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Abstract. In this paper, we define a Γ-BCI-algebra as a subclass of Γ-BCK-algebras and obtain its various properties. Next, we propose the notion of Γ-ideals and deal with some of its properties. Finally, we study some topological structures on Γ-BCI-algebras and quotient Γ-BCI-algebras respectively.

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Keywords: Γ-BCI-algebra, Γ-ideal, Topological Γ-BCI-algebra, Quotient Γ-BCI-algebra, Topological Quotient Γ-BIK-algebra.

Corresponding Author: J. I. Baek (jibaek@wku.ac.kr),
Samy M. Mostafa (samymostafa@yahoo.com)

1. Introduction

In 1980, Iséki [1] proposed the concept of BCI-algebras as a generalization of BCK-algebras introduced by Iséki and Tanaka [2]. After then, Some researchers introduced and studied some proper subclasses of BCK-algebras, for example, BCC-algebras (Dudek, [3]), BCH-algebras (Hu and Li [4]), BE-algebras (Kim and Kim [5]) and BRK-algebras (Bandaru [6]). In particular, Dong and Ryu [7], Roudabri and Torkzadeh [8] and Mohammed et al. [9] studied topological structures on BCK-algebras respectively. Jun et al. [10] and Hasankhani et al. [11] applied BCI-algebras to topology respectively. Ahn and Kwon [12], and Setudeh and Kouhestani [13] dealt with topological properties on BCC-algebras respectively. Mehrshad and Golzarpoor [14] studied some topological structures on BE-algebras. Jansi and Thiruveni [15, 16] applied BCH-algebras to topology and topological group. Mostafa et al. [17] discussed topological properties on KU-algebras proposed by Prabpayak and Leerawat [18]. Sivakumar et al. [19] investigated topological structures on BRK-algebras

In 2022, Saeid et al. [20] proposed the concept of Γ-BCK-algebras and studied some of its properties. By modifying a Γ-BCK-algebra proposed by Saeid et al., Shi et al. [21] redefined a Γ-BCK-algebra introduced by Saeid et al. [20] and investigated its various properties.

The purpose of our study is to introduce the concept of Γ-BCI-algebras as a subclass of Γ-BCK-algebras and study its topological structures. To accomplish our purpose, our research proceeds as follows: First, we define a Γ-BCI-algebra and obtain its various properties. Next, we define a Γ-ideal and investigate some of its properties. Also, we obtain some properties of the quotient Γ-BCI-algebra and the kernel of a Γ-homomorphism respectively. Finally, we discuss some of topological properties on Γ-BCI-algebras and quotient Γ-BCI-algebras respectively.

2. Preliminaries

We recall some definitions needed in next sections.

Definition 2.1 ([1, 2])**.** Let X be a nonempty set with a constant 0 and a binary operation $*$. Consider the following axioms: for all $x,\ y,\ z \in X$,

(A_1) $[(x*y)*(x*z)]*(z*y)=0$,
(A_2) $[x*(x*y)]*y=0$,
(A_3) $x*x=0$,
(A_4) $x*y=0$ and $y*x=0$ imply $x=y$,
(A_5) $0*x=0$..

Then X is called a:
(i) *BCI-algebra*, if it satisfies axioms (A_1)–(A_4),
(ii) *BCK-algebra*, if it satisfies axioms (A_1)–(A_5).

In BCI-algebra or BCK-algebra X, we define a binary relation $\leq$ on X as follows: for all $x\ , y \in X$,

$$x \leq y \text{ if and only if } x*y=0.$$

Definition 2.2 ([22])**.** Let X and Γ be two nonempty sets. Then X is called a Γ-*semigroup*, if there is a mapping $f: X\times\Gamma\times X \to X$, denoted by $f(x,\alpha,y)=x\alpha y$ for each $(x,\alpha,y)\in X\times\Gamma\times X$, such that it satisfies the following condition: for all $x,\ y,\ z\in X$ and all $\alpha,\ \beta\in\Gamma$,

$$x\alpha(y\beta z)=(x\alpha y)\beta z. \tag{2.1}$$

Definition 2.3 ([21])**.** Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is called a Γ-*BCK-algebra*, if there is a mapping $f: X\times\Gamma\times X\to X$, denoted by $f(x,\alpha,y)=x\alpha y$ for each $(x,\alpha,y)\in X\times\Gamma\times X$, satisfying the following axioms: for all $x,\ y,\ z\in X$ and all $\alpha,\ \beta\in\Gamma$,

(ΓA_1) $[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y)=0$,
(ΓA_2) $[x\alpha(x\beta y)]\alpha y=0$,
(ΓA_3) if $x\alpha y=0=y\alpha x$, then $x=y$,
(ΓA_4) $x\alpha x=0$,
(ΓA_5) $0\alpha x=0$.

For a Γ-BCK-algebra X and a fixed $\alpha\in\Gamma$, we define the operation $*: X\times X\to X$ as follows: for all $x,\ y\in X$,

$$x*y=x\alpha y.$$

Then it is clear $(X,*,0)$ is a BCK-algebra and is denoted by X_α.

3. Γ-BCI-algebras

Definition 3.1. Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is called a Γ-*BCI-algebra*, if it satisfies the axioms (ΓA_1)–(ΓA_4).

If Γ is a singleton set, then a Γ-BCI/BCK-algebra is a classical BCK/BCI-algebra.

In a Γ-BCI-algebra X, we define a binary relation $\leq$ on X as follows (See [20]): for all $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$x \leq y \text{ if and only if } x\alpha y = 0. \tag{3.1}$$

In this case, $\leq$ is called a Γ-BCI *ordering*. Then from the definition of $\leq$, we obtain a characterization of a Γ-BCI-algebra.

Theorem 3.2 (See Theorem 3.3, [21]). *X is a Γ-BCI-algebra if and only if it satisfies the following conditions: for all $x,\ y,\ z \in X$ and all $\alpha,\ \beta \in \Gamma$,*

(1) $(x\alpha y)\beta(x\alpha z) \leq z\alpha y$,
(2) $x\alpha(x\beta y) \leq y$,
(3) *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$,
(4) $x \leq x$.

Example 3.3. (1) Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $X = \{0, 1, 2\}$ be a set with the ternary operation defined as the following table:

α	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

β	0	1	2
0	0	2	1
1	1	0	0
2	2	0	0

γ	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

Table 3.1

Then clearly, X is a Γ-BCI-algebra.

(2) Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3\}$ be a set with the ternary operation defined as the following table:

α	0	1	2	3
0	0	0	0	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

β	0	1	2	3
0	0	0	0	3
1	1	0	1	3
2	2	1	0	1
3	3	2	1	0

Table 3.2

Then we can easily check that X is a Γ-BCI-algebra.

Proposition 3.4. *Let X be an algebra satisfying the axioms (ΓA_3), (ΓA_4). If $x \leq 0$ for each $x \in X$, then $x = 0$.*

Proof. Suppose $x \leq 0$ for each $x \in X$. Then clearly, $x\alpha 0 = 0$ for each $\alpha \in \Gamma$. By the axiom (ΓA_4), $0\alpha 0$. Thus $x\alpha 0 = 0 = 0\alpha 0 = 0$. So by the axiom ($\Gamma A_3$), $x = 0$. □

Proposition 3.5. *Let X be a Γ-BCI-algebra and let $x,\ y,\ z \in X$.*
(1) *If $x \leq y$, then $z\alpha y \leq z\alpha x$ for each $\alpha \in \Gamma$.*
(2) *If $x \leq y$ and $y \leq z$, then $x \leq z$.*

Proof. (1) Suppose $x \leq y$ and let $\alpha,\ \beta \in \Gamma$. Then by Theorem 3.2(1) and the hypothesis, we get

$$(z\alpha y)\beta(z\alpha x) \leq x\alpha y = 0.$$

Thus by Proposition 3.4, $(z\alpha y)\beta(z\alpha x) = 0$. So $z\alpha y \leq z\alpha x$.

(2) Suppose $x \leq y$, $y \leq z$ and let $\alpha \in \Gamma$. Since $y \leq z$, by (1), $x\alpha z \leq x\alpha y$. Since $x \leq y$, $x\alpha y = 0$. Then $x\alpha z \leq 0$. Thus by Proposition 3.4, $x\alpha z = 0$. So $x \leq z$. □

Corollary 3.6. *Let X be a Γ-BCI-algebra and let $x,\ y,\ z \in X$, $\alpha \in \Gamma$. If $x\alpha y \leq z$, then $x\alpha z \leq y$.*

Proof. Suppose $x\alpha y \leq z$. Then by (3.1) and Proposition 3.5(1),

$$x\alpha z \leq x\beta(x\alpha y) \leq y \text{ for each } \beta \in \Gamma.$$

Thus by Proposition 3.5(2), $x\alpha z \leq y$. □

Proposition 3.7. *Let X be a Γ-BCI-algebra. If the following condition holds:*

$$(x\alpha y)\beta z \leq (x\alpha z)\beta y \textit{ for all } x,\ y,\ z \in X \textit{ and all } \alpha,\ \beta \in \Gamma, \tag{3.2}$$

then the axiom ($\Gamma\mathrm{A}_2$) *holds.*

Proof. Suppose (3.2) holds and let $x,\ y,\ z \in X$, $\alpha,\ \beta \in \Gamma$. Then by Theorem 3.2 (1), Proposition 3.5(1), the hypothesis and the axiom ($\Gamma\mathrm{A}_3$),

$$[x\alpha(x\beta y)]\alpha y \leq (x\alpha y)\beta(x\alpha y) = 0.$$

Thus by Proposition 3.4, $[x\alpha(x\beta y)]\alpha y = 0$. So the axiom ($\Gamma\mathrm{A}_2$) holds. □

Proposition 3.8. *Let X be a Γ-BCI-algebra. Then*

$$(x\alpha y)\beta z = (x\alpha z)\beta y \textit{ for all } x,\ y,\ z \in X,\ \alpha,\ \beta \in \Gamma. \tag{3.3}$$

Proof. Let $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$. Then by Theorem 3.2(2), we get

$$x\alpha(x\alpha z) \leq z.$$

Thus by Theorem 3.2(2) and Proposition 3.5(1), we have

$$(x\alpha y)\beta z \leq (x\alpha y)\beta[x\alpha(x\alpha z)] \leq (x\alpha z)\beta y.$$

Since $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$ are arbitrary, we obtain the following inequality:

$$(x\alpha z)\beta y \leq (x\alpha y)\beta z.$$

So by Theorem 3.2(3), $(x\alpha y)\beta z = (x\alpha z)\beta y$. □

Corollary 3.9. *Let X be a Γ-BCI-algebra. Then the followings are equivalent: for all $x,\ y,\ z \in X$ and all $\alpha,\ \beta \in \Gamma$,*
(1) $(x\alpha y)\beta(x\alpha z) \leq z\alpha y$,
(2) $(x\alpha z)\beta(y\alpha z) \leq x\alpha y$.

Proof. The proof follows from Propositions 3.8 and 3.5(1). □

Proposition 3.10. *Let X be a Γ-BCI-algebra. Then the followings hold: for all $x,\ y,\ z,\ u \in X$ and all $\alpha,\ \beta \in \Gamma$,*
(1) $x \leq y$ *implies* $x\alpha z \leq y\alpha z$,
(2) $x\alpha[x\beta(x\alpha y)] = x\alpha y$,
(3) $(0\alpha x)\beta(0\alpha y) = 0\beta(x\beta y)$ *or* $(0\alpha x)\beta(0\alpha y) = 0\beta(x\alpha y)$,
(4) $[(x\alpha y)\beta z]\alpha(u\beta z) \leq (x\alpha u)\beta y$,
(5) $[(x\alpha y)\beta z]\alpha[(x\alpha u)\beta y] \leq u\beta y$,

Proof. (1) Suppose $x \leq y$. Then clearly, $x\alpha y = 0$. Thus by Corollary 3.9(2), we have

$$(x\alpha z)\beta(y\alpha z) \leq x\alpha y = 0.$$

So by Proposition 3.4, $(x\alpha z)\beta(y\alpha z) = 0$. Hence $x\alpha z \leq y\alpha z$.
(2) By Theorem 3.2(1) and the axiom (ΓA_2), we get

$$(x\alpha y)\beta[x\alpha(x\beta(x\alpha y))] \leq [x\beta(x\alpha y)]\beta y = 0.$$

Then by Proposition 3.4, we have

$$(x\alpha y)\beta[x\alpha(x\beta(x\alpha y))] = 0.$$

Also by the axiom (ΓA_2), we have

$$[x\alpha(x\beta(x\alpha y))]\beta(x\alpha y) = 0.$$

Thus by the axiom (ΓA_3), $x\alpha[x\beta(x\alpha y)] = x\alpha y$.
(3) By the axiom (ΓA_4) and Proposition 3.8, we have
$(0\alpha x)\beta(0\alpha y) = [((x\beta y)\alpha(x\beta y))\alpha x]\beta(0\alpha y)$
$= [((x\beta y)\alpha x)\alpha(x\beta y)]\beta(0\alpha y)$
$= [((x\beta x)\alpha y)\alpha(x\beta y)]\beta(0\alpha y)$
$= [(0\alpha y)\alpha(x\beta y)]\beta(0\alpha y)$
$= [(0\alpha y)\alpha(0\alpha y)]\beta(x\beta y)$
$= 0\beta(x\beta y)$.
Also, we have
$(0\alpha x)\beta(0\alpha y) = [((x\beta y)\alpha(x\beta y))\alpha x]\beta(0\alpha y)$
$= [((x\alpha y)\alpha x)\alpha(x\alpha y)]\beta(0\alpha y)$
$= [((x\alpha x)\alpha y)\alpha(x\alpha y)]\beta(0\alpha y)$
$= [(0\alpha y)\alpha(x\alpha y)]\beta(0\alpha y)$
$= [(0\alpha y)\alpha(0\alpha y)]\beta(x\alpha y)$
$= 0\beta(x\alpha y)$.
(4) By Corollary 3.9(2) and Proposition 3.8, we have

$$[(x\alpha y)\beta z]\alpha(u\beta z) \leq (x\alpha y)\beta u = (x\alpha u)\beta y.$$

Then the result holds.
(5) The proof follows from Theorem 3.2(2) and (4).

□

Lemma 3.11. *Let X be an algebra satisfying the axioms (ΓA_3), (ΓA_4) and Proposition 3.4. If Proposition 3.10(4) holds, then Propositions 3.10(5) and 3.8 hold.*

Proof. Suppose Proposition 3.10(5) holds. Then clearly, Proposition 3.10(5) holds. For all $x,\ y,\ z,\ u \in X$ and all $\alpha,\ \beta \in \Gamma$, let $u = z$. Then from Proposition 3.10(5) and the axiom (ΓA_4), we have

$$[(x\alpha y)\beta z]\alpha[(x\alpha z)\beta y] \leq z\alpha z = 0.$$

Thus by Proposition 3.4, $[(x\alpha y)\beta z]\alpha[(x\alpha z)\beta y] = 0$. Similarly, from Proposition 3.10(5) and Proposition 3.4, we get

$$[(x\alpha z)\beta y]\alpha[(x\alpha y)\beta z] = 0.$$

So by the axiom (ΓA_3), $(x\alpha z)\beta y = (x\alpha y)\beta z$. Hence Proposition 3.8 hold. □

Lemma 3.12. *Let X be an algebra satisfying the axioms (ΓA_3), (ΓA_4) and Proposition 3.4. If Proposition 3.10 (5) holds, then* Propositions 3.8 *and Proposition* 3.10(4) *hold.*

Proof. The proof is straightforward from Lemma 3.11. □

We give a characterization of Γ-BCI-algebras.

Theorem 3.13. *X is a Γ-BCI-algebra if and only if it satisfies the axioms (ΓA_3), (ΓA_4), Proposition 3.4 and Proposition 3.10(4) or (5).*

Proof. It is obvious that the necessary conditions hold. Suppose the axioms (ΓA_3), (ΓA_4) and Proposition 3.10(4) hold. For all $x,\ y,\ z,\ u \in X$ and all $\alpha,\ \beta \in \Gamma$, let $y = x\alpha u$. Then from Proposition 3.10(4) and (ΓA_4), we have

$$[x\alpha(x\alpha u))\beta z]\alpha(u\beta z) \leq (x\alpha u)\beta(x\alpha u) = 0.$$

Thus by Proposition 3.4, $[x\alpha(x\alpha u))\beta z]\alpha(u\beta z) = 0$. From Lemma 3.11 or 3.12, since Proposition 3.8 holds, we get

$$[(x\alpha z)\beta(x\alpha u)]\alpha(u\beta z) = 0.$$

So $[(x\alpha y)\beta(x\alpha z)]\alpha(z\beta y) = 0$. Hence the axiom ($\Gamma A_1$) holds.

Now for all $x,\ y,\ z,\ u \in X$, $\alpha,\ \beta \in \Gamma$, let $y = x\alpha y$, $u = z = y$. Then from Proposition 3.10(4) and the axiom (ΓA_4), we have

$$[(x\beta(x\alpha y))\beta y]\alpha(y\beta y) \leq (x\beta y)\alpha(x\beta y) = 0.$$

Thus by Proposition 3.4, $[(x\beta(x\alpha y))\beta y]\alpha(y\beta y) = 0$. Since $y\beta y = 0$, by by Proposition 3.4, we get

$$(x\beta(x\alpha y))\beta y = 0.$$

So the axiom (ΓA_2) holds. This completes the proof. □

From Theorem 3.13 and the definition of Γ-BCK-algebra, we obtain the following.

Corollary 3.14. *X is a Γ-BCK-algebra if and only if it satisfies the axioms (ΓA_3), (ΓA_4), (ΓA_5) and Proposition 3.10(4) or (5).*

Now we give another characterization of Γ-BCI-algebra.

Theorem 3.15. *X is a Γ-BCI-algebra if and only if it satisfies the axioms (ΓA_1), (ΓA_3) and the following condition:*

$$x\alpha 0 = x \text{ for each } x \in X \text{ and each } \alpha \in \Gamma. \tag{3.4}$$

Proof. Suppose X is a Γ-BCI-algebra. It is sufficient to show that (3.4) holds. For all $x,\ y,\ z \in X$ and all $\alpha,\ \beta \in \Gamma$, let $y = 0$. Then by the axiom (ΓA_2), we have

$$[x\alpha(x\beta 0)]\alpha 0 = 0. \tag{3.5}$$

On the other hand, let $y = x\beta 0$ and let $z = x$. Then from the axiom (ΓA_1), we have

$$[(x\alpha(x\beta 0))\beta(x\alpha x)]\beta[x\alpha(x\beta 0)] = 0.$$

Thus by the axiom (ΓA_4), we get

$$[(x\alpha(x\beta 0))\beta 0]\beta[x\alpha(x\beta 0)] = 0. \tag{3.6}$$

From (3.5), (3.6) and the axiom (ΓA_1), we have

$$x\alpha(x\beta 0) = 0. \tag{3.7}$$

Also by the axioms (ΓA_2) and (ΓA_4), we get

$$(x\alpha 0)\beta x = [x\alpha(x\beta x)]\alpha x = 0. \tag{3.8}$$

So by (3.7), (3.8) and the axiom (ΓA_3), $x\alpha 0 = x$. Hence (3.5) holds.

Suppose the necessary conditions hold. It is sufficient to prove that the axioms (ΓA_2) and (ΓA_4) hold. Let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_1$) and (3.6), we have

$$[x\alpha(x\beta y)]\alpha y = [(x\beta 0)\alpha(x\beta y)]\alpha(y\beta 0) = 0.$$

Thus the axiom (ΓA_2) holds.

Now let $x \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by (3.6) and the axiom (ΓA_1), we get

$$x\alpha x = (x\alpha x)\beta 0 = [(x\beta 0)\alpha(x\beta 0)]\beta(0\alpha 0) = 0.$$

Thus the axiom (ΓA_4) holds. This completes the proof. □

The following is an immediate consequence of Theorems 3.2 and 3.15.

Corollary 3.16. *X is a Γ-BCI-algebra if and only if there is a partial order $\leq$ on X satisfying the following conditions: for all $x,\ y,\ z \in X$ and all $\alpha,\ \beta \in \Gamma$,*
(1) $(x\alpha y)\beta(x\alpha z) \leq z\alpha y$,
(2) $x\alpha(x\beta y) \leq y$,
(3) $x\alpha y = 0$ *if and only if* $x \leq y$.

Definition 3.17. X is a Γ-BCI-algebra and let $x \in X$. Then x is called a *positive element* of X, if $0\alpha x = 0$, i.e., $x \geq 0$ for each $\alpha \in \Gamma$. We will denote the set of all positive elements of X as $P(X)$.

Example 3.18. Let X be the Γ-BCI-algebra given in Example 3.3. Then we can easily see that $P(X) = \{0, 1\}$.

Proposition 3.19. *Let X be the Γ-BCI-algebra. Then $x\alpha[0\beta(0\alpha x)] \in P(X)$ for each $x \in X$ and all $\alpha,\ \beta \in \Gamma$.*

Proof. Let $x \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have
$0\beta[x\alpha(0\beta(0\alpha x))] = (0\alpha x)\beta[0\alpha(0\beta(0\alpha x))]$
[By the second part of Proposition 3.10 (3)]
$= (0\alpha x)\beta(0\alpha x)$ [By Proposition 3.10 (2)]
$= 0.$
Thus $x\alpha[0\beta(0\alpha x)] \in P(X)$. □

Definition 3.20. X is a Γ-BCI-algebra and let $a \in X$. Then a is said to be:
(i) *minimal*, if $x\alpha a = 0$ $(x \in X)$ implies $x = a$ for each $\alpha \in \Gamma$,
(ii) the *least element* of X, if $a\alpha x = 0$ for each $x \in X$ and each $\alpha \in \Gamma$,
(iii) *maximal*, if $a\alpha x = 0$ $(x \in X)$ implies $a = x$ for each $\alpha \in \Gamma$,
(iv) the *greatest element* of X, if $x\alpha a = 0$ for each $x \in X$ and each $\alpha \in \Gamma$
We will denote the set of all minimal [resp. maximal] elements of X as $Min(X)$ [resp. $Max(X)$].

It is obvious that 0 is a minimal element of X and if there is the least element a of X, then $a = 0$.

Example 3.21. (1) Let X be the Γ-BCI-algebra given in Example 3.3(1). Then we can easily check that $Min(X) = \{0\}$ and $Max(X) = \{2\}$. In particular, 0 is the greatest element and 2 the least element of X.

(2) Let X be the Γ-BCI-algebra given in Example 3.3(2). Then clearly, $Min(X) = \varnothing$ and $Max(X) = \{\}$.

Proposition 3.22. *Let X be the Γ-BCI-algebra and let $a \in X$. Then the followings are equivalent: for all $\alpha,\ \beta \in \Gamma$,*
(1) $a \in Min(X)$,
(2) $0\alpha(0\beta a) = a$,
(3) *there is $x \in X$ such that $a = 0\alpha x$.*

Proof. (1)⇒(2): Suppose $a \in Min(X)$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_2$), $[0\alpha(0\beta a)]\alpha a = 0$. Thus by the hypothesis, $0\alpha(0\beta a) = a$.

(2)⇒(3): Suppose $0\alpha(0\beta a) = a$ for any $\alpha,\ \beta \in \Gamma$ and let $x = 0\beta a$. Then clearly, $x \in X$. Moreover, by the hypothesis, $a = 0\alpha(0\beta a) = 0\alpha x$.

(3)⇒(1): Suppose the condition (3) holds and suppose $y\beta a = 0$ for each $\beta \in \Gamma$. Then clearly, we have

$$y\beta(0\alpha x) = 0. \tag{3.10}$$

On the other hand, we get
$a\beta y = (0\alpha x)\beta y$
$= [0\alpha(0\beta(0\alpha x))]\beta y$ [By Proposition 3.10(2)]
$= (0\alpha y)\beta[0\beta(0\alpha x)]$ [By Proposition 3.8]
$= 0\beta[y\beta(0\alpha x)]$ [By Proposition 3.10(3)]
$= 0\beta 0$ [By (3.10)]
$= 0.$
Thus $a\beta y = 0 = y\beta a$. So by the axiom (ΓA_3), $y = a$. Hence $a \in MIn(X)$. □

Definition 3.23. Let X be a Γ-BCI-algebra and let S be a nonempty subset of X. Then S is called a Γ-*subalgebra* of X, if S itself is a Γ-BCI-algebra.

It is obvious that X and $\{0\}$ are Γ-subalebras of X. In this case, X and $\{0\}$ will be called the *trivial* Γ-*subalgebras* of X. A nonempty subset S is called a *proper* Γ-*subalgebra* of X, if S is a Γ-subalgebra of X and $S \subsetneqq X$. It is clear that $\{0\}$ is a proper Γ-subalgebra of X.

From the above definition, we obtain easily the following.

Theorem 3.24. *Let X be a Γ-BCI-algebra and let S be a nonempty subset of X. Then S is a Γ-subalgebra of X if and only if $x\alpha y \in X$ for all $x,\ y \in S$ and each $\alpha \in \Gamma$.*

Proposition 3.25. *Let X be a Γ-BCI-algebra. Then $P(X)$ and $Min(X)$ are Γ-subalgebra of X.*

Proof. It is clear that $0\alpha 0 = 0$ for each $\alpha \in \Gamma$. Then $P(X) \neq \varnothing$. Let $x,\ y \in P(X)$ and let $\alpha,\ \beta \in \Gamma$. Then $0\alpha x = 0 = 0\alpha y$. Thus by Proposition 3.10(3), we have

$$0\alpha(x\alpha y) = (0\beta x)\alpha(0\beta y) = 0\alpha 0 = 0$$

or

$$0\beta(x\alpha y) = (0\alpha x)\beta(0\alpha y) = 0\beta 0 = 0.$$

So $x\alpha y \in P(X)$. Hence $P(X)$ is a Γ-subalgebra of X. □

4. Γ-ideals of Γ-BCI-algebras

Definition 4.1. Let X be a Γ-BCI-algebra and let I be a nonempty subset of X. Then I is called a Γ-*ideal* of X, if it satisfies the following conditions: for all $x,\ y \in X$ and $\alpha \in \Gamma$,

(i) $0 \in I$,

(ii) if $x\alpha y \in I$ and $y \in I$, then $x \in I$, equivalently, if $x \leq y$ and $y \in I$, then $x \in I$.

We will denote the set of all Γ-ideals of X by $\Gamma\mathcal{I}(X)$.

Example 4.2. (1) Let X be the Γ-BCI-algebra given in Example 3.3(1). Then we can easily check that $\{0,1\},\ \{0,2\} \notin \Gamma\mathcal{I}(X)$.

(2) Let X be the Γ-BCI-algebra given in Example 3.3(2). The we can see that

$$\{0,1\},\ \{0,2\},\ \{0,3\} \in \Gamma\mathcal{I}(X).$$

From Definition 4.1, we obtain easily the following characterization of Γ-ideals.

Theorem 4.3. *Let X be a Γ-BCI-algebra and let I be a nonempty subset of X. Then $I \in \Gamma\mathcal{I}(X)$ if and only if it satisfies the condition (i) and the following condition:*

(4.1) *if $x\alpha y \leq z$ and $y,\ z \in I$, then $x \in I$ for all $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$.*

Lemma 4.4. *Let X be a Γ-BCI-algebra. If $I \in \Gamma\mathcal{I}(X)$, then*

$$I = \bigcup_{x,\ y \in I} A(x,y),$$

where $A(x,y) = \{z \in X : (z\alpha x)\beta y = 0$ for all $x,\ y \in X$ and all $\alpha,\ \beta \in \Gamma\}$.

Proof. Suppose I is a Γ-ideal of X and let $z \in I$. Then clearly, for all $\alpha,\ \beta \in \Gamma$,

$$(z\alpha 0)\beta z = z\alpha z)\beta 0 = 0\beta 0 = 0.$$

Since $0 \in I$, $z \in A(0,z)$. Thus we have

$$I \subset \bigcup_{z \in I} A(0,z) \subset \bigcup_{x,\ y \in I} A(x,y).$$

Now let $z \in \bigcup_{x,\ y \in I} A(x, y)$. Then there are $a,\ b \in I$ such that $z \in A(a, b)$. Thus $(z\alpha a)\beta b = 0$. Since $I \in \Gamma\mathcal{I}(X)$, by Theorem 4.3, $z \in I$. So $\bigcup_{x,\ y \in I} A(x, y) \subset I$. Hence $I = \bigcup_{x,\ y \in I} A(x, y)$. $\square$

Corollary 4.5. *Let X be a Γ-BCI-algebra. If $I \in \Gamma\mathcal{I}(X)$, then*

$$I = \bigcup_{x \in I} A(0, x).$$

Proof. From Lemma 4.4, it is clear that $\bigcup_{x \in I} A(0, x) \subset \bigcup_{x,\ y \in I} A(x, y) = I$. Let $x \in I$ and let $\alpha,\ \beta \in \Gamma$. Then $(x\alpha 0)\beta x = 0$. Thus $x \in A(0, x)$. So $I \subset \bigcup_{x \in I} A(0, x)$. Hence $I = \bigcup_{x \in I} A(0, x)$. $\square$

Lemma 4.6. *Let I be a subset of a Γ-BCI-algebra X such that $0 \in I$. If $I = \bigcup_{x,\ y \in I} A(x, y)$, then $I \in \Gamma\mathcal{I}(X)$.*

Proof. Suppose $I = \bigcup_{x,\ y \in I} A(x, y)$ and $x\alpha y,\ y \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then by the axiom the axiom (ΓA_2), $[x\beta(x\alpha y)]\beta y = 0$. Thus $x \in A(x\alpha y,\ y) \subset I$. So $I \in \Gamma\mathcal{I}(X)$. $\square$

From Lemmas 4.4 and 4.6, we have a characterization of Γ-ideals.

Theorem 4.7. *Let I be a subset of a Γ-BCI-algebra X such that $0 \in I$. Then $I \in \Gamma\mathcal{I}(X)$ if and only if $I = \bigcup_{x,\ y \in I} A(x, y)$.*

Definition 4.8. Let X be a Γ-BCI-algebra X and let $I \in \Gamma\mathcal{I}(X)$. Then I is called a *closed Γ-ideal* of X, if $x \in I$ implies $0\alpha x \in I$ for each $\alpha \in \Gamma$.

We will denote the set of all closed Γ-ideals of X by $\Gamma\mathcal{I}_c(X)$.

Example 4.9. Let X be the Γ-BCI-algebra given in Example 3.3(2). The we can see that $\{0, 1\},\ \{0, 2\},\ \{0, 3\} \in \Gamma\mathcal{I}_c(X)$.

Proposition 4.10. *Every closed Γ-ideal of a Γ-BCI-algebra X is a Γ-subalgebra of X.*

Proof. The proof follows from Definitions 3.23 and 4.8, and Proposition 3.10(3). $\square$

The following is a characterization of closed Γ-ideals.

Theorem 4.11. *Let X be a Γ-BCI-algebra and let I be a subset of X. Then $I \in \Gamma\mathcal{I}_c(X)$ if and only if it satisfies the following conditions: for all $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,*

(1) $0 \in I$,

(2) *if $x\alpha z,\ y\alpha z,\ z \in I$, then $x\alpha y \in I$.*

Proof. Suppose $I \in \Gamma\mathcal{I}_c(X)$ and $x\alpha z,\ y\alpha z,\ z \in I$ for all $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$. Then clearly, $0 \in I$ and $x,\ y \in I$. Thus by Proposition 4.10, $x\alpha y \in I$.

Conversely, suppose the necessary conditions hold and $x\alpha y,\ y \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. It is obvious that $0\alpha 0,\ y\alpha 0,\ 0 \in I$. Then by (2), $0\alpha y \in I$. Also, by (2), $x = x\alpha 0 \in I$. Thus $I \in \mathcal{I}(X)$. Now let $x \in X$ and let $\alpha \in \Gamma$. Then clearly, $0\alpha 0,\ x\alpha 0,\ 0 \in I$. Thus by (2), $0\alpha x \in I$. So $I \in \Gamma\mathcal{I}_c(X)$. $\square$

Proposition 4.12. *Let I be a Γ-ideal of a Γ-BCI-algebra X. Then the subset I^0_Γ of X defined by*

$$I^0_\Gamma = \{x \in X : 0\alpha x \in I \ for\ each\ \alpha \in \Gamma\}$$

is the greatest closed Γ-ideal of X such that $I^0_\Gamma \subset I$.

Proof. By the definition of I^0_Γ, it is clear that $0 \in I^0_\Gamma$. Suppose $x\alpha y,\ y \in I^0_\Gamma$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$. Then by the definition of I^0_Γ and Proposition 3.10(3), we have: for each $\beta \in \Gamma$,

$$0\alpha y,\ (0\alpha x)\beta(0\alpha y) = 0\beta(x\alpha y) \in I.$$

Since $I \in \Gamma\mathcal{I}(X)$, $0\alpha x \in I$. Thus $x \in I^0_\Gamma$. So $I^0_\Gamma \in \Gamma\mathcal{I}(X)$.

Now let $x \in I^0_\Gamma$. Then by the definition of I^0_Γ and the axiom (ΓA_2), we get: for any $\alpha,\ \beta \in \Gamma$,

$$0\alpha x \in I,\ [0\beta(0\alpha x)]\beta x = 0.$$

Thus $0\beta(0\alpha x) \in I$. So $0\alpha x \in I^0_\Gamma$. Hence $I^0_\Gamma \in \Gamma\mathcal{I}_c(X)$.

Finally let $J \in \Gamma\mathcal{I}_c(X)$ such that $J \subset I$ and let $x \in J,\ \alpha \in \Gamma$. Then $0\alpha x \in J$. Since $J \subset I$, $0\alpha x \in I$. Thus $x \in I^0_\Gamma$. So $J \subset I^0_\Gamma$. So I^0_Γ is the greatest closed Γ-ideal of X contained in I. □

Now we discuss some properties of commutative Γ-ideals of a Γ-BCI-algebra.

Definition 4.13. Let X be a Γ-BCI-algebra and let I be a nonempty subset of X. Then I is called a *commutative* Γ*-ideal* of X, if it satisfies the following conditions: for all $x,\ y,\ z, \in X$ and all $\alpha,\ \beta \in \Gamma$,

(i) $0 \in I$,

(ii) if $(x\alpha y)\beta z,\ z \in I$, then $x\alpha[(y\beta(y\alpha x))\beta(0\alpha(0\beta(x\alpha y)))] \in I$.

We will denote the set of all commutative Γ-ideals and the set of all commutative closed Γ-ideals of X by $\Gamma\mathcal{CI}(X)$ and $\Gamma\mathcal{CI}_c(X)$ respectively.

Proposition 4.14. *Every commutative* Γ*-ideal is a* Γ*-ideal.*

Proof. Let X be a Γ-BCI-algebra and let $I \in \Gamma\mathcal{CI}(X)$. Suppose $x\alpha y,\ y \in I$ for all $x,\ y \in X$ and each $\alpha \in \gamma$. Then by the axiom (ΓA_4), $(x\beta 0)\alpha y \in I$ and $y \in I$. Thus by the condition (ii), we have

$$x\alpha[(0\beta(0\alpha x))\beta(0\alpha(0\beta(x\alpha 0)))] \in I.$$

By Theorem 3.15 and Proposition 3.10(2), $x = x\alpha[(0\beta(0\alpha x))\beta(0\alpha(0\beta(x\alpha 0)))]$. So $x \in I$. Hence $I \in \Gamma\mathcal{I}(X)$. □

The converse of Proposition 4.14 does not hold in general (See Example 4.15).

Example 4.15. Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3, 4\}$ be a set with the ternary operation defined as the following table:
Then we can easily check that X is a Γ-BCI-algebra and $\{0, 1\} \in \Gamma\mathcal{I}_c(X)$ but $\{0, 1\} \notin \Gamma\mathcal{CI}(X)$.

We give a characterization of Γ-ideals.

α	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

β	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	0	0
2	2	3	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Table 4.1

Theorem 4.16. *Let I be a Γ-ideal of a Γ-BCI-algebra X. Then $I \in \Gamma\mathcal{CI}(X)$ is commutative if and only if it satisfies the following condition: for all $x,\ y \in X$ and all $\alpha,\ \beta \in \Gamma$,*

$$\text{if } x\alpha y \in I, \text{ then } x\alpha[(y\beta(y\alpha x))\beta(0\alpha(0\beta(x\alpha y)))] \in I. \tag{4.2}$$

Proof. The proof is straightforward from Definition 4.13. □

The following is another characterization of commutative Γ-ideals.

Theorem 4.17. *Let I be a closed Γ-ideal of a Γ-BCI-algebra X. Then $I \in \Gamma\mathcal{CI}(X)$ if and only if it satisfies the following condition: for all $x,\ y \in X$ andall $\alpha,\ \beta \in \Gamma$,*

$$\text{if } x\alpha y \in I, \text{ then } x\alpha[y\beta(y\alpha x)] \in I. \tag{4.3}$$

Proof. Suppose $I \in \Gamma\mathcal{CI}(X)$ and $x\alpha y \in I$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$. Since I is closed, $0\beta(x\alpha y) \in I$ for each $\beta \in \Gamma$. Since $I \in \Gamma\mathcal{CI}(X)$, by (4.2), we get

$$x\alpha[(y\beta(y\alpha x))\beta(0\alpha(0\beta(x\alpha y)))] \in I.$$

On the other hand, we have

$[x\alpha(y\beta(y\alpha x))]\beta[x\alpha((y\beta(y\alpha x))\beta(0\alpha(0\beta(x\alpha y))))]$
$= [x\alpha(y\beta(y\alpha x))]\beta[x\alpha(0\alpha(0\beta(x\alpha y))))]\beta(y\beta(y\alpha x))$ [By Proposition 3.8]
$\leq (0\beta(x\alpha y)))\alpha(y\beta(y\alpha x))\beta(y\beta(y\alpha x))$ [By Theorem 3.2(1)]
$= (0\alpha(0\beta(x\alpha y))))\beta 0$ [By the axiom (ΓA_4)]
$= 0\alpha(0\beta(x\alpha y))$. [By Theorem 3.15]
$= 0\beta(x\alpha y) \in I$.

Then by Theorem 4.3, $x\alpha(y\beta(y\alpha x)) \in I$. Thus the condition (4.3) holds.

Conversely, suppose the condition (4.3) holds and let $x\alpha y \in I$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$. Then clearly, $x\alpha(y\beta(y\alpha x)) \in I$ for each $\beta \in \Gamma$. From Definition 4.1, it is obvious that $0\alpha(0\beta(x\alpha y)) \in I$. On the other hand, we have

$[(x\alpha(y\beta(y\alpha x)))\beta(0\alpha(0\beta(x\alpha y)))]\beta[x\alpha(y\beta(y\alpha x))]$
$= [(x\alpha(y\beta(y\alpha x)))\beta(x\alpha(y\beta(y\alpha x)))]\beta(0\alpha(0\beta(x\alpha y)))$
$\leq [(y\beta(y\alpha x))\beta(y\beta(y\alpha x))]\beta(0\alpha(0\beta(x\alpha y)))$
$\leq 0\alpha(0\beta(x\alpha y)) \in I$.

Thus by Theorem 4.3, $(x\alpha(y\beta(y\alpha x)))\beta(0\alpha(0\beta(x\alpha y))) \in I$. So by Theorem 4.16, $I \in \Gamma\mathcal{CI}(X)$. □

Proposition 4.18. *Let I, J be two Γ-ideals of a Γ-BCI-algebra X such that $I \subset J$. If $J \in \Gamma\mathcal{I}_c(X)$ and $I \in \Gamma\mathcal{CI}(X)$, then $J \in \Gamma\mathcal{CI}(X)$.*

Proof. Suppose $x\alpha y \in J$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$ and let $u = x\alpha y$. Since $J \in \Gamma\mathcal{I}_c(X)$, $0\beta u \in J$ for each $\beta \in \Gamma$. Then by Proposition 3.8 and the axiom (ΓA_4),

$(x\alpha u)\beta y = 0 \in I$. Since $I \in \Gamma\mathcal{CI}(X)$, by Theorem 4.16, we have

$$(x\alpha u)\beta[y\alpha(y\beta(x\alpha u))] = (x\alpha y)\beta[(y\alpha(y\beta(x\alpha u)))\beta(0\alpha(0\beta((x\alpha u)\beta y)))] \in I.$$

Since $I \subset J$, by Proposition 3.8, we get

$$(x\alpha u)\beta[y\alpha(y\beta(x\alpha u))] = [x\alpha(y\alpha(y\beta(x\alpha u)))]\beta u \in J.$$

Since $u \in J$, $x\alpha[y\alpha(y\beta(x\alpha u))] \in J$. On the other hand, we have

$[x\alpha(y\alpha(y\beta x))]\beta[x\alpha(y\alpha(y\beta(x\alpha u)))]$
$\leq [y\alpha(y\beta(x\alpha u))]\alpha[y\alpha(y\beta x)]$ [By Corollary 3.9(1)]
$\leq (y\beta x)\alpha(y\beta(x\alpha u))]$
$\leq (x\alpha u)\alpha x$
$= (x\alpha x)\alpha u$ [By Proposition 3.8]
$= 0\alpha u \in J$. [By the axiom (ΓA_4)]

Thus by Theorem 4.3, $x\alpha(y\alpha(y\beta x)) \in J$. So by Theorem 4.17, $J \in \Gamma\mathcal{CI}(X)$. □

Definition 4.19. A Γ-BCI-algebra X is said to be commutative, if it satisfies the following condition: for all $x,\ y \in X$ and all $\alpha,\ \beta \in \Gamma$,

$$\text{(4.4)} \qquad if\ x \leq y,\ then\ x = y\alpha(y\beta x).$$

We have a similar result to Theorem 3.18 in [21].

Theorem 4.20. *A Γ-BCI-algebra X is commutative if and only if it satisfies the following condition: for all $x,\ y \in X$ and all $\alpha,\ \beta \in \Gamma$,*

$$\text{(4.5)} \qquad x\alpha(x\beta y) = y\alpha[y\beta(x\alpha(x\beta y))].$$

Theorem 4.21. *Let X be A Γ-BCI-algebra. The the followings are equivalent:*
(1) *X is commutative,*
(2) *every closed Γ-ideal of X is commutative,*
(3) *$\{0\} \in \Gamma\mathcal{CI}(X)$.*

Proof. (1)⇒(2): Suppose X is commutative and let $I \in \Gamma\mathcal{I}_c(X)$. For all $x,\ y \in X$ and each $\alpha \in \Gamma$, suppose $x\alpha y \in I$. Then clearly, $0\beta(x\alpha y) \in I$ for each $\beta \in \Gamma$. On the other hand, we get

$[x\alpha(y\beta(y\alpha x))]\beta(x\alpha y)$
$= [x\alpha(x\alpha y)]\beta[y\beta(y\alpha x)]$ [By Proposition 3.8]
$= [y\alpha(y\alpha(x\alpha(x\alpha y)))]\beta[y\beta(y\alpha x)]$ [By Theorem 4.20]
$= [y\alpha(y\beta(y\alpha x))]\beta[y\alpha(x\alpha(x\alpha y))]$ [By Proposition 3.8]
$= (y\alpha x)\beta[y\alpha(x\alpha(x\alpha y))]$ [By Proposition 3.10(2)]
$\leq [x\alpha(x\alpha y)]\alpha x$ [By Corollary 3.9 (1)]
$= 0\alpha(x\alpha y) \in I$.

Thus by Theorem 4.3, $x\alpha(y\beta(y\alpha x)) \in I$. So $I \in \Gamma\mathcal{CI}(X)$.

(2)⇒(3): The proof is straightforward.

(3)⇒(1): Suppose $\{0\} \in \Gamma\mathcal{CI}(X)$ and suppose $x \leq y$ for all $x,\ y \in X$. Then clearly, $x\beta y = 0 \in \{0\}$. for each $\beta \in \Gamma$. Since $\{0\} \in \Gamma\mathcal{CI}_c(X)$, by Theorem 4.17, we have

$$x\beta[y\alpha(y\beta x)] \in \{0\},\ \text{i.e.,}\ x\beta[y\alpha(y\beta x)] = 0,\ \text{i.e.,}\ x \leq y\alpha(y\beta x) \text{ for all } \alpha,\ \beta \in \Gamma.$$

Since X is a Γ-BCI-algebra, by Theorem 3.2(2), we get

$$y\alpha(y\beta x) \leq x \text{ for all } \alpha,\ \beta \in \Gamma.$$

Thus by Theorem 3.2(3), $x = y\alpha(y\beta x)$. So X is commutative. This completes the proof. □

Lemma 4.22 (See Proposition 4.1, [23]). *Let X be A Γ-BCI-algebra and let $I \in \Gamma\mathcal{I}(X)$. Let $\sim$ be the relation on X define as follows: for all $x,\ y \in X$,*

$$x \sim y \text{ if and only if } x\alpha y,\ y\alpha x \in I \text{ for each } \alpha \in \Gamma.$$

Then $\sim$ is a congruence relation on X, i.e., it satisfies the following conditions: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,
(1) *$x \sim x$, i.e., $\sim$ is reflexive,*
(2) *if $x \sim y$, then $y \sim x$, i.e., $\sim$ is symmetric,*
(3) *if $x \sim y$ and $y \sim z$, then $\sim z$, i.e., $\sim$ is transitive,*
(4) *if $x \sim u$ and $y \sim v$, then $x\alpha y \sim u\alpha v$.*

Proof. The proof is similar to Proposition 4.1 in [23]. □

For a congruence relation $\sim$ on a Γ-BCI-algebra X and each $x \in X$, a subset $I[x]$ of X defined by

$$I[x] = \{y \in X : x \sim y\}$$

is called the *congruence class* in X determined by x with respect to $\sim$. The set of all congruence classes in X is denoted by X/I. It is obvious that if $I \in \Gamma\mathcal{I}_c(X)$, then $I[0] = I$ but if $I \notin \Gamma\mathcal{I}_c(X)$, then $I[0] \neq I$.

Example 4.23. Let X be the Γ-BCI-algebra and let $I = \{0, 1\}$ be the closed Γ-ideal of X given in Example 4.15. Then by the calculation, we have

$$I[0] = I = I[1],\ I[2] = \{2\},\ I[3] = \{3\},\ I[4] = \{4\}.$$

Thus $X/I = \{I, I[2], I[3], I[4]\}$.

We obtain a similar consequence of Proposition 4.2 in [23].

Lemma 4.24. *Let X be a Γ-BCI-algebra, $I \in \Gamma\mathcal{I}_c(X)$ and let $\sim$ be a congruence relation on X. We define a mapping $f : X/I \times \Gamma \times X/I \to X/I$ as follows: for each $(I[x], \alpha, I[y]) \in X/I \times \Gamma \times X/I$,*

$$f(I[x], \alpha, I[y]) = I[x]\alpha I[y] = I[x\alpha y].$$

Then X/I is a Γ-BCI-algebra. In this case, X/I is called the *quotient Γ-BCI-algebra of X by I.*

We define a partial ordering $\leq$ on X/I as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$I[x] \leq I[y] \text{ if and only if } I[x]\alpha I[y] = I[0] = I.$$

Then we have a similar consequence of Theorem 3.2.

Proposition 4.25. *Let X be a Γ-BCI-algebra and let X/I be the quotient Γ-BCI-algebra of X by $I \in \mathcal{I}_c(X)$. Then the followings hold: for all $x,\ y,\ z \in X$ and all $\alpha,\ \beta \in \Gamma$,*
(1) *$(I[x]\alpha I[y])\beta(I[x]\alpha I[z]) \leq I[z]\alpha I[y]$,*
(2) *$I[x]\alpha(I[x]\beta I[y]) \leq I[y]$,*
(3) *if $I[x] leq I[y]$ and $I[y] \leq I[x]$, then $I[x] = I[y]$,*
(4) *$I[x] \leq I[x]$.*

Theorem 4.26. *Let X be a Γ-BCI-algebra and let $I \in \Gamma\mathcal{I}_c(X)$. Then $I \in \Gamma\mathcal{CI}(X)$ if and only if X/I is a commutative Γ-BCI-algebra.*

Proof. Suppose $I \in \Gamma\mathcal{CI}_c(X)$. It is clear that $I[0] = I$ and $\{I[0]\}$ is the zero Γ-ideal of X/I. Suppose $I[x]\alpha I[y] \in \{I[0]\}$, $I[x]\alpha I[y] = I[0]$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$. Then $x\alpha y \in I$. Thus by Theorem 4.17, $x\alpha[y\beta(y\alpha x)] \in I$. So we have

$$I[x]\alpha[I[y]\beta(I[y]\alpha I[x])] = I[x\alpha[y\beta(y\alpha x)]] = I = I[0] \in \{I[0]\}.$$

Hence $\{I[0]\} \in \Gamma\mathcal{CI}(X/I)$. Therefore by Theorem 4.21, X/I is commutative.

Conversely, suppose X/I is a commutative Γ-BCI-algebra. Then by Theorem 4.21, $\{I[0]\} \in \Gamma\mathcal{CI}(X/I)$. Suppose $x\alpha y \in I$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$. Then we have

$$I[x]\alpha I[y] = I[x\alpha y] = I - I[0] \in \{I[0]\}.$$

Thus $I[x\alpha[y\beta(y\alpha x)]] = I[x]\alpha[I[y]\beta(I[y]\alpha I[x])] \in \{I[0]\}$. So $x\alpha[y\beta(y\alpha x)] \in I$. Hence $I \in \Gamma\mathcal{CI}(X)$. □

Definition 4.27. Let X and Y be Γ-BCI-algebras. Then a mapping $f : X \to Y$ is called a Γ-*homomorphism*, if $f(x\alpha y) = f(x)\alpha f(y)$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$. In this case, the subset $ker(f)$ (called *the kernel of f*) of X and the subset $Im(f)$ (called *the image of f*) of Y are defined as follows respectively:

$$ker(f) = \{x \in X : f(x) = 0\},\ Im(f) = \{f(x) \in Y : x \in X\}.$$

We have easily similar consequences of some properties given in [20].

Proposition 4.28. *Let X and Y be Γ-BCI-algebras and let $f : X \to Y$ be a Γ-homomorphism. Then*
(1) *$Im(f)$ is a Γ-subalgebra of Y* (See Theorem 3.18, [20]),
(2) *$ker(f)$ is a Γ-subalgebra of X* (See Lemma 3.19, [20]).

Proposition 4.29 (See Lemma 3.20, [20])**.** *Let X and Y be Γ-BCI-algebras and let $f : X \to Y$ be a Γ-homomorphism.*
(1) $f(0) = 0$.
(2) *If $x\alpha y = 0$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$, then $f(x)\alpha f(y) = 0$.*

Theorem 4.30 (See Theorem 3.21, [20])**.** *Let X and Y be Γ-BCI-algebras and let $f : X \to Y$ be a Γ-homomorphism. Then f is injective if and only if $ker(f) = \{0\}$.*

Proposition 4.31 (See Theorem 4.10, [20])**.** *Let X and Y be Γ-BCI-algebras and let $f : X \to Y$ be a Γ-homomorphism. Then $ker(f) \in \Gamma\mathcal{I}_c(X)$.*

Proof. Since f is a a Γ-homomorphism, $f(0) = f(0\alpha 0) = f(0)\alpha f(0) = 0$ for each $\alpha \in \Gamma$. Then $0 \in ker(f)$. Now suppose $x\alpha y,\ y \in ker(f)$ for all $x,\ y \in X$ and each $\alpha \in \Gamma$. Then we have

$$0 = f(x\alpha y) = f(x\alpha y) = f(x)\alpha f(y) = f(x)\alpha 0 = f(x).$$

Thus $x \in ker(f)$. So $ker(f) \in \Gamma\mathcal{I}(X)$. Finally, let $x \in ker(f)$. Then $f(x) = 0$. On the other hand, we get

$$f(0\alpha x) = f(0)\alpha f(x) = 0\alpha 0 = 0 \text{ for each } \alpha \in \Gamma.$$

Thus $0\alpha x \in ker(f)$ for each $\alpha \in \Gamma$. So $ker(f) \in \Gamma\mathcal{I}_c(X)$. □

Theorem 4.32. *Let X and Y be Γ-BCI-algebras and let $f : X \to Y$ be a Γ-epimorphism. Then $ker(f) \in \Gamma\mathcal{CI}(X)$ if and only if Y is a commutative Γ-BCI-algebra.*

Proof. By Proposition 4.31, $ker(f) \in \Gamma\mathcal{I}_c(X)$. Then by Theorem 4.26, $ker(f) \in \Gamma\mathcal{CI}(X)$ if and only if $X/ker(f)$ is a commutative Γ-BCI-algebra. Since f is surjective, it is obvious that $X/ker(f)$ is isomorphic to Y. Thus the result holds. □

Remark 4.33. Let X be a Γ-BCI-algebra and let $I \in \Gamma\mathcal{I}_c(X)$. We define the mapping $\pi : X \to X/I$ as follows:

$$\pi(x) = I[x] \text{ for each } x \in X.$$

Then we can easily check that π is a surjective homomorphism. In this case, π is called the *natural homomorphism*.

Proposition 4.34. *Let X be a Γ-BCI-algebra and let $\pi : X \to X/I$ be the natural homomorphism, where $I \in \Gamma\mathcal{I}_c(X)$. If $J \in \Gamma\mathcal{I}_c(X/I)$, then $\pi^{-1}(J) \in \Gamma\mathcal{I}_c(X)$ such that $I \subset \pi^{-1}(J)$.*

Proof. The proof is straightforward. □

5. Topological structures on Γ-BCI-algebras

We recall some terms and notations related to a general topology (See [24, 25]). For a subset A of a topological space (X, τ), we denote the closure and the interior of A as $cl_\tau(A)$, $cl(A)$ or $\overline{A}$ and $int_\tau(A)$, $int(A)$ or A°. A subfamily $\mathcal{B}$ of τ is called a *base* for τ, if for each $U \in \tau$ either $U = \varnothing$ or there is $\mathcal{B}' \subset \mathcal{B}$ such that $U = \bigcup \mathcal{B}'$. A subset A of X is called a *neighborhood* of $x \in X$, if there is $U \in \tau$ such that $x \in U \subset A$. We denote the set of all neighborhoods of x as $N_\tau(x)$ or $N(x)$ and $N(x)$ is called the *neighborhood filter* of $x \in X$. A subfamily $\mathcal{N}(x)$ of $N(x)$ is called a *fundamental system of neighborhoods* of x, if for each $U \in N(x)$ there is $V \in \mathcal{N}(x)$ such that $V \subset U$. In fact, $\mathcal{N}(x)$ is a filter base of $N(x)$. In particular, it is well-known ([24]) that $N_\tau(x)$ satisfies the following properties:

(N_1) $x \in U$ for each $U \in N_\tau(x)$,
(N_2) if $U \in N_\tau(x)$ and $U \subset V \subset X$, then $V \in N_\tau(x)$,
(N_3) if $U_1,\ U_2 \in N_\tau(x)$, then $U_1 \cap U_2 \in N_\tau(x)$,
(N_4) if $V \in N_\tau(x)$, there is $W \in N_\tau(x)$ such that $V \in N_\tau(x)$ for each $y \in W$.

Furthermore, it is well-known (Proposition 1.1.2, [24]) that for each $x \in X$ if $\mathcal{B}(x)$ be a set of subsets of X satisfying the properties (N_1)–(N_4), then a unique topology on X such that $\mathcal{B}(x) = N_\tau(x)$. In fact,

$$\tau = \{V \subset X : \forall x \in V,\ \exists U \in \mathcal{B}(x) \text{ such that } U \subset V\}.$$

Definition 5.1 (See Theorem 3.3, [10])**.** Let X be a BCI-algebra and let τ be a topology on X. Then X is called a *topological BCI-algebra* (briefly, $TBCI$-algebra), if $* : (X \times X, \tau \times \tau) \to (X, \tau)$ is continuous, i.e., for all $x,\ y \in X$ and each $W \in N(x * y)$, there are $U \in N(x)$ and $V \in N(y)$ such that $U * V \subset W$, where $U * V = \{x * y \in X : x \in U,\ y \in V\}$.

Definition 5.2. Let X be a Γ-BCI-algebra and let τ be a topology on X. Then X is called a *topological* Γ-BCI*-algebra* (briefly, $T\Gamma$-BCI-algebra), if the mapping $f : (X, \tau) \times \Gamma \times (X, \tau) \to (X, \tau)$ is continuous at each $(x, \alpha, y) \in X \times \Gamma \times X$, i.e., for each $\alpha \in \Gamma$, all $x,\ y \in X$ and each $W \in N(x\alpha y)$, there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset W$, where $U\alpha V \subset W = \{x\alpha y : x \in U,\ y \in V\}$.

It is obvious that if X is a $T\Gamma$-BCI-algebra, then X_α is a $TBCI$-algebra for each $\alpha \in \Gamma$.

Example 5.3. Let $X = \{0, 1, 2, 3, 4\}$, let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3, 4\}$ be the Γ-BCI-algebra having the the ternary operation defined as the following table:

α	0	1	2	3	4	β	0	1	2	3	4
0	0	0	0	0	4	0	0	1	0	0	4
1	1	0	0	1	4	1	1	0	2	1	4
2	2	2	0	2	4	2	2	2	0	3	4
3	3	3	3	0	4	3	3	1	3	0	4
4	4	4	4	4	0	4	4	4	4	4	0

Table 5.1

Consider the topology τ on X given by:

$$\tau = \{\varnothing, \{4\}, \{0, 1, 2, 3\}, X\}.$$

Then we can easily check that (X, τ) is a $T\Gamma$-BCI-algebra. Moreover, X_α and X_β are $TBCI$-algebras.

Proposition 5.4. *Let X be a $T\Gamma$-BCI-algebra. If $\{0\}$ is open in X, then X is discrete.*

Proof. Suppose $\{0\}$ is open in X and let $x \in X$. Then clearly, $x\alpha x = 0 \in \{0\}$. for each $\alpha \in \Gamma$. Thus by the hypothesis, there are $U,\ V \in N(x)$ such that $U\alpha V \subset \{0\}$, i.e., $U\alpha V = \{0\}$. Let $W = U \cap V$. Then $W\alpha W \subset U\alpha V$, i.e., $W\alpha W = \{0\}$. Since $U,\ V \in N(x)$, $x \in U \cap V$. Thus $x \in W$. So $W = \{x\}$ and W is open in X. Hence X is discrete. □

The following is an immediate consequence of Proposition 5.4.

Corollary 5.5 (See Proposition 3.5, [10])**.** *Let X be a $T\Gamma$-BCI-algebra. If $\{0\}$ is open in X, then each X_α is discrete.*

Theorem 5.6. *Let X be a $T\Gamma$-BCI-algebra. Then $\{0\}$ is closed in X if and only if X is Hausdorff.*

Proof. Suppose $\{0\}$ is closed in X, let $x,\ y \in X$ such that $x \neq y$. and let $\alpha \in \Gamma$. Then $x\alpha y \neq 0$ or $y\alpha x \neq 0$, say $x\alpha y \neq 0$ for each $\alpha \in \Gamma$. Since $\{0\}$ is closed in X and $x\alpha y \neq 0$, $\{0\}^c$ is open in X and $x\alpha y \in \{0\}^c$. Thus $\{0\}^c \in N(x\alpha y)$. Since X is a $T\Gamma$-BCI-algebra, by Definition 5.2, there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset \{0\}^c$. So $U \cap V = \varnothing$. Hence X is Hausdorff.

Conversely, suppose X is Haousdorff and let $x \in \{0\}^c$. Then $x \neq 0$. By the hypothesis, there are $U \in N(x)$ and $V \in N(0)$ such that $U \cap V = \varnothing$. Thus $0 \notin U$. So $U \subset \{0\}^c$. Hence $\{0\}^c$ is open in X. Therefore $\{0\}$ is closed in X. □

The following is an immediate consequence of Theorem 5.6.

Corollary 5.7 (See Proposition 3.6, [10]). *Let X be a $T\Gamma$-BCI-algebra. Then $\{0\}$ is closed in X if and only if each X_α is Hausdorff.*

Proposition 5.8. *Let X be a $T\Gamma$-BCI-algebra and let A be open in X. If A is Γ-subalgebra of X, then A is a $T\Gamma$-BCI-algebra.*

Proof. Let τ be the topology on X and let τ_A be the subspace topology on A with respect to τ. Let $x,\ y \in A$. Since A is a Γ-subalgebra of X, $x\alpha y \in A$ for each $\alpha \in \Gamma$. Let $W_A \in N_{\tau_A}(x\alpha y)$, where $N_{\tau_A}(x\alpha y)$ denotes the neighborhood of $x\alpha y$ in the subspace $(A, \tau_A$ of (X, τ). Then there is $W \in N(x\alpha y)$ such that $W_A = A \cap W$. Since X is a $T\Gamma$-BCI-algebra, there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset W$. Thus $U_A = A \cap U \in N_{\tau_A}(x)$ and $V_A = A \cap V \in N_{\tau_A}(x)$. It is clear that

$$U_A\alpha V_A = (A \cap U)\alpha(A \cap V) \subset W \text{ and } U_A\alpha V_A \subset A.$$

So $U_A\alpha V_A \subset A \cap W = W_A$. Hence A is a $T\Gamma$-BCI-algebra. □

We have an immediate consequence of Proposition 5.8.

Corollary 5.9. *Let X be a $T\Gamma$-BCI-algebra and let A be open in X_α for each $\alpha \in \Gamma$. If A is subalgebra of X_α, then A is a $TBCI$-algebra.*

Proposition 5.10. *Let X be a $T\Gamma$-BCI-algebra and let I be open in X. If I is a Γ-ideal of X, then I is closed in X.*

Proof. Let $x \in I^c$ and let $\alpha \in \Gamma$. Since $x\alpha x = 0 \in I$ and I is open, $I \in N(0)$. Since X is a $T\Gamma$-BCI-algebra, there is $U \in N(x)$ such that $U\alpha U \subset I$. Assume that $U \not\subset I^c$. Then there is $y \in X$ such that $y \in U \cap I$. It is obvious that $z\alpha y \in U\alpha U \subset I$ for each $z \in U$. Since I is a Γ-ideal of X and $y \in I$, $z \in I$. Thus $U \subset I$. This is a contradiction. So $U \subset I^c$, i.e., I^c is open in X. Hence I is closed in X. □

We obtain an immediate consequence of Proposition 5.10.

Corollary 5.11 (See Proposition 3.8, [10]). *Let X be a $T\Gamma$-BCI-algebra and let I be open in X_α for each $\alpha \in \Gamma$. If I is an ideal of X_α, then I is closed in X_α.*

Proposition 5.12. *Let X be a $T\Gamma$-BCI-algebra and let I be a Γ-ideal of X. If $0 \in int(I)$, then I is open in X.*

Proof. Let $x \in I$ and let $\alpha \in \Gamma$. Since $0 \in int(I)$ and $x\alpha x = 0 \in I$, there is $W \in N(0) = N(x\alpha x)$ such that $W \subset I$. Since X is a $T\Gamma$-BCI-algebra, by Definition 5.2, there are $U,\ V \in N(x)$ such that $U\alpha V \subset W \subset I$. It is clear that $y\alpha x \in U\alpha V \subset I$ for each $y \in U$. Since I is a Γ-ideal of X and $x \in I$, $y \in I$. Then $y \in I$. Thus $U \subset I$. So I is open in X. □

In Proposition 5.12, when $0 \neq x \in int(I)$, I need not open in X (See Example 3.12, [23]).

Proposition 5.13. *Let X be a $T\Gamma$-BCI-algebra. Then $\bigcap N(0) = \{0\}$ and thus $\bigcap \mathcal{N}(0) = \{0\}$.*

Proof. The proof is similar to Proposition 3.13 in [23]. □

Proposition 5.14. *Let (X, τ) be a $T\Gamma$-BCI-algebra and let $\mathcal{B}_1,\ \mathcal{B}_2$ be the families of subsets of X given by:*

$$\mathcal{B}_1 = \{x\alpha U : x \in X,\ \alpha \in \Gamma,\ U \in \mathcal{N}(0)\},\ \mathcal{B}_2 = \{U\alpha x : x \in X,\ \alpha \in \Gamma,\ U \in \mathcal{N}(0)\},$$

where $x\alpha U = \{x\alpha u : u \in U\}$ and $U\alpha x = \{u\alpha x : u \in U\}$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for τ.

Proof. The proof is similar to Proposition 3.14 in [23]. □

To give a filter base on X generating a topology on a Γ-BCI-algebra, let us define a subset $U(a)$ of X generated by each $a \in X$ and each $U \in P(X)$ as follows:

$$U(a) = \{x \in X : x\alpha a \in U,\ a\alpha x \in U,\ \alpha \in \Gamma\}.$$

It is obvious that $U(a) \subset V(a)$ for $U,\ V \in P(X)$ such that $U \subset V$.

Proposition 5.15. *Let X be a Γ-BCI-algebra. Suppose $\mathcal{B}$ is a filter base on X satisfying the following condition: for any $a,\ b \in U \in \mathcal{B}$, each $x \in X$ and any $\alpha,\ \beta \in \Gamma$,*

(1) $0\alpha a \in U$,

(2) $(x\alpha a)\beta b = 0$ *implies* $x \in U$.

Then there is a unique topology τ on X such that $\mathcal{B} = \mathcal{N}_\tau(0)$ and (X, τ) is a $T\Gamma$-BCI-algebra.

Proof. Let $\tau = \{O \in P(X) :$ for each $a \in O$ there is $B \in \mathcal{B}$ such that $B(a) \subset O\}$.

Claim 1: τ is a topology on X. The proof is same as Claim 1 of Proposition 3.15 in [21].

Claim 2: $\mathcal{B} = \mathcal{N}_\tau(0)$. Let $a \in B \in \mathcal{B}$ and let $\alpha \in \Gamma$. Then by (1), $0\alpha a \in B$. Thus by Proposition 3.10(2) and the axiom (ΓA_4), $(0\alpha a)\beta(0\alpha a) = 0$. So by (2), $0 \in B$.

Let $x \in B(a)$. Then $x\alpha a,\ a\alpha x \in B$ and thus there is $u \in U$ such that $x\alpha a = u$. By the the axiom (ΓA_4), $(x\alpha a)\beta u = 0$ for each $\beta \in \Gamma$. By (2), $x \in B$. So $B(a) \subset B$. By Claim 1, $B \in \tau$. Since $0 \in B$, $B \in N_\tau(0)$. Hence $\mathcal{B} \subset N_\tau(0)$. Now let $V \in N_\tau(0)$. Then there is $B \in \mathcal{B}$ such that $B(0) \subset V$. It is clear that $0\alpha a,\ a\alpha 0 \in B$ for each $a \in B$ and each $\alpha \in \Gamma$. Thus $a \in B(0)$ and $0 \in B \subset B(0) \subset V$. So $\mathcal{B} = \mathcal{N}_\tau(0)$.

Claim 3: $B(a) \in \tau$ for each $a \in X$ and each $B \in \mathcal{B}$. Let $x \in B(a)$. Then $x\alpha a,\ a\alpha x \in B$ for each $\alpha \in \Gamma$. Note that there are $B_1,\ B_2 \in \mathcal{B}$ such that $B_1(x\alpha a) \subset B$ and $B_2(a\alpha x) \subset B$. Since $\mathcal{B}$ is a filter base on X, there is $U \in \mathcal{B}$ such that $U \in B_1 \cap B_2$. Thus we have

$$U(x\alpha a) \subset B_1(x\alpha a) \subset B \text{ and } U(a\alpha x) \subset B_2(a\alpha x) \subset B.$$

Let $y \in B(x)$. Then $x\alpha y,\ y\alpha x \in B$. By Corollary 3.9(2), we have

$$(x\alpha a)\beta(y\alpha a) \leq x\alpha y,\ (y\alpha a)\beta(x\alpha y) \leq y\alpha x \text{ for all } \alpha,\ \beta \in \Gamma, \text{ i.e.,}$$

$$[(x\alpha a)\beta(y\alpha a)]\beta(x\alpha y) = 0,\ [(y\alpha a)\beta(x\alpha y)]\beta(y\alpha x) = 0.$$

By (2), $(x\alpha a)\beta(y\alpha a),\ (y\alpha a)\beta(x\alpha y) \in U$. Thus $y\alpha a \in U(x\alpha a) \subset B_1(x\alpha a) \subset B$. So $y\alpha a \in B$. Similarly, we can show that $a\alpha y \in B$. Hence $y \in B(a)$, i.e., $U(x) \subset B(a)$. Therefore $B(a) \in \tau$.

Claim 4: A mapping $f : (X, \tau) \times \Gamma \times (X, \tau) \to (X, \tau)$ is continuous at each $(x, \alpha, y) \in X \times \Gamma \times X$. Let $x,\ y \in X$ and let $x\alpha y \in W \in \tau$ for each $\alpha \in \Gamma$. Then there is $V \in \mathcal{B}$ such that $V(x\alpha y) \subset W$. Let $a \in V(x)$ and let $b \in V(y)$. Then we have

$[(x\alpha y)\beta(a\alpha b)]\beta(x\alpha a) = [(x\alpha y)\beta(x\alpha a)]\beta(a\alpha b)$ [By Proposition 3.8]
$\leq (a\alpha y)\beta(a\alpha b)$ [By Theorem 3.2(1)]
$\leq b\alpha y$. [By Theorem 3.2(1)]

Thus $([(x\alpha y)\beta(a\alpha b)]\beta(a\alpha b))\beta(b\alpha y) = 0$. By (2), $(x\alpha y)\beta(a\alpha b) \in V$. From Proposition 3.8 and the axiom (ΓA_4), we have: for all $u,\ v \in V$ and any $\alpha,\ \beta \in \Gamma$,

$$[(u\alpha v)\beta u]\alpha(0\beta b) = 0.$$

So by (1) and (2), we get

$$u\alpha v \in V \text{ for any } u,\ v \in V \text{ each } \alpha \in \Gamma. \tag{5.1}$$

Since $(x\alpha y)\beta(a\alpha b), x\alpha a \in V$, by (5.1), we have

$$[(x\alpha y)\beta(a\alpha b)]\beta(x\alpha a),\ (x\alpha a)\beta[(x\alpha y)\beta(a\alpha b)] \in V.$$

Thus $(x\alpha y)\beta(a\alpha b) \in V(x\alpha a)$. By (2), $V(x\alpha a) \subset V$. So $(x\alpha y)\beta(a\alpha b) \in V$. Hence $a\alpha b \in V(x\alpha y)$, i.e., $V(x)\alpha V(y) \subset V(x\alpha y) \subset W$. Therefore by Claim 3, f is continuous. The proof of uniqueness for τ is easy. This completes the proof. □

Corollary 5.16. *Let X be a Γ-BCI-algebra. Then $(X, \tau_{\Gamma\mathcal{I}_c(X)})$ is a $T\Gamma$-BCI-algebra.*

Proof. We can easily prove that $\Gamma\mathcal{I}_c(X)$ is a filter base in X. By Definition 4.8, it is obvious that $0\alpha a \in I$ for each $a \in I \in \Gamma\mathcal{I}_c(X)$ and each $\alpha \in \Gamma$. Then the condition 1 of Proposition 5.15 holds. Suppose $(x\alpha a)\beta b = 0$ for all $a,\ b \in I \in \Gamma\mathcal{I}_c(X)$, all $\alpha,\ \beta \in \Gamma$ and each $x \in X$. Then $x\alpha a \leq b$ and $b \in I$. Thus by Definition 4.1, $x \in I$. So the condition 2 of Proposition 5.15 holds. Hence by Proposition 5.15, there is a unique topology $\tau_{\Gamma\mathcal{I}_c(X)}$ on X. Therefore $(X, \tau_{\Gamma\mathcal{I}_c(X)})$ is a $T\Gamma$-BCI-algebra. □

Example 5.17. Let $X = \{0, 1, 2, 3\}$ be the Γ-BCI-algebra given in Example 5.3 and let $\mathcal{B} = \{\{0, 1\}, \{0, 1, 2\}, \{0, 1, 3\}\}$. Then we can easily check that $\mathcal{B}$ is a filter base on X satisfying the conditions (1) and (2) of Proposition 5.15. Thus the topology $\tau_{\mathcal{B}}$ on X generated by $\mathcal{B}$ is given as follows:

$$\tau_{\mathcal{B}} = \{\varnothing, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \{0, 1, 3, 4\}, X\}.$$

So $(X, \tau_{\mathcal{B}})$ is a $T\Gamma$-BCI-algebra.

Unless otherwise specified, $\mathcal{B}$ denotes a filter base on a Γ-BCI-algebra X satisfying the conditions 1 and 2 of Proposition 5.15.

Lemma 5.18 (See Lemma 3.17, [23]; Lemma 3.14, [10])**.** *Let $(X, \tau_{\mathcal{B}})$ be a $T\Gamma$-BCI-algebra. Then for each $B \in \mathcal{B}$,*
(1) *$B(a) \in N_\tau(a)$ for each $a \in X$,*
(2) *$B(A) = \bigcup_{a\in A} B(a) \in N_\tau(A) \in \tau_{\mathcal{B}}$ such that $A \subset B(A)$ for each $A \in P(X)$.*

Proof. The proof is obvious. □

Proposition 5.19 (See Proposition 3.18, [23]; Theorem 3.15, [10])**.** *If $(X, \tau_{\mathcal{B}})$ is a $T\Gamma$-BCI-algebra, then $\overline{A} = \bigcap_{B\in\mathcal{B}} B(A)$ for each $A \subset X$.*

Proof. The proof is similar to Proposition 3.18 in [23]. □

Corollary 5.20 (See Corollary 3.19, [23]; Corollary 3.16, [10]). *Let $(X, \tau_{\mathcal{B}})$ be a $T\Gamma$-BCI-algebra. Then every $B \in \mathcal{B}$ is closed in X, i.e., $\mathcal{B}$ is a collection of clopen subsets of X.*

Proof. The proof is similar to Corollary 3.19 in [23]. □

Proposition 5.21. *Let $(X, \tau_{\mathcal{B}})$ be a $T\Gamma$-BCI-algebra. If A is a compact subset of X and $U \in \tau_{\mathcal{B}}$ such that $A \subset U$, then there is $B \in \mathcal{B}$ such that $A \subset B(A) \subset U$.*

Proof. Suppose A is a compact subset of X and $U \in \tau_{\mathcal{B}}$ such that $A \subset U$ and let $a \in A$. Then there is $B_a \in \mathcal{B}$ such that $B_a(a) \subset U$. It is clear that $A \subset \bigcup_{a \in A} B_a(a)$. Since A is a compact subset of X, there are $a_1,\ a_2,\ \cdots,\ a_n \in A$ such that $A \subset \bigcup_{i=1}^n B_{a_i}(a_i)$. Let $B = \bigcap_{i=1}^n B_{a_i}$ and $a \in A$. It is obvious that there is $i \in \{1, 2, \cdots, n\}$ such that $a \in B_{a_i}(a_i)$. Then $a\alpha a_i,\ a_i\alpha a \in B_{a_i}$ for each $\alpha \in \Gamma$. Now let $x \in B(a)$ and let $\alpha,\ \beta \in \Gamma$. Then by Corollary 3.9(2), we have

$$(a\alpha a_i)\beta(x\alpha a_i) \leq a\alpha x,\ \text{i.e.},\ [(a\alpha a_i)\beta(x\alpha a_i)]\beta(a\alpha x) = 0.$$

Since $a,\ x \in B$, by the condition (2) of Proposition 5.15, $(a\alpha a_i)\beta(x\alpha a_i) \in B$. Similarly, $(x\alpha a_i)\beta(a\alpha a_i) \in B$. Thus we get

$$x\alpha a_i \in B(a\alpha a_i) \subset B_{a_i}(a\alpha a_i) \subset B_{a_i}(B_{a_i}) \subset B_{a_i}.$$

Similarly, $a_i\alpha x \in B_{a_i}$. Thus $y \in B_{a_i} \subset U$. So $B(a) \subset U$ for each $a \in A$, i.e., $B(A) \subset U$. Since $\mathcal{B}$ is a filter base on X, there is $V \in \mathcal{B}$ such that $V \subset \bigcap_{i=1}^n B_{a_i} = B$. Hence $V(a) \subset B(a) \subset U$ for each $a \in A$. Therefore $V(A) \subset B(A) \subset U$. This completes the proof. □

Proposition 5.22. *Let $(X, \tau_{\mathcal{B}})$ be a $T\Gamma$-BCI-algebra, let A be a compact subset of X and let F is closed in X. If $A \cap F = \varnothing$, then there is $B \in \mathcal{B}$ such that $B(A) \cap B(F) = \varnothing$.*

Proof. Suppose $A \cap F = \varnothing$. Then clearly, $F^c \in \tau_{\mathcal{B}}$ and $A \subset F^c$. Thus by Proposition 5.21, there is $B \in \mathcal{B}$ such that $A \subset B(A) \subset F^c$. Assume that $B(A) \cap B(F) \neq \varnothing$. Then there are $x \in X$, $a \in A$ and $f \in F$ such that $x \in B(a)$ and $y \in B(f)$. By Theorem 3.2(1), we have: for any $\alpha,\ \beta \in \Gamma$,

$$(a\alpha x)\beta(a\alpha f) \leq f\alpha x \in B,\ (a\alpha f)\beta(a\alpha x) \leq a\alpha f \in B.$$

Thus $a\alpha f \in B(a\alpha x) \subset B(B) \subset B$, i.e., $a\alpha f \in B$. Similarly, $f\alpha x \in B$. So $f \in B(a)$. This contradicts to $B(A) \subset F^c$. Hence $B(A) \cap B(F) = \varnothing$. □

Now we discuss topological properties on quotient Γ-BCI-algebras. To do this, we denote subsets of X/I as $\dot{A},\ \dot{B},\ \dot{C}$, etc. and $\dot{\varnothing} = \varnothing$, $\dot{X} = X/I$. All the proofs of propositions, lemmas and corollaries listed below are almost same as these corresponding to [23] respectively, so they are omitted.

Proposition 5.23 (See Proposition 4.13, [23]). *Let (X, τ) be a $T\Gamma$-BCK-algebra, $I \in \Gamma\mathcal{I}_c(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. We define a collection τ_π of subsets of X/I as follows:*

$$\tau_\pi = \{\dot{A} \in P(X/I) : \pi^{-1}(\dot{A}) \in \tau\},$$

where $\dot{A} = \{I[a] : a \in A\}$ for some $A \subset X$. Then

(1) *τ_π is a topology on X/I,*

(2) $\pi : (X, \tau) \to (X/I, \tau_\pi)$ *is continuous, open and closed,*
(3) τ_π *is the finest topology on* X/I *with respect to which* π *is continuous,*
(4) $(X/I, \tau_\pi)$ *is a* $T\Gamma$-BCI*-algebra.*

In this case, τ_π is called the *quotient topology on* X/I *induced by* π and $(X/I, \tau_\pi)$ is called a *quotient* $T\Gamma$-BCI*-algebra* and π is called a *quotient mapping.*

Proposition 5.24 (See Proposition 4.14, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* $\{I\}$ *is open in* $(X/I, \tau_\pi)$*, then* X/I *is discrete.*

The following is an immediate consequence of Propositions 5.23 and 5.24.

Corollary 5.25 (See Corollary 4.15, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* $\{0\}$ *is open in* X*, then* X/I *is discrete.*

Proposition 5.26 (See Proposition 4.16, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* $(X/I, \tau_\pi)$ *is a* T_1*-space, then* $\{0\}$ *is closed in* X*.*

Theorem 5.27 (See Theorem 4.17, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. Then* $\{I\}$ *is closed in* $(X/I, \tau_\pi)$ *if and only if* X/I *is Hausdorff.*

The following is an immediate consequence of Proposition 5.23 and Theorem 5.27.

Corollary 5.28 (See Corollary 4.18, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. Then* $\{0\}$ *is closed in* X *if and only if* X/I *is Hausdorff.*

Proposition 5.29 (See Proposition 4.19, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* $\dot{A}$ *is a* Γ*-ideal of* X/I *and* $I \in int_{\tau_\pi}(\dot{A})$*, then* $\dot{A}$ *is open in* X/I*.*

Lemma 5.30 (See Lemma 4.20, [23]). *Let* X *be a* Γ-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* A *is a* Γ*-ideal of* X*, then* $\pi(A)$ *is a* Γ*-ideal of* X/I*.*

The following is an immediate consequence of Propositions 5.23, 5.29 and Lemma 5.30.

Corollary 5.31 (See Corollary 4.21, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCK*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* A *is an ideal of* X *and* $I \in int_\tau(\pi(A))$*, then* $\pi(A)$ *is open in* X/I*.*

Proposition 5.32 (See Proposition 4.22, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* $\dot{A}$ *is a* Γ*-ideal of* X/I *and is open in* X/I*, then* $\dot{A}$ *is closed in* X/I*.*

The following is an immediate consequence of Propositions 5.23 and 5.32.

Corollary 5.33 (See Corollary 4.23, [23]). *Let* (X, τ) *be a* $T\Gamma$-BCI*-algebra,* $I \in \Gamma\mathcal{I}_c(X)$ *and let* $\pi : X \to X/I$ *be the natural homomorphism. If* A *is a* Γ*-ideal of* X *and is open in* X*, then* $\pi(A)$ *is closed in* X/I*.*

Proposition 5.34 (See Proposition 4.24, [23]). *Let (X, τ) be a $T\Gamma$-BCI-algebra, $I \in \Gamma\mathcal{I}_c(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If $(X/I, \tau_\pi)$ is Hausdorff, then $\bigcap_{\dot{U} \in N_{\tau_\pi}(I)} \dot{U} = \{I\}$. Moreover, $\bigcap_{\dot{U} \in \mathcal{N}_{\tau_\pi}(I)} \dot{U} = \{I\}$.*

Lemma 5.35 (See Lemma 4.25, [23]). *Let (X, τ) be a $T\Gamma$-BCI-algebra, $I \in \Gamma\mathcal{I}_c(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. Then $\pi(\mathcal{N}_\tau(0) = \mathcal{N}_{\tau_\pi}(I)$.*

Lemma 5.36 (See Lemma 4.26, [23]). *Let (X, τ) be a $T\Gamma$-BCI-algebra, $I \in \Gamma\mathcal{I}_c(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. Then If X is Hausdorff, then $(X/I, \tau_\pi)$ is Hausdorff.*

The following is an immediate consequence of Propositions 5.23, 5.34 and Lemmas 5.35, 5.36.

Proposition 5.37 (See Proposition 4.27, [23]). *Let (X, τ) be a $T\Gamma$-BCI-algebra, $I \in \Gamma\mathcal{I}_c(X)$ and let $\pi : X \to X/I$ be the natural homomorphism. If X is Hausdorff, then $\bigcap_{\dot{U} \in \mathcal{N}_{\tau_\pi}(I)} \dot{U} = \{I\}$.*

6. Conclusions

We obtained various properties of Γ-BCI-algebras. Also, we dealt with some properties of Γ-ideals, quotient Γ-BCI-algebras and the kernel of a Γ-BCI-homomorphism. Moreover, we studied some of topological properties on Γ-BCI-algebras and quotient Γ-BCI-algebras.

In the future, we will define various types of logical Γ-algebras and discuss their properties, and apply them to topology.

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D. L. Shi (shidali@gzgs.edu.cn)
School of Big Data and Financial Statistics, Wonkwang University, Korea

J. I. Baek (jibaek@wku.ac.kr)
School of Big Data and Financial Statistics, Wonkwang University, Korea

Samy M. Mostafa (samymostafa@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt

S. H. Han (shhan235@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, Korea

K. Hur (kulhur@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, Korea