w_1) - P(w_2), h) > \rho_j(w_1 - w_2, h)\ \forall \rho_j \in D$. If $h > 0$ and $w_1, w_2 < 0, w_1 \neq w_2$ and $\forall \rho_j \in D$, then

$$\rho_j(P(w_1) - P(w_2), h) = \frac{h}{h + |(w_1 - w_2)(1 + \frac{w_1+w_2}{(w_1^2+1)(w_2^2+1)})|}.$$

Since $|(1 + \frac{w_1+w_2}{(w_1^2+1)(w_2^2+1)}| < 1,\ \rho_j(P(w_1) - P(w_2), h) > \rho_j(w_1 - w_2, h)\ \forall \rho_j \in D$. In the end, If $h > 0$ and $w_1 \geq 0, w_2 < 0$ and $\rho_j \in D$, then we get

$$\rho_j(P(w_1) - P(w_2), h) = \frac{h}{h + |w_1 - \frac{w_1^2}{w_1^2+1} - w_2 - \frac{w_2^2}{w_2^2+1}|}.$$

As $2u - \frac{u^2}{u^2+1} > 0, \forall u > 0,\ |w_1 - w_2| > |w_1 - \frac{w_1^2}{w_1^2+1} - w_2 + \frac{w_2^2}{w_2^2+1}|$. Thus

$$\rho_j(P(w_1) - P(w_2), h) > \rho_j(w_1 - w_2, h)\ \forall \rho_j \in D.$$

In the subsequent theorem, the property of fixed point continuous dependence on a parameter is examined.

Theorem 3.3. *For any $\rho_j \in D$ and $v \neq 0$, let $\rho_j(v, .)$ be strictly increasing. Let φ be Hausdorff fuzzy topological space. If $g : W \times \varphi \longrightarrow \varphi$ has the following properties:*

- *the mapping $g_r : \varphi \longrightarrow W, g_v(u) = g(v,u)$ is fuzzy continuous $\forall v \in W$,*
- *$\exists \lambda \in (0, \frac{1}{2}) \ni \rho_j(g(v_1,u) - g(v_2,u), h) > \rho_j(v_1 - v_2, \frac{h}{\lambda}))\ \forall v_1, v_2 \in W, \forall u \in \varphi, \forall h > 0$ and $\forall \rho_j \in D$.*

Then $\forall u \in \varphi$, the mapping $g_u : W \longrightarrow W,\ g_u(v) = g(v,w)$, has a unique fixed point v_u and $f : \varphi \longrightarrow W,\ f(u) = v_u$, is fuzzy continuous.

Proof. Since, for any $u \in \varphi, g_u$ is fuzzy contraction , then by Theorem 2.3, we get g_u has a unique fixed point v_u. Consider $u_0, u \in \varphi,\ h > 0$ and $\rho_j \in D$. Then we have

$$\begin{aligned}\rho_j(f(u) - f(u_0), h) &= \rho_j(v_u - v_{u_0}) \\ &= \rho_j(g_u(v_u) - g_{u_0}(v_{u_0}), h) \\ &= \rho_j(g_u(v_u) - g_u(v_{u_0}) + g_u(v_{u_0}) - g_{u_0}(v_{u_0}), h) \\ &\geq \min\{\rho_j(g_u(v_u) - g_u(v_{u_0}), \frac{h}{2}), \rho_j(g_u(v_{u_0}) - g_{u_0}(v_{u_0}), \frac{h}{2})\} \\ &\geq \min\{\rho_j(v_u - v_{u_0}, \frac{h}{2\lambda}), \rho_j(g_u(v_{u_0}) - g_{u_0}(v_{u_0}), \frac{h}{2})\}.\end{aligned}$$

Now, considering the hypothesis is regarding the fuzzy seminorms $\rho_j \in D$. We have that $\rho_j(f(u) - f(u_0), h) \geq \rho_j(g_u(v_{u_0}) - g_{u_0}(v_{u_0}), \frac{h}{2})\ \forall u_0,\ u \in \varphi,\ h > 0$ and $\forall \rho_j \in D$. We demonstrate f is fuzzy continuous. Let $u_0 \in \varphi$. Since $g_{v_{u_0}}$ is fuzzy continuous at u_0, $\forall \alpha \in (0,1)$ and $\forall h > 0,\ \exists M$ is a neighborhood of u_0 such that

$\rho_j(g_u(v_{u_0}) - g_{u_0}(v_{u_0}), h) = \rho_j(g_u(v_{u_0}(u)) - g_{u_0}(v_{u_0}), h) > 1 - \alpha\ \forall u \in M$ and $\forall \rho_j \in D$.

Thus f is fuzzy continuous at u_0. So f is fuzzy continuous on φ. □

Theorem 3.4. *For any $y \neq 0$, let $\rho_j(y, .)$ be strictly increasing $\forall \rho_j \in D$. If $f : W \longrightarrow W$ is a fuzzy continuous such that there are $d,\ e \in (0,1),\ d + e < 1$: $\rho_j(f_y - f_x, h) \geq \min\{\rho_j(y - f_x, \frac{h}{d}), \rho_j(x - f_y, \frac{h}{e})\},\ \forall x,\ y \in W,\ \forall h > 0$ and $\forall \rho_j \in D$. Then f has a fixed point.*

Proof. Assume that $d \leq e$. Let $y \in W$ and $x = f_y$. Then for $h > 0$, we get

$$\begin{aligned}\rho_j(f_y - f_y^2, h) &\geq \min\{\rho_j(y - f_y^2, \frac{h}{d}), \rho_j(f_y - f_y, \frac{h}{e})\} \\ &= \rho_j(y - f_y^2, \frac{h}{d}) = \rho_j(y - f_y + f_y - f_y^2, \frac{h}{d+e} + \frac{h}{\frac{d(d+e)}{e}})\ \forall \rho_j \in D.\end{aligned}$$

Since $d + e < 1$ and the condition of ρ_j, we get

$$\rho_j(f_y - f_y^2, h) < \rho_j(f_y - f_y^2, \frac{h}{\frac{d(d+e)}{e}})\ \forall \rho_j \in D \text{ for } f_y \neq f_y^2.$$

Thus we have

$$\rho_j(f_y - f_y^2, h) \geq \rho_j(y - f_y -, \frac{h}{d+e})\ \forall h > 0 \text{ and } \forall \rho_j \in D.$$

In the same way, for $e < d$, we get

$$\rho_j(f_y - f_y^2, h) \geq \rho_j(y - f_y -, \frac{h}{d+e})\ \forall h > 0 \text{ and } \forall \rho_j \in D.$$

It is now straight forward to see that f has a fixed point. □

We will now demonstrate the latter theorem in this paper.

Theorem 3.5. *For any $y \neq 0$, let $\rho_j(y, .)$ be strictly increasing $\forall \rho_j \in D$. If $f : W \longrightarrow W$ is a fuzzy continuous such that there are $d,\ e,\ s \in (0,1) : \rho_j(f_y - f_x, h) \geq \min\{\rho_j(y - x, \frac{h}{d}), \rho_j(y - f_y, \frac{h}{e}), \rho_j(x - f_x, \frac{h}{s})\}\ \forall x,\ y \in W,\ \forall h > 0$ and $\forall \rho_j \in D$. Then f has a unique fixed point.*

Proof. Assume that $y \in W$ and $x = f_y$. Then for $h > 0$, we get

$$\rho_j(f_y - f_y^2, h) \geq \min\{\rho_j(y - f_y, \frac{h}{d}), \rho_j(y - f_y, \frac{h}{e}), \rho_j(f_y - f_y^2, \frac{h}{s})\} \ \forall \rho_j \in D$$

Since $\rho_j(f_y - f_y^2, \frac{h}{s}) > \rho_j(f_y - f_y^2, h)$ for $f_y \neq f_y^2$, we have

$$\rho_j(f_y - f_y^2, h) \geq \rho_j(y - f(y), \min\{\frac{h}{d}, \frac{h}{e}\}) \ \forall \rho_j \in D.$$

As the proof of theorem 3.4, we get f has a fixed point $x \in W$.

Now, we suppose that $f_x = x$ and $f_m = m$. Then we get

$$\begin{aligned}\rho_j(x - m, h) &= \rho_j(f_x - f_m, h)\\ &\geq \min\{\rho_j(x - m, \frac{h}{d}), \rho_j(x - f_x, \frac{h}{e}), \rho_j(m - f_m, \frac{h}{s})\}\\ &= \min\{\rho_j(x - m, \frac{h}{d}, 1, 1)\} = \rho_j(x - m, \frac{h}{d}) \ \forall h > 0, n \in N \text{ and } \forall \rho_j \in D.\end{aligned}$$

Thus $\rho_j(x - m, h) \geq \rho_j(x - m, \frac{h}{d^n}) \ \forall h > 0, n \in N$ and $\forall \rho_j \in D$. As $n \longrightarrow \infty$, it follows that $\rho_j(x - m, h) = 1 \ \forall h > 0$ and $\forall \rho_j \in D$. So $x = m$. □

4. Application

Our goal is to devise an optimal strategy for minimizing the total expected cost by solving a differential equation.

We encounter in dynamic programming g the following functional equations

$$g(y) = \max_x G(y, x, g(T(y, x))), \tag{4.1}$$

where y is the state variable, the decision variable is X, and the optimum mapping is g.

Fixed point theorems in fuzzy Fréchet space can be utilized in the analysis of these equations. For example, we will examine the following equation, referred to as the "optimal supply distribution equation"

$$g(y) = \inf_{x \geq y} G(y, x, g), \tag{4.2}$$

where

(4.3)

$$G(y,x,g) = F(y-x) + r\int_x^\infty q(\alpha - x)S(\alpha)d\alpha + rg(0)\int_x^\infty S(\alpha)d\alpha + r\int_0^x g(x-\alpha)S(\alpha)d\alpha,$$

such that, g is the anonymous mapping, F, q, S are given mappings and $0 \neq r \in R^+$.
Let $C(R^+)$ is the space of continuous mapping on R^+, and let $BC(R^+)$ the subspace of bounded mappings within $C(R^+)$

Consider the family of seminorms $B = \{N_j\}_{j \in J}$ on $BC(R^+)$, defined as

$$N_j(g) = \sup_{v \in R^+} |g(v)|,$$

and there the family of fuzzy seminorms $D = \{\rho_j\}_{j \in J}$ such that

$$rho_j(g,h) = \begin{cases} 0 & h \leq 0 \\ \frac{g}{h+N_j(g)} & h > 0. \end{cases}$$

Then $BC(R^+)$ forms a fuzzy Fréchet space.
We will now define the operator $\phi : BC(R^+) \longrightarrow BC(R^+)$ such that

$$(\phi_g)(y) = \inf_{x \geq y} G(y,x,g).$$

To ensure that $BC(R^+)$ is an invariant subset for ϕ, we will impose the following assumption based on the problem's hypothesis:

- $\int_0^\infty S(\alpha)d\alpha = 1$ and $\int_{R^+} |S|$ finite,
- $\int_0^\infty q(\alpha)S(\alpha)d\alpha < \infty, S \in C(R^+)$.

It follows from equation (4.3) that ϕ is well-defined. We will now, check whether ϕ satisfies the fuzzy contraction condition.

$$\rho_j(\phi(g_1) - \phi(g_2), h) = \frac{h}{h + \sup_{y\in R^+}(\phi(g_1(y)) - \phi(g_2(y)))}$$

$$= \frac{h}{h + \sup_{y\in R^+} |\inf_{x\geq y} G(y,x,g_1) - \inf_{x\geq y} G(y,x,g_2)}$$

$$\geq \frac{h}{h + \sup_{y\in R^+}\sup_{x\geq y} |(G(y,x,g_1) - G(y,x,g_2)|}$$

$$\geq \frac{h}{h + r + \sup_{y\in R^+}\sup_{x\geq y} |(g_1(0) - g_2(0))| \int_x^\infty S(\alpha)d\alpha + \int_0^x |g_1(x-\alpha) - g_2(x-\alpha)g(\alpha)d\alpha}$$

$$\geq \frac{h}{h + r + \sup_{y\in R^+}\sup_{x\geq y} |(g_1(0) - g_2(0))| \int_x^\infty S(\alpha)d\alpha + \sup_{y\in R^+} |g_1(x-y) - g_2(x-y)| \int_0^x S(\alpha)d\alpha}$$

$$\geq \frac{h}{h + r + \sup_{y\in R^+}\sup_{x\geq y} |(g_1(0) - g_2(0))| \int_x^\infty S(\alpha)d\alpha + \sup_{y\in R^+} |g_1(y_1) - g_2(y_2)| \int_0^x S(\alpha)d\alpha}$$

$$\geq \frac{h}{h + r.N_j(g_1 - g_2)} = \rho_j(g_1 - g_2, \frac{h}{r})\ \forall \rho_j \in D,\ \forall h > 0 \text{ and } g_1, g_2 \in BC(R^+).$$

Then for $0 \neq r \in R^+$, we get

$$\rho_j(\phi(g_1) - \phi(g_2), h) \geq \rho_j(g_1 - g_2), \frac{h}{r})\ \forall \rho_j \in D,\ \forall h > 0, \text{ and } g_1,\ g_2 \in BC(R^+).$$

If $r \in (0,1)$ and the application ϕ is a fuzzy contraction, then from above theorems of fixed point in fuzzy Fréchet space, the equation (4.2) has a unique solution in $BC(R^+)$, which can be obtained using the method of successive approximation.

5. Conclusion

By applying fixed-point theorems in fuzzy Fréchet space, previously solved real-life problems, as well as some unresolved ones, can now be examined within a new framework. Such an approach was explored in the context mentioned above by applying the fixed-point theorems in FFS to a problem in dynamic programming.

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