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Abstract. The target of this article is to extend five styles of single-granulation approximation spaces to multi-granulation approximation spaces utilizing ideals. We discuss the main properties of the deduced approximation spaces and clarify that these spaces shrink the boundary regions and improve the accuracy values. The submitted definitions is more accurate than some already defined definitions of approximation spaces. A comparison of these methods with the prior ones was also done, demonstrating that the current study is more generic. At the end of this work, we present an applicable example in the field of Chemistry by using the current methodologies to interpret the applications of our definitions in decision making problems.

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1. Introduction

A brief review on rough set theory and its applications. The idea of rough sets is usually based on an equivalence relation between universal objects. Pawlak [1, 2] introduced this theory in 1982 to handle faulty information from real difficulties in disciplines like incomplete information systems and decision analysis. The original rough model (also known as the Pawlak rough model) is divided into two halves by a pair of operators known as lower and upper approximation operators. These operations provide two categories for the information subsets: internally (externally, fully) indefinable subsets and approximately defined subsets. Two further key ideas in rough set theory that are derived from the approximation operators are the accuracy measure and boundary region. These concepts provide information on the organisation and comprehensiveness of the knowledge of the information being studied.

As a result of the incapacity to manage actual facts that aren't connected to an equivalency relation, several shortcomings eventually surfaced. Yao [3, 4] was inspired by this to suggest a unique way to divide blocks (or do granular computing) that he named "right and left neighbourhoods induced from any relation". Next, Allam et al. [5, 6] used the minimal right neighbourhoods to generate the approximation operators.Afterwards, Kozae et al. [7] used the intersection between minimal left and right neighbourhoods in defining the lower and upper approximations. Certain relations, such as similarity [8, 9] and tolerance [8, 9], were the basis for the introduction of some of them. Novel paradigms of generalized rough models have been presented through the use of new types of neighbourhood systems, whose aim is to maximize the subset's accuracy measure. Among these are core neighbourhoods [10], sometimes referred to as j-adhesion neighbourhoods and thoroughly examined in [11]. Dai et al. [12] constructed the crude models generated by maximum right neighbourhoods under a similarity relation. Al-shami [13] then went on to show the different types of maximum neighbourhoods and described how they were used to rank COVID-19 suspects. The confinement neighbourhoods [14] and subset neighbourhoods [15] systems were provided in order to enhance the rough paradigms' characteristics with regard to boundary region, accuracy measures, monotonic property, and so forth. In Abu-Donia [16], new approximation operators were introduced and their primary benefits were demonstrated by substituting a family of relations for a relation.

The notion of an ideal was composed to a topological space in [17], and consequently it was defined the notion of a bi-ideals in [18]. There is an important target in studying roughness of a set. The target is to increase the accuracy value or shrink the boundary region of a set, and this is could be found in [19, 18, 20]. Kandil et al. [21] created "ideal approximation spaces" by combining the concept of ideals with rough neighbourhoods. These models improve accuracy by shrinking upper approximations and enlarging lower approximations. Meanwhile, Kandil et al. [18] suggested the "minimal intersection neighborhoods" with ideals to define approximation spaces. Others who pursued this line of inquiry focused on certain phenomena that were discussed in [22, 23, 24, 25, 26]. Many research works were introduced to study the topological structures via ordinary and soft sets [27, 28, 29, 30].

From this point of view, the previous models are referred to as a single-granulation rough set approaches. Nevertheless, single-granulation is not good enough for real life problems solving. For example:

(1) A map may be described from many perspectives, with the coarser perspective focusing on larger information granules and the finer perspective on smaller information granules. This allows us to study a map at different degrees of granulation,
(2) Due to the vulnerability of single feature information (for example, fingerprints to theft and duplication by criminals), multi-granulations authentication is required,
(3) Because it must perform the intersection on several binary relations, the single-granulation technique in rough set theory is exceedingly time-consuming.

A short review on the multi- granulation approximations To enhance the efficacy of Pawlak's rough set theory by addressing shortcomings in the single-granulation approach, Qian et al. [31, 32, 33] proposed the idea of the multi-granulation rough set. It differs from Pawlak's rough set model in that it is characterised by a family of equivalence relations rather than just one. Both optimistic and pessimistic multi-granulation rough sets are included in the multi-granulation rough set model. The notion that at least one granular structure in a multi-independent granular structure has to satisfy the inclusion relation between the equivalency class and the undefinable set is referred to as "optimistic." Conversely, the term "pessimistic" refers to the notion that every granular structure needs to fulfil the inclusion relation between the undefinable set and the equivalency class. Numerous investigations on multi-granulation rough set models based on different kinds of relations have produced a variety of interesting concepts, including those shown in [34, 35, 36, 37, 38, 39, 40, 41, 42].

The motivation of this paper Although many scholars have published a lot of papers to develop many concepts of rough set content, there has been a lot of development for this set. The essential motivation of this paper is to improve the approximations and accuracy measure of a set by deleting some objects from the upper approximations of decision categories and/or adding new objects to the lower approximations. This target can be established by using the proposed notion "multi-granulations and ideals" where the approximation sets are defined by multi-binary relations on the universal set upon the existence of ideals. This paper is organized as follows: Section 1 is assigned to the introductory text. Section 2 stands for the basics and the notations will be used in the article. In Section 3, there are five methods based on multi-granulations and ideals are suggested to achieve the main target of this paper, that is, to shrink the boundary region and to improve the accuracy value of some rough set. The relevant properties and results of these methods are instituted. In the first method we generalized the model of multi-granulation rough set given in [31] using an ideal defined on the multiple equivalence relations. In the rest four methods, we used multi-granulations and ideals to generalize the preceding methods that use single granulation defined on the universe [1, 6, 18, 21, 43, 44]. More importantly, it is proved that the first method is monotonic. Section 4 is the most important section in the paper. However, studying which one of the five methods is the best was submitted, and it was shown that the fifth one is the best. Also, it is proved that these recent methods are better than their corresponding already introduced definitions given in [1, 6, 18, 21, 31, 44]. In section 5, it is introduced a new technique of approximation spaces based on multi-granulations via the notion of bi-ideals, called bi-ideals multi-granulations approximation spaces. It is submitted in two different forms. Some results and properties of these multi-granulation approximation spaces via bi-ideals are submitted. The comparisons between these two forms and the previous forms [18] are analyzed. Moreover, the in between relations of these two forms are discussed. In Section 6, it is submitted a real-life example to ensure that this article is applicable. A chemical application about amino acids (see [45, 46]) using the recent methods is discussed to establish the best method. The conclusion of this manuscript is submitted as Section 7.

2. Preliminaries

Definition 2.1 ([1])**.** Let Π be a universe of objects and $\Re$ be an equivalence relation on it, $[\zeta]_{\Re}$ be the equivalence class containing $\zeta \in \Pi$. For any subset Γ of Π, the *lower* and *upper approximations*, $\underline{Paw}(\Gamma)$ and $\overline{Paw}(\Gamma)$, the *boundary region*, $BND(\Gamma)$ and the *accuracy measure*, $ACC(\Gamma)$ of Γ are defined respectively by:

$$\underline{Paw}(\Gamma) = \{\zeta \in \Pi : [\zeta]_{\Re} \subseteq \Gamma\},$$
$$\overline{Paw}(\Gamma) = \{\zeta \in \Pi : [\zeta]_{\Re} \cap \Gamma \neq \phi\},$$
$$BND(\Gamma) = \overline{Paw}(\Gamma) - \underline{Paw}(\Gamma),\ ACC(\Gamma) = \frac{|\underline{Paw}(\Gamma)|}{|\overline{Paw}(\Gamma)|},\ \Gamma \neq \phi.$$

Proposition 2.2 ([1, 4])**.** *The following is a list of the main characterizations of the approximation operators defined in Definition 2.1*

(1) $\underline{Paw}(\Pi) = \overline{Paw}(\Pi) = \Pi$ *and* $\underline{Paw}(\phi) = \overline{Paw}(\phi) = \phi$,
(2) $\underline{Paw}(\Gamma) \subseteq \Gamma \subseteq \overline{Paw}(\Gamma)$,
(3) *if* $\Gamma \subseteq \Upsilon$, *then* $\underline{Paw}(\Gamma) \subseteq \underline{Paw}(\Upsilon)$ *and* $\overline{Paw}(\Gamma) \subseteq \overline{Paw}(\Upsilon)$,
(4) $\underline{Paw}(\Gamma^c) = [\overline{Paw}(\Gamma)]^c$ *and* $\overline{Paw}(\Gamma^c) = [\underline{Paw}(\Gamma)]^c$,
(5) $\overline{Paw}[\overline{Paw}(\Gamma)] = \overline{Paw}(\Gamma)$ *and* $\underline{Paw}[\underline{Paw}(\Gamma)] = \underline{Paw}(\Gamma)$,
(6) $\underline{Paw}(\Gamma \cap \Upsilon) = \underline{Paw}(\Gamma) \cap \underline{Paw}(\Upsilon)$ *and* $\underline{Paw}(\Gamma) \cup \underline{Paw}(\Upsilon) \subseteq \underline{Paw}(\Gamma \cup \Upsilon)$,
(7) $\overline{Paw}(\Gamma \cap \Upsilon) \subseteq \overline{Paw}(\Gamma) \cap \overline{Paw}(\Upsilon)$ *and* $\overline{Paw}(\Gamma) \cup \overline{Paw}(\Upsilon) = \overline{Paw}(\Gamma \cup \Upsilon)$,
(8) $\underline{Paw}[\overline{Paw}(\Gamma)] = \overline{Paw}(\Gamma)$ *and* $\underline{Paw}(\Gamma) = \overline{Paw}[\underline{Paw}(\Gamma)]$.

Definition 2.3 ([3])**.** Let Π be a universe of objects and $\Re$ be a binary relation on it. For any subset Γ of Π , the *lower* and *upper approximations*, $\underline{Yao}(\Gamma)$ and $\overline{Yao}(\Gamma)$ are defined respectively by:

$$\underline{Yao}(\Gamma) = \{\zeta \in \Pi : \zeta\Re \subseteq \Gamma\}$$
$$\overline{Yao}(\Gamma) = \{\zeta \in \Pi : \zeta\Re \cap \Gamma \neq \phi\},$$

where $\zeta\Re$ is the right neighborhood of ζ and it is given by $\zeta\Re = \{\xi \in \Pi : \zeta\Re\xi\}$.

Theorem 2.4 ([6])**.** *If $\Re$ is a transitive and reflexive relation on Π. Then, the approximations defined in Definition 2.3 satisfy the properties (1)-(7) in Proposition 2.2*

Definition 2.5 ([6])**.** A set $\prec \zeta \succ \Re$ is the intersection of all right neighborhoods containing ζ, where $\Re$ is any binary relation on Π, i.e.,

$$\prec \zeta \succ \Re = \begin{cases} \cap_{\zeta \in \xi\Re}(\xi\Re) & \text{if } \exists \xi : \zeta \in \xi\Re \\ \phi & \textit{otherwise}. \end{cases}$$

Also, $\Re \prec \zeta \succ$ is the intersection of all left neighborhoods containing ζ, i.e.,

$$\Re \prec \zeta \succ = \begin{cases} \cap_{\zeta \in \xi\Re}(\Re\xi) & \text{if } \exists \xi : \zeta \in \Re\xi \\ \phi & \textit{otherwise}, \end{cases}$$

where $\Re\xi$ is the left neighborhood of ξ and it is given by $\Re\xi = \{\zeta \in \Pi : \zeta\Re\xi\}$.

Definition 2.6 ([6])**.** Let $\Re$ be binary relation on Π. For any subset Γ of Π, a pair of *lower* and *upper approximations*, $Aprox_*(\Gamma)$ and $Aprox^*(\Gamma)$ are defined respectively by:

$$Aprox_*(\Gamma) = \{\zeta \in \Pi \ :\prec \zeta \succ \Re \subseteq \Gamma\},$$
$$Aprox^*(\Gamma) = \{\zeta \in \Pi \ :\prec \zeta \succ \Re \cap \Gamma \neq \phi\}.$$

Definition 2.7 ([5])**.** Let $\Re$ be binary relation on Π. For any subset Γ of Π, a pair of *lower* and *upper approximations*, $\underline{Aprox}(\Gamma)$ and $\overline{Aprox}(\Gamma)$ are defined by:

$$\underline{Aprox}(\Gamma) = \{\zeta \in \Gamma \ :\prec \zeta \succ \Re \subseteq \Gamma\},$$
$$\overline{Aprox}(\Gamma) = \Gamma \cup Aprox^*(\Gamma).$$

Theorem 2.8 ([5])**.** *Let $\Re$ be reflexive relation on Π. Then, the approximations defined in Definition 2.6 satisfy the properties (1)-(7) in Proposition 2.2*

Definition 2.9 ([7])**.** Let $\Re$ be binary relation on Π. For any subset Γ of Π, a pair of *lower* and *upper approximations*, $\underline{Aprox}(\Gamma)$ and $\overline{Aprox}(\Gamma)$ are defined by:

$$\underline{Aprox}(\Gamma) = \{\zeta \in \Pi \ : \Re \prec \zeta \succ \Re \subseteq \Gamma\},$$
$$\overline{Aprox}(\Gamma) = \{\zeta \in \Pi \ : \Re \prec \zeta \succ \Re \cap \Gamma \neq \phi\},$$

where $\Re \prec \zeta \succ \Re \ = \prec \zeta \succ \Re \ \cap \ \Re \prec \zeta \succ$.

Theorem 2.10 ([7])**.** *Let $\Re$ be reflexive relation on Π. Then, the approximations defined in Definition 2.10 satisfy the properties (1)-(7) in Proposition 2.2*

Lemma 2.11 ([5, 7])**.** *Let $\Re$ be a binary relation on Π. Then we have:*

(1) *if $\zeta \in < \xi > \Re$, then $\prec \zeta \succ \Re \subseteq < \xi > \Re$,*
(2) *if $\zeta \in \Re < \xi > \Re$, then $\Re \prec \zeta \succ \Re \subseteq \Re < \xi > \Re$.*

Definition 2.12 ([19])**.** Let Π be a non-empty set. Then a non-empty collection $\eth$ of subsets of Π is called an *ideal* on Π, if it satisfies the following conditions:

(i) if $\Gamma \in \eth$ and $\Upsilon \subseteq \Gamma$, then $\Upsilon \in \eth$,
(ii) if $\Gamma, \Upsilon \in \eth$, then $\Gamma \cup \Upsilon \in \eth$.

Definition 2.13 ([20])**.** If $\eth_1$, $\eth_2$ are two ideals on a non-empty set Π. The collection generated by $\eth_1$ and $\eth_2$, denoted by $\prec \eth_1, \eth_2 \succ$, is defined as follows:

$$\prec \eth_1, \eth_2 \succ = \{G_1 \cup G_2 : \ G_1 \in \eth_1, G_2 \in \eth_2\}.$$

Proposition 2.14 ([20])**.** *Let $\eth_1, \eth_2$ are two ideals on a non-empty set Π and $\Gamma, \Upsilon \subseteq \Pi$. Then the collection $\prec \eth_1, \eth_2 \succ$ has the following proprieties:*

(1) $\prec \eth_1, \eth_2 \succ \neq \phi$,
(2) *if $\Gamma \in \prec \eth_1, \eth_2 \succ$, $\Upsilon \subseteq \Gamma$, then $\Upsilon \in \prec \eth_1, \eth_2 \succ$,*
(3) *if $\Gamma, \Upsilon \in \prec \eth_1, \eth_2 \succ$, Υ, then $\Gamma \cup \Upsilon \in \prec \eth_1, \eth_2 \succ$.*

Definition 2.15 ([20])**.** A quadrable $(\Pi, \Re, \eth_1, \eth_2)$ is said to be a *bi-ideal approximation space* or a *bi-ideal knowledge base*, if $\Re$ is a binary relation on Π and $\eth_1$, $\eth_2$ are two ideals on Π. The *lower* and *upper approximations* of $\Gamma \subseteq \Pi$ are defined respectively by:

$$\underline{\Re}_{\prec \eth_1, \eth_2 \succ}(\Gamma) = \{\zeta \in \Gamma : \Re \prec \zeta \succ \Re \cap \Gamma^c \in \prec \eth_1, \eth_2 \succ\}, \tag{2.1}$$

$$\overline{\Re}_{\prec \eth_1, \eth_2 \succ}(\Gamma) = \Gamma \cup \{\zeta \in \Pi : \Re \prec \zeta \succ \Re \cap \Gamma \notin \prec \eth_1, \eth_2 \succ\}. \tag{2.2}$$

Definition 2.16 ([20])**.** Let $(\Pi, \Re, \eth_1, \eth_2)$ be a bi-ideal knowledge base.The *lower* and *upper approximations*, $\underline{\Re}_{\eth_1,\eth_2}(\Gamma)$ and $\overline{\Re}_{\eth_1,\eth_2}(\Gamma)$ of $\Gamma \subseteq \Pi$ are defined respectively by:

$$\underline{\Re}_{\eth_1,\eth_2}(\Gamma) = \underline{\Re}_{\eth_1}(\Gamma) \cup \underline{\Re}_{\eth_2}(\Gamma),$$
$$\overline{\Re}_{\eth_1,\eth_2}(\Gamma) = \overline{\Re}_{\eth_1}(\Gamma) \cap \overline{\Re}_{\eth_2}(\Gamma),$$

where $\underline{\Re}_{\eth_\iota}(\Gamma)$ and $\overline{\Re}_{\eth_\iota}(\Gamma)$ are the lower and upper approximations of Γ with respect to $\eth_\iota$, $\iota \in \{1, 2\}$ as in Definition 2.21.

Definition 2.17 ([23])**.** Let $\Re$ be a binary relation on Π, $\eth$ an ideal defined on Π and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations*, $Aprox_*(\Gamma)$ and $Aprox^*(\Gamma)$ $Aprox_*(\Gamma)$ and $Aprox^*(\Gamma)$ of Γ are defined by:

$$Aprox_*(\Gamma) = \{\zeta \in \Pi \ : \prec \zeta \succ \Re \cap \Gamma^c \in \eth\},$$
$$Aprox^*(\Gamma) = \{\zeta \in \Pi \ : \prec \zeta \succ \Re \cap \Gamma \notin \eth\}.$$

Definition 2.18 ([23])**.** Let $\Re$ be a binary relation on Π, $\eth$ an ideal defined on Π and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations*, $\underline{Aprox}(\Gamma)$ and $\overline{Aprox}(\Gamma)$, the *boundary region*, $BND(\Gamma)$ and the *accuracy measure*, $\overline{ACC(}\Gamma)$ of Γ are defined respectively by:

$$\underline{Aprox}(\Gamma) = \{\zeta \in \Gamma \ : \prec \zeta \succ \Re \cap \Gamma^c \in \eth\},$$
$$\overline{Aprox}(\Gamma) = \Gamma \cup \{\zeta \in \Pi \ : \prec \zeta \succ \Re \cap \Gamma \notin \eth\},$$
$$BND(\Gamma) = \overline{Aprox}(\Gamma) - \underline{Aprox}(\Gamma),\ ACC(\Gamma) = \frac{|\underline{Aprox}(\Gamma)|}{|\overline{Aprox}(\Gamma)|},\ |\overline{Aprox}(\Gamma)| \neq 0,$$

where $0 \leq ACC(\Gamma) \leq 1$.

Theorem 2.19 ([23])**.** *Let $\Re$ be a binary relation on Π. Then the approximations defined in Definition 2.18 satisfy the properties (1)-(7) in Proposition 2.2*

Definition 2.20 ([20])**.** Let $\Re$ be a binary relation on Π, $\eth$ an ideal defined on Π and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations*, $Aprox_*(\Gamma)$ and $Aprox^*(\Gamma)$ of Γ are defined by:

$$Aprox_*(\Gamma) = \{\zeta \in \Pi \ : \Re \prec \zeta \succ \Re \cap \Gamma^c \in \eth\},$$
$$Aprox^*(\Gamma) = \{\zeta \in \Pi \ : \Re \prec \zeta \succ \Re \cap \Gamma \notin \eth\}.$$

Definition 2.21 ([20])**.** Let $\Re$ be a binary relation on Π, $\eth$ an ideal defined on Π and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations*, $\underline{Aprox}(\Gamma)$ and $\overline{Aprox}(\Gamma)$, the *boundary region*, $BND(\Gamma)$ and the *accuracy measure*, $\overline{ACC(}\Gamma)$ of Γ are defined respectively by:

$$\underline{Aprox}(\Gamma) = \{\zeta \in \Gamma \ : \Re \prec \zeta \succ \Re \cap \Gamma^c \in \eth\},$$
$$\overline{Aprox}(\Gamma) = \Gamma \cup Aprox^*(\Gamma),$$
$$BND(\Gamma) = \overline{Aprox}(\Gamma) - \underline{Aprox}(\Gamma),\ ACC(\Gamma) = \frac{|\underline{Aprox}(\Gamma)|}{|\overline{Aprox}(\Gamma)|},\ |\overline{Aprox}(\Gamma)| \neq 0,$$

where $0 \leq ACC(\Gamma) \leq 1$.

Theorem 2.22 ([20]). *Let $\Re$ be a binary relation on Π. Then, the approximations defined in Definition 2.21 satisfy the properties (1)-(7) in Proposition 2.2*

Definition 2.23 ([34]). A pair $(\Pi, \mathcal{R})$ is said to be a *Knowledge base*, if $\mathcal{R}$ is a family of equivalence relations on Π. Let $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations*, $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)$ and $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)$ of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) = \bigcup\{\zeta \in \Pi : [\zeta]_{\Re_\lambda} \subseteq \Gamma, \lambda \leq \Lambda\}$$

$$\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) = [\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma^c)]^c.$$

3. Some new methods based on multi-granulations and ideals

3.1. **Method (1).** In granular computing, the approximations of a set is describes by using a single equivalence relation (granulation) on the universe. Firstly, we discuss the approximations of a set by using two equivalence relations under a given ideal on the universe.

Definition 3.1. A triple $(\Pi, \mathcal{R}, \eth)$ is said to be an *ideal Knowledge bas*, if $\mathcal{R}$ is a family of equivalence relations on Π and $\eth$ is an ideal on Π. Let $\Re_1, \Re_2 \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the $(\Re_1 + \Re_2)_*$-*lower* and $(\Re_1 + \Re_2)^*$-*upper approximations* of Γ are defined respectively by:

$$(\Re_1 + \Re_2)_*(\Gamma) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma^c \in \eth \text{ or } [\zeta]_{\Re_2} \cap \Gamma^c \in \eth\} \tag{3.1}$$

$$(\Re_1 + \Re_2)^*(\Gamma) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma \notin \eth \text{ and } [\zeta]_{\Re_2} \cap \Gamma \notin \eth\}. \tag{3.2}$$

Proposition 3.2. *Let $(\Pi, \mathcal{R}, \eth)$ be an ideal Knowledge base, $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \Re_2 \in \mathcal{R}$. Then the following properties hold:*

(1) $(\Re_1 + \Re_2)_*(\Pi) = \Pi$ *and* $(\Re_1 + \Re_2)^*(\phi) = \phi$,
(2) $(\Re_1 + \Re_2)_*(\Gamma^c) = [(\Re_1 + \Re_2)^*(\Gamma)]^c$ *and* $(\Re_1 + \Re_2)^*(\Gamma^c) = [(\Re_1 + \Re_2)_*(\Gamma)]^c$,
(3) $(\Re_1 + \Re_2)_*(\Gamma) = (\Re_1)_*(\Gamma) \cup (\Re_2)_*(\Gamma)$ *and* $(\Re_1 + \Re_2)^*(\Gamma) = (\Re_1)^*(\Gamma) \cap (\Re_2)^*(\Gamma)$,
(4) *if* $\Gamma \subseteq \Upsilon$, *then* $(\Re_1 + \Re_2)_*(\Gamma) \subseteq (\Re_1 + \Re_2)_*(\Upsilon)$ *and* $(\Re_1 + \Re_2)^*(\Gamma) \subseteq (\Re_1 + \Re_2)^*(\Upsilon)$,
(5) $(\Re_1 + \Re_2)^*[(\Re_1 + \Re_2)^*(\Gamma)] = (\Re_1 + \Re_2)^*(\Gamma)$,
(6) $(\Re_1 + \Re_2)_*(\Gamma) = (\Re_1 + \Re_2)_*[(\Re_1 + \Re_2)_*(\Gamma)]$,
(7) $(\Re_1 + \Re_2)_*(\Gamma \cap \Upsilon) = [(\Re_1)_*(\Gamma) \cap (\Re_1)_*(\Upsilon)] \cup [(\Re_2)_*(\Gamma) \cap (\Re_2)_*(\Upsilon)]$,
(8) $(\Re_1 + \Re_2)^*(\Gamma \cup \Upsilon) = [(\Re_1)^*(\Gamma) \cup (\Re_1)^*(\Upsilon)] \cap [(\Re_2)^*(\Gamma) \cup (\Re_2)^*(\Upsilon)]$,
(9) $(\Re_1 + \Re_2)_*(\Gamma \cap \Upsilon) \subseteq (\Re_1 + \Re_2)_*(\Gamma) \cap (\Re_1 + \Re_2)_*(\Upsilon)$ *and* $(\Re_1 + \Re_2)_*(\Gamma) \cup (\Re_1 + \Re_2)_*(\Upsilon) \subseteq (\Re_1 + \Re_2)_*(\Gamma \cup \Upsilon)$,
(10) $(\Re_1 + \Re_2)^*(\Gamma \cap \Upsilon) \subseteq (\Re_1 + \Re_2)^*(\Gamma) \cap (\Re_1 + \Re_2)^*(\Upsilon)$ *and* $(\Re_1 + \Re_2)^*(\Gamma) \cup (\Re_1 + \Re_2)^*(\Upsilon) \subseteq (\Re_1 + \Re_2)^*(\Gamma \cup \Upsilon)$.

Proof. (1) $(\Re_1 + \Re_2)_*(\Pi) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Pi^c \in \eth \text{ or } [\zeta]_{\Re_2} \cap \Pi^c \in \eth\}$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \phi \in \eth \text{ or } [\zeta]_{\Re_2} \cap \phi \in \eth\} = \Pi$.
Clearly, $(\Re_1 + \Re_2)^*(\phi) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \phi \notin \eth \text{ and } [\zeta]_{\Re_2} \cap \phi \notin \eth\} = \phi$.

(2) $[(\Re_1+\Re_2)^*(\Gamma)]^c = [\{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma \notin \eth \textit{ and } [\zeta]_{\Re_2} \cap \Gamma \notin \eth\}]^c$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma \in \eth \textit{ or } [\zeta]_{\Re_2} \cap \Gamma \in \eth\}$
$= (\Re_1+\Re_2)_*(\Gamma^c)$.
Clearly, $[(\Re_1+\Re_2)_*(\Gamma)]^c = (\Re_1+\Re_2)^*(\Gamma^c)$.

(3) $(\Re_1+\Re_2)_*(\Gamma) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma^c \in \eth \textit{ or } [\zeta]_{\Re_2} \cap \Gamma^c \in \eth\}$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma^c \in \eth\} \ \cup \ \{\zeta \in \Pi : [\zeta]_{\Re_2} \cap \Gamma^c \in \eth\}$
$= (\Re_1)_*(\Gamma) \cup (\Re_2)_*(\Gamma)$.

(4) Let $\zeta \in (\Re_1+\Re_2)_*(\Gamma)$. Then $[\zeta]_{\Re_1} \cap \Gamma^c \in \eth$ *or* $[\zeta]_{\Re_2} \cap \Gamma^c \in \eth$. $\Gamma \subseteq \Upsilon$ implies that $[\zeta]_{\Re_1} \cap \Upsilon^c \in \eth$ *or* $[\zeta]_{\Re_2} \cap \Upsilon^c \in \eth$. Thus $\zeta \in (\Re_1+\Re_2)_*(\Upsilon)$.

(5) Let $\zeta \in (\Re_1+\Re_2)^*[(\Re_1+\Re_2)^*(\Gamma)]$. Then we have
$[\zeta]_{\Re_1} \cap (\Re_1+\Re_2)^*(\Gamma) \notin \eth$ and $[\zeta]_{\Re_2} \cap (\Re_1+\Re_2)^*(\Gamma) \notin \eth$
$\Rightarrow [\zeta]_{\Re_1} \cap (\Re_1+\Re_2)^*(\Gamma) \neq \phi$ and $[\zeta]_{\Re_2} \cap (\Re_1+\Re_2)^*(\Gamma) \neq \phi$
$\Rightarrow \exists\ \xi \in [\zeta]_{\Re_1} \cap (\Re_1+\Re_2)^*(\Gamma)$ and $\xi \in [\zeta]_{\Re_2} \cap (\Re_1+\Re_2)^*(\Gamma)$
$\Rightarrow \xi \in [\zeta]_{\Re_1}, \xi \in (\Re_1+\Re_2)^*(\Gamma)$ and $\xi \in [\zeta]_{\Re_2}, \xi \in (\Re_1+\Re_2)^*(\Gamma)$
$\Rightarrow [\zeta]_{\Re_1} = [\xi]_{\Re_1}, [\xi]_{\Re_1} \cap (\Gamma) \notin \eth$ and $[\zeta]_{\Re_2} = [\xi]_{\Re_2}, [\xi]_{\Re_2} \cap (\Gamma) \notin \eth$
$\Rightarrow \zeta \in (\Re_1+\Re_2)^*(\Gamma)$
$\Rightarrow (\Re_1+\Re_2)^*[(\Re_1+\Re_2)^*(\Gamma)] \subseteq (\Re_1+\Re_2)^*(\Gamma)$.
Conversely, let $\zeta \in (\Re_1+\Re_2)^*(\Gamma)$. Then $[\zeta]_{\Re_1} \cap \Gamma \notin \eth, [\zeta]_{\Re_2} \cap \Gamma \notin \eth$. Thus $[\zeta]_{\Re_1} \subseteq (\Re_1)^*(\Gamma)$ and $[\zeta]_{\Re_2} \subseteq (\Re_2)^*(\Gamma)$. Now, if $[\zeta]_{\Re_1} \cap \Gamma \notin \eth$, then $\exists \xi \in [\zeta]_{\Re_1}, \xi \in \Gamma, \{\xi\} \notin \eth$. Thus $[\xi]_{\Re_2} \cap \Gamma \notin \eth$ and $\xi \in (\Re_2)^*(\Gamma)$. So
$$[\zeta]_{\Re_1} \cap (\Re_1)^*(\Gamma) \cap (\Re_2)^*(\Gamma) = [\zeta]_{\Re_1} \cap (\Re_1+\Re_2)^*(\Gamma) \notin \eth.$$
By the same way, $[\zeta]_{\Re_2} \cap (\Re_1+\Re_2)^*(\Gamma) \notin \eth$. Hence w get
$$\zeta \in (\Re_1+\Re_2)^*[(\Re_1+\Re_2)^*(\Gamma)].$$

(6) Directly, by putting $\Gamma = \Gamma^c$ in (7) and using (2).

(7) From (3), we have
$(\Re_1+\Re_2)_*(\Gamma \cap \Upsilon) = (\Re_1)_*(\Gamma \cap \Upsilon) \cup (\Re_2)_*(\Gamma \cap \Upsilon)$
$= [(\Re_1)_*(\Gamma) \cap (\Re_1)_*(\Upsilon)] \cup [(\Re_2)_*(\Gamma) \cap (\Re_2)_*(\Upsilon)]$.

(8) From (4), we have
$(\Re_1+\Re_2)^*(\Gamma \cup \Upsilon) = (\Re_1)^*(\Gamma \cup \Upsilon) \cap (\Re_2)^*(\Gamma \cup \Upsilon)$
$= [(\Re_1)^*(\Gamma) \cup (\Re_1)^*(\Upsilon)] \cap [(\Re_2)^*(\Gamma) \cup (\Re_2)^*(\Upsilon)]$.

(9) Direct from (5).

(10) Direct from (6).

□

Remark 3.3. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal Knowledge base, $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \Re_2 \in \mathcal{R}$. Then Example 3.4 shows that in general:

(1) $(\Re_1+\Re_2)^*(\Pi) \neq \Pi$ and $(\Re_1+\Re_2)_*(\phi) \neq \phi$,

(2) $(\Re_1+\Re_2)_*(\Gamma) \nsubseteq \Gamma$ and $\Gamma \nsubseteq (\Re_1+\Re_2)^*(\Gamma)$,

(3) $(\Re_1+\Re_2)_*(\Gamma) \subseteq (\Re_1+\Re_2)_*(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$ and $(\Re_1+\Re_2)^*(\Gamma) \subseteq (\Re_1+\Re_2)^*(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$,

(4) $(\Re_1+\Re_2)^*(\Gamma) \nsubseteq (\Re_1+\Re_2)_*[(\Re_1+\Re_2)^*(\Gamma)]$ and $(\Re_1+\Re_2)^*(\Gamma) \nsupseteq (\Re_1+\Re_2)_*[(\Re_1+\Re_2)^*(\Gamma)]$,

(5) $(\Re_1+\Re_2)_*(\Gamma \cap \Upsilon) \neq (\Re_1+\Re_2)_*(\Gamma) \cap (\Re_1+\Re_2)_*(\Upsilon)$ and $(\Re_1+\Re_2)^*(\Gamma \cup \Upsilon) \neq (\Re_1+\Re_2)^*(\Gamma) \cup (\Re_1+\Re_2)^*(\Upsilon)$,

(6) $(\Re_1+\Re_2)_*(\Gamma) \nsubseteq (\Re_1+\Re_2)^*[(\Re_1+\Re_2)_*(\Gamma)]$ and $(\Re_1+\Re_2)_*(\Gamma) \nsupseteq (\Re_1+\Re_2)^*[(\Re_1+\Re_2)_*(\Gamma)]$.

Example 3.4. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal Knowledge base, $\Re_1, \Re_2 \in \mathcal{R}$, where $\Pi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$, $\Pi/\Re_1 = \{\{\zeta_1, \zeta_6\}, \{\zeta_2, \zeta_3, \zeta_5\}, \{\zeta_4\}\}$, $\Pi/\Re_2 = \{\{\zeta_1, \zeta_2\}, \{\zeta_3, \zeta_4, \zeta_5\}, \{\zeta_6\}\}$ *and* $\eth = \{\phi, \{\zeta_4\}, \{\zeta_6\}\{\zeta_4, \zeta_6\}\}$.

(1) $(\Re_1+\Re_2)_*(\phi) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Pi \in \eth \text{ or } [\zeta]_{\Re_2} \cap \Pi \in \eth\} = \{\zeta_4\} \cup \{\zeta_6\} = \{\zeta_4, \zeta_6\} \neq \phi$. Also, $(\Re_1+\Re_2)^*(\Pi) = \{\zeta_1, \zeta_2, \zeta_3, \zeta_5, \zeta_6\} \cap \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_5\} \neq \Pi$.

(2) If $\Gamma = \{\zeta_1, \zeta_2\}$, then $(\Re_1+\Re_2)_*(\Gamma) = \{\zeta_1, \zeta_4, \zeta_6\} \cup \{\zeta_1, \zeta_2, \zeta_6\} = \{\zeta_1, \zeta_2, \zeta_4, \zeta_6\} \nsubseteq \Gamma$. Also, if $\Gamma = \{\zeta_4, \zeta_6\}$, then $(\Re_1+\Re_2)^*(\Gamma) = \phi \cap \phi = \phi \nsupseteq \Gamma$.

(3) Let $\Gamma = \{\zeta_1\}, \Upsilon = \{\zeta_2\}$. Then $(\Re_1+\Re_2)_*(\Gamma) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \{\zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\} \in \eth \text{ or } [\zeta]_{\Re_2} \cap \{\zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\} \in \eth\} = \{\zeta_1, \zeta_4, \zeta_6\} \cup \{\zeta_6\} = \{\zeta_1, \zeta_4, \zeta_6\}$. Also, $(\Re_1+\Re_2)_*(\Upsilon) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \{\zeta_1, \zeta_3, \zeta_4, \zeta_5, \zeta_6\} \in \eth \text{ or } [\zeta]_{\Re_2} \cap \{\zeta_1, \zeta_3, \zeta_4, \zeta_5, \zeta_6\} \in \eth\} = \{\zeta_4\} \cup \{\zeta_6\} = \{\zeta_4, \zeta_6\}$. Thus $(\Re_1+\Re_2)_*(\Gamma) \subseteq (\Re_1+\Re_2)_*(\Upsilon)$ but $\Gamma \nsubseteq \Upsilon$.

(4) Let $\Gamma = \{\zeta_1\}, \Upsilon = \{\zeta_4\}$. Then $(\Re_1+\Re_2)^*(\Gamma) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \{\zeta_1\} \notin \eth \text{ and } [\zeta]_{\Re_2} \cap \{\zeta_1\} \notin \eth\} = \{\zeta_1, \zeta_6\} \cap \{\zeta_1, \zeta_2\} = \{\zeta_1\}$. Also, $(\Re_1+\Re_2)^*(\Upsilon) = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \{\zeta_4\} \notin \eth \text{ and } [\zeta]_{\Re_2} \cap \{\zeta_4\} \notin \eth\} = \phi \cap \phi = \phi$. Thus $(\Re_1+\Re_2)^*(\Gamma) \subseteq (\Re_1+\Re_2)^*(\Upsilon)$ but $\Gamma \nsubseteq \Upsilon$.

(5) From (2), $\Gamma = \{\zeta_4, \zeta_6\}$ and $(\Re_1+\Re_2)^*(\Gamma) = \phi$. But $(\Re_1+\Re_2)_*[(\Re_1+\Re_2)^*(\Gamma)] = \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Pi \in \eth \text{ or } [\zeta]_{\Re_2} \cap \Pi \in \eth\} = \{\zeta_4, \zeta_6\}$. Then $(\Re_1+\Re_2)^*(\Gamma) \nsupseteq (\Re_1+\Re_2)_*[(\Re_1+\Re_2)^*(\Gamma)]$.

(6) Consider $\eth = \{\phi, \{\zeta_3\}, \{\zeta_5\}, \{\zeta_3, \zeta_5\}\}$. If $\Gamma = \{\zeta_1, \zeta_4\}$ and $\Upsilon = \{\zeta_2\}$, then $(\Re_1+\Re_2)_*(\Gamma) \cap (\Re_1+\Re_2)_*(\Upsilon) = \{\zeta_3, \zeta_4, \zeta_5\} \cap \{\zeta_2, \zeta_3, \zeta_5\} = \{\zeta_3, \zeta_5\}$ but $(\Re_1+\Re_2)_*(\Gamma \cap \Upsilon) = (\Re_1+\Re_2)_*(\phi) = \phi$. Also, if $\Gamma = \{\zeta_2, \zeta_3\}$ and $\Upsilon = \{\zeta_4, \zeta_5\}$, then $(\Re_1+\Re_2)^*(\Gamma) \cup (\Re_1+\Re_2)^*(\Upsilon) = \{\zeta_2\} \cup \{\zeta_4\} = \{\zeta_2, \zeta_4\}$ but $(\Re_1+\Re_2)^*(\Gamma \cup \Upsilon) = (\Re_1+\Re_2)^*(\{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}) = \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$.

(7) Consider $\eth = \{\phi, \{\zeta_3\}\}$ and $\Gamma = \{\zeta_3, \zeta_5\}$. Then, $(\Re_1+\Re_2)^*(\Gamma) = \{\zeta_2, \zeta_3, \zeta_5\} \cap \{\zeta_3, \zeta_4, \zeta_5\} = \{\zeta_3, \zeta_5\}$. But $(\Re_1+\Re_2)_*[(\Re_1+\Re_2)^*(\Gamma)] = \phi \cup \phi = \phi$. So, $(\Re_1+\Re_2)^*(\Gamma) \nsubseteq (\Re_1+\Re_2)_*[(\Re_1+\Re_2)^*(\Gamma)]$.

(8) From (5), $\Gamma = \phi$ and $(\Re_1+\Re_2)_*(\Gamma) = \{\zeta_4, \zeta_6\}$. But $(\Re_1+\Re_2)^*[(\Re_1+\Re_2)_*(\Gamma)] = \phi$. Thus, $(\Re_1+\Re_2)_*(\Gamma) \nsubseteq (\Re_1+\Re_2)^*[(\Re_1+\Re_2)_*(\Gamma)]$.

(9) Consider $\Pi/\Re_1 = \{\{\zeta_1, \zeta_6\}, \{\zeta_2, \zeta_3, \zeta_5\}, \{\zeta_4\}\}$, $\Pi/\Re_2 = \{\{\zeta_1, \zeta_6\}, \{\zeta_3, \zeta_4, \zeta_5\}, \{\zeta_2\}\}$ and $\eth = \{\phi, \{\zeta_5\}\}$. If $\Gamma = \{\zeta_2, \zeta_4\}$, then $(\Re_1+\Re_2)_*(\Gamma) = \{\zeta_2, \zeta_4\}$. But $(\Re_1+\Re_2)^*[(\Re_1+\Re_2)_*(\Gamma)] = \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\} \cap \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\} = \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$. Thus

$$(\Re_1+\Re_2)_*(\Gamma) \nsupseteq (\Re_1+\Re_2)^*[(\Re_1+\Re_2)_*(\Gamma)].$$

Definition 3.5. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal Knowledge base, $\mathcal{R}$ be a family of equivalence relations, $\eth$ be an ideal on Π, $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_*(\Gamma) = \{\zeta \in \Pi : \bigcup([\zeta]_{\Re_\lambda} \cap \Gamma^c \in \eth), \lambda \leq \Lambda\}, \tag{3.3}$$

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^*(\Gamma) = \{\zeta \in \Pi : \bigcap([\zeta]_{\Re_\lambda} \cap \Gamma \notin \eth), \lambda \leq \Lambda\}. \tag{3.4}$$

Proposition 3.6. *Let* $(\Pi, \mathcal{R}, \eth)$ *be an ideal Knowledge base,* $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \cdots, \Re_\Lambda \in \mathcal{R}$. *Then the following properties hold:*

(1) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Pi) = \Pi$ *and* $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\phi) = \phi$,

(2) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma^c) = [\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)]^c$ *and* $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma^c) = [\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)]^c$,

(3) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma) = \bigcup_{\lambda=1}^{\Lambda}(\Re_\lambda)_*(\Gamma)$ *and* $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) = \bigcap_{\lambda=1}^{\Lambda}(\Re_\lambda)^*(\Gamma)$,

(4) *if* $\Gamma \subseteq \Upsilon$, *then* $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Upsilon)$ *and* $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Upsilon)$,

(5) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)] = \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)$,

(6) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)] = \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)$,

(7) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma \cap \Upsilon) = \bigcup_{\lambda=1}^{\Lambda}[(\Re_\lambda)_*(\Gamma) \cap (\Re_\lambda)_*(\Upsilon)]$,

(8) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma \cup \Upsilon) = \bigcap_{\lambda=1}^{\Lambda}[(\Re_\lambda)^*(\Gamma) \cup (\Re_\lambda)^*(\Upsilon)]$,

(9) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma\cap\Upsilon) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)\cap\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Upsilon)$ *and* $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)\cup\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Upsilon) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma \cup \Upsilon)$,

(10) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma\cap\Upsilon) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)\cap\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Upsilon)$ *and* $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)\cup\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Upsilon) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma \cup \Upsilon)$.

Proof. (1) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Pi)$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Pi^c \in \eth \text{ or } [\zeta]_{\Re_2} \cap \Pi^c \in \eth \text{ or } \cdots \text{ or } [\zeta]_{\Re_\Lambda} \cap \Pi^c \in \eth\}$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \phi \in \eth \text{ or } [\zeta]_{\Re_2} \cap \phi \in \eth \text{ or } \cdots \text{ or } [\zeta]_{\Re_\Lambda} \cap \phi \in \eth\}$
$= \Pi \cup \Pi \cup ... \cup \Pi$
$= \Pi$.
Clearly, $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\phi) = \phi$.

(2) $[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)]^c$
$= [\{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma \notin \eth \text{ and } [\zeta]_{\Re_2} \cap \Gamma \notin \eth \text{ and } \cdots \text{ and } [\zeta]_{\Re_\Lambda} \cap \Gamma \notin \eth\}]^c$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma \in \eth \text{ or } [\zeta]_{\Re_2} \cap \Gamma \in \eth \text{ or } \cdots \text{ or } [\zeta]_{\Re_\Lambda} \cap \Gamma \in \eth\}\}$
$= \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma^c)$.
Clearly, $[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)]^c = \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma^c)$.

(3) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma^c \in \eth \text{ or } [\zeta]_{\Re_2} \cap \Gamma^c \in \eth \text{ or } \cdots \text{ or } [\zeta]_{\Re_\Lambda} \cap \Gamma^c \in \eth\}$
$= \{\zeta \in \Pi : [\zeta]_{\Re_1} \cap \Gamma^c \in \eth\} \cup \{\zeta \in \Pi : [\zeta]_{\Re_2} \cap \Gamma^c \in \eth\} \cup \cdots \cup \{\zeta \in \Pi : [\zeta]_{\Re_\Lambda} \cap \Gamma^c \in \eth\}$
$= (\Re_1)_*(\Gamma) \cup (\Re_2)_*(\Gamma) \cup \cdots \cup (\Re_\Lambda)_*(\Gamma)$
$= \bigcup_{\lambda=1}^{\Lambda}(\Re_\lambda)_*(\Gamma)$.

(4) Let $\zeta \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma)$. Then $[\zeta]_{\Re_1} \cap \Gamma^c \in \eth$ *or* $[\zeta]_{\Re_2} \cap \Gamma^c \in \eth$ *or* $\cdots$ *or* $[\zeta]_{\Re_\Lambda} \cap \Gamma^c \in \eth$. $\Gamma \subseteq \Upsilon$ implies that $[\zeta]_{\Re_1} \cap \Upsilon^c \in \eth$ *or* $[\zeta]_{\Re_2} \cap \Upsilon^c \in \eth$ *or* $\cdots$ *or* $[\zeta]_{\Re_\Lambda} \cap \Upsilon^c \in \eth$. Thus $\zeta \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Upsilon)$.

(5) Let $\zeta \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)]$. Then we have

$[\zeta]_{\Re_1} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \notin \eth,\ [\zeta]_{\Re_2}\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \notin \eth, \cdots$

$\Rightarrow [\zeta]_{\Re_1} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \neq \phi$ *and* $[\zeta]_{\Re_2} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \neq \phi$ *and* $\cdots$

$\Rightarrow \exists\ \xi_1 \in [\zeta]_{\Re_1} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)$ *and* $\xi_2 \in [\zeta]_{\Re_2} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)$ *and* $\cdots$

$\Rightarrow \xi_1 \in [\zeta]_{\Re_1}, \xi_1 \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)$ *and* $\xi_2 \in [\zeta]_{\Re_2}, \xi_2 \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)$ *and* $\cdots$

$\Rightarrow [\zeta]_{\Re_1} = [\xi_1]_{\Re_1}, [\xi_1]_{\Re_1} \cap (\Gamma) \notin \eth$ *and* $[\zeta]_{\Re_2} = [\xi_2]_{\Re_2}, [\xi_2]_{\Re_2} \cap (\Gamma) \notin \eth$ *and* $\cdots$

$\Rightarrow \zeta \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \Rightarrow \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)] \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)$.

Conversely, let $\zeta \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)$. Then $[\zeta]_{\Re_1} \cap \Gamma \notin \eth$ and $[\zeta]_{\Re_2} \cap \Gamma \notin \eth$ and $\cdots$ and $[\zeta]_{\Re_\Lambda} \cap \Gamma \notin \eth$. Thus $[\zeta]_{\Re_1} \subseteq (\Re_1)^*(\Gamma)$ and $[\zeta]_{\Re_2} \subseteq (\Re_2)^*(\Gamma)$ and $\cdots$ and $[\zeta]_{\Re_\Lambda} \subseteq (\Re_\Lambda)^*(\Gamma)$. Now, if $[\zeta]_{\Re_1} \cap \Gamma \notin \eth$, then $\exists \xi, \in [\zeta]_{\Re_1}, \xi \in \Gamma, \{\xi\} \notin \eth$. Thus $[\xi]_{\Re_2} \cap \Gamma \notin \eth$ and $\cdots$ and $[\xi]_{\Re_\Lambda} \cap \Gamma \notin \eth$. So $\xi \in (\Re_2)^*(\Gamma)$ and $\cdots$ and $\xi \in (\Re_\Lambda)^*(\Gamma)$. Hence, $[\zeta]_{\Re_1} \cap (\Re_1)^*(\Gamma) \cap (\Re_2)^*(\Gamma) \cap \cdots \cap (\Re_\Lambda)^*(\Gamma) = [\zeta]_{\Re_1} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \notin \eth$. By the same way, $[\zeta]_{\Re_2} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \notin \eth$ and $\cdots$ and $[\zeta]_{\Re_\Lambda} \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma) \notin \eth$. Therefore $\zeta \in \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma)]$.

(6) Directly, by putting $\Gamma = \Gamma^c$ in (7) and using (2).

(7) From (3), we get

$$\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_*(\Gamma \cap \Upsilon) = \bigcup_{\lambda=1}^{\Lambda} [(\Re_\lambda)_*(\Gamma \cap \Upsilon)] = \bigcup_{\lambda=1}^{\Lambda} [(\Re_\lambda)_*(\Gamma) \cap (\Re_\lambda)_*(\Upsilon)].$$

(8) From (4), we have

$$\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma \cup \Upsilon) = \bigcup_{\lambda=1}^{\Lambda} [(\Re_\lambda)^*(\Gamma \cup \Upsilon)] = \bigcup_{\lambda=1}^{\Lambda} [(\Re_\lambda)^*(\Gamma) \cup (\Re_\lambda)^*(\Upsilon)].$$

(9) Direct from (5).

(10) Direct from (6). □

Definition 3.7. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\mathcal{R}$ be a family of equivalence relations, $\eth$ be an ideal on Π, $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\underline{\sum_{\lambda=1}^{\Lambda}} \Re_\lambda(\Gamma) = \{\zeta \in \Gamma : \bigcup([\zeta]_{\Re_\lambda} \cap \Gamma^c \in \eth), \lambda \leq \Lambda\}, \tag{3.5}$$

$$\overline{\sum_{\lambda=1}^{\Lambda}} \Re_\lambda(\Gamma) = \Gamma \cup \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^*(\Gamma). \tag{3.6}$$

Proposition 3.8. *Let* $(\Pi, \mathcal{R}, \eth)$ *be an ideal knowledge base,* $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \cdots, \Re_\Lambda \in \mathcal{R}$. *Then the following properties hold:*

(1) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Pi) = \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Pi) = \Pi$ *and* $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\phi) = \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\phi) = \phi$,

(2) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \subseteq \Gamma \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)$,

(3) *if* $\Gamma \subseteq \Upsilon$, *then* $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Upsilon)$ *and* $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Upsilon)$.

(4) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma^c) = [\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)]^c$ *and* $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma^c) = [\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)]^c$,

(5) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) = \bigcup_{\lambda=1}^{\Lambda} \underline{\Re_\lambda}(\Gamma)$ *and* $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) = \bigcap_{\lambda=1}^{\Lambda} \overline{\Re_\lambda}(\Gamma)$,

(6) $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}[\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)] = \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)$ *and* $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}[\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)] = \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)$,

(7) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}[\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)] \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)$ *and* $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}[\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)]$,

(8) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma \cap \Upsilon) = \bigcup_{\lambda=1}^{\Lambda} [\underline{\Re_\lambda}(\Gamma) \cap \underline{\Re_\lambda}(\Upsilon)]$ *and* $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma \cup \Upsilon) = \bigcap_{\lambda=1}^{\Lambda} [\overline{\Re_\lambda}(\Gamma) \cup \overline{\Re_\lambda}(\Upsilon)]$,

(9) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma \cap \Upsilon) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \cap \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Upsilon)$ *and* $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \cup \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Upsilon) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma \cup \Upsilon)$,

(10) $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma \cap \Upsilon) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \cap \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Upsilon)$ *and* $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \cup \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Upsilon) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma \cup \Upsilon)$.

Proof. The proof is similar to that of Proposition 3.6. □

Definition 3.9. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\mathcal{R}$ be a family of equivalence relations, $\eth$ be an ideal on Π, $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *boundary region* $BND_{\mathcal{R}}(\Gamma)$ and the *accuracy measure* $\mu_{\mathcal{R}}(\Gamma)$ are defined respectively by:

$$BND_{\mathcal{R}}(\Gamma) = \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) - \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma), \ \mu_{\mathcal{R}}(\Gamma) = \frac{|\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)|}{|\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)|}.$$

where $\Gamma \neq \phi, |A|$ denotes the cardinality of set A.

Proposition 3.10. *Let* $(\Pi, \mathcal{R}, \eth)$ *be an ideal knowledge base,* $\mathcal{R} = \{\Re_1, \Re_2, \cdots, \Re_\Lambda\}$ *be a family of equivalence relations,* $\check{\mathcal{R}} \subseteq \mathcal{R}$, $\eth$ *be an ideal on* Π, $\Gamma \subseteq \Pi$ $\forall \Re_\lambda \in \mathcal{R}(\lambda \leq \Lambda)$. *Then* $\mu_{\mathcal{R}}(\Gamma) \geq \mu_{\check{\mathcal{R}}}(\Gamma) \geq \mu_{\Re_\lambda}(\Gamma)$.

Proof. From Definition 3.7, we have

$$\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \supseteq \underline{\sum_{\Re_\lambda \in \check{\mathcal{R}}} \Re_\lambda}(\Gamma) \supseteq \underline{\Re_\lambda}(\Gamma) \ and \ \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \subseteq \overline{\sum_{\Re_\lambda \in \check{\mathcal{R}}} \Re_\lambda}(\Gamma) \subseteq \overline{\Re_\lambda}(\Gamma).$$

Then we get

$$\mu_{\mathcal{R}}(\Gamma) = \frac{|\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)|}{|\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)|} \geq \mu_{\check{\mathcal{R}}}(\Gamma) = \frac{|\underline{\sum_{\Re_\lambda \in \check{\mathcal{R}}} \Re_\lambda}(\Gamma)|}{|\overline{\sum_{\Re_\lambda \in \check{\mathcal{R}}} \Re_\lambda}(\Gamma)|} \geq \mu_{\Re_\lambda}(\Gamma) = \frac{|\underline{\Re_\lambda}(\Gamma)|}{|\overline{\Re_\lambda}(\Gamma)|}$$

Thus $\mu_{\mathcal{R}}(\Gamma) \geq \mu_{\check{\mathcal{R}}}(\Gamma) \geq \mu_{\Re_\lambda}(\Gamma)$ hold for arbitrary $\check{\mathcal{R}} \subseteq \mathcal{R}$ and every $\Re_\lambda \in \check{\mathcal{R}}$. □

Remark 3.11. Proposition 3.10 shows that the accuracy measure of a set is improved whenever the number of granulations is increased as explained in this example.

Example 3.12. Continue from Example 3.4. Suppose $\Gamma = \{\zeta_2, \zeta_4, \zeta_6\}$. By computing, we have

$$\underline{\Re_1}(\Gamma) = \{\zeta_4\}, \overline{\Re_1}(\Gamma) = \{\zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}.$$

Also, $\underline{\Re_2}(\Gamma) = \{\zeta_6\}, \overline{\Re_2}(\Gamma) = \{\zeta_1, \zeta_2, \zeta_4, \zeta_6\}$. Then

$$\mu_{\mathcal{R}}(\Gamma) = \frac{|\underline{\Re_1 + \Re_2}(\Gamma)|}{|\overline{\Re_1 + \Re_2}(\Gamma)|} = \frac{2}{3}, \mu_{\Re_1}(\Gamma) = \frac{|\underline{\Re_1}(\Gamma)|}{|\overline{\Re_1}(\Gamma)|} = \frac{1}{5}, \mu_{\Re_2}(\Gamma) = \frac{|\underline{\Re_2}(\Gamma)|}{|\overline{\Re_2}(\Gamma)|} = \frac{1}{4}.$$

Obviously, it follows from above computations that:

$$\mu_{\mathcal{R}}(\Gamma) > \mu_{\Re_1}(\Gamma) \textit{ and } \mu_{\mathcal{R}}(\Gamma) > \mu_{\Re_2}(\Gamma).$$

Proposition 3.13. *Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\mathcal{R}$ be a family of equivalence relations $\mathcal{R} = \{\Re_1, \Re_2, \cdots, \Re_\Lambda\}$ with $\Re_1 \subseteq \Re_2 \subseteq \cdots \subseteq \Re_\Lambda$. Then*

(1) $\underline{\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}}(\Gamma) = \underline{\Re_1}(\Gamma)$,
(2) $\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) = \overline{\Re_1}(\Gamma)$.

Proof. (1) From Definition 3.7, we have $\underline{\Re_1}(\Gamma) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma)$. Let $\zeta \notin \underline{\Re_1}(\Gamma)$. Then $[\zeta]_{\Re_1} \cap \Gamma^c \notin \eth$ and $[\zeta]_{\Re_\lambda} \cap \Gamma^c \notin \eth, i \in \{2, 3, \cdots, \Lambda\}$,
where $\Re_1 \subseteq \Re_2 \subseteq \cdots \subseteq \Re_\Lambda$. This means that $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) \subseteq \underline{\Re_1}(\Gamma)$. Thus $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}(\Gamma) = \underline{\Re_1}(\Gamma)$.
(2) Similarly. □

Remark 3.14. It should be noted that Definition 3.7 is more accurate than Definitions 2.1, 2.23. In particular,

(1) if $\eth = \{\phi\}$, then the recent definition is equivalent to Definition 2.23.
(2) if $\eth = \{\phi\}$ and $\Lambda = 1$, then the recent definition is equivalent to Definition 2.1.

3.2. **Method (2).**

Definition 3.15. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\mathcal{R}$ be a family of binary relations $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma) = \{\zeta \in \Pi : \bigcup(\prec \zeta \succ \Re_\lambda \subseteq \Gamma), \lambda \leq \Lambda\}, \tag{3.7}$$

$$\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma) = \{\zeta \in \Pi : \bigcap(\prec \zeta \succ \Re_\lambda \cap \Gamma \neq \phi), \lambda \leq \Lambda\}. \tag{3.8}$$

Proposition 3.16. *Let $(\Pi, \mathcal{R})$ be a knowledge base, $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, ..\Re_\Lambda \in \mathcal{R}$. Then, the lower and upper approximations of Γ defined in Eq. (3.7) and Eq. (3.8) satisfy the properties (1)-(10) in Proposition 3.6.*

Proof. Direct as shown in Proposition 3.6. □

Remark 3.17. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\Gamma, \Upsilon \subseteq \Pi$ and $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$. Then Examples 3.18 and 3.19 show that in general:

(1) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Pi) \neq \Pi$ and $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\phi) \neq \phi$,

(2) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma) \nsubseteq \Gamma$ and $\Gamma \nsubseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma)$,

(3) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$ and $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma) \subseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$,

(4) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma \cap \Upsilon) \neq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma) \cap \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Upsilon)$ and $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma \cup \Upsilon) \neq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma) \cup \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Upsilon)$,

(5) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma) \nsubseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma)]$ and $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma) \nsupseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}(\Gamma)]$,

(6) $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma) \nsubseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma)]$ and $\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma) \nsupseteq \left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)^{*1}[\left(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\right)_{*1}(\Gamma)]$.

Example 3.18. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\Re_1, \Re_2 \in \mathcal{R}$. Consider

$$\Pi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\},\ \Re_1 = \{(\zeta_1, \zeta_1), (\zeta_1, \zeta_4), (\zeta_4, \zeta_4)\},$$

$$\Re_2 = \{(\zeta_1, \zeta_1), (\zeta_1, \zeta_2), (\zeta_2, \zeta_2), (\zeta_3, \zeta_4), (\zeta_4, \zeta_4)\}.$$

Then $< \zeta_1 > \Re_1 = \{\zeta_1, \zeta_4\}, < \zeta_2 > \Re_1 = \phi, < \zeta_3 > \Re_1 = \phi, < \zeta_4 > \Re_1 = \{\zeta_4\}$. Also, $< \zeta_1 > \Re_2 = \{\zeta_1, \zeta_2\}, < \zeta_2 > \Re_2 = \{\zeta_2\}, < \zeta_3 > \Re_2 = \phi, < \zeta_4 > \Re_2 = \{\zeta_4\}$.

(1) $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*1}(\phi) = \{\zeta \in \Pi : \prec \zeta \succ \Re_1 \subseteq \phi \text{ } or \text{ } \prec \zeta \succ \Re_2 \subseteq \phi\} = \{\zeta_2, \zeta_3\} \cup \{\zeta_3\} = \{\zeta_2, \zeta_3\} \neq \phi$. Also, $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*1}(\Pi) = \{\zeta_1, \zeta_4\} \cap \{\zeta_1, \zeta_2, \zeta_4\} = \{\zeta_1, \zeta_4\} \neq \Pi$.

(2) If $\Gamma = \{\zeta_1, \zeta_2\}$, then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*1}(\Gamma) = \{\zeta_2, \zeta_3\} \cup \{\zeta_1, \zeta_2, \zeta_3\} = \{\zeta_1, \zeta_2, \zeta_3\} \nsubseteq \Gamma$. Also, if $\Gamma = \{\zeta_2, \zeta_3\}$, then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*1}(\Gamma) = \phi \cap \{\zeta_1, \zeta_2\} = \phi \nsupseteq \Gamma$.

(3) Let $\Gamma = \{\zeta_1\}, \Upsilon = \{\zeta_2\}$. Then we have

$$\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*1}(\Gamma) = \{\zeta_2, \zeta_3\} \cup \{\zeta_3\} = \{\zeta_2, \zeta_3\}.$$

Also, we get

$$\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*1}(\Upsilon) = \{\zeta_2, \zeta_3\} \cup \{\zeta_2, \zeta_3\} = \{\zeta_2, \zeta_3\}.$$

Thus $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*1}(\Gamma) \subseteq \left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*1}(\Upsilon)$ but $\Gamma \nsubseteq \Upsilon$.

(4) Let $\Gamma=\{\zeta_2\},\Upsilon=\{\zeta_1\}$. Then $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)=\phi\cap\{\zeta_1,\zeta_2\}=\phi$. Also, $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Upsilon)=\{\zeta_1\}\cap\{\zeta_1\}=\{\zeta_1\}$. Thus $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)\subseteq\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Upsilon)$ but $\Gamma\nsubseteq\Upsilon$.

(5) If $\Gamma=\{\zeta_1,\zeta_2\}$ and $\Upsilon=\{\zeta_1,\zeta_4\}$, then $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma\cap\Upsilon)=\{\zeta_2,\zeta_3\}$ but $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)\cap\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Upsilon)=\{\zeta_1,\zeta_2,\zeta_3\}$. Also, if $\Gamma=\{\zeta_3,\zeta_4\}$ and $\Upsilon=\{\zeta_2,\zeta_3\}$, then we have $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)\cup\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Upsilon)=\{\zeta_4\}$ but $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma\cup\Upsilon)=\{\zeta_1,\zeta_4\}$.

(6) From (2), $\Gamma=\{\zeta_2,\zeta_3\}$ and $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)=\phi$. But we get

$$\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)]=\{\zeta_2,\zeta_3\}.$$

Then $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)\nsupseteq\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)]$.

(7) From (5), $\Gamma=\{\zeta_1\}$ and $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)=\{\zeta_2,\zeta_3\}$. But we get

$$\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)]=\phi.$$

Then we have

$$\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)\nsubseteq\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)].$$

Example 3.19. Let $(\Pi,\mathcal{R})$ be a knowledge base, $\Re_1,\Re_2\in\mathcal{R}$, where $\Pi=\{\zeta_1,\zeta_2,\zeta_3\}$, $\Re_1=\{(\zeta_1,\zeta_1),(\zeta_1,\zeta_2),(\zeta_2,\zeta_2),(\zeta_3,\zeta_2),(\zeta_3,\zeta_3)\}$, $\Re_2=\{(\zeta_1,\zeta_1),(\zeta_1,\zeta_2),(\zeta_2,\zeta_2),(\zeta_2,\zeta_3),(\zeta_3,\zeta_3)\}$. Then we have

$$<\zeta_1>\Re_1=\{\zeta_1,\zeta_2\},<\zeta_2>\Re_1=\{\zeta_2\},<\zeta_3>\Re_1=\{\zeta_2,\zeta_3\}.$$

Also, $<\zeta_1>\Re_2=\{\zeta_1,\zeta_2\},<\zeta_2>\Re_2=\{\zeta_2\},<\zeta_3>\Re_2=\{\zeta_3\}$.

(1) Consider $\Gamma=\{\zeta_1\}$. Then $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)=\{\zeta_1\}\cap\{\zeta_1\}=\{\zeta_1\}$. But $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)]=\phi$. Thus we get

$$\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)\nsubseteq\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}(\Gamma)].$$

(2) Consider $\Gamma=\{\zeta_2\}$. Then $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)=\{\zeta_2\}$. But we get

$$\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)]=\{\zeta_1,\zeta_2,,\zeta_3\}\cap\{\zeta_1,\zeta_2\}=\{\zeta_1,\zeta_2\}.$$

Thus $\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)\nsupseteq\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)^{*1}[\Big(\sum_{\lambda=1}^{2}\Re_\lambda\Big)_{*1}(\Gamma)]$.

Definition 3.20. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\mathcal{R}$ be a family of binary relations $\Re_1, \Re_2, ...\Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_{\,1} (\Gamma) = \{\zeta \in \Gamma : \bigcup (\prec \zeta \succ \Re_\lambda \subseteq \Gamma), \lambda \leq \Lambda\}, \tag{3.9}$$

$$\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^{\,1} (\Gamma) = \Gamma \cup \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*1} (\Gamma). \tag{3.10}$$

Proposition 3.21. *Let $(\Pi, \mathcal{R})$ be a knowledge base, $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, ..\Re_\Lambda \in \mathcal{R}$. Then, the lower and upper approximations of Γ defined in Eq. (3.9) and Eq. (3.10) satisfy the properties* (1)-(10) *in Proposition 3.8.*

Proof. Similar to that of Proposition 3.6. □

Definition 3.22. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\mathcal{R}$ be a family of binary relations $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *boundary region* $BND^1_{\mathcal{R}}(\Gamma)$ and the *accuracy measure* $\mu^1_{\mathcal{R}}(\Gamma)$ are defined respectively by:

$$BND^1_{\mathcal{R}}(\Gamma) = \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^{\,1} (\Gamma) - \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_{\,1} (\Gamma), \;\; \mu^1_{\mathcal{R}}(\Gamma) = \frac{|\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_{\,1}(\Gamma)|}{|\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^{\,1}(\Gamma)|}, \;\; \Gamma \neq \phi.$$

Proposition 3.23. *Let $(\Pi, \mathcal{R})$ be a knowledge base, $\mathcal{R} = \{\Re_1, \Re_2, \cdots, \Re_\Lambda\}$ be a family of binary relations, $\check{\mathcal{R}} \subseteq \mathcal{R}$, $\Gamma \subseteq \Pi$ $\forall \Re_\lambda \in \mathcal{R}(\lambda \leq \Lambda)$. Then*

$$\mu^1_{\mathcal{R}}(\Gamma) \geq {\mu^1}_{\check{\mathcal{R}}}(\Gamma) \geq {\mu^1}_{R\lambda}(\Gamma).$$

Proof. As shown in Proposition 3.10. □

Remark 3.24. Proposition 3.23 ensures that Definition 3.20 is more accurate than Definition 2.7. In particular, if $\Lambda = 1$, then both of definitions are equivalent.

3.3. Method (3).

Definition 3.25. Let $(\Pi, \mathcal{R}, \eth)$ be a knowledge base, $\mathcal{R}$ be a family of binary relations, $\eth$ be an ideal on Π, $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*2} (\Gamma) = \{\zeta \in \Pi : \bigcup (\prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \eth), \lambda \leq \Lambda\}, \tag{3.11}$$

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*2} (\Gamma) = \{\zeta \in \Pi : \bigcap (\prec \zeta \succ \Re_\lambda \cap \Gamma \notin \eth), \lambda \leq \Lambda\}. \tag{3.12}$$

Proposition 3.26. *Let $(\Pi, \mathcal{R}, \eth)$ be a knowledge base, $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \cdots, \Re_\Lambda \in \mathcal{R}$. Then, the lower and upper approximations of Γ defined in Eq. (3.11) and Eq. (3.12) satisfy the properties* (1)-(10) *in Proposition 3.6.*

Proof. It is following the technique used in Proposition 3.6. □

Remark 3.27. Any one can give different ideals to extend Example 3.18 and Example 3.19 to show that in general:

(1) $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Pi)\neq\Pi$ and $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\phi)\neq\phi$,

(2) $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)\nsubseteq\Gamma$ and $\Gamma\nsubseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)$,

(3) $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)\subseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Upsilon)\nRightarrow\Gamma\subseteq\Upsilon$ and $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)\subseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Upsilon)\nRightarrow\Gamma\subseteq\Upsilon$,

(4) $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma\cap\Upsilon)\neq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)\cap\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Upsilon)$ and $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma\cup\Upsilon)\neq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)\cup\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Upsilon)$,

(5) $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)\nsubseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}[\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)]$ and $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)\nsupseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}[\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)]$,

(6) $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)\nsubseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}[\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)]$ and $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)\nsupseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}[\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)]$,

(7) $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)\nsubseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}[\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma)]$ and $\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}[\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)]\nsubseteq\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)_{*2}(\Gamma)$.

Definition 3.28. Let $(\Pi,\mathcal{R},\eth)$ be a knowledge base, $\mathcal{R}$ be a family of binary relations $\Re_1,\Re_2,\cdots,\Re_\Lambda\in\mathcal{R}$ and $\Gamma\subseteq\Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1,\Re_2,\cdots,\Re_\Lambda$ are defined respectively by:

$$\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_{2}(\Gamma)=\{\zeta\in\Gamma:\bigcup(\prec\zeta\succ\Re_\lambda\cap\Gamma^c\in\eth),\lambda\leq\Lambda\},\tag{3.13}$$

$$\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^{2}(\Gamma)=\Gamma\cup\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*2}(\Gamma).\tag{3.14}$$

Proposition 3.29. *Let* $(\Pi,\mathcal{R},\eth)$ *be a knowledge base,* $\Gamma,\Upsilon\subseteq\Pi$, $\Re_2,\cdots,\Re_\Lambda\in\mathcal{R}$. *Then the lower and upper approximations of* Γ *defined in Eq. (3.13) and Eq. (3.14) satisfy the properties* (1)-(10) *in Proposition 3.8.*

Proof. The results are similar to that of Proposition 3.6. □

Definition 3.30. Let $(\Pi,\mathcal{R},\eth)$ be an ideal knowledge base, $\Re_1,\Re_2,\cdots,\Re_\Lambda\in\mathcal{R}$ and $\Gamma\subseteq\Pi$. Then the *boundary region* $BND^2_{\mathcal{R}}(\Gamma)$ and the *accuracy measure* $\mu^2_{\mathcal{R}}(\Gamma)$ are defined respectively by:

$$BND^2_{\mathcal{R}}(\Gamma)=\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^{2}(\Gamma)-\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_{2}(\Gamma),\ \mu^2_{\mathcal{R}}(\Gamma)=\frac{|\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_{2}(\Gamma)|}{|\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^{2}(\Gamma)|}\text{ for }\Gamma\neq\phi.$$

Proposition 3.31. *Let* $(\Pi, \mathcal{R})$ *be a knowledge base,* $\mathcal{R} = \{\Re_1, \Re_2, \cdots, \Re_\Lambda\}$ *be a family of binary relations,* $\check{\mathcal{R}} \subseteq \mathcal{R}$, $\Gamma \subseteq \Pi$ $\forall \Re_\lambda \in \mathcal{R}(\lambda \leq \Lambda)$. *Then*

$$\mu^2_{\mathcal{R}}(\Gamma) \geq \mu^2{}_{\check{\mathcal{R}}}(\Gamma) \geq \mu^2{}_{R\lambda}(\Gamma).$$

Proof. Similar to that of Proposition 3.10. □

Remark 3.32. Proposition 3.31 ensures that Definition 3.28 is more accurate than Definition 2.18. In particular, if $\Lambda = 1$, then both definitions are equivalent.

3.4. **Method (4).**

Definition 3.33. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\mathcal{R}$ be a family of binary relations $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma) = \{\zeta \in \Pi : \bigcup (\Re_\lambda \prec \zeta \succ \Re_\lambda \subseteq \Gamma), \lambda \leq \Lambda\}, \tag{3.15}$$

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma) = \{\zeta \in \Pi : \bigcap (\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma \neq \phi), \lambda \leq \Lambda\}. \tag{3.16}$$

Proposition 3.34. *Let* $(\Pi, \mathcal{R})$ *be a knowledge base,* $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \cdots, \Re_\Lambda \in \mathcal{R}$. *Then the lower and upper approximations of* Γ *defined in Eq. (3.15) and Eq. (3.16) satisfy the properties* (1)-(10) *in Proposition 3.6.*

Proof. The results are similar to that of Proposition 3.6. □

Remark 3.35. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$. Then Examples 3.36 and 3.37 show that in general:

(1) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Pi) \neq \Pi$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\phi) \neq \phi$,

(2) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma) \nsubseteq \Gamma$ and $\Gamma \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma)$,

(3) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma) \subseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma) \subseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$,

(4) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma) \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma)]$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma) \nsupseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma)]$,

(5) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma) \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma)]$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma) \nsupseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma),.$

(6) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma \cap \Upsilon) \neq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Gamma) \cap \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*3}(\Upsilon)$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma \cup \Upsilon) \neq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Gamma) \cup \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*3}(\Upsilon)$.

Example 3.36. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\Re_1, \Re_2 \in \mathcal{R}$. Consider

$$\Pi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\},\ \Re_1 = \{(\zeta_1, \zeta_1), (\zeta_1, \zeta_4), (\zeta_4, \zeta_4)\},$$
$$\Re_2 = \{(\zeta_1, \zeta_2), (\zeta_2, \zeta_2), (\zeta_2, \zeta_1), (\zeta_3, \zeta_3)\}.$$

Then we have

$\Re_1 < \zeta_1 > \Re_1 = \{\zeta_1\}, \Re_1 < \zeta_2 > \Re_1 = \phi, \Re_1 < \zeta_3 > \Re_1 = \phi, \Re_1 < \zeta_4 > \Re_1 = \{\zeta_4\}.$

Also, we get

$$\Re_2 < \zeta_1 > \Re_2 = \{\zeta_1, \zeta_2\},\ \Re_2 < \zeta_2 > \Re_2 = \{\zeta_2\},$$
$$\Re_2 < \zeta_3 > \Re_2 = \{\zeta_3\},\ \Re_2 < \zeta_4 > \Re_2 = \phi.$$

(1) $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\phi) = \{\zeta \in \Pi : \Re_1 \prec \zeta \succ \Re_1 \subseteq \phi\ or\ \Re_1 \prec \zeta \succ \Re_1 \subseteq \phi\}$
$= \{\zeta_2, \zeta_3\} \cup \{\zeta_4\}$
$= \{\zeta_2, \zeta_3, \zeta_4\}$
$\neq \phi.$

Also, $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Pi) = \{\zeta_1, \zeta_4\} \cap \{\zeta_1, \zeta_2, \zeta_3\} = \{\zeta_1\} \neq \Pi.$

(2) If $\Gamma = \{\zeta_1, \zeta_2\}$, then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Gamma) = \{\zeta_1, \zeta_2, \zeta_3\} \cup \{\zeta_2, \zeta_4\} = \Pi \nsubseteq \Gamma$. Also, if $\Gamma = \{\zeta_2, \zeta_3\}$, then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Gamma) = \phi \cap \{\zeta_1, \zeta_2, \zeta_3\} = \phi \nsupseteq \Gamma$.

(3) Let $\Gamma = \{\zeta_1\}, \Upsilon = \{\zeta_2\}$. Then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Gamma) = \{\zeta_1, \zeta_2, \zeta_3\} \cup \{\zeta_4\} = \Pi$. Also, $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Upsilon) = \{\zeta_2, \zeta_3\} \cup \{\zeta_2, \zeta_4\} = \{\zeta_2, \zeta_3, \zeta_4\}$. Thus $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Upsilon) \subseteq \left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Gamma)$ but $\Upsilon \nsubseteq \Gamma$.

(4) Let $\Gamma = \{\zeta_2\}, \Upsilon = \{\zeta_1\}$. Then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Gamma) = \phi \cap \{\zeta_1, \zeta_2\} = \phi$. Also, $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Upsilon) = \{\zeta_1\} \cap \{\zeta_1\} = \{\zeta_1\}$. Thus $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Gamma) \subseteq \left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Upsilon)$ but $\Gamma \nsubseteq \Upsilon$.

(5) From (2), $\Gamma = \{\zeta_2, \zeta_3\}$ and $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Gamma) = \phi$. But we have

$$\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}\left[\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Gamma)\right] = \{\zeta_2, \zeta_3, \zeta_4\}.$$

Then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Gamma) \nsupseteq \left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}\left[\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}(\Gamma)\right]$.

(6) From (5), $\Gamma = \phi$ and $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Gamma) = \{\zeta_2, \zeta_3, \zeta_4\}$. But we get

$$\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}\left[\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Gamma)\right] = \phi.$$

Then $\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Gamma) \nsubseteq \left(\sum_{\lambda=1}^{2} \Re_\lambda\right)^{*3}\left[\left(\sum_{\lambda=1}^{2} \Re_\lambda\right)_{*3}(\Gamma)\right]$.

Example 3.37. Let $(\Pi, \mathcal{R})$ be a knowledge base, $\Re_1, \Re_2 \in \mathcal{R}$. Consider $\Pi = \{\zeta_1, \zeta_2, \zeta_3\}$. Suppose $\Re_1 = \{(\zeta_1, \zeta_1), (\zeta_1, \zeta_2), (\zeta_2, \zeta_1), (\zeta_2, \zeta_2), (\zeta_3, \zeta_3)\}$, $\Re_2 = \{(\zeta_1, \zeta_1), (\zeta_2, \zeta_2), (\zeta_2, \zeta_3), (\zeta_3, \zeta_2), (\zeta_3, \zeta_3)\}$. Then $\Re_1 < \zeta_1 > \Re_1 = \{\zeta_1, \zeta_2\}, \Re_1 < \zeta_2 > \Re_1 =$

$\{\zeta_1,\zeta_2\}, \Re_1 < \zeta_3 > \Re_1 = \{\zeta_3\}$. Also, $\Re_2 < \zeta_1 > \Re_2 = \{\zeta_1\}, \Re_2 < \zeta_2 > \Re = \{\zeta_2,\zeta_3\}, \Re_2 < \zeta_3 > \Re_2 = \{\zeta_2,\zeta_3\}$.

(1) Consider $\Gamma = \{\zeta_2\}$. Then $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*3}(\Gamma) = \{\zeta_1,\zeta_2\}\cap\{\zeta_2,\zeta_3\} = \{\zeta_2\}$. But $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*3}[\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*3}(\Gamma)] = \phi\cup\phi = \phi$. Thus we have

$$\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*3}(\Gamma) \nsubseteq \left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*3}[\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*3}(\Gamma)].$$

(2) Consider $\Gamma = \{\zeta_1,\zeta_3\}$. Then. $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*3}(\Gamma) = \{\zeta_1,\zeta_3\}$. But $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*3}[\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*3}(\Gamma)] = \{\zeta_1,\zeta_2,,\zeta_3\}\cap\{\zeta_1,\zeta_2,\zeta_3\} = \{\zeta_1,\zeta_2,\zeta_3\} = \Pi$. Thus $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*3}(\Gamma) \nsupseteq \left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*3}[\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*3}(\Gamma)]$.

(3) If $\Gamma = \{\zeta_1,\zeta_2\}$ and $\Upsilon = \{\zeta_2,\zeta_3\}$, then $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*1}(\Gamma\cap\Upsilon) = \phi$ but $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*1}(\Gamma)\cap\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)_{*1}(\Upsilon) = \{\zeta_1,\zeta_2\}\cap\{\zeta_2,\zeta_3\} = \{\zeta_2\}$. Also, if $\Gamma = \{\zeta_1\}$ and $\Upsilon = \{\zeta_3\}$, then $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*1}(\Gamma)\cup\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*1}(\Upsilon) = \{\zeta_1\}\cup\{\zeta_3\} = \{\zeta_1,\zeta_3\}$ but $\left(\sum_{\lambda=1}^{2}\Re_\lambda\right)^{*1}(\Gamma\cup\Upsilon) = \Pi$.

Definition 3.38. Let $(\Pi,\mathcal{R},\eth)$ be an ideal knowledge base, $\mathcal{R}$ be a family of binary relations $\Re_1,\Re_2,\cdots,\Re_\Lambda\in\mathcal{R}$ and $\Gamma\subseteq\Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1,\Re_2,\cdots,\Re_\Lambda$ are defined respectively by:

$$\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_{3}(\Gamma) = \{\zeta\in\Gamma : \bigcup(\Re_\lambda \prec \zeta \succ \Re_\lambda\subseteq\Gamma), \lambda\leq\Lambda\}, \tag{3.17}$$

$$\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^{3}(\Gamma) = \Gamma\cup\left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*3}(\Gamma). \tag{3.18}$$

Proposition 3.39. *Let $(\Pi,\mathcal{R},\eth)$ be an ideal knowledge base, $\Gamma,\Upsilon\subseteq\Pi$, $\Re_1,\cdots,\Re_\Lambda\in\mathcal{R}$. Then the lower and upper approximations of Γ defined in Eq. (3.17) and Eq. (3.18) satisfy the properties (1)-(10) in Proposition 3.8.*

Proof. The results are similar to that of Proposition 3.6. □

Definition 3.40. Let $(\Pi,\mathcal{R})$ be a knowledge base, $\mathcal{R}$ be a family of equivalence relations $\Re_1,\Re_3,\cdots,\Re_\Lambda\in\mathcal{R}$ and $\Gamma\subseteq\Pi$. Then the *boundary region* $BND^3_{\mathcal{R}}(\Gamma)$ and the *accuracy measure* $\mu^3_{\mathcal{R}}(\Gamma)$ are defined respectively by:

$$BND^3_{\mathcal{R}}(\Gamma) = \overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^{3}(\Gamma) - \underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_{3}(\Gamma),\ \mu^3_{\mathcal{R}}(\Gamma) = \frac{|\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_{3}(\Gamma)|}{|\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^{3}(\Gamma)|},\ \Gamma\neq\phi.$$

Proposition 3.41. *Let $(\Pi,\mathcal{R})$ be a knowledge base, $\mathcal{R} = \{\Re_1,\Re_3,\cdots,\Re_\Lambda\}$ be a family of binary relations, $\check{\mathcal{R}}\subseteq\mathcal{R}$, $\Gamma\subseteq\Pi\ \forall\Re_\lambda\in\mathcal{R}(\lambda\leq\Lambda)$. Then*

$$\mu^3_{\mathcal{R}}(\Gamma)\geq\mu^3{}_{\check{\mathcal{R}}}(\Gamma)\geq\mu^3{}_{R\lambda}(\Gamma).$$

Proof. Similar to that of Proposition 3.10. □

3.5. **Method (5).**

Definition 3.42. Let $(\Pi, \mathcal{R}, \eth)$ be a knowledge base, $\mathcal{R}$ be a family of binary relations, $\eth$ be an ideal on Π, $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma) = \{\zeta \in \Pi : \bigcup(\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \eth), \lambda \leq \Lambda\}, \tag{3.19}$$

$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma) = \{\zeta \in \Pi : \bigcap(\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma \notin \eth), \lambda \leq \Lambda\}. \tag{3.20}$$

Proposition 3.43. *Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\Gamma, \Upsilon \subseteq \Pi$, $\Re_1, \cdots, \Re_\Lambda \in \mathcal{R}$. Then the lower and upper approximations of Γ defined in Eq. (3.19) and Eq. (3.20) satisfy the properties* (1)-(10) *in Proposition 3.8.*

Proof. The results are similar to that of Proposition 3.6. □

Remark 3.44. In a similar way, we can give different ideals to extend Examples 3.36 and 3.37 to show that in general:

(1) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Pi) \neq \Pi$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\phi) \neq \phi$,

(2) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma) \nsubseteq \Gamma$ and $\Gamma \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma)$,

(3) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma) \subseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma) \subseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Upsilon) \nRightarrow \Gamma \subseteq \Upsilon$,

(4) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma) \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma)]$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma) \nsupseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma)]$,

(5) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma) \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma)]$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma) \nsupseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma)]$,

(6) $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma) \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma)]$ and $\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}[\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma)] \nsubseteq \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4}(\Gamma)$.

Definition 3.45. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\mathcal{R}$ be a family of binary relations, $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *lower* and *upper approximations* of Γ related to $\Re_1, \Re_2, \cdots, \Re_\Lambda$ are defined respectively by:

$$\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_4(\Gamma) = \{\zeta \in \Gamma : \bigcup(\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \eth), \lambda \leq \Lambda\}, \tag{3.21}$$

$$\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^4(\Gamma) = \Gamma \cup \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}(\Gamma). \tag{3.22}$$

Proposition 3.46. *Let* $(\Pi, \mathcal{R}, \eth)$ *be an ideal knowledge base,* $\Gamma, \Upsilon \subseteq \Pi$, $\Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$. *Then, the lower and upper approximations of* Γ *defined in Eq. (3.21) and Eq. (3.22) satisfy the properties* (1)-(10) *in Proposition 3.8.*

Proof. The results are similar to that of Proposition 3.6. □

Definition 3.47. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\Re_1, \Re_4, \cdots, \Re_\Lambda \in \mathcal{R}$ and $\Gamma \subseteq \Pi$. Then the *boundary region* $BND^4_{\mathcal{R}}(\Gamma)$ and the *accuracy measure* $\mu^4_{\mathcal{R}}(\Gamma)$ are defined respectively by:

$$BND^4_{\mathcal{R}}(\Gamma) = \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^4(\Gamma) - \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_4(\Gamma),\ \mu^4_{\mathcal{R}}(\Gamma) = \frac{|\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_4(\Gamma)|}{|\overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^4(\Gamma)|},\ \Gamma \neq \phi.$$

Proposition 3.48. *Let* $(\Pi, \mathcal{R}, \eth)$ *be an ideal knowledge base,* $\mathcal{R} = \{\Re_1, \Re_4, \cdots, \Re_\Lambda\}$ *be a family of binary relations,* $\check{\mathcal{R}} \subseteq \mathcal{R}$, $\Gamma \subseteq \Pi$ $\forall \Re_\lambda \in \mathcal{R}(\lambda \leq \Lambda)$. *Then*

$$\mu^4_{\mathcal{R}}(\Gamma) \geq \mu^4{}_{\check{\mathcal{R}}}(\Gamma) \geq \mu^4{}_{R\lambda}(\Gamma).$$

Proof. Similar to that of Proposition 3.10. □

Remark 3.49. Proposition 3.48 ensures that Definition 3.45 produces better accuracy values. This definition is different from Definition 2.21. In particular, if $\Lambda = 1$, then the recent definition is identical with Definition 2.21.

4. Comparison of the presented methods

Theorem 4.1. *Let* $(\Pi, \mathcal{R}, \eth)$ *be an ideal knowledge base,* $\Re_1, \Re_4, \cdots, \Re_\Lambda \in \mathcal{R}$ *and* $\Gamma \subseteq \Pi$. *Then*

(1) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_1(\Gamma) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_2(\Gamma) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_4(\Gamma) \subseteq \Gamma \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^4(\Gamma) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^2(\Gamma) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^1(\Gamma)$,

(2) $\underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_1(\Gamma) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_3(\Gamma) \subseteq \underline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}_4(\Gamma) \subseteq \Gamma \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^4(\Gamma) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^3(\Gamma) \subseteq \overline{\sum_{\lambda=1}^{\Lambda} \Re_\lambda}^1(\Gamma)$,

(3) $BND^4_{\mathcal{R}}(\Gamma) \subseteq BND^2_{\mathcal{R}}(\Gamma) \subseteq BND^1_{\mathcal{R}}(\Gamma)$ *and* $BND^4_{\mathcal{R}}(\Gamma) \subseteq BND^3_{\mathcal{R}}(\Gamma) \subseteq BND^1_{\mathcal{R}}(\Gamma)$,

(4) $\mu^4_{\mathcal{R}}(\Gamma) \geq \mu^2_{\mathcal{R}}(\Gamma) \geq \mu^1_{\mathcal{R}}(\Gamma)$ *and* $\mu^4_{\mathcal{R}}(\Gamma) \geq \mu^3_{\mathcal{R}}(\Gamma) \geq \mu^1_{\mathcal{R}}(\Gamma)$.

Proof. Straightforward from Definitions 3.20, 3.28, 3.38 and 3.45 using Lemma 2.11. □

Remark 4.2. The method given in Definition 3.45 is the best way to enhance approximations and raise the accuracy measure, according to Theorem 4.1. By decreasing the upper approximation and raising the lower approximation, this approach reduces or cancels the boundary region. This method is different from the other approaches given in Definitions 3.7, 3.20, 3.28 and 3.38. In particular,

(1) if $\eth = \{\phi\}$, then the recent approximations coincide with the approximations given in Definition 3.38,
(2) if $\mathcal{R}$ is a family of symmetric relations, then the recent approximations coincide with the approximations given in Definition 3.28,
(3) if $\eth = \{\phi\}$ and $\mathcal{R}$ is a family of symmetric relations, then the recent approximations coincide with the approximations given in Definition 3.38,
(4) if $\eth = \{\phi\}$ and $\mathcal{R}$ is a family of equivalence relations, then the recent approximations coincide with the approximations given in Definition 3.7.

Remark 4.3. In the following example, the methods in Definitions 3.20, 3.28, 3.38 and 3.45 are used to compare the accuracy values of subsets of Π . Moreover, the example shows that Definition 3.45 refers to the best method to improve the accuracy values.

Example 4.4. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\Re_1, \Re_2 \in \mathcal{R}$, where $\Pi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. Consider $\eth = \{\phi, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_2, \zeta_3\}\}$, $\Re_1 = \{(\zeta_1, \zeta_1), (\zeta_1, \zeta_2), (\zeta_1, \zeta_4), (\zeta_2, \zeta_2), (\zeta_2, \zeta_3), (\zeta_2, \zeta_4), (\zeta_3, \zeta_2), (\zeta_3, \zeta_3)\}$, $\Re_2 = \{(\zeta_1, \zeta_1), (\zeta_1, \zeta_2), (\zeta_1, \zeta_4), (\zeta_2, \zeta_2), (\zeta_2, \zeta_3), (\zeta_3, \zeta_3), (\zeta_3, \zeta_2), (\zeta_4, \zeta_4)\}$. Then we have

$$< \zeta_1 > \Re_1 = \{\zeta_1, \zeta_2, \zeta_4\}, \ < \zeta_2 > \Re_1 = \{\zeta_2\},$$
$$< \zeta_3 > \Re_1 = \{\zeta_2, \zeta_3\}, \ < \zeta_4 > \Re_1 = \{\zeta_2, \zeta_4\},$$
$$\Re_1 < \zeta_1 > \Re_1 = \{\zeta_1\}, \ \Re_1 < \zeta_2 > \Re_1 = \{\zeta_2\},$$
$$\Re_1 < \zeta_3 > \Re_1 = \{\zeta_2, \zeta_3\}, \Re_1 < \zeta_4 > \Re_1 = \phi.$$

Also, we get

$$< \zeta_1 > \Re_2 = \{\zeta_1, \zeta_2, \zeta_4\}, \ < \zeta_2 > \Re_2 = \{\zeta_2\},$$
$$< \zeta_3 > \Re_2 = \{\zeta_2, \zeta_3\}, \ < \zeta_4 > \Re_2 = \{\zeta_4\},$$
$$\Re_2 < \zeta_1 > \Re_2 = \{\zeta_1\}, \ \Re_2 < \zeta_2 > \Re_2 = \{\zeta_2\},$$
$$\Re_2 < \zeta_3 > \Re_2 = \{\zeta_2, \zeta_3\}, \ \Re_2 < \zeta_4 > \Re_2 = \{\zeta_4\}.$$

The comparison between the introduced methods is shown in Tables 1 and 2. For example, take $\Gamma = \{\zeta_1, \zeta_3\}$. Then the boundary region ($BND^4_{\mathcal{R}}$) and accuracy measure ($\mu^4_{\mathcal{R}}$) by Definition 3.45 are ϕ and 1, respectively. Whereas $BND^1_{\mathcal{R}}$ and $\mu^1_{\mathcal{R}}$ are $\{\zeta_1, \zeta_3\}$ and 0, respectively. Also, $BND^2_{\mathcal{R}}$ and $\mu^2_{\mathcal{R}}$ are $\{\zeta_1\}$ and $\frac{1}{2}$, respectively,$BND^3_{\mathcal{R}}$ and $\mu^3_{\mathcal{R}}$ are $\{\zeta_3\}$ and $\frac{1}{2}$, respectively. Additionally, it is clear that the methods in Definition 3.28 and Definition 3.38 are independent methods.

TABLE 1. Comparison between lower and upper approximations and boundary regions of a set $\Gamma \subseteq \Pi$ by using the proposed methods in Definitions 3.20, 3.28, 3.38 and 3.45

	Method in Definition 3.20			Method in Definition 3.28			Method in Definition 3.38			Method in Definition 3.45		
$\Gamma \subseteq \Pi$	$\underline{\sum_{\lambda=1}^{2} \Re_{\lambda}}_{1}(\Gamma)$	$\overline{\sum_{\lambda=1}^{2} \Re_{\lambda}}^{1}(\Gamma)$	$BND^{1}_{\mathcal{R}}$	$\underline{\sum_{\lambda=1}^{2} \Re_{\lambda}}_{2}(\Gamma)$	$\overline{\sum_{\lambda=1}^{2} \Re_{\lambda}}^{2}(\Gamma)$	$BND^{2}_{\mathcal{R}}$	$\underline{\sum_{\lambda=1}^{2} \Re_{\lambda}}_{3}(\Gamma)$	$\overline{\sum_{\lambda=1}^{2} \Re_{\lambda}}^{3}(\Gamma)$	$BND^{3}_{\mathcal{R}}$	$\underline{\sum_{\lambda=1}^{2} \Re_{\lambda}}_{4}(\Gamma)$	$\overline{\sum_{\lambda=1}^{2} \Re_{\lambda}}^{4}(\Gamma)$	$BND^{4}_{\mathcal{R}}$
Π	Π	Π	ϕ	Π	Π	ϕ	Π	Π	ϕ	Π	Π	ϕ
$\{\zeta_1\}$	ϕ	$\{\zeta_1\}$	$\{\zeta_1\}$	ϕ	$\{\zeta_1\}$	$\{\zeta_1\}$	$\{\zeta_1\}$	$\{\zeta_1\}$	ϕ	$\{\zeta_1\}$	$\{\zeta_1\}$	ϕ
$\{\zeta_2\}$	$\{\zeta_2\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_2\}$	$\{\zeta_2\}$	ϕ	$\{\zeta_2\}$	$\{\zeta_2, \zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_2\}$	$\{\zeta_2\}$	ϕ
$\{\zeta_3\}$	ϕ	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	ϕ	ϕ	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	ϕ
$\{\zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_1, \zeta_4\}$	$\{\zeta_1\}$	$\{\zeta_4\}$	$\{\zeta_1, \zeta_4\}$	$\{\zeta_1\}$	$\{\zeta_4\}$	$\{\zeta_4\}$	ϕ	$\{\zeta_4\}$	$\{\zeta_4\}$	ϕ
$\{\zeta_1, \zeta_2\}$	$\{\zeta_2\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_2\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$	ϕ
$\{\zeta_1, \zeta_3\}$	ϕ	$\{\zeta_1, \zeta_3\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_1\}$	$\{\zeta_1\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_1, \zeta_3\}$	ϕ
$\{\zeta_1, \zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_1, \zeta_4\}$	$\{\zeta_1\}$	$\{\zeta_1, \zeta_4\}$	$\{\zeta_1, \zeta_4\}$	ϕ	$\{\zeta_1, \zeta_4\}$	$\{\zeta_1, \zeta_4\}$	ϕ	$\{\zeta_1, \zeta_4\}$	$\{\zeta_1, \zeta_4\}$	ϕ
$\{\zeta_2, \zeta_3\}$	$\{\zeta_2, \zeta_3\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_1\}$	$\{\zeta_2, \zeta_3\}$	$\{\zeta_2, \zeta_3\}$	ϕ	$\{\zeta_2, \zeta_3\}$	$\{\zeta_2, \zeta_3\}$	ϕ	$\{\zeta_2, \zeta_3\}$	$\{\zeta_2, \zeta_3\}$	ϕ
$\{\zeta_2, \zeta_4\}$	$\{\zeta_2, \zeta_4\}$	Π	$\{\zeta_1, \zeta_3\}$	$\{\zeta_2, \zeta_4\}$	$\{\zeta_1, \zeta_2, \zeta_4\}$	$\{\zeta_1\}$	$\{\zeta_2, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_3\}$	$\{\zeta_2, \zeta_4\}$	$\{\zeta_2, \zeta_4\}$	ϕ
$\{\zeta_3, \zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_3, \zeta_4\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	$\{\zeta_1\}$	$\{\zeta_4\}$	$\{\zeta_3, \zeta_4\}$	$\{\zeta_3\}$	$\{\zeta_3, \zeta_4\}$	$\{\zeta_3, \zeta_4\}$	ϕ
$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_2, \zeta_3\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_1\}$	$\{\zeta_2, \zeta_3\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_1\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	ϕ	$\{\zeta_1, \zeta_2, \zeta_3\}$	$\{\zeta_1, \zeta_2, \zeta_3\}$	ϕ
$\{\zeta_1, \zeta_2, \zeta_4\}$	$\{\zeta_1, \zeta_2, \zeta_4\}$	Π	$\{\zeta_3\}$	$\{\zeta_1, \zeta_2, \zeta_4\}$	$\{\zeta_1, \zeta_2, \zeta_4\}$	ϕ	$\{\zeta_1, \zeta_2, \zeta_4\}$	Π	$\{\zeta_3\}$	$\{\zeta_1, \zeta_2, \zeta_4\}$	$\{\zeta_1, \zeta_2, \zeta_4\}$	ϕ
$\{\zeta_1, \zeta_3, \zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	$\{\zeta_1, \zeta_3\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	ϕ	$\{\zeta_1, \zeta_4\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	$\{\zeta_3\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	$\{\zeta_1, \zeta_3, \zeta_4\}$	ϕ
$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	Π	$\{\zeta_1\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	Π	$\{\zeta_1\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	ϕ	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	ϕ

Table 2. Accuracy measure of a set $\Gamma \subseteq \Pi$ by using the proposed methods.

$\Gamma \subseteq \Pi$	First approach ($\mu^1_{\mathcal{R}}$)	Second approach ($\mu^2_{\mathcal{R}}$)	Third approach ($\mu^3_{\mathcal{R}}$)	Fourth approach ($\mu^4_{\mathcal{R}}$)
Π	1	1	1	1
$\{\zeta_1\}$	0	0	1	1
$\{\zeta_2\}$	1/3	1	1/2	1
$\{\zeta_3\}$	0	1	0	1
$\{\zeta_4\}$	1/2	1/2	1	1
$\{\zeta_1,\zeta_2\}$	1/3	1/2	2/3	1
$\{\zeta_1,\zeta_3\}$	0	1/2	1/2	1
$\{\zeta_1,\zeta_4\}$	1/2	1	1	1
$\{\zeta_2,\zeta_3\}$	2/3	1	1	1
$\{\zeta_2,\zeta_4\}$	1/2	2/3	2/3	1
$\{\zeta_3,\zeta_4\}$	1/3	2/3	1/2	1
$\{\zeta_1,\zeta_2,\zeta_3\}$	2/3	2/3	1	1
$\{\zeta_1,\zeta_2,\zeta_4\}$	3/4	1	3/4	1
$\{\zeta_1,\zeta_3,\zeta_4\}$	1/3	1	2/3	1
$\{\zeta_2,\zeta_3,\zeta_4\}$	3/4	3/4	1	1

5. Multi-granulations via bi-ideals

Here, a new generalization via two ideals of the best method that is given in Definition 3.45, called bi-ideals multi-granulations approximation spaces is submitted. This generalization is established using two distinct methods.

Definition 5.1. A triple $(\Pi, \mathcal{R}, \prec \eth_1, \eth_2 \succ)$ is called an *ideal knowledge base associated with* $(\Pi, \mathcal{R}, \eth_1, \eth_2)$, if $(\Pi, \mathcal{R}, \eth_1, \eth_2)$ is a bi-ideal knowledge base. Let $\Re_1, \Re_2, \cdots, \Re_\Lambda \in \mathcal{R}$. Then the *lower* and *upper approximations* of $\Gamma \subseteq \Pi$ are defined respectively by:

(5.1)
$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)_{*4\prec\eth_1,\eth_2\succ}(\Gamma) = \{\zeta \in \Pi : \bigcup(\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \prec \eth_1, \eth_2 \succ), \lambda \leq \Lambda\},$$

(5.2)
$$\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}_{\prec\eth_1,\eth_2\succ}(\Gamma) = \{\zeta \in \Pi : \bigcap(\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma \notin \prec \eth_1, \eth_2 \succ), \lambda \leq \Lambda\}$$

Remark 5.2. The recent approximations given in Definition 5.1 are identical to these approximations given in Definition 3.42 whenever $\eth_1 = \eth_2$. Also, the resulting properties of the approximations of Definition 5.1 are identical to those given in Proposition 3.43.

Definition 5.3. Let $(\Pi, \mathcal{R}, \eth_1, \eth_2)$ be a bi-ideal knowledge base. Then the *lower* and *upper approximations* of $\Gamma \subseteq \Pi$ are defined respectively by:

(5.3)
$$\underline{\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)}_{4\,\prec\eth_1,\eth_2\succ}(\Gamma) = \{\zeta \in \Gamma : \bigcup(\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \prec \eth_1, \eth_2 \succ), \lambda \leq \Lambda\},$$

(5.4)
$$\overline{\Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)}^{4}_{\prec\eth_1,\eth_2\succ}(\Gamma) = \Gamma \cup \Big(\sum_{\lambda=1}^{\Lambda} \Re_\lambda\Big)^{*4}_{\prec\eth_1,\eth_2\succ}(\Gamma).$$

Remark 5.4. The resulting approximations of Definition 5.3 are identical to these approximations of Definition 3.45 whenever $\eth_1 = \eth_2$.. Also, the resulting properties of the approximations in Definition 5.3 are identical to those given in Proposition 3.46.

Definition 5.5. Let $(\Pi, \mathcal{R}, \eth_1, \eth_2)$ be a bi-ideal knowledge base. Then the *lower* and *upper approximations* of $\Gamma \subseteq \Pi$ are given respectively by:

$$\left(\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_4\right)_{\eth_1,\eth_2}(\Gamma) = \left(\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_4\right)_{\eth_1}(\Gamma) \cup \left(\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_4\right)_{\eth_2}(\Gamma), \tag{5.5}$$

$$\left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1,\eth_2}(\Gamma) = \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1}(\Gamma) \cap \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_2}(\Gamma), \tag{5.6}$$

where $\left(\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_4\right)_{\eth_\iota}(\Gamma)$ and $\left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_\iota}(\Gamma)$ are the lower and upper approximations of Γ with respect to $\eth_\iota$, $\iota \in \{1, 2\}$ as in Definition 3.45.

Remark 5.6. The resulting properties of the approximations in Definition 5.5 are identical with those given in Proposition 3.46.

Remark 5.7. Definitions 5.3 and 5.5 are new types of multi-granulation approximation spaces via two ideals. These definitions are different from the previous Definitions 2.15 and 2.16 given in [20]. In particular,

(1) if $\Lambda = 1$, the, Definition 5.3 coincides with the previous Definition 2.15,
(2) if $\Lambda = 1$, then Definition 5.5 coincides with the previous Definition 2.16.

Theorem 5.8. *Let* $(\Pi, \mathcal{R}, \eth_1, \eth_2)$ *be a bi-ideal knowledge base and* $\Gamma \subseteq \Pi$. *Then*

(1) $\left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\prec\eth_1,\eth_2\succ}(\Gamma) \subseteq \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1,\eth_2}(\Gamma)$,
(2) $\left(\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_4\right)_{\eth_1,\eth_2}(\Gamma) \subseteq \left(\underline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}_4\right)_{\prec\eth_1,\eth_2\succ}(\Gamma)$,
(3) $BND_{\prec\eth_1,\eth_2\succ}(\Gamma) \subseteq BND_{\eth_1,\eth_2}(\Gamma)$,
(4) $\mu_{\eth_1,\eth_2}(\Gamma) \subseteq \mu_{\prec\eth_1,\eth_2\succ}(\Gamma)$.

Proof. (1) Let $\zeta \in \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\prec\eth_1,\eth_2\succ}(\Gamma)$. Then $\zeta \in \Gamma$ or $\zeta \in \left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*4}_{\prec\eth_1,\eth_2\succ}(\Gamma)$. Thus $\zeta \in \Gamma$ or $\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma \notin \prec \eth_1, \eth_2 \succ \forall\ \lambda \in \{1, 2, \cdots, \Lambda\}$.

Case 1: Suppose $\zeta \in \Gamma$. Then $\zeta \in \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1}(\Gamma)$ and $\zeta \in \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_2}(\Gamma)$. Thus $\zeta \in \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1,\eth_2}(\Gamma)$.

Case 2: Suppose $\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma \notin \prec \eth_1, \eth_2 \succ \forall\ \lambda \in \{1, 2, \cdots, \Lambda\}$. Then $\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma \notin \eth_1$ and $\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma \notin \eth_2\ \forall\ \lambda \in \{1, 2, \cdots, \Lambda\}$. Thus $\zeta \in \left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*4}_{\eth_1}(\Gamma)$ and $\zeta \in \left(\sum_{\lambda=1}^{\Lambda}\Re_\lambda\right)^{*4}_{\eth_2}(\Gamma)$. So $\zeta \in \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1}(\Gamma)$ and $\zeta \in \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_2}(\Gamma)$. This means that $\zeta \in \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1,\eth_2}(\Gamma)$. Hence we have

$$\left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\prec\eth_1,\eth_2\succ}(\Gamma) \subseteq \left(\overline{\sum_{\lambda=1}^{\Lambda}\Re_\lambda}^4\right)_{\eth_1,\eth_2}(\Gamma).$$

(2) Let $\zeta \in \left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\eth_1,\eth_2}(\Gamma)$. Then we have

$$\zeta \in \left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\eth_1}(\Gamma) \text{ or } \zeta \in \left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\eth_2}(\Gamma).$$

Thus $\exists \Re_\lambda \prec \zeta \succ \Re_\lambda$ such that $\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \eth_1$ or $\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \eth_2$, $\lambda \in \{1, 2, \cdots, \Lambda\}$. Since $\eth_1, \eth_2 \subseteq \prec \eth_1, \eth_2 \succ$, $\Re_\lambda \prec \zeta \succ \Re_\lambda \cap \Gamma^c \in \prec \eth_1, \eth_2 \succ$. So $\zeta \in \left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\prec\eth_1,\eth_2\succ}(\Gamma)$. Hence we get

$$\left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\eth_1,\eth_2}(\Gamma) \subseteq \left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\prec\eth_1,\eth_2\succ}(\Gamma).$$

(3) Straightforward.

(4) Directly, from (1) and (2). □

Proposition 5.9. *Let* $(\Pi, \mathcal{R}, \eth_1, \eth_2)$ *be a bi-ideal knowledge base and* $\Gamma \subseteq \Pi$. *Then*

(1) $\left(\overline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}^{4}\right)_{\prec\eth_1,\eth_2\succ}(\Gamma) \subseteq \left(\overline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}^{4}\right)_{\eth_1,\eth_2}(\Gamma) \subseteq \left(\overline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}^{4}\right)_{\eth_\iota}(\Gamma)\ \forall \iota \in \{1, 2\}$,

(2) $\left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\eth_\iota}(\Gamma) \subseteq \left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\eth_1,\eth_2}(\Gamma) \subseteq \left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\prec\eth_1,\eth_2\succ}(\Gamma)\ \forall j \in \{1, 2\}$,

(3) $BND_{\prec\eth_1,\eth_2\succ}(\Gamma) \subseteq BND_{\eth_1,\eth_2}(\Gamma) \subseteq BND_{\eth_\iota}(\Gamma)\ \forall j \in \{1, 2\}$,

(4) $\mu^{4}_{\eth_\iota}(\Gamma) \subseteq \mu_{\eth_1,\eth_2}(\Gamma) \subseteq \mu_{\prec\eth_1,\eth_2\succ}(\Gamma)\ \forall j \in \{1, 2\}$.

Proof. Direct from Definition 5.5 and Theorem 5.8. □

Remark 5.10. Theorem 5.8 ensures that Definition 5.3 is accurate more than Definition 5.5 as shown in the next example.

Example 5.11. Let $(\Pi, \mathcal{R}, \eth)$ be an ideal knowledge base, $\Re_1, \Re_2 \in \mathcal{R}$, where $\Pi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. Consider $\eth_1 = \{\phi, \{\zeta_1\}\}, \eth_2 = \{\phi, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_2, \zeta_3\}\}$. Then

$$\prec \eth_1, \eth_2 \succ = \{\phi, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}\}.$$

Let $\Re_1 = \{(\zeta_1, \zeta_1), (\zeta_1, \zeta_2), (\zeta_1, \zeta_3), (\zeta_2, \zeta_1), (\zeta_2, \zeta_2), (\zeta_2, \zeta_3)\}$ and
$\Re_2 = \{(\zeta_2, \zeta_1), (\zeta_2, \zeta_2), (\zeta_2, \zeta_3), (\zeta_3, \zeta_1), (\zeta_3, \zeta_2), (\zeta_3, \zeta_3), (\zeta_4, \zeta_4)\}$.
Then we have

$$\Re_1 < \zeta_1 > \Re_1 = \{\zeta_1, \zeta_2, \zeta_3\},\ \Re_1 < \zeta_2 > \Re_1 = \{\zeta_1, \zeta_2, \zeta_3\},$$
$$\Re_1 < \zeta_3 > \Re_1 = \{\zeta_1, \zeta_2, \zeta_3\},\ \Re_1 < \zeta_4 > \Re_1 = \phi.$$

Also, we get

$$\Re_2 < \zeta_1 > \Re_2 = \{\zeta_1, \zeta_2, \zeta_3\},\ \Re_2 < \zeta_2 > \Re_2 = \{\zeta_1, \zeta_2, \zeta_3\},$$
$$\Re_2 < \zeta_3 > \Re_2 = \{\zeta_1, \zeta_2, \zeta_3\},\ \Re_2 < \zeta_4 > \Re_2 = \{\zeta_4\}.$$

Thus The comparison between the methods introduced in Definitions 5.3 and 5.5 is shown in Table 3.

TABLE 3. Comparison between lower and upper approximations and boundary regions of a set $\Gamma \subseteq \Pi$ by using the two different methods in Definitions 5.3, 5.5

	Method in Definition 5.5				Method in Definition 5.3			
$\Gamma \subseteq \Pi$	$\left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\eth_1,\eth_2}(\Gamma)$	$\left(\overline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}^{4}\right)_{\eth_1,\eth_2}(\Gamma)$	$BND_{\eth_1,\eth_2}(\Gamma)$	$\mu_{\eth_1,\eth_2}(\Gamma)$	$\left(\underline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}_{4}\right)_{\prec\eth_1,\eth_2\succ}(\Gamma)$	$\left(\overline{\sum_{\lambda=1}^{\Lambda} \Re_{\lambda}}^{4}\right)_{\prec\eth_1,\eth_2\succ}(\Gamma)$	$BND_{\prec\eth_1,\eth_2\succ}(\Gamma)$	$\mu_{\prec\eth_1,\eth_2\succ}(\Gamma)$
Π	Π	Π	Φ	1	Π	Π	Φ	1
$\{\zeta_1\}$	$\{\zeta_1\}$	$\{\zeta_1\}$	ϕ	1	$\{\zeta_1\}$	$\{\zeta_1\}$	ϕ	1
$\{\zeta_2\}$	ϕ	$\{\zeta_2\}$	$\{\zeta_2\}$	0	$\{\zeta_2\}$	$\{\zeta_2\}$	ϕ	1
$\{\zeta_3\}$	ϕ	$\{\zeta_3\}$	$\{\zeta_3\}$	0	$\{\zeta_3\}$	$\{\zeta_3\}$	ϕ	1
$\{\zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_4\}$	ϕ	1	$\{\zeta_4\}$	$\{\zeta_4\}$	ϕ	1
$\{\zeta_1,\zeta_2\}$	$\{\zeta_1,\zeta_2\}$	$\{\zeta_1,\zeta_2,\zeta_3\}$	$\{\zeta_3\}$	2/3	$\{\zeta_1,\zeta_2\}$	$\{\zeta_1,\zeta_2\}$	ϕ	1
$\{\zeta_1,\zeta_3\}$	$\{\zeta_1,\zeta_3\}$	$\{\zeta_1,\zeta_2,\zeta_3\}$	$\{\zeta_2\}$	2/3	$\{\zeta_1,\zeta_3\}$	$\{\zeta_1,\zeta_3\}$	ϕ	1
$\{\zeta_1,\zeta_4\}$	$\{\zeta_1,\zeta_4\}$	$\{\zeta_1,\zeta_4\}$	ϕ	1	$\{\zeta_1,\zeta_4\}$	$\{\zeta_1,\zeta_4\}$	ϕ	1
$\{\zeta_2,\zeta_3\}$	$\{\zeta_2,\zeta_3\}$	$\{\zeta_2,\zeta_3\}$	ϕ	1	$\{\zeta_2,\zeta_3\}$	$\{\zeta_2,\zeta_3\}$	ϕ	1
$\{\zeta_2,\zeta_4\}$	$\{\zeta_2,\zeta_4\}$	$\{\zeta_2,\zeta_4\}$	ϕ	1	$\{\zeta_2,\zeta_4\}$	$\{\zeta_2,\zeta_4\}$	ϕ	1
$\{\zeta_3,\zeta_4\}$	$\{\zeta_3,\zeta_4\}$	$\{\zeta_3,\zeta_4\}$	ϕ	1	$\{\zeta_3,\zeta_4\}$	$\{\zeta_3,\zeta_4\}$	ϕ	1
$\{\zeta_1,\zeta_2,\zeta_3\}$	$\{\zeta_1,\zeta_2,\zeta_3\}$	$\{\zeta_1,\zeta_2,\zeta_3\}$	ϕ	1	$\{\zeta_1,\zeta_2,\zeta_3\}$	$\{\zeta_1,\zeta_2,\zeta_3\}$	ϕ	1
$\{\zeta_1,\zeta_2,\zeta_4\}$	$\{\zeta_1,\zeta_2,\zeta_4\}$	Π	$\{\zeta_3\}$	3/4	$\{\zeta_1,\zeta_2,\zeta_4\}$	$\{\zeta_1,\zeta_2,\zeta_4\}$	ϕ	1
$\{\zeta_1,\zeta_3,\zeta_4\}$	$\{\zeta_1,\zeta_3,\zeta_4\}$	Π	$\{\zeta_2\}$	3/4	$\{\zeta_1,\zeta_3,\zeta_4\}$	$\{\zeta_1,\zeta_3,\zeta_4\}$	ϕ	1
$\{\zeta_2,\zeta_3,\zeta_4\}$	$\{\zeta_2,\zeta_3,\zeta_4\}$	$\{\zeta_2,\zeta_3,\zeta_4\}$	ϕ	1	$\{\zeta_2,\zeta_3,\zeta_4\}$	$\{\zeta_2,\zeta_3,\zeta_4\}$	ϕ	1

6. Chemical application

Finally, this part introduces an applicable case in the field of chemistry using the approximations given in Definitions 5.3 and 5.5 to emphasize our recent models.

Example 6.1. Let $\Pi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ be five amino acids (AAs). The (AAs) are described in terms of six attributes: u_1= PIE, u_2= PIF (two measures of the side chain lipophilicity), u_3=SAC=surface area, u_4= Vol=molecular volume, u_5= LAM=the side chain polarity and u_6 =MR=molecular refractivity [46, 47]. Table 4 shows all quantitative attributes of five AAs.

Now, we consider six reflexive relations on Π defined as follows:

$$\Re_\lambda = \{(\zeta_\iota, \zeta_\kappa) : \zeta_\iota(u_\lambda) - \zeta_\kappa(u_\lambda) \leq \sigma_\lambda,\ \lambda = 1, 2, \cdots 6, j, \kappa = 1, 2, \cdots 5\},$$

where σ_λ represents the standard deviation of the quantitative attributes u_λ, $\lambda = 1, 2, \cdots, 6$. The right and left neighborhoods, the minimal right and left neighborhoods and the intersection of minimal left and right neighborhoods for all elements of $\Pi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ with respect to the relations $\Re_\lambda$, $\lambda = 1, 2, \cdots, 6$ are shown in the next tables.

By Chemistry's expert, if $\eth_1 = \{\phi, \{\zeta_4\}, \{\zeta_5\}, \{\zeta_4, \zeta_5\}\}$ and $\eth_2 = \{\phi, \{\zeta_3\}\}$ are the selected two ideals on Π, then $\prec \eth_1, \eth_2 \succ = \{\phi, \{\zeta_3\}, \{\zeta_4\}, \{\zeta_5\}, \{\zeta_3, \zeta_4\}, \{\zeta_3, \zeta_5\}, \{\zeta_4, \zeta_5\}, \{\zeta_3, \zeta_4, \zeta_5\}\}$ is the generated bi-ideal.

Observation: Anyone can offer two ideals to demonstrate that the approximations in Definition 5.3 are superior to the other in Definition 5.5 by extending an example similar to the one in Table 3 by comparing the resultant accuracy. For $\Gamma = \{\zeta_2\}$, we have $\left(\underline{\sum_{\lambda=1}^{6} \Re_\lambda}_4\right)_{\eth_1,\eth_2}(\Gamma) = \phi$, $\left(\overline{\sum_{\lambda=1}^{6} \Re_\lambda}^4\right)_{\eth_1,\eth_2}(\Gamma) = \{\zeta_2\}$ and $\mu_{\eth_1,\eth_2}(\Gamma) = 0$. Also, $\left(\underline{\sum_{\lambda=1}^{6} \Re_\lambda}_4\right)_{\prec\eth_1,\eth_2\succ}(\Gamma) = \{\zeta_2\}$, $\left(\overline{\sum_{\lambda=1}^{6} \Re_\lambda}^4\right)_{\prec\eth_1,\eth_2\succ}(\Gamma) = \{\zeta_2\}$ and $\mu_{\eth_1,\eth_2}(\Gamma) = 1$. The resulting accuracy measure demonstrates that the approximations in Definition 5.3 are superior than those in Definition 5.5. Therefore, the proposed technique in Definition 5.3 will be more accurate than the other proposed types in decision-making for extracting the information and aid in reducing the data ambiguity in real-life problems.

Table 4. Quantitative attributes of five amino acids.

	u_1	u_2	u_3	u_4	u_5	u_6
ζ_1	1.56	1,79	336.1	155.8	-0.05	4.638
ζ_2	0.38	0.49	228.5	106.7	-0.31	2.876
ζ_3	0.00	-0.04	266.7	88.5	-0.40	2.279
ζ_4	0.17	0.26	282.9	105.3	-0.53	2.743
ζ_5	1.85	2.25	401.8	185.9	-0.31	5.755

TABLE 5. Right neighborhoods of seven reflexive relations.

	$\zeta_\iota \Re_1$	$\zeta_\iota \Re_2$	$\zeta_\iota \Re_3$	$\zeta_\iota \Re_4$	$\zeta_\iota \Re_5$	$\zeta_\iota \Re_6$
ζ_1	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_4, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1\}$	$\{\zeta_1, \zeta_5\}$
ζ_2	Π	Π	Π	Π	$\{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$	Π
ζ_3	Π	Π	Π	Π	Π	Π
ζ_4	Π	Π	Π	Π	Π	Π
ζ_5	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$

TABLE 6. Left neighborhoods of seven reflexive relations.

	$\Re_1 \zeta_\iota$	$\Re_2 \zeta_\iota$	$\Re_3 \zeta_\iota$	$\Re_4 \zeta_\iota$	$\Re_5 \zeta_\iota$	$\Re_6 \zeta_\iota$
ζ_1	Π	Π	Π	Π	Π	Π
ζ_2	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_3	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_4	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_5	Π	Π	Π	Π	$\{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$	Π

TABLE 7. Minimal right neighborhoods of seven reflexive relations.

	$< \zeta_\iota > \Re_1$	$< \zeta_\iota > \Re_2$	$< \zeta_\iota > \Re_3$	$< \zeta_\iota > \Re_4$	$< \zeta_\iota > \Re_5$	$< \zeta_\iota > \Re_6$
ζ_1	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1\}$	$\{\zeta_1, \zeta_5\}$
ζ_2	Π	Π	Π	Π	$\{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$	Π
ζ_3	Π	Π	Π	Π	$\{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$	Π
ζ_4	Π	$\{\zeta_1, \zeta_4, \zeta_5\}$	Π	Π	Π	Π
ζ_5	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$	$\{\zeta_5\}$

TABLE 8. Minimal left neighborhoods of seven reflexive relations.

	$\Re_1 < \zeta_\iota >$	$\Re_2 < \zeta_\iota >$	$\Re_3 < \zeta_\iota >$	$\Re_4 < \zeta_\iota >$	$\Re_5 < \zeta_\iota >$	$\Re_6 < \zeta_\iota >$
ζ_1	Π	Π	Π	Π	Π	Π
ζ_2	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_3	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_4	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_5	Π	Π	Π	Π	$\{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$	Π

TABLE 9. The intersection between minimal left and right neighborhoods of seven reflexive relations.

	$\Re_1 < \zeta_\iota > \Re_1$	$\Re_2 < \zeta_\iota > \Re_2$	$\Re_3 < \zeta_\iota > \Re_3$	$\Re_4 < \zeta_\iota > \Re_4$	$\Re_5 < \zeta_\iota > \Re_5$	$\Re_6 < \zeta_\iota > \Re_6$
ζ_1	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1\}$	$\{\zeta_1, \zeta_5\}$
ζ_2	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_5\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_3	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_3\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_4	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$	$\{\zeta_3, \zeta_4\}$	$\{\zeta_2, \zeta_3, \zeta_4\}$
ζ_5	$\{\zeta_1, \zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_5\}$	$\{\zeta_1, \zeta_5\}$	$\{\zeta_2, \zeta_3, \zeta_5\}$	$\{\zeta_5\}$

7. Conclusion

The main target and motivation of this paper is to investigate more generalized models improving the accuracy values of some rough set. These new models depends on both of the notion of "multi-granulations" and the notion of "ideals". The paper includes most of properties of these new models. Lots of examples are included to emphasize the effectiveness of using our methods in building our models. Also, some examples are submitted to ensure that our models are more accurate than those defined in before as in given [34, 1, 6, 23, 7, 20]. Also, the comparison between these methods is presented and we obtained that the bi-ideals multi-granulations approximation defined in Definition 5.3 is the best. It should be noted that this method can be extended similarly by employing n-ideals. At last, as an application, we studied a case involving amino acids in the field of chemistry to demonstrate our approximation approaches. The current study is significant because it introduces new types of multi-granulation approximation spaces via bi-ideals, increases the accuracy measures and shrinks the boundary region of a rough set. This is a goal in rough set theory, moreover it helps to address more real-life problems. For example, the present techniques were successful and powerful techniques to reduce the boundary region and improve the accuracy measure. This allows the medical staff to decide the impact factors of COVID-19 infections, heart attacks, etc. They can handle any imperfect data for symptoms of the diseases, and this automatically make the diagnosis of patients easy and accurate. Consequently, this can help the medical staff to make a precise decision about the diagnosis of patients.

The five proposed methods depend on ideals and minimal left and right neighborhoods. In future work, we will introduce the five methods of multi-granulation approximation spaces based on ideals and maximal left and right neighborhoods, in the rough and soft rough sets.

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