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Abstract. Using the concepts of Łukasiewicz t-norm and Łukasiewicz t-conorm, the Łukasiewicz intuitionistic fuzzy set based on an intuitionistic fuzzy set is introduced and applied it to BCK-algebras and BCI-algebras. The notion of (strong) Łukasiewicz intuitionistic fuzzy subalgebra is introduced and its various properties are investigated. Characterizations of Łukasiewicz intuitionistic fuzzy subalgebras are discussed. Conditions for Łukasiewicz intuitionistic fuzzy set to be a Łukasiewicz intuitionistic fuzzy subalgebra are explored, and the conditions under which Łukasiewicz intuitionistic fuzzy subalgebra becomes strong are provided. Three types of subsets so called Łukasiewicz $\in$-set, Łukasiewicz q-set and Łukasiewicz O-set are established, and the conditions under which they can be subalgebras are displayed.

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Keywords: Łukasiewicz fuzzy set, Łukasiewicz $\in$-sets, (Strong) Łukasiewicz fuzzy subalgebra, Łukasiewicz q-sets, Łukasiewicz O-sets.

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1. Introduction

The intuitionistic fuzzy set, which was introduced by Krassimir Atanassov in 1986 (See [1]), is an extension of a fuzzy set that accounts for both membership and non-membership degrees, as well as the hesitation or indeterminacy associated with an element's belongingness. The intuitionistic fuzzy set is particularly valuable in situations where partial and contradictory information is present, allowing for a nuanced representation of uncertainty, and it is applied to various fields such as decision-making under uncertainty, pattern recognition, medical diagnosis, knowledge representation in artificial intelligence, etc. The Łukasiewicz t-norm is a popular triangular norm (t-norm) used in fuzzy logic and multi-valued logic systems.

It defines a mathematical framework for combining fuzzy truth values in the unit interval $[0, 1]$. Jun constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set, and applied it to BCK-algebras, BCI-algebras, BE-algebras, hoops, Sheffer stroke Hilbert algebras, etc. (See [2, 3, 8, 9, 10, 11, 12, 13, 14, 15]). The Łukasiewicz t-conorm is the dual operation to the Łukasiewicz t-norm under the De Morgan laws and it is also used in fuzzy logic.

In this paper, using the idea of Łukasiewicz t-norm and Łukasiewicz t-conorm, we introduce the concept of Łukasiewicz intuitionistic fuzzy set based on an intuitionistic fuzzy set, and apply it to BCK-algebras and BCI-algebras. We define the concepts of (strong) Łukasiewicz intuitionistic fuzzy subalgebras, and investigate several properties. We explore conditions for Łukasiewicz intuitionistic fuzzy set to be a Łukasiewicz intuitionistic fuzzy subalgebra, and provide the conditions under which Łukasiewicz intuitionistic fuzzy subalgebra becomes strong. We disuss characterizations of Łukasiewicz intuitionistic fuzzy subalgebras. We construct a three kind of subsets so called Łukasiewicz $\in$-set, Łukasiewicz q-set and Łukasiewicz O-set, and we find the conditions under which they can be subalgebras.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (See [6] and [7]) and was extensively investigated by several researchers. We recall the definitions and basic results required in this paper. See the books [5, 16] for further information regarding BCK-algebras and BCI-algebras.

If a set X has a special element "0" and a binary operation "$*$" satisfying the conditions:

- (I_1) $(\forall \mathfrak{a}, \mathfrak{b}, \in X)$ $(((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0)$,
- (I_2) $(\forall \mathfrak{a}, \mathfrak{b} \in X)$ $((\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0)$,
- (I_3) $(\forall \mathfrak{a} \in X)$ $(\mathfrak{a} * \mathfrak{a} = 0)$,
- (I_4) $(\forall \mathfrak{a}, \mathfrak{b} \in X)$ $(\mathfrak{a} * \mathfrak{b} = 0,\ \mathfrak{b} * \mathfrak{a} = 0 \ \Rightarrow\ \mathfrak{a} = \mathfrak{b})$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

- (K) $(\forall \mathfrak{a} \in X)$ $(0 * \mathfrak{a} = 0)$,

then X is called a *BCK-algebra.*

A BCI-algebra X is said to be *p-semisimple* (See [5]), if $0 * (0 * \mathfrak{a}) = \mathfrak{a}$ for all $\mathfrak{a} \in X$.

The order relation "$\leq$" in a BCK/BCI-algebra X is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \leq \mathfrak{b} \ \Leftrightarrow\ \mathfrak{a} * \mathfrak{b} = 0). \tag{2.1}$$

Every BCK/BCI-algebra X satisfies the following conditions (See [5, 16]):

$$(\forall \mathfrak{a} \in X)\,(\mathfrak{a} * 0 = \mathfrak{a})\,, \tag{2.2}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,(\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a} * \mathfrak{c} \leq \mathfrak{b} * \mathfrak{c},\ \mathfrak{c} * \mathfrak{b} \leq \mathfrak{c} * \mathfrak{a})\,, \tag{2.3}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = (\mathfrak{a} * \mathfrak{c}) * \mathfrak{b})\,. \tag{2.4}$$

Every BCI-algebra X satisfies (See [5]):

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,(\mathfrak{a} * (\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) = \mathfrak{a} * \mathfrak{b})\,, \tag{2.5}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,(0 * (\mathfrak{a} * \mathfrak{b}) = (0 * \mathfrak{a}) * (0 * \mathfrak{b}))\,. \tag{2.6}$$

A subset K of a BCK/BCI-algebra X is called a *subalgebra* of X (See [5, 16]), if $\mathfrak{a} * \mathfrak{b} \in K$ for all $\mathfrak{a}, \mathfrak{b} \in K$.

Let X be a set. An *intuitionistic fuzzy set* $\mathcal{K}^*$ in X (See [1]) is an object having the form

$$\mathcal{K}^* := \{\langle \mathfrak{a}, f_K(\mathfrak{a}), g_K(\mathfrak{a})\rangle \mid f_K(\mathfrak{a}) + g_K(\mathfrak{a}) \leq 1,\ \mathfrak{a} \in X\},$$

which is simply denoted by $\mathcal{K}^* := (X; f_K, g_K)$ where f_K and g_K are fuzzy sets in X, The intuitionistic fuzzy set $\mathcal{K}^* := (X; f_K, g_K)$ in X can be represented as follows:

$$\mathcal{K}^* := (X; f_K, g_K) : X \to [0,1] \times [0,1], \mathfrak{a} \mapsto (f_K(\mathfrak{a}), g_K(\mathfrak{a}))$$

such that $f_K(\mathfrak{a}) + g_K(\mathfrak{a}) \leq 1$.

An intuitionistic fuzzy set $\mathcal{K}^* := (X; f_K, g_K)$ in a set X of the form:

$$\mathcal{K}^* := (X; f_K, g_K) : X \to [0,1] \times [0,1],$$
$$\mathfrak{b} \mapsto \begin{cases} (s,t) \in (0,1] \times [0,1) & \text{if } \mathfrak{b} = \mathfrak{a}, \\ (0,1) & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be an *intuitionistic fuzzy point* with the support and the value (s,t) such that $s + t \leq 1$, and is denoted by $\mathfrak{a}_{(s,t)}$.

Given an intuitionistic fuzzy set $\mathcal{K}^* := (X; f_K, g_K)$ and intuitionistic fuzzy point $\mathfrak{a}_{(s,t)}$ in X, we say

$$\mathfrak{a}_{(s,t)} \in \mathcal{K}^* \text{ if } f_K(\mathfrak{a}) \geq s \text{ and } g_K(\mathfrak{a}) \leq t. \tag{2.7}$$

$$\mathfrak{a}_{(s,t)}\, q\, \mathcal{K}^* \text{ if } f_K(\mathfrak{a}) + s > 1 \text{ and } g_K(\mathfrak{a}) + t < 1. \tag{2.8}$$

Given $(s,t) \in (0,1] \times [0,1)$ and an intuitionistic fuzzy set $\mathcal{K}^* := (X; f_K, g_K)$ in X, consider the following sets:

$$\begin{aligned} (f_K, s)_\in &:= \{\mathfrak{a} \in X \mid f_K(\mathfrak{a}) \geq s\}, \\ (g_K, t)_\in &:= \{\mathfrak{a} \in X \mid g_K(\mathfrak{a}) \leq t\}, \\ (f_K, s)_q &:= \{\mathfrak{a} \in X \mid f_K(\mathfrak{a}) + s > 1\}, \\ (g_K, t)_q &:= \{\mathfrak{a} \in X \mid g_K(\mathfrak{a}) + t < 1\}. \end{aligned}$$

Also, we consider the sets below.

$$\begin{aligned} (\mathcal{K}^*, (s,t))_\in &:= (f_K, s)_\in \cap (g_K, t)_\in, \\ (\mathcal{K}^*, (s,t))_q &:= (f_K, s)_q \cap (g_K, t)_q, \end{aligned}$$

which are called the *intuitionistic level set* and *intuitionistic* q*-set* of $\mathcal{K}^* := (X; f_K, g_K)$, respectively.

3. Łukasiewicz intuitionistic fuzzy sets

Using the ideas of Łukasiewicz t-norm and Łukasiewicz t-conorm and the intuitionistic fuzzy set, we build Łukasiewicz intuitionistic fuzzy set on a set. From now on, all elements (s,t) in $[0,1]\times[0,1]$ are elements that satisfy $s+t\in[0,1]$. Also the intuitionistic fuzzy set $\mathcal{K}^* := (X; f_K, g_K)$ in X is simply denoted by $(X; f, g)$.

Definition 3.1. Let $(X; f, g)$ be an intuitionistic fuzzy set in a set X and $\varepsilon, \delta \in [0,1]$ be such that $\varepsilon+\delta \leq 1$. A *Łukasiewicz intuitionistic fuzzy set* Ł of $(X; f, g)$ is defined as an object in the form below

$$Ł := \{\langle x, Ł_f^\delta, Ł_g^\varepsilon\rangle \mid x \in X\}, \tag{3.1}$$

where

$$Ł_f^\delta : X \to [0,1],\ x \mapsto \max\{0, f(x)+\delta-1\},$$
$$Ł_g^\varepsilon : X \to [0,1],\ x \mapsto \min\{1, g(x)+\varepsilon\}$$

such that $Ł_f^\delta(x) + Ł_g^\varepsilon(x) \in [0,1]$ for all $x \in X$.

We use the simple notation $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ instead of the Łukasiewicz intuitionistic fuzzy set $Ł := \{\langle x, Ł_f^\delta, Ł_g^\varepsilon\rangle \mid x \in X\}$ of $(X; f, g)$, and it can be represented as follows:

$$Ł := (Ł_f^\delta, Ł_g^\varepsilon) : X \to [0,1]\times[0,1],$$
$$x \mapsto (\max\{0, f(x)+\delta-1\}, \min\{1, g(x)+\varepsilon\})$$

such that $\max\{0, f(x)+\delta-1\} + \min\{1, g(x)+\varepsilon\} \in [0,1]$ for all $x \in X$.

Let $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ be a Łukasiewicz intuitionistic fuzzy set of $(X; f, g)$. If $(\delta,\varepsilon) = (1,0)$, then $\max\{0, f(x)+\delta-1\} = \max\{0, f(x)+1-1\} = \max\{0, f(x)\} = f(x)$ and $\min\{1, g(x)+\varepsilon\} = \min\{1, g(x)+0\} = g(x)$. This shows that if $(\delta,\varepsilon) = (1,0)$, then the Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ of (X, f, g) is the classical intuitionistic fuzzy set (X, f, g) in X. If $(\delta,\varepsilon) = (0,1)$, then $\max\{0, f(x)+\delta-1\} = \max\{0, f(x)+0-1\} = \max\{0, f(x)-1\} = 0$ and $\min\{1, g(x)+\varepsilon\} = \min\{1, g(x)+1\} = 1$. Thus, if $(\delta,\varepsilon) = (0,1)$, then the Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ of (X, f, g) is the constant function with the value $(0,1)$. Therefore, in handling the Łukasiewicz intuitionistic fuzzy set, the value of (δ,ε) can always be considered to be in $(0,1)\times(0,1)$.

Proposition 3.2. *Let $(X; f, g)$ be an intuitionistic fuzzy set and let $(\delta,\varepsilon) \in (0,1)\times(0,1)$. For the Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ of $(X; f, g)$, we have*

1. *if $f(x) \geq f(y)$ and $g(x) \geq g(y)$, then $Ł_f^\delta(x) \geq Ł_f^\delta(y)$ and $Ł_g^\varepsilon(x) \geq Ł_g^\varepsilon(y)$ for all $x, y \in X$,*
2. *if $x_{(\delta,\varepsilon)}\, q\, Ł$, then $Ł_f^\delta(x) = f(x)+\delta-1$ and $Ł_g^\varepsilon(x) = g(x)+\varepsilon$ for all $x \in X$,*
3. *$Ł_f^{\delta_1}(x) \geq Ł_f^{\delta_2}(x)$ and $Ł_g^{\varepsilon_1}(x) \geq Ł_g^{\varepsilon_2}(x)$ for all $x \in X$ and $(\delta_1,\varepsilon_1), (\delta_2,\varepsilon_2) \in (0,1)\times(0,1)$ with $\delta_1 \geq \delta_2$ and $\varepsilon_1 \geq \varepsilon_2$.*

Proof. Straightforward. □

Proposition 3.3. *Let $(X; f, g)$ and $(X; \xi, \vartheta)$ be intuitionistic fuzzy sets in a set X and let $(\delta,\varepsilon) \in (0,1)\times(0,1)$. If $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ and $Ł := (Ł_\xi^\delta, Ł_\vartheta^\varepsilon)$ are Łukasiewicz* *intuitionistic fuzzy sets of* $(X; f, g)$ *and* $(X; \xi, \vartheta)$*, respectively, then* $Ł^{\delta}_{f\cap\xi} = Ł^{\delta}_{f} \cap Ł^{\delta}_{\xi}$ *and* $Ł^{\varepsilon}_{g\cap\vartheta} = Ł^{\varepsilon}_{g} \cap Ł^{\varepsilon}_{\vartheta}$*.*

Proof. For every $x \in X$, we have

$$\begin{aligned} Ł^{\delta}_{f\cap\xi}(x) &= \max\{0, (f \cap \xi)(x) + \delta - 1\} \\ &= \max\{0, \min\{f(x), \xi(x)\} + \delta - 1\} \\ &= \max\{0, \min\{f(x) + \delta - 1, \xi(x) + \delta - 1\}\} \\ &= \min\{\max\{0, f(x) + \delta - 1\}, \max\{0, \xi(x) + \delta - 1\}\} \\ &= \min\{Ł^{\delta}_{f}(x), Ł^{\delta}_{\xi}(x)\} \\ &= (Ł^{\delta}_{f} \cap Ł^{\delta}_{\xi})(x) \end{aligned}$$

and

$$\begin{aligned} Ł^{\varepsilon}_{g\cap\vartheta}(x) &= \min\{1, (g \cap \vartheta)(x) + \varepsilon\} \\ &= \min\{1, \min\{g(x), \vartheta(x)\} + \varepsilon\} \\ &= \min\{1, \min\{g(x) + \varepsilon, \vartheta(x) + \varepsilon\}\} \\ &= \min\{\min\{1, g(x) + \varepsilon\}, \min\{1, \vartheta(x) + \varepsilon\}\} \\ &= \min\{Ł^{\varepsilon}_{g}(x), Ł^{\varepsilon}_{\vartheta}(x)\} \\ &= (Ł^{\varepsilon}_{g} \cap Ł^{\varepsilon}_{\vartheta})(x). \end{aligned}$$

This completes the proof. □

4. Łukasiewicz intuitionistic fuzzy subalgebras

In what follows, let X be a BCK-algebra or a BCI-algebra unless otherwise specified. We start with the concept of intuitionistic fuzzy subalgebra in BCK/BCI-algebras. An intuitionistic fuzzy set $(X; f, g)$ is called an *intuitionistic fuzzy subalgebra* of X (See [4]), if it satisfies:

$$f(x * y) \geq \min\{f(x), f(y)\},\ g(x * y) \leq \max\{g(x), g(y)\} \tag{4.1}$$

for all $x, y \in X$.

Lemma 4.1 ([17])**.** *An intuitionistic fuzzy set* $(X; f, g)$ *is an intuitionistic fuzzy subalgebra of* X *if and only if it satisfies:*

$$\begin{aligned} &x_{(s_x,\, t_x)} \in (X; f, g),\ y_{(s_y,\, t_y)} \in (X; f, g) \\ &\quad \Rightarrow\ (x * y)_{(\min\{s_x, s_y\}, \max\{t_x, t_y\})} \in (X; f, g) \end{aligned} \tag{4.2}$$

for all $x, y \in X$ *and* $(s_x,\, t_x), (s_y,\, t_y) \in [0, 1] \times [0, 1]$*.*

Definition 4.2. Given an intuitionistic fuzzy set $(X; f, g)$, its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł^{\delta}_{f}, Ł^{\varepsilon}_{g})$ is called a *Łukasiewicz intuitionistic fuzzy subalgebra* of X, if it satisfies:

$$x_{(s_{\mathfrak{a}},\, t_{\mathfrak{a}})} \in Ł,\ y_{(s_{\mathfrak{b}},\, t_{\mathfrak{b}})} \in Ł\ \Rightarrow\ (x * y)_{(\min\{s_{\mathfrak{a}}, s_{\mathfrak{b}}\}, \max\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\})} \in Ł \tag{4.3}$$

for all $x, y \in X$ and $(s_{\mathfrak{a}},\, t_{\mathfrak{a}}), (s_{\mathfrak{b}},\, t_{\mathfrak{b}}) \in (0, 1] \times [0, 1)$.

TABLE 1. Cayley table for the operation "$*$"

$*$	c_0	c_1	c_2	c_3	c_4
c_0	c_0	c_0	c_0	c_0	c_0
c_1	c_1	c_0	c_1	c_0	c_0
c_2	c_2	c_2	c_0	c_0	c_0
c_3	c_3	c_3	c_3	c_0	c_0
c_4	c_4	c_3	c_4	c_1	c_0

Example 4.3. Consider a set $X = \{c_0, c_1, c_2, c_3, c_4\}$, and define a binary operation "$*$" by Table 1.
Then $(X, *, c_0)$ is a BCK-algebra (See [16]). Define an intuitionistic fuzzy set $(X; f, g)$ in X as follows:

$$(X; f, g) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.75, 0.17) & \text{if } x = c_0, \\ (0.68, 0.29) & \text{if } x = c_1, \\ (0.62, 0.23) & \text{if } x = c_2, \\ (0.56, 0.36) & \text{if } x = c_3, \\ (0.41, 0.48) & \text{if } x = c_4. \end{cases}$$

If we take $(\delta, \varepsilon) = (0.53, 0.38)$, then the Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ of $(X, *, c_0)$ is given as follows:

$$Ł := (Ł_f^\delta, Ł_g^\varepsilon) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.28, 0.55) & \text{if } x = c_0, \\ (0.21, 0.67) & \text{if } x = c_1, \\ (0.15, 0.61) & \text{if } x = c_2, \\ (0.09, 0.74) & \text{if } x = c_3, \\ (0.00, 0.86) & \text{if } x = c_4. \end{cases}$$

It is simple to check that $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X.

Theorem 4.4. *If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $(s_{\mathfrak{a}}, t_{\mathfrak{a}}), (s_{\mathfrak{b}}, t_{\mathfrak{b}}) \in (0,1] \times [0,1)$ be such that $x_{(s_{\mathfrak{a}}, t_{\mathfrak{a}})} \in Ł$ and $y_{(s_{\mathfrak{b}}, t_{\mathfrak{b}})} \in Ł$. Then $Ł_f^\delta(x) \geq s_{\mathfrak{a}}$, $Ł_g^\varepsilon(x) \leq t_{\mathfrak{a}}$, $Ł_f^\delta(y) \geq s_{\mathfrak{b}}$, and $Ł_g^\varepsilon(y) \leq t_{\mathfrak{b}}$. Thus

$$\begin{aligned} Ł_f^\delta(x * y) &= \max\{0, f(x * y) + \delta - 1\} \\ &\geq \max\{0, \min\{f(x), f(y)\} + \delta - 1\} \\ &= \max\{0, \min\{f(x) + \delta - 1, f(y) + \delta - 1\}\} \\ &= \min\{\max\{0, f(x) + \delta - 1\}, \max\{0, f(y) + \delta - 1\}\} \\ &= \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} \geq \min\{s_{\mathfrak{a}}, s_{\mathfrak{b}}\} \end{aligned}$$

and

$$\begin{aligned}Ł_g^{\varepsilon}(x*y) &= \min\{1, g(x*y)+\varepsilon\}\\ &\leq \min\{1, \max\{g(x), g(y)\}+\varepsilon\}\\ &= \min\{1, \max\{g(x)+\varepsilon, g(y)+\varepsilon\}\}\\ &= \max\{\min\{1, g(x)+\varepsilon\}, \min\{1, g(y)+\varepsilon\}\}\\ &= \max\{Ł_g^{\varepsilon}(x), Ł_g^{\varepsilon}(y)\} \leq \max\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\}\end{aligned}$$

which shows that $(x*y)_{(\min\{s_{\mathfrak{a}},s_{\mathfrak{b}}\},\max\{t_{\mathfrak{a}},t_{\mathfrak{b}}\})} \in Ł$. So $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X. □

The converse of Theorem 4.4 is not true in general as seen in the following example.

Example 4.5. Let $X = \{c_0, c_1, c_2, c_3, c_4\}$ be a set with a binary operation "$*$" given by Table 2.

Table 2. Cayley table for the operation "$*$"

$*$	c_0	c_1	c_2	c_3	c_4
c_0	c_0	c_0	c_2	c_3	c_4
c_1	c_1	c_0	c_2	c_3	c_4
c_2	c_2	c_2	c_0	c_4	c_3
c_3	c_3	c_3	c_4	c_0	c_2
c_4	c_4	c_4	c_3	c_2	c_0

Then $(X, *, c_0)$ is a BCI-algebra (See [5]). Define an intuitionistic fuzzy set $(X; f, g)$ as follows:

$$(X; f, g) : X \to [0,1]\times[0,1],\ x \mapsto \begin{cases} (0.72, 0.28) & \text{if } x = c_0,\\ (0.68, 0.32) & \text{if } x = c_1,\\ (0.61, 0.39) & \text{if } x = c_2,\\ (0.57, 0.43) & \text{if } x = c_3,\\ (0.39, 0.61) & \text{if } x = c_4.\end{cases}$$

If we take $(\delta, \varepsilon) = (0.41, 0.54)$, then the Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ of $(X; f, g)$ is given as follows:

$$Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon}) : X \to [0,1]\times[0,1],\ x \mapsto \begin{cases} (0.14, 0.86) & \text{if } x = c_0,\\ (0.10, 0.90) & \text{if } x = c_1,\\ (0.03, 0.97) & \text{if } x = c_2,\\ (0.00, 1.00) & \text{if } x = c_3,\\ (0.00, 1.00) & \text{if } x = c_4.\end{cases}$$

It is routine to check that $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X. But $(X; f, g)$ is not an intuitionistic fuzzy subalgebra of X since $f(c_2 * c_3) = f(c_4) = 0.39 \ngeq 0.57 = \min\{f(c_2), f(c_3)\}$ and/or $g(c_2 * c_3) = g(c_4) = 0.61 \nleq 0.43 = \max\{g(c_2), g(c_3)\}$.

We find a characterization of Łukasiewicz intuitionistic fuzzy subalgebra.

Theorem 4.6. *If $(X; f, g)$ is an intuitionistic fuzzy set, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X if and only if the following is valid:*

$$(\forall x, y \in X)\left(\begin{array}{l} Ł_f^\delta(x*y) \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} \\ Ł_g^\varepsilon(x*y) \leq \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \end{array}\right). \tag{4.4}$$

Proof. Assume that $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X and let $x, y \in X$. It is clear that $x_{(s_\mathfrak{a}, t_\mathfrak{a})} \in Ł$ and $y_{(s_\mathfrak{b}, t_\mathfrak{b})} \in Ł$ for $(s_\mathfrak{a}, t_\mathfrak{a}) = (Ł_f^\delta(x), Ł_g^\varepsilon(x))$ and $(s_\mathfrak{b}, t_\mathfrak{b}) = (Ł_f^\delta(y), Ł_g^\varepsilon(y))$. It follows from (4.3) that $(x*y)_{(\min\{s_\mathfrak{a}, s_\mathfrak{b}\}, \max\{t_\mathfrak{a}, t_\mathfrak{b}\})} \in Ł$. Then

$$Ł_f^\delta(x*y) \geq \min\{s_\mathfrak{a}, s_\mathfrak{b}\} = \min\{Ł_f^\delta(x), Ł_f^\delta(y)\}$$

and $Ł_g^\varepsilon(x*y) \leq \max\{t_\mathfrak{a}, t_\mathfrak{b}\} = \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\}$.

Conversely, suppose that $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies (4.4). Let $x, y \in X$ and $(s_\mathfrak{a}, t_\mathfrak{a})$, $(s_\mathfrak{b}, t_\mathfrak{b}) \in (0,1] \times [0,1)$ be such that $x_{(s_\mathfrak{a}, t_\mathfrak{a})} \in Ł$ and $y_{(s_\mathfrak{b}, t_\mathfrak{b})} \in Ł$. Then $Ł_f^\delta(x) \geq s_\mathfrak{a}$, $Ł_g^\varepsilon(x) \leq t_\mathfrak{a}$, $Ł_f^\delta(y) \geq s_\mathfrak{b}$, and $Ł_g^\varepsilon(y) \leq t_\mathfrak{b}$. It follows from (4.4) that $Ł_f^\delta(x*y) \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} \geq \min\{s_\mathfrak{a}, s_\mathfrak{b}\}$ and

$$Ł_g^\varepsilon(x*y) \leq \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \leq \max\{t_\mathfrak{a}, t_\mathfrak{b}\}.$$

Thus $(x*y)_{(\min\{s_\mathfrak{a}, s_\mathfrak{b}\}, \max\{t_\mathfrak{a}, t_\mathfrak{b}\})} \in Ł$. So $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X. □

Lemma 4.7. *If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies:*

$$(\forall x \in X)(Ł_f^\delta(x) \leq Ł_f^\delta(0),\ Ł_g^\varepsilon(x) \geq Ł_g^\varepsilon(0)). \tag{4.5}$$

Proof. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then

$$Ł_f^\delta(0) = Ł_f^\delta(x*x) \geq \min\{Ł_f^\delta(x), Ł_f^\delta(x)\} = Ł_f^\delta(x)$$

and $Ł_g^\varepsilon(0) = Ł_g^\varepsilon(x*x) \leq \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(x)\} = Ł_g^\varepsilon(x)$ for all $x \in X$. □

Proposition 4.8. *If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} Ł_f^\delta(x) = Ł_f^\delta(0) \ \Leftrightarrow \ Ł_f^\delta(x*y) \geq Ł_f^\delta(y) \\ Ł_g^\varepsilon(x) = Ł_g^\varepsilon(0) \ \Leftrightarrow \ Ł_g^\varepsilon(x*y) \leq Ł_g^\varepsilon(y) \end{array}\right). \tag{4.6}$$

Proof. Assume that $Ł_f^\delta(x) = Ł_f^\delta(0)$ and $Ł_g^\varepsilon(x) = Ł_g^\varepsilon(0)$ for all $x \in X$. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X by Theorem 4.4. Thus the combination of Theorem 4.6 and Lemma 4.7 induces

$$Ł_f^\delta(x*y) \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} = \min\{Ł_f^\delta(0), Ł_f^\delta(y)\} = Ł_f^\delta(y)$$

and $Ł_g^\varepsilon(x*y) \leq \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} = \max\{Ł_g^\varepsilon(0), Ł_g^\varepsilon(y)\} = Ł_g^\varepsilon(y)$ for all $y \in X$.

Conversely, suppose that $Ł_f^\delta(x*y) \geq Ł_f^\delta(y)$ and $Ł_g^\varepsilon(x*y) \leq Ł_g^\varepsilon(y)$ for all $x, y \in X$. It follows from (2.2) that $Ł_f^\delta(x) = Ł_f^\delta(x*0) \geq Ł_f^\delta(0)$ and $Ł_g^\varepsilon(x) = Ł_g^\varepsilon(x*0) \leq Ł_g^\varepsilon(0)$.

Using Lemma 4.7, we conclude that $Ł_f^\delta(x) = Ł_f^\delta(0)$ and $Ł_g^\varepsilon(x) = Ł_g^\varepsilon(0)$ for all $x \in X$. □

Proposition 4.9. *If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of a BCI-algebra X, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies:*

$$L_f^\delta(0 * x) \geq L_f^\delta(x),\ L_g^\varepsilon(0 * x) \leq L_g^\varepsilon(x), \tag{4.7}$$

$$x_{(s_\mathfrak{a},\, t_\mathfrak{a})} \in L,\ y_{(s_\mathfrak{b},\, t_\mathfrak{b})} \in L \ \Rightarrow\ (x * (0 * y))_{(\min\{s_\mathfrak{a}, s_\mathfrak{b}\}, \max\{t_\mathfrak{a}, t_\mathfrak{b}\})} \in L \tag{4.8}$$

for all $x, y \in X$ and $(s_\mathfrak{a},\, t_\mathfrak{a}),\ (s_\mathfrak{b},\, t_\mathfrak{b}) \in (0, 1] \times [0, 1)$.

Proof. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of a BCI-algebra X, then $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X by Theorem 4.4. Thus Theorem 4.6 and Lemma 4.7 induce

$$Ł_f^\delta(0 * x) \geq \min\{Ł_f^\delta(0), Ł_f^\delta(x)\} = Ł_f^\delta(x)$$

and $Ł_g^\varepsilon(0 * x) \leq \max\{Ł_g^\varepsilon(0), Ł_g^\varepsilon(x)\} = Ł_g^\varepsilon(x)$ for all $x \in X$. Let $x, y \in X$ and $(s_\mathfrak{a},\, t_\mathfrak{a})$, $(s_\mathfrak{b},\, t_\mathfrak{b}) \in (0, 1] \times [0, 1)$ be such that $x_{(s_\mathfrak{a},\, t_\mathfrak{a})} \in Ł$ and $y_{(s_\mathfrak{b},\, t_\mathfrak{b})} \in Ł$. Then $Ł_f^\delta(x) \geq s_\mathfrak{a}$, $Ł_g^\varepsilon(x) \leq t_\mathfrak{a}$, $Ł_f^\delta(y) \geq s_\mathfrak{b}$, and $Ł_g^\varepsilon(y) \leq t_\mathfrak{b}$. Thus it follows that

$$\begin{aligned}
Ł_f^\delta(x * (0 * y)) &= \max\{0, f(x * (0 * y)) + \delta - 1\} \\
&\geq \max\{0, \min\{f(x), f(0 * y)\} + \delta - 1\} \\
&\geq \max\{0, \min\{f(x), \min\{f(0), f(y)\}\} + \delta - 1\} \\
&= \max\{0, \min\{f(x), f(y)\} + \delta - 1\} \\
&= \max\{0, \min\{f(x) + \delta - 1, f(y) + \delta - 1\}\} \\
&= \min\{\max\{0, f(x) + \delta - 1\}, \max\{0, f(y) + \delta - 1\}\} \\
&= \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} \\
&\geq \min\{s_\mathfrak{a}, s_\mathfrak{b}\}
\end{aligned}$$

and

$$\begin{aligned}
Ł_g^\varepsilon(x * (0 * y)) &= \min\{1, g(x * (0 * y)) + \varepsilon\} \\
&\leq \min\{1, \max\{g(x), g(0 * y)\} + \varepsilon\} \\
&\leq \min\{1, \max\{g(x), \max\{g(0), g(y)\}\} + \varepsilon\} \\
&= \min\{1, \max\{g(x), g(y)\} + \varepsilon\} \\
&= \min\{1, \max\{g(x) + \varepsilon, g(y) + \varepsilon\}\} \\
&= \max\{\min\{1, g(x) + \varepsilon\}, \min\{1, g(y) + \varepsilon\}\} \\
&= \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \leq \max\{t_\mathfrak{a}, t_\mathfrak{b}\}.
\end{aligned}$$

So $(x * (0 * y))_{(\min\{s_\mathfrak{a}, s_\mathfrak{b}\}, \max\{t_\mathfrak{a}, t_\mathfrak{b}\})} \in Ł$. □

We explore conditions for a Łukasiewicz intuitionistic fuzzy set to be a Łukasiewicz intuitionistic fuzzy subalgebra.

Theorem 4.10. *Let $(X; f, g)$ be an intuitionistic fuzzy set. If its Łukasiewicz intuitionistic fuzzy set $L := (L_f^\delta, L_g^\varepsilon)$ satisfies:*

$$y_{(s_\mathfrak{a},\, t_\mathfrak{a})} \in L,\ z_{(s_\mathfrak{b},\, t_\mathfrak{b})} \in L \ \Rightarrow\ (x * y)_{(\min\{s_\mathfrak{a}, s_\mathfrak{b}\}, \max\{t_\mathfrak{a}, t_\mathfrak{b}\})} \in L \tag{4.9}$$

for all $x, y, z \in X$ with $z \leq x$ and $(s_{\mathfrak{a}}, t_{\mathfrak{a}}), (s_{\mathfrak{b}}, t_{\mathfrak{b}}) \in (0,1] \times [0,1)$, then $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X.

Proof. It is verified right from (I_3), (2.1) and (4.9). □

Proposition 4.11. *Let X be a BCI-algebra. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ satisfies:*

$$x_{(s_{\mathfrak{a}},\, t_{\mathfrak{a}})} \in Ł,\ y_{(s_{\mathfrak{b}},\, t_{\mathfrak{b}})} \in Ł \ \Rightarrow\ (x*(0*y))_{(\min\{s_{\mathfrak{a}},s_{\mathfrak{b}}\},\max\{t_{\mathfrak{a}},t_{\mathfrak{b}}\})} \in Ł \tag{4.10}$$

for all $x, y \in X$ and $(s_{\mathfrak{a}}, t_{\mathfrak{a}}), (s_{\mathfrak{b}}, t_{\mathfrak{b}}) \in (0,1] \times [0,1)$.

Proof. Assume that $(X; f, g)$ is an intuitionistic fuzzy subalgebra of a BCI-algebra X. Then $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X by Theorem 4.4. Let $x, y \in X$ and $(s_{\mathfrak{a}}, t_{\mathfrak{a}}), (s_{\mathfrak{b}}, t_{\mathfrak{b}}) \in (0,1] \times [0,1)$ be such that $x_{(s_{\mathfrak{a}},\, t_{\mathfrak{a}})} \in Ł$ and $y_{(s_{\mathfrak{b}},\, t_{\mathfrak{b}})} \in Ł$. Then $Ł_f^{\delta}(x) \geq s_{\mathfrak{a}}$, $Ł_g^{\varepsilon}(x) \leq t_{\mathfrak{a}}$, $Ł_f^{\delta}(y) \geq s_{\mathfrak{b}}$, and $Ł_g^{\varepsilon}(y) \leq t_{\mathfrak{b}}$. It follows from Theorem 4.6 and Lemma 4.7 that

$$\begin{aligned} Ł_f^{\delta}(x*(0*y)) &\geq \min\{Ł_f^{\delta}(x), Ł_f^{\delta}(0*y)\} \\ &\geq \min\{Ł_f^{\delta}(x), \min\{Ł_f^{\delta}(0), Ł_f^{\delta}(y)\}\} \\ &= \min\{Ł_f^{\delta}(x), Ł_f^{\delta}(y)\} \geq \min\{s_{\mathfrak{a}}, s_{\mathfrak{b}}\} \end{aligned}$$

and

$$\begin{aligned} Ł_g^{\varepsilon}(x*(0*y)) &\leq \max\{Ł_g^{\varepsilon}(x), Ł_g^{\varepsilon}(0*y)\} \\ &\leq \max\{Ł_g^{\varepsilon}(x), \max\{Ł_g^{\varepsilon}(0), Ł_g^{\varepsilon}(y)\}\} \\ &= \max\{Ł_g^{\varepsilon}(x), Ł_g^{\varepsilon}(y)\} \leq \max\{t_{\mathfrak{a}}, t_{\mathfrak{b}}\}. \end{aligned}$$

This shows that $(x*(0*y))_{(\min\{s_{\mathfrak{a}},s_{\mathfrak{b}}\},\max\{t_{\mathfrak{a}},t_{\mathfrak{b}}\})} \in Ł$. □

Let X be a BCI-algebra. For the Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ of an intuitionistic fuzzy set $(X; f, g)$, consider the following condition:

$$(\forall x \in X)(Ł_f^{\delta}(0*x) = Ł_f^{\delta}(x) \text{ and } Ł_f^{\varepsilon}(0*x) = Ł_f^{\varepsilon}(x)). \tag{4.11}$$

The question arises: Will a Łukasiewicz intuitionistic fuzzy subalgebra $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ satisfy with the condition (4.11). We can look at an example that show that the answer to this question is negative. In fact, Consider the Łukasiewicz intuitionistic fuzzy subalgebra $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ of X in Example 4.5. If we take $(\delta, \varepsilon) = (0.42, 0.53)$, then $Ł_f^{0.42}(c_0 * c_1) = Ł_f^{0.42}(c_0) = 0.14 \neq 0.10 = Ł_f^{0.42}(c_1)$ and/or $Ł_g^{0.53}(c_0 * c_1) = Ł_g^{0.53}(c_0) = 0.86 \neq 0.90 = Ł_g^{0.53}(c_1)$.

If a Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ of an intuitionistic fuzzy set $(X; f, g)$ in a BCI-algebra X satisfies the condition (4.11), we say it is *strong*.

We provide a condition for a Łukasiewicz intuitionistic fuzzy subalgebra to be strong.

Theorem 4.12. *Let X be a p-semisimple BCI-algebra. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra X, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^{\delta}, Ł_g^{\varepsilon})$ is a strong Łukasiewicz intuitionistic fuzzy subalgebra of X.*

Proof. Let X be a p-semisimple BCI-algebra. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra X, then its Łukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X by Theorem 4.4. In the same way as the proof in Theorem 4.6, we can check that the condition (4.10) is equivalent to the following:

$$(\forall x, y \in X)\left(\begin{array}{l} Ł_f^\delta(x*(0*y)) \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} \\ Ł_g^\varepsilon(x*(0*y)) \leq \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \end{array}\right).$$

Since X is p-semisimple, it follows from Theorem 4.6 and Lemma 4.7 that

$$\begin{aligned} Ł_f^\delta(x) &= Ł_f^\delta(0*(0*x)) \geq \min\{Ł_f^\delta(0), Ł_f^\delta(0*x)\} \\ &= Ł_f^\delta(0*x) = Ł_f^\delta(0*(0*(0*x))) \\ &\geq \min\{Ł_f^\delta(0), Ł_f^\delta(0*(0*x))\} \\ &= Ł_f^\delta(0*(0*x)) \geq Ł_f^\delta(x) \end{aligned}$$

and

$$\begin{aligned} Ł_g^\varepsilon(x) &= Ł_g^\varepsilon(0*(0*x)) \leq \max\{Ł_g^\varepsilon(0), Ł_g^\varepsilon(0*x)\} \\ &= Ł_g^\varepsilon(0*x) = Ł_g^\varepsilon(0*(0*(0*x))) \\ &\leq \max\{Ł_g^\varepsilon(0), Ł_g^\varepsilon(0*(0*x))\} \\ &= Ł_g^\varepsilon(0*(0*x)) \leq Ł_g^\varepsilon(x) \end{aligned}$$

for all $x \in X$. Thus $Ł_f^\delta(x) = Ł_f^\delta(0*x)$ and $Ł_g^\varepsilon(x) = Ł_g^\varepsilon(0*x)$ for all $x \in X$. So $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a strong Łukasiewicz intuitionistic fuzzy subalgebra of X. □

Corollary 4.13. *Let X be a BCI-algebra which satisfies any one of the following conditions:*

$$\begin{aligned} &X = \{0*x \mid x \in X\}, \\ &(\forall x, y \in X)(x*(0*y) = y*(0*x)), \\ &(\forall x \in X)(0*x = 0 \ \Rightarrow \ x = 0), \\ &(\forall x, y \in X)(0*(y*x) = x*y), \\ &(\forall x, y, z \in X)(z*x = z*y \ \Rightarrow \ x = y), \\ &(\forall x, y, z \in X)((x*y)*(x*z) = z*y). \end{aligned}$$

If $(X; f, g)$ is an intuitionistic fuzzy set in X, then its Łukasiewicz intuitionistic fuzzy subalgebra $L := (Ł_f^\delta, Ł_g^\varepsilon)$ of X is strong.

Theorem 4.14. *Let $L := (Ł_f^\delta, Ł_g^\varepsilon)$ be a Łukasiewicz intuitionistic fuzzy set of an intuitionistic fuzzy set $(X; f, g)$. Then the nonempty sets $(Ł_f^\delta, s)_\in$ and $(Ł_g^\varepsilon, t)_\in$, called Łukasiewicz $\in$-sets, are subalgebras of X for all $(s, t) \in (0.5, 1] \times [0, 0.5)$ if and only if $L := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} \max\{Ł_f^\delta(x*y), 0.5\} \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} \\ \min\{Ł_g^\varepsilon(x*y), 0.5\} \leq \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \end{array}\right). \tag{4.12}$$

Proof. Let $(s,t) \in (0.5,1] \times [0,0.5)$ be such that $(Ł_f^\delta, s)_\in$ and $(Ł_g^\varepsilon, t)_\in$ are nonempty subalgebras of X. Suppose that (4.12) is not valid. Then there exist $\mathfrak{a}, \mathfrak{b} \in X$ such that $\max\{Ł_f^\delta(\mathfrak{a} * \mathfrak{b}), 0.5\} < \min\{Ł_f^\delta(\mathfrak{a}), Ł_f^\delta(\mathfrak{b})\}$ or

$$\min\{Ł_g^\varepsilon(\mathfrak{a} * \mathfrak{b}), 0.5\} > \max\{Ł_g^\varepsilon(\mathfrak{a}), Ł_g^\varepsilon(\mathfrak{b})\}.$$

If $\max\{Ł_f^\delta(\mathfrak{a} * \mathfrak{b}), 0.5\} < \min\{Ł_f^\delta(\mathfrak{a}), Ł_f^\delta(\mathfrak{b})\}$, then $\mathfrak{a}, \mathfrak{b} \in (Ł_f^\delta, s)_\in$ and $\mathfrak{a} * \mathfrak{b} \notin (Ł_f^\delta, s)_\in$ for $s := \min\{Ł_f^\delta(\mathfrak{a}), Ł_f^\delta(\mathfrak{b})\} \in (0.5, 1]$. This is a contradiction. If

$$\min\{Ł_g^\varepsilon(\mathfrak{a} * \mathfrak{b}), 0.5\} > \max\{Ł_g^\varepsilon(\mathfrak{a}), Ł_g^\varepsilon(\mathfrak{b})\},$$

then $\mathfrak{a}, \mathfrak{b} \in (Ł_g^\varepsilon, t)_\in$ and $\mathfrak{a} * \mathfrak{b} \notin (Ł_g^\varepsilon, t)_\in$ for $t := \max\{Ł_g^\varepsilon(\mathfrak{a}), Ł_g^\varepsilon(\mathfrak{b})\} \in [0, 0.5)$, which is a contradiction. Thus $\max\{Ł_f^\delta(x * y), 0.5\} \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\}$ and $\min\{Ł_g^\varepsilon(x * y), 0.5\} \leq \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\}$ for all $x, y \in X$.

Conversely, suppose that $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies (4.12). Let $(s,t) \in (0.5,1] \times [0,0.5)$. If $x, y \in (Ł_f^\delta, s)_\in$ and $\mathfrak{a}, \mathfrak{b} \in (Ł_g^\varepsilon, t)_\in$, then $Ł_f^\delta(x) \geq s$, $Ł_f^\delta(y) \geq s$, $Ł_g^\varepsilon(\mathfrak{a}) \leq t$, and $Ł_g^\varepsilon(\mathfrak{b}) \leq t$. It follows from (4.12) that

$$\max\{Ł_f^\delta(x * y), 0.5\} \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} \geq s > 0.5$$

and $\min\{Ł_g^\varepsilon(\mathfrak{a} * \mathfrak{b}), 0.5\} \leq \max\{Ł_g^\varepsilon(\mathfrak{a}), Ł_g^\varepsilon(\mathfrak{b})\} \leq t < 0.5$. Thus $Ł_f^\delta(x * y) \geq s$ and $Ł_g^\varepsilon(\mathfrak{a} * \mathfrak{b}) \leq t$, i.e., $x * y \in (Ł_f^\delta, s)_\in$ and $\mathfrak{a} * \mathfrak{b} \in (Ł_g^\varepsilon, t)_\in$. So $(Ł_f^\delta, s)_\in$ and $(Ł_g^\varepsilon, t)_\in$ are subalgebras of X. □

Corollary 4.15. *Let $(X; f, g)$ be an intuitionistic fuzzy set in X. If its Łukasiewicz intuitionistic fuzzy $L := (L_f^\delta, L_g^\varepsilon)$ in X satisfies (4.12), then the nonempty intuitionistic level set $(L, (s,t))_\in$ of $L := (L_f^\delta, L_g^\varepsilon)$ is a subalgebra of X for all $(s,t) \in (0.5,1] \times [0,0.5)$.*

Theorem 4.16. *Let $L := (L_f^\delta, L_g^\varepsilon)$ be a Łukasiewicz intuitionistic fuzzy set of an intuitionistic fuzzy set $(X; f, g)$. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then the nonempty sets $(L_f^\delta, s)_q$ and $(L_g^\varepsilon, t)_q$, called Łukasiewicz q-sets, are subalgebras of X for all $(s,t) \in (0,1] \times [0,1)$.*

Proof. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X by Theorem 4.4. Let $(s,t) \in (0,1] \times [0,1)$. If $x, y \in (Ł_f^\delta, s)_q$ and $\mathfrak{a}, \mathfrak{b} \in (Ł_g^\varepsilon, t)_q$, then $Ł_f^\delta(x) + s > 1$, $Ł_f^\delta(y) + s > 1$, $Ł_g^\varepsilon(\mathfrak{a}) + t < 1$, and $Ł_g^\varepsilon(\mathfrak{b}) + t < 1$. Using Theorem 4.6, we have

$$Ł_f^\delta(x * y) + s \geq \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} + s = \min\{Ł_f^\delta(x) + s, Ł_f^\delta(y) + s\} > 1$$

and

$$Ł_g^\varepsilon(\mathfrak{a} * \mathfrak{b}) + t \leq \max\{Ł_g^\varepsilon(\mathfrak{a}), Ł_g^\varepsilon(\mathfrak{b})\} + t = \max\{Ł_g^\varepsilon(\mathfrak{a}) + t, Ł_g^\varepsilon(\mathfrak{b}) + t\} < 1$$

Thus $x * y \in (Ł_f^\delta, s)_q$ and $\mathfrak{a} * \mathfrak{b} \in (Ł_g^\varepsilon, t)_q$. So $(Ł_f^\delta, s)_q$ and $(Ł_g^\varepsilon, t)_q$ are subalgebras of X. □

Corollary 4.17. *Let $L := (L_f^\delta, L_g^\varepsilon)$ be a Łukasiewicz intuitionistic fuzzy set of an intuitionistic fuzzy set $(X; f, g)$. If $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X, then the nonempty intuitionistic q-set $(L, (s,t))_q$ of $L := (L_f^\delta, L_g^\varepsilon)$ is a subalgebra of X for all $(s,t) \in (0,1] \times [0,1)$.*

Proposition 4.18. *Let* $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ *be a Łukasiewicz intuitionistic fuzzy set of an intuitionistic fuzzy set* $(X; f, g)$*. If the nonempty sets* $(Ł_f^\delta, s)_q$ *and* $(Ł_g^\varepsilon, t)_q$ *are subalgebras of* X*, then* $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ *satisfies:*

(4.13) $$x_{(s_\mathfrak{a},\, t_\mathfrak{a})}\, q\, Ł,\ y_{(s_\mathfrak{b},\, t_\mathfrak{b})}\, q\, Ł \ \Rightarrow\ (x * y)_{(\max\{s_\mathfrak{a}, s_\mathfrak{b}\}, \min\{t_\mathfrak{a}, t_\mathfrak{b}\})} \in Ł$$

for all $x, y \in X$ *and* $(s_\mathfrak{a},\, t_\mathfrak{a}), (s_\mathfrak{b},\, t_\mathfrak{b}) \in (0, 0.5] \times [0.5, 1)$*.*

Proof. Let $x, y \in X$ and $(s_\mathfrak{a},\, t_\mathfrak{a}), (s_\mathfrak{b},\, t_\mathfrak{b}) \in (0, 0.5] \times [0.5, 1)$ be such that $x_{(s_\mathfrak{a},\, t_\mathfrak{a})}\, q\, Ł$ and $y_{(s_\mathfrak{b},\, t_\mathfrak{b})}\, q\, Ł$. Then

$$\begin{aligned} 1 &< Ł_f^\delta(x) + s_\mathfrak{a} \le Ł_f^\delta(x) + \max\{s_\mathfrak{a}, s_\mathfrak{b}\}, \\ 1 &< Ł_f^\delta(y) + s_\mathfrak{b} \le Ł_f^\delta(y) + \max\{s_\mathfrak{a}, s_\mathfrak{b}\}, \\ 1 &> Ł_g^\varepsilon(x) + t_\mathfrak{a} \ge Ł_g^\varepsilon(x) + \min\{t_\mathfrak{a}, t_\mathfrak{b}\}, \\ 1 &> Ł_g^\varepsilon(y) + t_\mathfrak{b} \ge Ł_g^\varepsilon(y) + \min\{t_\mathfrak{a}, t_\mathfrak{b}\}. \end{aligned}$$

Thus $x, y \in (Ł_f^\delta, \max\{s_\mathfrak{a}, s_\mathfrak{b}\})_q \cap (Ł_g^\varepsilon, \min\{t_\mathfrak{a}, t_\mathfrak{b}\})_q$. So

$$x * y \in (Ł_f^\delta, \max\{s_\mathfrak{a}, s_\mathfrak{b}\})_q \cap (Ł_g^\varepsilon, \min\{t_\mathfrak{a}, t_\mathfrak{b}\})_q.$$

Since $(\max\{s_\mathfrak{a}, s_\mathfrak{b}\}, \min\{t_\mathfrak{a}, t_\mathfrak{b}\}) \in (0, 0.5] \times [0.5, 1)$, it follows that

$$Ł_f^\delta(x * y) > 1 - \max\{s_\mathfrak{a}, s_\mathfrak{b}\} \ge \max\{s_\mathfrak{a}, s_\mathfrak{b}\}$$

and $Ł_g^\varepsilon(x * y) < 1 - \min\{t_\mathfrak{a}, t_\mathfrak{b}\} \le \min\{t_\mathfrak{a}, t_\mathfrak{b}\}$. Hence

$$(x * y)_{(\max\{s_\mathfrak{a}, s_\mathfrak{b}\}, \min\{t_\mathfrak{a}, t_\mathfrak{b}\})} \in Ł.$$

This completes the proof. □

Let $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ be a Łukasiewicz intuitionistic fuzzy set of an intuitionistic fuzzy set $(X; f, g)$. Consider the following sets:

(4.14) $$O(Ł_f^\delta) := \{x \in X \mid Ł_f^\delta(x) > 0\},\ O(Ł_g^\varepsilon) := \{x \in X \mid Ł_g^\varepsilon(x) < 1\}$$

which are called the Łukasiewicz O-sets of $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$. It is observed that

$$O(Ł_f^\delta) = \{x \in X \mid f(x) + \delta - 1 > 0\}$$

and $O(Ł_g^\varepsilon) = \{x \in X \mid g(x) + \varepsilon < 1\}$.

We explore conditions for the Łukasiewicz O-sets to be subalgebras.

Theorem 4.19. *Let* $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ *be a Łukasiewicz intuitionistic fuzzy set of an intuitionistic fuzzy set* $(X; f, g)$*. If* $(X; f, g)$ *is an intuitionistic fuzzy subalgebra of* X*, then the nonempty Łukasiewicz* O*-sets of* $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ *are subalgebras of* X*.*

Proof. Assume that $(X; f, g)$ is an intuitionistic fuzzy subalgebra of X. Then $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ is a Łukasiewicz intuitionistic fuzzy subalgebra of X by Theorem 4.4. If $x, y \in O(Ł_f^\delta) \cap O(Ł_g^\varepsilon)$, then $f(x) + \delta - 1 > 0$, $f(y) + \delta - 1 > 0$, $g(x) + \varepsilon < 1$, and $g(y) + \varepsilon < 1$. It follows from Theorem 4.6 that

$$Ł_f^\delta(x * y) \ge \min\{Ł_f^\delta(x), Ł_f^\delta(y)\} = \min\{f(x) + \delta - 1, f(y) + \delta - 1\} > 0$$

and $Ł_g^\varepsilon(x * y) \le \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} = \max\{g(x) + \varepsilon, g(y) + \varepsilon\} < 1$. Thus $x * y \in O(Ł_f^\delta) \cap O(Ł_g^\varepsilon)$. So $O(Ł_f^\delta)$ and $O(Ł_g^\varepsilon)$ are subalgebras of X. □

Theorem 4.20. *Let $(X; f, g)$ be an intuitionistic fuzzy set. If its Lukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies:*

$$x_{(s_\mathfrak{a},\, t_\mathfrak{a})} \in Ł,\ y_{(s_\mathfrak{b},\, t_\mathfrak{b})} \in Ł \ \Rightarrow\ (x * y)_{(\max\{s_\mathfrak{a}, s_\mathfrak{b}\}, \min\{t_\mathfrak{a}, t_\mathfrak{b}\})}\, q\, Ł \tag{4.15}$$

for all $x, y \in X$ and $(s_\mathfrak{a},\, t_\mathfrak{a}), (s_\mathfrak{b},\, t_\mathfrak{b}) \in (0,1] \times [0,1)$, then the nonempty Lukasiewicz O-sets of $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ are subalgebras of X.

Proof. Assume that $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies (4.15) for all $x, y \in X$ and $(s_\mathfrak{a},\, t_\mathfrak{a}), (s_\mathfrak{b},\, t_\mathfrak{b}) \in (0,1] \times [0,1)$. Let $x, y \in O(Ł_f^\delta) \cap O(Ł_g^\varepsilon)$. Then $f(x) + \delta - 1 > 0$, $f(y) + \delta - 1 > 0$, $g(x) + \varepsilon < 1$, and $g(y) + \varepsilon < 1$. If we take $(s_\mathfrak{a},\, t_\mathfrak{a}) = (Ł_f^\delta(x), Ł_g^\varepsilon(x))$ and $(s_\mathfrak{b},\, t_\mathfrak{b}) = (Ł_f^\delta(y), Ł_g^\varepsilon(y))$, then $x_{(s_\mathfrak{a},\, t_\mathfrak{a})} \in Ł$ and $y_{(s_\mathfrak{b},\, t_\mathfrak{b})} \in Ł$. Using (4.15), we have $(x * y)_{(\max\{s_\mathfrak{a}, s_\mathfrak{b}\}, \min\{t_\mathfrak{a}, t_\mathfrak{b}\})}\, q\, Ł$. If $x * y \notin O(Ł_f^\delta)$ or $x * y \notin O(Ł_g^\varepsilon)$, then $Ł_f^\delta(x * y) = 0$ or $Ł_g^\varepsilon(x * y) = 1$. Thus

$$\begin{aligned}
&Ł_f^\delta(x * y) + \max\{s_\mathfrak{a}, s_\mathfrak{b}\} = \max\{s_\mathfrak{a}, s_\mathfrak{b}\} = \max\{Ł_f^\delta(x), Ł_f^\delta(y)\} \\
&= \max\{\max\{0, f(x) + \delta - 1\}, \max\{0, f(y) + \delta - 1\}\} \\
&= \max\{f(x) + \delta - 1, f(y) + \delta - 1\} \\
&= \max\{f(x), f(y)\} + \delta - 1 \leq 1 + \delta - 1 = \delta \leq 1
\end{aligned}$$

or

$$\begin{aligned}
&Ł_g^\varepsilon(x * y) + \min\{t_\mathfrak{a}, t_\mathfrak{b}\} = 1 + \min\{t_\mathfrak{a}, t_\mathfrak{b}\} = 1 + \min\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \\
&= 1 + \min\{\min\{1, g(x) + \varepsilon\}, \min\{1, g(y) + \varepsilon\}\} \\
&= 1 + \min\{g(x) + \varepsilon, g(y) + \varepsilon\} \\
&= 1 + \min\{g(x), g(y)\} + \varepsilon \geq 1 + \varepsilon > 1.
\end{aligned}$$

This is a contradiction, and therefore $x * y \in O(Ł_f^\delta) \cap O(Ł_g^\varepsilon)$. So $O(Ł_f^\delta)$ and $O(Ł_g^\varepsilon)$ are subalgebras of X. □

Theorem 4.21. *Let $(X; f, g)$ be an intuitionistic fuzzy set. If its Lukasiewicz intuitionistic fuzzy set $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies (4.13) for all $x, y \in X$ and $(s_\mathfrak{a},\, t_\mathfrak{a}), (s_\mathfrak{b},\, t_\mathfrak{b}) \in (0,1] \times [0,1)$, then the nonempty Lukasiewicz O-sets of $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ are subalgebras of X.*

Proof. Assume that $Ł := (Ł_f^\delta, Ł_g^\varepsilon)$ satisfies (4.13) for all $x, y \in X$ and $(s_\mathfrak{a},\, t_\mathfrak{a}), (s_\mathfrak{b},\, t_\mathfrak{b}) \in (0,1] \times [0,1)$. Let $x, y \in O(Ł_f^\delta) \cap O(Ł_g^\varepsilon)$. Then $f(x) + \delta - 1 > 0$, $f(y) + \delta - 1 > 0$, $g(x) + \varepsilon < 1$, and $g(y) + \varepsilon < 1$. Thus

$$\begin{aligned}
&Ł_f^\delta(x) + 1 = \max\{0, f(x) + \delta - 1\} + 1 = f(x) + \delta - 1 + 1 = f(x) + \delta > 1, \\
&Ł_f^\delta(y) + 1 = \max\{0, f(y) + \delta - 1\} + 1 = f(y) + \delta - 1 + 1 = f(y) + \delta > 1, \\
&Ł_g^\varepsilon(x) + 0 = \min\{1, g(x) + \varepsilon\} + 0 = g(x) + \varepsilon < 1, \\
&Ł_g^\varepsilon(y) + 0 = \min\{1, g(y) + \varepsilon\} + 0 = g(y) + \varepsilon < 1.
\end{aligned}$$

So $x_{(1,0)}\, q\, Ł$ and $y_{(1,0)}\, q\, Ł$. It follows from (4.13) that

$$(x * y)_{(1,0)} = (x * y)_{(\max\{1,1\}, \min\{0,0\})} \in Ł.$$

Hence $Ł_f^\delta(x * y) = 1 > 0$ and $Ł_g^\varepsilon(x * y) = 0 < 1$, and thus $x * y \in O(Ł_f^\delta)$ and $x * y \in O(Ł_g^\varepsilon)$. Therefore $O(Ł_f^\delta)$ and $O(Ł_g^\varepsilon)$ are subalgebras of X. □

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