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Translations and extensions of fuzzy Sheffer stroke BE-filters and BE-subalgebras

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Abstract. The first aim of this article is to investigate the further properties of fuzzy Sheffer stroke BE-filters/BE-subalgebras. Next, concepts of normalized fuzzy Sheffer stroke BE-filters/BE-subalgebras, translations and extensions are introduced and related properties are studied. Characterizations of a fuzzy Sheffer stroke BE-filters are considered, and the conditions under which the fuzzy set which is maded by the upper set can be a fuzzy Sheffer stroke BE-filter are explored. How to configure a fuzzy Sheffer stroke BE-filter/BE-subalgebra using a collection of Sheffer stroke BE-filters/BE-subalgebras is displayed. The methods of constructing the normalization of a given fuzzy Sheffer stroke BE-filter/BE-subalgebra are given. Relations between a fuzzy Sheffer stroke BE-filter/BE-subalgebra and its normalization are discussed, and extensions of a fuzzy Sheffer stroke BE-filter/BE-subalgebra are established.

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1. Introduction

In the late 19th century and early 20th century, Charles S. Peirce and H. M. Sheffer independently discovered that a single binary logical connective suffices to define all logical connectives, the Sheffer stroke (denoted by $|$ or $\uparrow$) and the Peirce arrow (denoted by $\downarrow$). The concept of Sheffer operation (the so-called Sheffer stroke in [6]) was first introduced by Sheffer [21] in 1913. The Sheffer stroke is defined by the truth table given in Table 1. The Sheffer stroke has been applied to several algebraic structures, for example, Boolean algebra, MV-algebra, BL-algebra, BCK-algebra, ortholattices, and Hilbert algebra, etc., and it is also being dealt with in the fuzzy

TABLE 1. Truth table for the classical Sheffer stroke

p	q	$p \uparrow q$
0	0	1
0	1	1
1	0	1
1	1	0

environment (See [7, 11, 12, 13, 14, 16, 17]). BE-algebras, which are first introduced in [10], are a generalization of BCK-algebras. Since then, several studies have been conducted on BE-algebras (See [1, 2, 3, 4, 5, 18, 19, 20]). Katican et al. [9] first applied the Sheffer stroke to BE-algebras. They introduced the notions of Sheffer stroke BE-algebras, Sheffer stroke BE-filters and Sheffer stroke BE-subalgebras, and investigated several properties (See also [8]). Oner et al. [15] dealt with the fuzzy notion of Sheffer stroke BE-algebras. They introduced the concepts of fuzzy Sheffer stroke BE-filters and fuzzy Sheffer stroke BE-subalgebras, and investigated several properties.

In this paper, we first investigate further properties of fuzzy Sheffer stroke BE-filters/BE-subalgebras. Using a collection of Sheffer stroke BE-filters/BE-subalgebras, we establish a fuzzy Sheffer stroke BE-filter/BE-subalgebra. We introduce the notion of normalized fuzzy Sheffer stroke BE-filters/BE-subalgebras, and investigate its properties. For a given fuzzy Sheffer stroke BE-filter/BE-subalgebra, we provide a way to normalize it. We discuss the translation and extension of fuzzy Sheffer stroke BE-filters/BE-subalgebras, and consider several properties. We establish the relationship between a fuzzy Sheffer stroke BE-filter/BE-subalgebra and its normalization. We introduce S-extension and F-extension, and investigate their properties related to a translation.

2. Preliminaries

Definition 2.1 ([21])**.** Let $\mathcal{A} := (A, \uparrow)$ be a groupoid. Then the operation " $\uparrow$ " is said to be *Sheffer stroke* or *Sheffer operation*, if it satisfies:

(s1) $(\forall a, b \in A)$ $(a \uparrow b = b \uparrow a)$,
(s2) $(\forall a, b \in A)$ $((a \uparrow a) \uparrow (a \uparrow b) = a)$,
(s3) $(\forall a, b, c \in A)$ $(a \uparrow ((b \uparrow c) \uparrow (b \uparrow c)) = ((a \uparrow b) \uparrow (a \uparrow b)) \uparrow c)$,
(s4) $(\forall a, b, c \in A)$ $((a \uparrow ((a \uparrow a) \uparrow (b \uparrow b))) \uparrow (a \uparrow ((a \uparrow a) \uparrow (b \uparrow b))) = a)$.

Definition 2.2 ([9, 13])**.** A groupoid $\mathcal{X} := (X, \uparrow)$ with a Sheffer stroke " $\uparrow$ " is called a *Sheffer stroke BE-algebra*, if it satisfies:

(sBE1) $a \uparrow (a \uparrow a) = 1$,
(sBE2) $a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) = b \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))$

for all $a, b, c \in X$.

Let $\mathcal{X} := (X, \uparrow)$ be a Sheffer stroke BE-algebra. Define a relation " $\preceq$ " on X by

$$(\forall a, b \in X)(a \preceq b \iff a \uparrow (b \uparrow b) = 1). \tag{2.1}$$

The relation " $\preceq$ " is not a partial order on X. It is only a reflexive relation on X (See [9]).

Proposition 2.3 ([9]). *Every Sheffer stroke BE-algebra $\mathcal{X} := (X, \uparrow)$ satisfies:*

$$(\forall a \in X)(a \uparrow (1 \uparrow 1) = 1), \tag{2.2}$$

$$(\forall a \in X)(1 \uparrow (a \uparrow a) = a), \tag{2.3}$$

Definition 2.4 ([9]). A Sheffer stroke BE-algebra $\mathcal{X} := (X, \uparrow)$ is said to be *self-distributive*, if it satisfies:

$$a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) = (a \uparrow (b \uparrow b)) \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c))) \tag{2.4}$$

for all $a, b, c \in X$.

Definition 2.5 ([9]). Let $\mathcal{X} := (X, \uparrow)$ be a Sheffer stroke BE-algebra. A subset F of X is called

- a *Sheffer stroke BE-subalgebra* of $\mathcal{X} := (X, \uparrow)$, if it satisfies:

$$(\forall a, b \in X)(a, b \in F \;\Rightarrow\; a \uparrow (b \uparrow b) \in F), \tag{2.5}$$

- a *Sheffer stroke BE-filter* of $\mathcal{X} := (X, \uparrow)$ if it satisfies:

$$1 \in F, \tag{2.6}$$

$$(\forall a, b \in X)(a, \in F,\, a \uparrow (b \uparrow b) \in F \;\Rightarrow\; b \in F). \tag{2.7}$$

Definition 2.6 ([15]). Let $\mathcal{X} := (X, \uparrow)$ be a Sheffer stroke BE-algebra. A fuzzy set ξ in X is called

- a *fuzzy Sheffer stroke BE-subalgebra* of $\mathcal{X} := (X, \uparrow)$ if it satisfies:

$$(\forall x, y \in X)(\xi(x \uparrow (y \uparrow y)) \geq \min\{\xi(x), \xi(y)\}), \tag{2.8}$$

- a *fuzzy Sheffer stroke BE-filter* of $\mathcal{X} := (X, \uparrow)$ if it satisfies:

$$(\forall x \in X)(\xi(1) \geq \xi(x)), \tag{2.9}$$

$$(\forall x, y \in X)(\xi(y) \geq \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\}). \tag{2.10}$$

3. Properties of fuzzy Sheffer stroke BE-filters

In what follows, $\mathcal{X} := (X, \uparrow)$ stands for a Sheffer stroke BE-algebra, unless otherwise stated.

Proposition 3.1. *Every fuzzy Sheffer stroke BE-filter ξ of $\mathcal{X} := (X, \uparrow)$ satisfies:*

$$(\forall x, y \in X)\left(\xi(x \uparrow (y \uparrow y)) = \xi(1) \;\Rightarrow\; \xi(x) \leq \xi(y)\right). \tag{3.1}$$

Proof. Let $x, y \in X$ be such that $\xi(x \uparrow (y \uparrow y)) = \xi(1)$. Then

$$\xi(y) \geq \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\} = \min\{\xi(x), \xi(1)\} = \xi(x)$$

by (2.10) and (2.9). □

The combination of (2.1) and (3.1) induces the following corollary.

Corollary 3.2 ([15]). *Every fuzzy Sheffer stroke BE-filter ξ of $\mathcal{X} := (X, \uparrow)$ satisfies:*

$$(\forall x, y \in X)\left(x \preceq y \;\Rightarrow\; \xi(x) \leq \xi(y)\right), \tag{3.2}$$

that is, ξ is order preserving.

We discuss a characterization of a fuzzy Sheffer stroke BE-filter.

Theorem 3.3. *A fuzzy set ξ in X is a fuzzy Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$ if and only if it satisfies (3.1) and*

$$\xi(x \uparrow (z \uparrow z)) \geq \min\{\xi(x \uparrow ((y \uparrow (z \uparrow z)) \uparrow (y \uparrow (z \uparrow z)))), \xi(y)\} \tag{3.3}$$

for all $x, y, z \in X$.

Proof. Let ξ be a fuzzy Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$. Then it satisfies the condition (3.1) by Proposition 3.1. Using (2.10) and (sBE2) leads to

$$\begin{aligned}\xi(x \uparrow (z \uparrow z)) &\geq \min\{\xi(y \uparrow ((x \uparrow (z \uparrow z)) \uparrow (x \uparrow (z \uparrow z)))), \xi(y)\}\\ &= \min\{\xi(x \uparrow ((y \uparrow (z \uparrow z)) \uparrow (y \uparrow (z \uparrow z)))), \xi(y)\}\end{aligned}$$

for all $x, y, z \in X$.

Conversely, suppose that ξ satisfies (3.1) and (3.3) for all $x, y, z \in X$. The combination of (2.2) and (3.1) derives to $\xi(x) \leq \xi(1)$ for all $x \in X$. If we take $x := 1$ in (3.3) and use (2.3), then

$$\begin{aligned}\xi(z) &= \xi(1 \uparrow (z \uparrow z))\\ &\geq \min\{\xi(1 \uparrow ((y \uparrow (z \uparrow z)) \uparrow (y \uparrow (z \uparrow z)))), \xi(y)\}\\ &= \min\{\xi(y \uparrow (z \uparrow z)), \xi(y)\}\end{aligned}$$

for all $y, z \in X$. Therefore ξ is a fuzzy Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$. □

Lemma 3.4 ([15])**.** *A fuzzy set ξ in X is a fuzzy Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$ if and only if the non-empty set*

$$\xi_t := \{x \in X \mid \xi(x) \geq t\}$$

is a Sheffer stroke BE-filter of $\mathcal{X}$ for all $t \in [0, 1]$.

Consider the following set for $x, y \in X$.

$$U_x^y := \{z \in X \mid x \preceq y \uparrow (z \uparrow z)\} \tag{3.4}$$

which is called the *upper set* of x and y (See [9]).

For every $x, y \in X$, we consider the following fuzzy set in X.

$$\xi_x^y : X \to [0, 1],\ z \mapsto \begin{cases} t_1 & \text{if } z \in U_x^y \\ t_2 & \text{otherwise,}\end{cases} \tag{3.5}$$

where $t_1 > t_2$. In the following example, we know that ξ_x^y is not a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$ in general.

Example 3.5. Consider a set $X = \{0, 1, 2, 3, 4, 5\}$, and define a Sheffer stroke " $\uparrow$ " by Table 2. Then $\mathcal{X} := (X, \uparrow)$ is a Sheffer stroke BE-algebra (See [9]). Note that $4, 5 \in U_1^2$ and $3 \notin U_1^2$. Hence $\xi_1^2(3) = t_2 < t_1 = \min\{\xi_1^2(4), \xi_1^2(4 \uparrow (5 \uparrow 5))\}$, and so ξ_1^2 is not a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$.

We explore the conditions under which the fuzz set ξ_x^y can be a fuzzy Sheffer stroke BE-filter.

Lemma 3.6 ([9])**.** *If $\mathcal{X} := (X, \uparrow)$ is a self-distributive Sheffer stroke BE-algebra, then the upper set of x and y is a Sheffer stroke BE-filter of $\mathcal{X}$ for all $x, y \in X$*

Table 2. Cayley table for the Sheffer stroke "$\uparrow$"

$\uparrow$	0	2	3	4	5	1
0	1	1	1	1	1	1
2	1	3	1	1	1	3
3	1	1	2	1	1	2
4	1	1	1	5	1	5
5	1	1	1	1	4	4
1	1	3	2	5	4	0

Theorem 3.7. *If $\mathcal{X} := (X, \uparrow)$ is a self-distributive Sheffer stroke BE-algebra, then the fuzzy set ξ_x^y is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$ for all $x, y \in X$.*

Proof. Note that $(\xi_x^y)_t = U_x^y$ or $(\xi_x^y)_t = X$ for all $x, y \in X$. Hence ξ_x^y is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$ for all $x, y \in X$ by Lemma 3.4 and Lemma 3.6. □

Given a subset F of X, we consider the following fuzzy set in X.

$$\xi_F : X \to [0,1],\ z \mapsto \begin{cases} t_1 & \text{if } z \in F \\ t_2 & \text{otherwise,} \end{cases} \tag{3.6}$$

where $t_1 > t_2$.

In the following example, we show that ξ_F is not a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$ in general.

Example 3.8. Consider a set $X = \{0, 1, 2, 3, 4, 5\}$, and define a Sheffer stroke "$\uparrow$" by Table 3.

Table 3. Cayley table for the Sheffer stroke "$\uparrow$"

$\uparrow$	0	2	3	4	5	1
0	1	1	1	1	1	1
2	1	5	4	1	1	5
3	1	4	4	1	1	4
4	1	1	1	3	2	3
5	1	1	1	2	2	2
1	1	5	4	3	2	0

Then $\mathcal{X} := (X, \uparrow)$ is a Sheffer stroke BE-algebra (See [9]). If we take $F := \{1, 2, 5\}$, then ξ_F is not a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$, since $\xi_F(4) = t_2 < t_1 = \min\{\xi_F(5), \xi_F(5 \uparrow (4 \uparrow 4))\}$.

Lemma 3.9 ([15])**.** *The fuzzy set ξ_F in (3.6) is a fuzzy Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$ if and only if F is a Sheffer stroke BE-filter of $\mathcal{X}$.*

Lemma 3.10 ([9])**.** *If a subset F of X satisfies:*

$$(\forall x, y \in X)(U_x^y \subseteq F), \tag{3.7}$$

then F is a Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$.

The combination of Lemmas 3.9 and 3.10 induces the following theorem.

Theorem 3.11. *If a subset F of X satisfies the condition (3.7), then the fuzzy set ξ_F is a fuzzy Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$.*

Theorem 3.12. *Let $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ be an ascending sequence of Sheffer stroke BE-filters/BE-subalgebras of $\mathcal{X} := (X, \uparrow)$ and let $\langle t_n \rangle_{n \in \mathbb{N}}$ be a strictly decreasing sequence in $(0, 1]$. If we define a fuzzy set ξ in X as follows:*

$$\xi : X \to [0,1] \ x \mapsto \begin{cases} 0 & \text{if } x \notin F_n \\ t_n & \text{if } x \in F_n \setminus F_{n-1} \end{cases}$$

for $n \in \mathbb{N}$, where $F_0 = \varnothing$, then ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$.

Proof. It is clear that $F := \bigcup_{n \in \mathbb{N}} F_n$ is a Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$, because it is the union of an ascending sequence of Sheffer stroke BE-filters/BE-subalgebras of $\mathcal{X} := (X, \uparrow)$. Let $x, y \in X$. Since $1 \in F_1$, we get $\xi(1) = t_1 \geq \xi(x)$ for all $x \in X$. If $y \notin F$, then $x \notin F$ or $x \uparrow (y \uparrow y) \notin F$. Thus

$$\xi(y) = 0 = \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\}.$$

Assume that $y \in F_n \setminus F_{n-1}$ for some $n \in \mathbb{N}$. Then $x \notin F_{n-1}$ or $x \uparrow (y \uparrow y) \notin F_{n-1}$. Thus $\xi(x) \leq t_n$ or $\xi(x \uparrow (y \uparrow y)) \leq t_n$. So

$$\xi(y) = t_n \geq \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\}.$$

Also, if $x \uparrow (y \uparrow y) \notin F$, then $x \notin F$ or $y \notin F$. Thus $\xi(x \uparrow (y \uparrow y)) = 0 = \min\{\xi(x), \xi(y)\}$. Assume that $x \uparrow (y \uparrow y) \in F_n \setminus F_{n-1}$ for some $n \in \mathbb{N}$. Then $x \notin F_{n-1}$ or $y \notin F_{n-1}$. Thus $\xi(x) \leq t_n$ or $\xi(y) \leq t_n$. So $\xi(x \uparrow (y \uparrow y)) = t_n \geq \min\{\xi(x), \xi(y)\}$. Hence ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. □

For a family $\{\xi_\alpha \mid \alpha \in \Gamma\}$ of fuzzy sets in X where Γ is any index set, we define two operations meet $\underset{\alpha \in \Gamma}{\sqcap} \xi_\alpha$ and join $\underset{\alpha \in \Gamma}{\sqcup} \xi_\alpha$ as follows:

$$\underset{\alpha \in \Gamma}{\sqcap} \xi_\alpha : X \to [0,1], \ x \mapsto \inf_{\alpha \in \Gamma} \xi_\alpha(x),$$
$$\underset{\alpha \in \Gamma}{\sqcup} \xi_\alpha : X \to [0,1], \ x \mapsto \sup_{\alpha \in \Gamma} \xi_\alpha(x).$$

Theorem 3.13. *The family of fuzzy Sheffer stroke BE-filters of $\mathcal{X} := (X, \uparrow)$ forms a completely distributive lattice with respect to the operations meet and join.*

Proof. Let $\{\xi_\alpha \mid \alpha \in \Gamma\}$ be the family of fuzzy Sheffer stroke BE-filters of $\mathcal{X}$. It is sufficient to show that $\underset{\alpha \in \Gamma}{\sqcap} \xi_\alpha$ and $\underset{\alpha \in \Gamma}{\sqcup} \xi_\alpha$ are fuzzy Sheffer stroke BE-filters of $\mathcal{X} := (X, \uparrow)$ because the unit interval $[0, 1]$ is a completely distributive lattice under the usual ordering in $[0, 1]$. For every $x \in X$, we have

$$(\underset{\alpha \in \Gamma}{\sqcap} \xi_\alpha)(1) = \inf_{\alpha \in \Gamma} \xi_\alpha(1) \geq \inf_{\alpha \in \Gamma} \xi_\alpha(x) = (\underset{\alpha \in \Gamma}{\sqcap} \xi_\alpha)(x)$$

and

$$(\underset{\alpha \in \Gamma}{\sqcup} \xi_\alpha)(1) = \sup_{\alpha \in \Gamma} \xi_\alpha(1) \geq \sup_{\alpha \in \Gamma} \xi_\alpha(x) = (\underset{\alpha \in \Gamma}{\sqcup} \xi_\alpha)(x).$$

For every $x, y \in X$, we get

$$\begin{aligned}(\underset{\alpha\in\Gamma}{\sqcap}\xi_\alpha)(y) &= \inf_{\alpha\in\Gamma}\xi_\alpha(y) \geq \inf_{\alpha\in\Gamma}\xi_\alpha\left\{\min\{\xi_\alpha(x), \xi_\alpha(x\uparrow(y\uparrow y))\}\right\}\\ &= \min\left\{\inf_{\alpha\in\Gamma}\xi(x), \inf_{\alpha\in\Gamma}\xi(x\uparrow(y\uparrow y))\right\}\\ &= \min\left\{(\underset{\alpha\in\Gamma}{\sqcap}\xi_\alpha)(x), (\underset{\alpha\in\Gamma}{\sqcap}\xi_\alpha)(x\uparrow(y\uparrow y))\right\}\end{aligned}$$

and

$$\begin{aligned}(\underset{\alpha\in\Gamma}{\sqcup}\xi_\alpha)(y) &= \sup_{\alpha\in\Gamma}\xi_\alpha(y) \geq \sup_{\alpha\in\Gamma}\xi_\alpha\left\{\min\{\xi_\alpha(x), \xi_\alpha(x\uparrow(y\uparrow y))\}\right\}\\ &= \min\left\{\sup_{\alpha\in\Gamma}\xi(x), \sup_{\alpha\in\Gamma}\xi(x\uparrow(y\uparrow y))\right\}\\ &= \min\left\{(\underset{\alpha\in\Gamma}{\sqcup}\xi_\alpha)(x), (\underset{\alpha\in\Gamma}{\sqcup}\xi_\alpha)(x\uparrow(y\uparrow y))\right\}.\end{aligned}$$

Then $\underset{\alpha\in\Gamma}{\sqcap}\xi_\alpha$ and $\underset{\alpha\in\Gamma}{\sqcup}\xi_\alpha$ are fuzzy Sheffer stroke BE-filters of $\mathcal{X}$. □

By the similar process to the proof of Theorem 3.13, we have the following assertion.

Theorem 3.14. *The family of fuzzy Sheffer stroke BE-subalgebras of $\mathcal{X} := (X, \uparrow)$ forms a completely distributive lattice with respect to the operations meet and join.*

Theorem 3.15. *Let $\{F_\alpha \mid \alpha \in \Lambda \subseteq [0,1]\}$ be a collection of Sheffer stroke BE-filters of $\mathcal{X} := (X, \uparrow)$ such that $X = \bigcup_{\alpha\in\Lambda} F_\alpha$ and*

$$(\forall \alpha, \beta \in \Lambda)\,(\alpha > \beta \ \Leftrightarrow F_\alpha \subseteq F_\beta)\,. \tag{3.8}$$

Define a fuzzy set ξ^* in X as follows:

$$\xi^* : X \to [0,1],\ x \mapsto \sup_{\gamma\in\Lambda} F_\gamma. \tag{3.9}$$

Then ξ^* is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$.

Proof. Given $\gamma \in [0,1]$, we consider the following two cases:

$$\gamma = \sup\{\alpha \in \Lambda \mid \alpha < \gamma\} \text{ and } \gamma \neq \sup\{\alpha \in \Lambda \mid \alpha < \gamma\}.$$

The first case implies that

$$x \in \xi^*_\gamma \ \Leftrightarrow\ (\forall \beta < \gamma)(x \in F_\beta) \ \Leftrightarrow\ x \in \bigcap_{\beta<\gamma} F_\beta.$$

Then $\xi^*_\gamma = \bigcap_{\beta<\gamma} F_\beta$ which is a Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$. If the second case is valid, then there exists $\varepsilon > 0$ such that $(\gamma - \varepsilon, \gamma) \cap \Lambda = \varnothing$. We claim that $\xi^*_\gamma = \bigcup_{\beta\geq\gamma} F_\beta$. If $x \in \bigcup_{\beta\geq\gamma} F_\beta$, then $x \in F_\beta$ for some $\beta \geq \gamma$. Thus $\xi^*(x) \geq \beta \geq \gamma$. So $x \in \xi^*_\gamma$. If $x \notin \bigcup_{\beta\geq\gamma} F_\beta$, then $x \notin F_\beta$ for all $\beta \geq \gamma$. Thus $x \notin F_\beta$ for all $\beta > \gamma - \varepsilon$, i.e., if $x \in F_\beta$, then $\beta \leq \gamma - \varepsilon$. So $\xi^*(x) \leq \gamma - \varepsilon$. Hence $x \notin \xi^*_\gamma$. Consequently,

$\xi^*_\gamma = \bigcup_{\beta\geq\gamma} F_\beta$ which is a Sheffer stroke BE-filter of $\mathcal{X} := (X, \uparrow)$. This completes the proof. □

By the similar process to the proof of Theorem 3.15, we have the following assertion.

Theorem 3.16. *If $\{F_\alpha \mid \alpha \in \Lambda \subseteq [0,1]\}$ is a collection of Sheffer stroke BE-subalgebras of $\mathcal{X} := (X, \uparrow)$ such that $X = \bigcup_{\alpha\in\Lambda} F_\alpha$ and satisfying (3.8), then the fuzzy set ξ^* in X given by (3.9) is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$.*

4. The normalizedized fuzzy Sheffer stroke BE-filters and BE-subalgebras

Definition 4.1. A fuzzy Sheffer stroke BE-filter/BE-subalgebra ξ of $\mathcal{X} := (X, \uparrow)$ is said to be *normalized*, if there exists $x \in X$ such that $\xi(x) = 1$.

Example 4.2. (1) Let $\mathcal{X} := (X, \uparrow)$ be the Sheffer stroke BE-algebra in Example 3.8. Define a fuzzy set ξ in X as follows:

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 1.00 & \text{if } x = 1 \\ 0.58 & \text{if } x = 5 \\ 0.43 & \text{if } x = 2 \\ 0.36 & \text{otherwise.} \end{cases}$$

It is routine to verify that ξ is a normalized fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$.

(2) Consider a set $X = \{1, 2, 3, 0\}$, and define a Sheffer stroke "$\uparrow$" by Table 4.

Table 4. Cayley table for the Sheffer stroke "$\uparrow$"

$\uparrow$	0	2	3	1
0	1	1	1	1
2	1	3	1	3
3	1	1	2	2
1	1	3	2	0

Then $\mathcal{X} := (X, \uparrow)$ is a Sheffer stroke BE-algebra (See [9]). A fuzzy set ξ in X defined by

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 1.00 & \text{if } x \in \{1,3\} \\ 0.52 & \text{if } x \in \{0,2\}. \end{cases}$$

It is routine to verify that ξ is a normalized fuzzy Sheffer stroke BE-filter of $\mathcal{X}$.

It is clear that if a fuzzy Sheffer stroke BE-filter/BE-subalgebra ξ of $\mathcal{X} := (X, \uparrow)$ is normalized, then $\xi(1) = 1$. Thus ξ is a normalized fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$ if and only if $\xi(1) = 1$.

Theorem 4.3. *Let ξ be a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X} := (X, \uparrow)$. Then the fuzzy set ξ^+ in X defined by*

$$\xi^+ : X \to [0,1],\ x \mapsto \xi(x) + 1 - \xi(1) \tag{4.1}$$

is a normalized fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$ which is greater than ξ, i.e, $\xi \subseteq \xi^+$.

Proof. It is clear that $\xi \subseteq \xi^+$. If ξ is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$, then

$$\begin{aligned}\xi^+(x \uparrow (y \uparrow y)) &= \xi(x \uparrow (y \uparrow y)) + 1 - \xi(1)\\ &\geq \min\{\xi(x), \xi(y)\} + 1 - \xi(1)\\ &= \min\{\xi(x) + 1 - \xi(1), \xi(y) + 1 - \xi(1)\}\\ &= \min\{\xi^+(x), \xi^+(y)\}\end{aligned}$$

for all $x, y \in X$. Thus ξ^+ is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$. Let ξ be a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$. Then

$$\xi^+(1) = \xi(1) + 1 - \xi(1) \geq \xi(x) + 1 - \xi(1) = \xi^+(x)$$

and

$$\begin{aligned}\xi^+(y) = \xi(y) + 1 - \xi(1) &\geq \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\} + 1 - \xi(1)\\ &= \min\{\xi(x) + 1 - \xi(1), \xi(x \uparrow (y \uparrow y)) + 1 - \xi(1)\}\\ &= \min\{\xi^+(x), \xi^+(x \uparrow (y \uparrow y))\}\end{aligned}$$

for all $x, y \in X$. Thus ξ^+ is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$. □

It is clear that a fuzzy Sheffer stroke BE-filter/BE-subalgebra ξ of $\mathcal{X} := (X, \uparrow)$ is normalized if and only if $\xi = \xi^+$.

Theorem 4.4. *Let ξ and f be fuzzy sets in X. Then*

(1) *If ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X} := (X, \uparrow)$, then $(\xi^+)^+ = \xi^+$.*
(2) *If ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$, then it is normalized if and only if there exists a fuzzy Sheffer stroke BE-filter/BE-subalgebra f of $\mathcal{X}$ such that $f^+ \subseteq \xi$.*
(3) *If ξ^+ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$, then so is ξ.*

Proof. (1) Suppose ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. Then $\xi^+(1) = 1$. Thus $(\xi^+)^+(x) = \xi^+(x) + 1 - \xi^+(1) = \xi^+(x)$ for all $x \in X$. So $(\xi^+)^+ = \xi^+$.

(2) Suppose ξ is a normalized fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. Then $\xi^+ = \xi$. Thus we are done by choosing $f = \xi$.

Conversely, suppose there exists a fuzzy Sheffer stroke BE-filter/BE-subalgebra f of $\mathcal{X}$ such that $f^+ \subseteq \xi$. Then $1 = \xi^+(1) \leq \xi(1)$. Thus $\xi(1) = 1$. So ξ is normalized.

(3) Suppose ξ^+ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. Then

$$\begin{aligned}\xi(x \uparrow (y \uparrow y)) + 1 - \xi(1) = \xi^+(x \uparrow (y \uparrow y)) &\geq \min\{\xi^+(x), \xi^+(y)\}\\ &= \min\{\xi(x) + 1 - \xi(1), \xi(y) + 1 - \xi(1)\}\\ &= \min\{\xi(x), \xi(y)\} + 1 - \xi(1).\end{aligned}$$

Thus $\xi(x \uparrow (y \uparrow y)) \geq \min\{\xi(x), \xi(y)\}$ for all $x, y \in X$. Also, we have

$$\xi(1) + 1 - \xi(1) = \xi^+(1) \geq \xi^+(x) = \xi(x) + 1 - \xi(1)$$

and

$$\begin{aligned}\xi(y) + 1 - \xi(1) = \xi^+(y) &\geq \min\{\xi^+(x), \xi^+(x \uparrow (y \uparrow y))\}\\ &= \min\{\xi(x) + 1 - \xi(1), \xi(x \uparrow (y \uparrow y)) + 1 - \xi(1)\}\\ &= \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\} + 1 - \xi(1)\end{aligned}$$

for all $x, y \in X$. So $\xi(1) \geq \xi(x)$ and $\xi(y) \geq \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\}$ for all $x, y \in X$. Hence ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. □

Theorem 4.5. *Let ξ be a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X} := (X, \uparrow)$ and let $\iota : [0, \xi(1)] \hookrightarrow [0, 1]$ be a non-decreasing inclusion map. Then a fuzzy set ξ_ι in X defined by*

$$\xi_\iota : X \to [0, 1],\ x \mapsto \iota(\xi(x))$$

is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. Moreover, if $\xi_\iota(1) = 1$, then ξ_ι is normalized and if $\iota(t) \geq t$ for all $t \in [0, \xi(1)]$, then $\xi \subseteq \xi_\iota$.

Proof. For every $x, y \in X$, we have

$$\begin{aligned}\xi_\iota(x \uparrow (y \uparrow y)) = \iota(\xi(x \uparrow (y \uparrow y))) &\geq \iota(\min\{\xi(x), \xi(y)\})\\ &= \min\{\iota(\xi(x)), \iota(\xi(y))\} = \min\{\xi_\iota(x), \xi_\iota(y)\}.\end{aligned}$$

Then ξ_ι is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$. Also, we have

$$\xi_\iota(1) = \iota(\xi(1)) \geq \iota(\xi(x)) = \xi_\iota(x)$$

and

$$\begin{aligned}\xi_\iota(y) = \iota(\xi(y)) &\geq \iota(\min\{\xi(x), \xi(x \uparrow (y \uparrow y))\})\\ &= \min\{\iota(\xi(x)), \iota(\xi(x \uparrow (y \uparrow y)))\}\\ &= \min\{\xi_\iota(x), \xi_\iota(x \uparrow (y \uparrow y))\}\end{aligned}$$

for all $x, y \in X$. Thus ξ_ι is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. It is clear that if $\xi_\iota(1) = 1$, then ξ_ι is normalized. Assume that $\iota(t) \geq t$ for all $t \in [0, \xi(1)]$. Then $\xi_\iota(x) = \iota(\xi(x)) \geq \xi(x)$ for all $x \in X$. Thus $\xi \subseteq \xi_\iota$. □

Theorem 4.6. *Let ξ be a normalized fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X} := (X, \uparrow)$ such that there exists at least one $x \in X$ such that $\xi(x) \neq \xi(1)$. Then every maximal element ξ of $(NF(X), \subseteq)$ is described as follows:*

$$\xi : X \to [0, 1],\ x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

where $NF(X)$ is the set of all normalized fuzzy Sheffer stroke BE-filters/BE-subalgebras of $\mathcal{X}$.

Proof. It is obvious that $(NF(X), \subseteq)$ is a poset. Let ξ be a maximal element of $(NF(X), \subseteq)$. Since ξ is normalized, we have $\xi(1) = 1$. Let $x \in X$ be such that

$\xi(x) \neq 1$. We will show that $\xi(x) = 0$ for such $x \in X$. If $\xi(x) \neq 0$, then $0 < \xi(a) < 1$ for some $a \in X$. Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \tfrac{1}{2}(\xi(x) + \xi(a))$$

for every $n(\neq 1) \in \mathbb{N}$. If $x_1 = x_2$ in X, then

$$f(x_1) = \tfrac{1}{2}(\xi(x_1) + \xi(a)) = \tfrac{1}{2}(\xi(x_2) + \xi(a)) = f(x_2).$$

Thus f is well-defined. For every $x, y \in X$, we have

$$\begin{aligned} f(x \uparrow (y \uparrow y)) &= \tfrac{1}{2}(\xi(x \uparrow (y \uparrow y)) + \xi(a)) \geq \tfrac{1}{2}(\min\{\xi(x), \xi(y)\} + \xi(a)) \\ &= \tfrac{1}{2}(\min\{\xi(x) + \xi(a), \xi(y) + \xi(a)\}) \\ &= \min\{\tfrac{1}{2}(\xi(x) + \xi(a)), \tfrac{1}{2}(\xi(y) + \xi(a))\} \\ &= \min\{f(x), f(y)\}. \end{aligned}$$

Then f is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$. Also, we have

$$f(1) = \tfrac{1}{2}(\xi(1) + \xi(a)) \geq \tfrac{1}{2}(\xi(x) + \xi(a)) = f(x)$$

and

$$\begin{aligned} f(y) &= \tfrac{1}{2}(\xi(y) + \xi(a)) \geq \tfrac{1}{2}(\min\{\xi(x), \xi(x \uparrow (y \uparrow y))\} + \xi(a)) \\ &= \tfrac{1}{2}(\min\{\xi(x) + \xi(a), \xi(x \uparrow (y \uparrow y)) + \xi(a)\}) \\ &= \min\{\tfrac{1}{2}(\xi(x) + \xi(a)), \tfrac{1}{2}(\xi(x \uparrow (y \uparrow y)) + \xi(a))\} \\ &= \min\{f(x), f(x \uparrow (y \uparrow y))\} \end{aligned}$$

for all $x, y \in X$. Then f is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$. Thus f^+ is a normalized fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$ by Theorem 4.3, i.e., $f^+ \in \mathcal{N}(X)$. We can observe that

$$\begin{aligned} f^+(x) &= f(x) + 1 - f(1) = \tfrac{1}{2}(\xi(x) + \xi(a)) + 1 - \tfrac{1}{2}(\xi(1) + \xi(a)) \\ &= \tfrac{1}{2}(\xi(x) + 1) \geq \xi(x) \end{aligned}$$

for all $x \in X$. So $\xi \subseteq f^+$. This shows that ξ is not a maximal element of $(NF(X), \subseteq)$, a contradiction. Hence $\xi(x) = 0$ for all $x \in X$ with $\xi(x) \neq 1$. Therefore ξ is described as (4.2). □

5. Translations and extensions

Given a fuzzy set ξ in X, we denote $\theta := 1 - \sup\{\xi(x) \mid x \in X\}$ unless otherwise specified.

Definition 5.1. Let ξ be a fuzzy set in X and $\gamma \in [0, \theta]$. If $\gamma \leq 1 - \xi(x)$ for all $x \in X$, then the fuzzy set ξ^θ_γ in X given by

$$\xi^\theta_\gamma : X \to [0,1],\ x \mapsto \xi(x) + \gamma \tag{5.1}$$

is called a *γ-translation* of ξ.

Example 5.2. Let $X = \mathbb{R}$ be the set of all real numbers. Define a fuzzy set ξ in X as follows:

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.23 & \text{if } x < 0 \\ 0.44 & \text{if } x = 0 \\ 0.56 & \text{if } x > 0. \end{cases}$$

Then $\theta = 0.44$. If we take $\gamma := 0.39$, then the γ-translation ξ_γ^θ of ξ is given as follows:

$$\xi_\gamma^\theta : X \to [0,1],\ x \mapsto \begin{cases} 0.62 & \text{if } x < 0 \\ 0.83 & \text{if } x = 0 \\ 0.95 & \text{if } x > 0. \end{cases}$$

Theorem 5.3. *If ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X} := (X, \uparrow)$, then so is its γ-translation for every $\gamma \in [0, \theta]$.*

Proof. Let $x, y \in X$. Then $\xi_\gamma^\theta(1) = \xi(1) + \gamma \geq \xi(x) + \gamma = \xi_\gamma^\theta(x)$ and

$$\begin{aligned} \xi_\gamma^\theta(y) = \xi(y) + \gamma &\geq \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\} + \gamma \\ &= \min\{\xi(x) + \gamma, \xi(x \uparrow (y \uparrow y)) + \gamma\} \\ &= \min\{\xi_\gamma^\theta(x), \xi_\gamma^\theta(x \uparrow (y \uparrow y))\}. \end{aligned}$$

Also, we have

$$\begin{aligned} \xi_\gamma^\theta(x \uparrow (y \uparrow y)) = \xi(x \uparrow (y \uparrow y)) + \gamma &\geq \min\{\xi(x), \xi(y)\} + \gamma \\ &= \min\{\xi(x) + \gamma, \xi(y) + \gamma\} \\ &= \min\{\xi_\gamma^\theta(x), \xi_\gamma^\theta(y)\}. \end{aligned}$$

Thus ξ_γ^θ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. □

Theorem 5.4. *Let ξ be a fuzzy set in X such that its γ-translation ξ_γ^θ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X} := (X, \uparrow)$ for $\gamma \in [0, \theta]$. Then ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$.*

Proof. Suppose ξ_γ^θ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$ for $\gamma \in [0, \theta]$. Then for every $x, y \in X$, we have

$$\begin{aligned} \xi(x \uparrow (y \uparrow y)) + \gamma = \xi_\gamma^\theta(x \uparrow (y \uparrow y)) &\geq \min\{\xi_\gamma^\theta(x), \xi_\gamma^\theta(y)\} \\ &= \min\{\xi(x) + \gamma, \xi(y) + \gamma\} \\ &= \min\{\xi(x), \xi(y)\} + \gamma, \end{aligned}$$

$$\begin{aligned} \xi(y) + \gamma = \xi_\gamma^\theta(y) &\geq \min\{\xi_\gamma^\theta(x), \xi_\gamma^\theta(x \uparrow (y \uparrow y))\} \\ &= \min\{\xi(x) + \gamma, \xi(x \uparrow (y \uparrow y)) + \gamma\} \\ &= \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\} + \gamma \end{aligned}$$

and

$$\xi(1) + \gamma = \xi_\gamma^\theta(1) \geq \xi_\gamma^\theta(x) = \xi(x) + \gamma.$$

It follows that

$$\xi(x \uparrow (y \uparrow y)) \geq \min\{\xi(x), \xi(y)\},$$

$\xi(1) \geq \xi(x)$ and $\xi(y) \geq \min\{\xi(x), \xi(x \uparrow (y \uparrow y))\}$ for all $x, y \in X$. Thus ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$. □

Definition 5.5. A fuzzy set f in X is called an *S-extension* (resp., *F-extension*) of a fuzzy set ξ in X if it satisfies:

(i) $\xi \subseteq f$, that is, $\xi(x) \leq f(x)$ for all $x \in X$,
(ii) If ξ is a fuzzy Sheffer stroke BE-subalgebra (resp., fuzzy Sheffer stroke BE-filter) of $\mathcal{X} := (X, \uparrow)$, then so is f.

Using the definition of γ-translation and Theorem 5.3, we have the following theorem

Theorem 5.6. *Let ξ be a fuzzy Sheffer stroke BE-filter (resp., fuzzy Sheffer stroke BE-subalgebra) of $\mathcal{X} := (X, \uparrow)$ for $\gamma \in [0, \theta]$. Then the γ-translation ξ^{θ}_{γ} of ξ is an F-extension (resp., S-extension) of ξ.*

The example below shows that the converse of Theorem 5.6 may not be true.

Example 5.7. Consider the Sheffer stroke BE-algebra $\mathcal{X}$ in Example 3.8. Define a fuzzy set ξ in X by

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.33 & \text{if } x \in \{0,2,3\} \\ 0.52 & \text{if } x \in \{4,5\} \\ 0.79 & \text{if } x = 1. \end{cases}$$

Then ξ is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$. Let f be a fuzzy set ξ in X defined by

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.39 & \text{if } x \in \{0,2,3\} \\ 0.59 & \text{if } x \in \{4,5\} \\ 0.81 & \text{if } x = 1. \end{cases}$$

It is routine to check that f is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$ and $\xi \subseteq f$. Thus f is an F-extension of ξ. But it is not the γ-translation ξ^{θ}_{γ} of ξ for all $\gamma \in [0, \theta]$.

Theorem 5.8. *Let ξ be a fuzzy Sheffer stroke BE-filter (resp., fuzzy Sheffer stroke BE-subalgebra) of $\mathcal{X} := (X, \uparrow)$. Then the intersection of F-extensions (resp., S-extensions) of ξ is an F-extension (resp., S-extension) of ξ.*

Proof. Let f and g be F-extensions of ξ. Then $\xi \subseteq f$ and $\xi \subseteq g$. Thus $\xi \subseteq f \cap g$. For every $x, y \in X$, we have

$$(f \cap g)(1) = \min\{f(1), g(1)\} \geq \min\{f(x), g(x)\} = (f \cap g)(x)$$

and

$$\begin{aligned}(f \cap g)(y) &= \min\{f(y), g(y)\} \\ &\geq \min\{\min\{f(x), f(x \uparrow (y \uparrow y))\}, \min\{g(x), g(x \uparrow (y \uparrow y))\}\} \\ &= \min\{\min\{f(x), g(x)\}, \min\{f(x \uparrow (y \uparrow y)), g(x \uparrow (y \uparrow y))\}\} \\ &= \min\{(f \cap g)(x), (f \cap g)(x \uparrow (y \uparrow y))\}.\end{aligned}$$

Also, we get

$$\begin{aligned}(f \cap g)(x \uparrow (y \uparrow y)) &= \min\{f(x \uparrow (y \uparrow y)), g(x \uparrow (y \uparrow y))\} \\ &\geq \min\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\} \\ &= \min\{(f \cap g)(x), (f \cap g)(y)\}.\end{aligned}$$

So $f \cap g$ is a fuzzy Sheffer stroke BE-filter (resp., fuzzy Sheffer stroke BE-subalgebra) of $\mathcal{X}$. Hence $f \cap g$ is an F-extension (resp., S-extension) of ξ. □

The following example shows that the union of F-extensions (resp., S-extensions) of ξ may not be an F-extension (resp., S-extension) of ξ.

Example 5.9. (1) Let $\mathcal{X} := (X, \uparrow)$ be the Sheffer stroke BE-algebra in Example 3.8. Define a fuzzy set ξ in X as follows:

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.7 & \text{if } x = 1 \\ 0.4 & \text{if } x = 2 \\ 0.5 & \text{if } x = 5 \\ 0.3 & \text{otherwise.} \end{cases}$$

It is routine to verify that ξ is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$. Let f and g be fuzzy sets in X defined by

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.73 & \text{if } x = 1 \\ 0.62 & \text{if } x \in \{3,4\} \\ 0.51 & \text{otherwise,} \end{cases}$$

and

$$g : X \to [0,1],\ x \mapsto \begin{cases} 0.77 & \text{if } x = 1 \\ 0.67 & \text{if } x = 5 \\ 0.46 & \text{otherwise,} \end{cases}$$

respectively. Then f and g are S-extensions of ξ. The union of f and g is given as follows:

$$f \cup g : X \to [0,1],\ x \mapsto \begin{cases} 0.77 & \text{if } x = 1 \\ 0.62 & \text{if } x \in \{3,4\} \\ 0.67 & \text{if } x = 5 \\ 0.51 & \text{otherwise,} \end{cases}$$

and it is not an S-extension of ξ, because $f \cup g$ is not a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$, since

$$(f \cup g)(5 \uparrow (3 \uparrow 3)) = 0.51 \ngeq 0.62 = \min\{(f \cup g)(5), (f \cup g)(3)\}.$$

(2) Let $X := \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set and $\uparrow$ be a Sheffer stroke on X given by Table 5. Then $\mathcal{X} := (X, \uparrow)$ is a Sheffer stroke BE-algebra (see [9]). Consider a fuzzy

Table 5. Cayley table for the Sheffer stroke $\uparrow$

$\uparrow$	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	0

Sheffer stroke BE-filter ξ and its F-extensions f and g which are given in Table 6. The union of f and g is given by Table 7. Since

Table 6. Tabular representation of ξ, f, and g

$x \in X$	0	2	3	4	5	6	7	1
$\xi(x)$	0.39	0.39	0.39	0.52	0.56	0.39	0.39	0.68
$f(x)$	0.42	0.42	0.42	0.54	0.63	0.42	0.42	0.72
$g(x)$	0.43	0.46	0.46	0.51	0.62	0.43	0.46	0.83

Table 7. Tabular representation of $f \cup g$

$x \in X$	0	2	3	4	5	6	7	1
$(f \cup g)(x)$	0.43	0.46	0.46	0.54	0.63	0.43	0.46	0.83

$$(f \cup g)(6) = 0.43 \ngeq 0.46 = \min\{(f \cup g)(2), (f \cup g)(2 \uparrow (6 \uparrow 6))\},$$

we know that $f \cup g$ is not an F-extension of ξ.

Theorem 5.10. *Let ξ be a fuzzy set in X and $\gamma \in [0, \theta]$. Then the γ-translation ξ^{θ}_{γ} of ξ is a fuzzy Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X} := (X, \uparrow)$ if and only if $U_\gamma(\xi, t)$ is a Sheffer stroke BE-filter/BE-subalgebra of $\mathcal{X}$ for all $t \in \operatorname{Im}(\xi)$ with $t \geq \gamma$.*

Proof. Suppose ξ^{θ}_{γ} is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$. It is clear that $1 \in U_\gamma(\xi, t)$. Let $x, y \in X$ be such that $x \in U_\gamma(\xi, t)$ and $x \uparrow (y \uparrow y) \in U_\gamma(\xi, t)$. Then $\xi^{\theta}_{\gamma}(x) = \xi(x) + \gamma \geq t$ and $\xi^{\theta}_{\gamma}(x \uparrow (y \uparrow y)) = \xi(x \uparrow (y \uparrow y)) + \gamma \geq t$. It follows from (2.10) that

$$\xi(y) + \gamma = \xi^{\theta}_{\gamma}(y) \geq \min\{\xi^{\theta}_{\gamma}(x), \xi^{\theta}_{\gamma}(x \uparrow (y \uparrow y))\} \geq t.$$

Thus $\xi(y) \geq t - \gamma$, i.e., $y \in U_\gamma(\xi, t)$. So $U_\gamma(\xi, t)$ is a Sheffer stroke BE-filter of $\mathcal{X}$.

The similar way is to show that if ξ^{θ}_{γ} is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$, then $U_\gamma(\xi, t)$ is a Sheffer stroke BE-subalgebra of $\mathcal{X}$.

Conversely, suppose $U_\gamma(\xi, t)$ is a Sheffer stroke BE-filter of $\mathcal{X}$ for all $t \in \operatorname{Im}(\xi)$ with $t \geq \gamma$. Assume that $\xi^{\theta}_{\gamma}(1) < \xi^{\theta}_{\gamma}(a) := t$ for some $a \in X$. Then $\xi(1) + \gamma < t$. Thus $1 \notin U_\gamma(\xi, t)$. This is a contradiction. So $\xi^{\theta}_{\gamma}(1) \geq \xi^{\theta}_{\gamma}(x)$ for all $x \in X$.

Assume that $\xi^{\theta}_{\gamma}(b) < \min\{\xi^{\theta}_{\gamma}(a), \xi^{\theta}_{\gamma}(a \uparrow (b \uparrow b))\}$ for some $a, b \in X$ and let us take

$$t := \min\{\xi^{\theta}_{\gamma}(a), \xi^{\theta}_{\gamma}(a \uparrow (b \uparrow b))\}.$$

Then $\xi(a) + \gamma = \xi^{\theta}_{\gamma}(a) \geq t$ and $\xi(a \uparrow (b \uparrow b)) + \gamma = \xi^{\theta}_{\gamma}(a \uparrow (b \uparrow b)) \geq t$, i.e., $a \in U_\gamma(\xi, t)$ and $a \uparrow (b \uparrow b) \in U_\gamma(\xi, t)$. Since $U_\gamma(\xi, t)$ is a Sheffer stroke BE-filter of $\mathcal{X}$, it follows that $b \in U_\gamma(\xi, t)$. Thus $\xi(b) \geq t - \gamma$, i.e., $\xi^{\theta}_{\gamma}(b) \geq t$. This is a contradiction. So $\xi^{\theta}_{\gamma}(y) \geq \min\{\xi^{\theta}_{\gamma}(x), \xi^{\theta}_{\gamma}(x \uparrow (y \uparrow y))\}$ for all $x, y \in X$. Hence ξ^{θ}_{γ} is a fuzzy Sheffer stroke BE-filter of $\mathcal{X}$. By the similar way, we can verify that if $U_\gamma(\xi, t)$ is a Sheffer stroke BE-subalgebra of $\mathcal{X} := (X, \uparrow)$ for all $t \in \operatorname{Im}(\xi)$ with $t \geq \gamma$, then ξ^{θ}_{γ} is a fuzzy Sheffer stroke BE-subalgebra of $\mathcal{X}$. □

Theorem 5.11. *Let ξ be a fuzzy Sheffer stroke BE-filter (resp., fuzzy Sheffer stroke BE-subalgebra) of $\mathcal{X} := (X, \uparrow)$ and $\delta \in [0, \theta]$. If f is an F-extension (resp., S-extension) of the δ-translation ξ^{θ}_{δ} of ξ, then there exists $\gamma \in [0, \theta]$ such that $\gamma \geq \delta$ and f is an F-extension (resp., S-extension) of the γ-translation ξ^{θ}_{γ} of ξ.*

Proof. Suppose f is an F-extension (resp., S-extension) of the δ-translation ξ^{θ}_{δ} of ξ. Since ξ is a fuzzy Sheffer stroke BE-filter (resp., fuzzy Sheffer stroke BE-subalgebra) of $\mathcal{X}$, by Theorem 5.3, its δ-translation ξ^{θ}_{δ} is a fuzzy Sheffer stroke BE-filter (resp., fuzzy Sheffer stroke BE-subalgebra) of $\mathcal{X}$ for every $\delta \in [0, \theta]$. Then by the hypothesis, $\xi^{\theta}_{\delta} \subseteq f$, i.e., $\xi(x)+\delta \leq f(x)$ for all $x \in X$ and f is a fuzzy Sheffer stroke BE-filter (resp., fuzzy Sheffer stroke BE-subalgebra) of $\mathcal{X}$. Thus there exists $\gamma \in [0, \theta]$ such that $\gamma \geq \delta$ and f is an F-extension (resp., S-extension) of the γ-translation ξ^{θ}_{γ} of ξ. □

The following example illustrates Theorem 5.11.

Example 5.12. Consider the Sheffer stroke BE-algebra $\mathcal{X} := (X, \uparrow)$ in Example 4.2(2). Let ξ be a fuzzy set in X defined by

$$\xi : X \to [0, 1],\ x \mapsto \begin{cases} 0.7 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2 \\ 0.6 & \text{if } x = 3 \\ 0.4 & \text{if } x = 0. \end{cases}$$

Then ξ is a Sheffer stroke BE-filter of $\mathcal{X}$ and $\theta = 0.3$ If we take $\delta := 0.2$, then the δ-translation ξ^{θ}_{δ} of ξ is give by

$$\xi^{\theta}_{\delta} : X \to [0, 1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = 1 \\ 0.7 & \text{if } x = 2 \\ 0.8 & \text{if } x = 3 \\ 0.6 & \text{if } x = 0. \end{cases}$$

Let f be a fuzzy set in X defined by

$$f : X \to [0, 1],\ x \mapsto \begin{cases} 0.97 & \text{if } x = 1 \\ 0.77 & \text{if } x = 2 \\ 0.89 & \text{if } x = 3 \\ 0.68 & \text{if } x = 0. \end{cases}$$

Then f is an F-extension of ξ^{θ}_{δ}. But f is not a γ-translation of ξ for all $\gamma \in [0, \theta] = [0, 0.3]$. If we take $\gamma := 0.26$, then $\gamma = 0.26 > 0.2 = \delta$ and the γ-translation ξ^{θ}_{γ} of ξ is provided as follows:

$$\xi^{\theta}_{\gamma} : X \to [0, 1],\ x \mapsto \begin{cases} 0.96 & \text{if } x = 1 \\ 0.76 & \text{if } x = 2 \\ 0.86 & \text{if } x = 3 \\ 0.66 & \text{if } x = 0 \end{cases}$$

which is a Sheffer stroke BE-filter of $\mathcal{X}$. Note that $f(x) \geq \xi^{\theta}_{\gamma}(x)$ for all $x \in X$, i.e., $\xi^{\theta}_{\gamma} \subseteq f$. Thus f is an F-extension of the γ-translation ξ^{θ}_{γ} of ξ.

6. Conclusion

In classical logic, Sheffer stroke, also called NAND or alternative denial, is one of the two operations that can be used by itself, without any other logical operations, to constitute a logical formal system. The stroke symbol is " $\uparrow$ " as in

$$(p \uparrow q) \leftrightarrow (\neg p \vee \neg q),$$

and it is a logical connective whose truth table is presented by Table 1. Fuzzy Sheffer stroke BE-algebras are first studied by Oner, Katican and Borumand Saeid. In this paper, we first investigated the further properties of fuzzy Sheffer stroke BE-filters/BE-subalgebras. Next, we introduced the concepts of normalized fuzzy Sheffer stroke BE-filters/BE-subalgebras, translations, S-extensions and F-extensions, and investigated related properties. We considered characterizations of a fuzzy Sheffer stroke BE-filters, and explored the conditions under which the fuzzy set which is maded by the upper set can be a fuzzy Sheffer stroke BE-filter. We displayed how to configure a fuzzy Sheffer stroke BE-filter/BE-subalgebra using a collection of Sheffer stroke BE-filters/BE-subalgebras. We provided the methods of constructing the normalization of a given fuzzy Sheffer stroke BE-filter/BE-subalgebra. We discussed relations between a fuzzy Sheffer stroke BE-filter/BE-subalgebra and its normalization, and established extensions of a fuzzy Sheffer stroke BE-filter/BE-subalgebra.

The contents and ideas of this paper will be applied to almost all applications, where fuzzy set theory is applied, including logical algebras, in the future. This will actively apply to the fuzzy set theory of the substructures in Sheffer stroke basic algebras, Sheffer stroke $BCK/BCI/BCH$-algebras, Sheffer stroke BL-algebras, Sheffer stroke MV-algebras, Sheffer stroke hoops, etc., when limited to the Sheffer stroke theory based on logical algebras.

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