Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 29, No. 2, (April 2025) pp. 191–213
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2025.29.2.191

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 29, No. 2, April 2025

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Received 8 July 2024; Revised 9 November 2024; Accepted 6 January 2025

Abstract. In this paper, we have proposed a method to solve fuzzy nonlinear optimization problems with inequality constraints problems. It is based on the pseudo-invex concept and Karush-Kuhn-Tucker optimality conditions. By using the α-couple level, we have obtained the objective functions with the interval values. In order to focus on this, we have suggested two kinds of optimal solutions according to the partial order defined on closed intervals. With this new approach, we have dealt with five didactic examples. This helped us show how our method works and compare it to other methods. Through a ranking function, we have shown that our method is the best option to solve nonlinear optimization problems with inequality constraints.

2020 AMS Classification: 03E72, 90C46, 90C70

Keywords: Fuzzy optimization, Optimality conditions,Ppseudo-invex function, Hukuhara's generalized subtraction.

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1. Introduction

Mathematical programming occupies a significant position in the domain of optimization. It starts by modeling situations that are usually real-life problems. It comes from many areas, such as economics, management, engineering, health, and transportation. Unfortunately, the data collection for the modeling is difficult due to lack of information and errors in estimation or prediction. Taking into account these difficulties, it is necessary to use fuzzy modeling. Let us remember that the fuzzy sets theory has been developed by Zadeh [1, 2, 3] and after results on fuzzy optimization by Seikkala [4], Delgado et al. [5] and Lodwick et al. [6].

In practice, the modeling of real-life problems can give a linear programming formula or a nonlinear programming formula. For each type of model, many methods have been developed to solve it. For example, for the single objective optimization problems, we have these main works : Sama et al. [7] proposed at first the solving of fuzzy nonlinear optimization problems by using the null set concept and then a hybrid approach for solving fuzzy fractional linear optimization problems [8]; Saad et al. [9] suggest a result on the solution of linear programming problems with rough interval coefficients in a fuzzy environment; Hu et al. [10] studied the duality theory in fuzzy linear programming problems with fuzzy coefficients; Dubois et al. [11] propose a work on fuzzy sets and systems: theory and applications; Abu et al. [12, 13, 14, 15, 16] worked on many topics such as adaptation of reproducing kernel algorithm for solving Fredholm-voltera Integrodiffential equations, the resolution of fuzzy M-fractional integrodifferential model and the Uncertain M-fractional differential problems; Seikh et al. [17, 18, 19, 20, 21] proposed some results with the using of matrix games such as matrix games in intuitionistic fuzzy environnement, solution of interval-valued matrix games using intuitionistic fuzzy optimisation technique and the application of the intuitionistic fuzzy mathematical programming with exponential membership and quadratic non-membership function. As far as instances for the multiple objectives problems, the followings are the main works for our paper : Sivakumar et al. [22] work on a Fuzzy mathematical approach for solving multi-objective fuzzy linear fractional programming problem with trapezoidal fuzzy numbers; Okumus et al. [23] suggested a power aggregation operators on trapezoidal fuzzy multi-numbers and theirs applications to zero-waste problem; Sama et al. [24] conceived some new approach to solving fuzzy multiobjective linear fractional optimization problems; Rommelfanger et al. [25] work on the fuzzy linear programming with single and multiple objective functions; Wu et al. [26] investigated on the Karush-Kuhn-Tucker optimality condition for multiobjective programming problems with fuzzy-valued objective function; Chalco-Cano et al. [42] work on the Karush-Kuhn-Tucker optimality conditions (briefly, KKT conditions) for fuzzy optimization problems. They ensure optimality by including stationarity (gradient balance), primal feasibility (compliance with constraints), dual feasibility (non-negativity of Lagrange multipliers), and complementarity (zero-product condition between constraints and their respective multipliers).

In the literature, there are other proposed methods that use Karush-Kuhn-Tucker optimality conditions to solve linear or nonlinear problems [28, 29, 30]. In practice, they are designed for convex cases where the objective functions and constraints must be differentiable [31, 32, 33, 34]. Therefore, for this work, we are investigating cases where the functions are not necessarily differentiable, but only invex. In this work, we propose a new method for solving fuzzy, nonlinear, single-objective optimization problems. With the method, we first transform the initial problem into the optimization of a real-interval objective function. Then we transform it into a deterministic, bi-objective optimization problem. Furthermore, we obtain the deterministic single objective optimization problem by using weighted sum aggregation. Finally, Karush-Kuhn-Tucker optimality conditions allow us to establish optimality conditions and reach optimal solutions to the problem. Some numerical results have been provided on five didactic examples, which have been compared to two other methods. These results indicate that our method is a good alternative to solving nonlinear optimization problems. In summary, we can say that the novelties of this work are:

(1) a formulation based on KKT conditions, adapted to fuzzy problems;
(2) the integration of pseudoinvexity to solve non-convex problems;
(3) bounding the optimal solutions through maximization and minimization of the bounds;
(4) the use of triangular fuzzy numbers to effectively model uncertainties.

The following is the structure of the remaining document: in Session 2, we have proposed some preliminaries focused on the concepts of fuzzy sets, fuzzy numbers, and fuzzy optimization; in Section 3, we have presented the main results of this work; in Section 4, we have given our conclusion.

2. Preliminaries

2.1. **Fuzzy sets and fuzzy numbers.** A fuzzy subset F from $\mathbb{K}$ is defined by a membership function μ_F that associates for all $x \in \mathbb{K}$ a real value $\mu_F(x)$ belonging to $[0, 1]$. In this work, we will note by $\mathbb{K}_C$ the family of bounded intervals in $\mathbb{R}$, with

$$\mathbb{K}_C = \left\{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R}\right\} \ \ \textit{and}\ \underline{a} \leq \overline{a}. \tag{2.1}$$

Let us consider $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ two bounded intervals in $\mathbb{R}$. That allows us to define the distance between A and B as follows :

$$H(A, B) = \max\left\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\right\}, \tag{2.2}$$

where $(\mathbb{K}_C, H)$ is a complete metric space. Now, we can clearly define the notion of fuzzy numbers.

Definition 2.1 ([35])**.** A fuzzy set $\tilde{A}$ defined on $\mathbb{R}$ is called a *fuzzy number*, when the following conditions are satisfied :

(i) all α-coupes of $\tilde{A}$ are not empty for $0 \leq \alpha \leq 1$,
(ii) all α-coupes of $\tilde{A}$ are bounded intervals of $\mathbb{R}$,
(iii) $Supp(\tilde{A}) = \left\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0\right\}$ is bounded.

Let us consider $\mathcal{F}_C$ as the family of fuzzy intervals. Then $\forall\ \tilde{A} \in \mathcal{F}_C$, we have $[\tilde{A}]^\alpha \in \mathbb{K}_C\ \forall\ \alpha\ \in [0, 1]$ with

$$[\tilde{A}]^\alpha = [\underline{\tilde{a}}^\alpha, \overline{\tilde{a}}^\alpha] \text{ with } \underline{\tilde{a}}^\alpha \text{ and } \overline{\tilde{a}}^\alpha\ \in\ \mathbb{R},\ \alpha\ \in [0, 1]. \tag{2.3}$$

Definition 2.2 ([36])**.** Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy intervals. The *Hukuhara's generalized difference*, noted by gh-difference, is a fuzzy interval $\tilde{C}$ such as:

$$\tilde{A} \ominus_{gH} \tilde{B} = \tilde{C} \Longleftrightarrow \begin{cases} (i)\ \tilde{A} = \tilde{B} + \tilde{C} \\ (ii)\ \tilde{B} = \tilde{A} + (-1)\tilde{C}. \end{cases} \tag{2.4}$$

Remark 2.3. To compute the gH-difference between two triangular fuzzy numbers, defined by their triplets $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$, several conditions must be met to ensure that the operation is well-defined and that the result remains a triangular fuzzy number:

- Interval Inclusion Condition [11, 37] The triplet $\tilde{A}$ must "contain" $\tilde{B}$ in terms of intervals, which implies:

$$a_1 \geq b_1 \quad \text{and} \quad a_3 \geq b_3. \tag{2.5}$$

- Formulation of the Resulting Triplets [38] The gH-difference produces a new triplet $\tilde{C} = (c_1, c_2, c_3)$, defined as:

$$c_1 = a_1 - b_1, \quad c_2 = a_2 - b_2, \quad c_3 = a_3 - b_3. \tag{2.6}$$

- Uniqueness of the Solution [39] The triplets $\tilde{A}$ and $\tilde{B}$ must satisfy:

$$a_1 \leq a_2 \leq a_3 \quad \text{and} \quad b_1 \leq b_2 \leq b_3. \tag{2.7}$$

- Preservation of the Triangular Structure [40] The resulting triplet $\tilde{C}$ must satisfy:

$$c_1 \leq c_2 \leq c_3. \tag{2.8}$$

If $\tilde{A} \ominus_{gH} \tilde{B} = \tilde{C}$ exists in terms of α-coupe, we can write

$$\begin{aligned} [\tilde{A} \ominus_{gH} \tilde{B}]^\alpha &= [\tilde{A}]^\alpha \ominus_{gH} [\tilde{B}]^\alpha \\ &= \Big[\min\big\{\underline{\tilde{a}}^\alpha - \underline{\tilde{b}}^\alpha, \overline{\tilde{a}}^\alpha - \overline{\tilde{b}}^\alpha\big\}, \max\big\{\underline{\tilde{a}}^\alpha - \underline{\tilde{b}}^\alpha, \overline{\tilde{a}}^\alpha - \overline{\tilde{b}}^\alpha\big\}\Big]. \end{aligned} \tag{2.9}$$

Let $\tilde{A}$ and $\tilde{B}$ be two elements of $\mathcal{F}_{\mathcal{C}}$. Then by defining the distance between A and B define as:

$$\begin{aligned} D(A, B) &= \sup_{\alpha \in [0,1]} H\big([A]^\alpha, [B]^\alpha\big) \\ &= \sup \max\big\{|\underline{\tilde{a}}^\alpha - \underline{\tilde{b}}^\alpha|, |\overline{\tilde{a}}^\alpha - \overline{\tilde{b}}^\alpha|\big\}, \end{aligned} \tag{2.10}$$

we can define (F_C, D) as a complete metric space.

Definition 2.4 ([41])**.** (Yager's Ranking Function) Let $\tilde{A} = (a_1, a_2, a_3)$ be a triangular fuzzy number. *Yager's ranking function*, denoted by $\mathcal{R}_1(\tilde{A})$, is defined as the arithmetic mean of the three parameters of the triangular fuzzy number:

$$\mathcal{R}_1(\tilde{A}) = \frac{a_1 + a_2 + a_3}{3}. \tag{2.11}$$

Remark 2.5. Let $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. We say that $\widetilde{A} \preceq \widetilde{B}$ if and only if $\mathcal{R}_1(\widetilde{A}) \leq \mathcal{R}_1(\widetilde{B})$.

2.2. **Fuzzy functions.** Let us consider $\mathbb{K} \subset \mathbb{R}^n$, $F : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ a fuzzy function. For all $\alpha \in [0, 1]$, we associate to F the family of functions with interval values. That means that $F_\alpha(x) : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ by $F_\alpha(x) = [F(x)]^\alpha$, $\alpha \in [0, 1]$ with $F_\alpha(x) = [\underline{F_\alpha}(x), \overline{F_\alpha}(x)]$, where $\underline{F_\alpha}(x)$ and $\overline{F_\alpha}(x)$ are respectively lower bound function and upper bound function of $F_\alpha(x)$ and are defined from $\mathbb{K} \to \mathbb{R}$.

Definition 2.6 ([42])**.** Let $F : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ be a fuzzy function, with $x_0 \in \mathbb{K}$ and h such as $x_0 + h \in \mathbb{K}$. The *gH-fuzzy derivative of F in* x_0 is defined as follows:

$$F'(x_0) = \lim_{h \to 0} \frac{F(x_0 + h) \ominus_{gH} F(x_0)}{h}. \tag{2.12}$$

If $F'(x_0) \in \mathcal{F}_{\mathcal{C}}$, then F is the Hukuhara's generalized differentiable function, gH-differentiable in x_0, where $\ominus_{gH}$ is the gH-difference.

Theorem 2.7 ([29])**.** *Let $F : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ be a fuzzy function. If F is gH-differentiable, then the function with interval values $F_\alpha : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ is differentiable for all $\alpha \in [0, 1]$, and we have:*

$$[F'(x)]^\alpha = F'_\alpha(x). \tag{2.13}$$

Let F be a function defined on $\mathbb{K} \subset \mathbb{R}^n$ i.e $F(x) = F(x_1, \ldots, x_n) \in \mathcal{F}_{\mathcal{C}}$ for $x = (x_1, \ldots, x_n)$. We note fuzzy interval $F(x)$ by $F(x) = [\underline{F}(x), \overline{F}(x)]$. And $\forall \alpha \in [0, 1]$ we have:

$$F_\alpha(x) = [\underline{F_\alpha}(x), \overline{F_\alpha}(x)] = [\underline{F}(\alpha, x), \overline{F}(\alpha, x)]. \tag{2.14}$$

Definition 2.8 ([29])**.** Let F be a function defined on $\mathbb{K} \subset \mathbb{R}^n$ and $x_0 = \big(x_1^{(0)}, \ldots, x_n^{(0)}\big)$ a fixed element of $\mathbb{K}$. Let us consider the fuzzy function defined for all i by $h_i(x_i) = F\big(x_1^{(0)}, \ldots, x_{i-1}^{(0)}, x_i^{(0)}, x_{i+1}^{(0)}, \ldots, x_n^{(0)}\big)$. If h_i is gH-differentiable in $x_i^{(0)}$, then F is said to be the i^e *gH-partial derivative in* x_0 and it is noted by $\dfrac{\partial F}{\partial x_i}(x_0)$. And, we have

$$\frac{\partial F}{\partial x_i}(x_0) = (h_i)'(x_i^{(0)}). \tag{2.15}$$

Definition 2.9 ([29])**.** Let F be a function defined on $\mathbb{K}$ and $x_0 \in \mathbb{K}$, where $x_0 = \big(x_1^{(0)}, \ldots, x_n^{(0)}\big)$. We say that F *is gH-differentiable in* x_0, if all of the gH-partial derivatives $\dfrac{\partial F}{\partial x_1}(x_0), \ldots, \dfrac{\partial F}{\partial x_n}(x_0)$ exist in a neighborhood of x_0 and are continuous in x_0.

If F is gH-differentiable in x_0, then $\dfrac{\partial F}{\partial x_i}(x_0)$ is a fuzzy interval. Thus $\forall\ \alpha\ \in [0, 1]$, we have:

$$\left[\frac{\partial F}{\partial x_i}(x_0)\right]^\alpha = \frac{\partial F_\alpha}{\partial x_i}(x_0) = \left[\frac{\partial \underline{F_\alpha}}{\partial x_i}(x_0), \frac{\partial \overline{F_\alpha}}{\partial x_i}(x_0)\right]. \tag{2.16}$$

Proposition 2.10 ([29])**.** *Let $F : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ a fuzzy function. If F is gH-differentiable in $x_0 \in \mathbb{K}\ \forall\ \alpha \in [0, 1]$, then the real value function $\underline{F_\alpha} + \overline{F_\alpha} : \mathbb{K} \to \mathbb{R}$ is differentiable in x_0 and we have*

$$\frac{\partial \underline{F_\alpha}}{\partial x_i}(x_0) + \frac{\partial \overline{F_\alpha}}{\partial x_i}(x_0) = \frac{\partial(\underline{F_\alpha} + \overline{F_\alpha})}{\partial x_i}(x_0). \tag{2.17}$$

Definition 2.11 ([29])**.** Let $F : \mathbb{K} \to \mathcal{F}_{\mathcal{C}}$ be a fuzzy function. The gradient of F in x_0 noted by $\tilde{\nabla} F(x_0)$ is defined by:

$$\tilde{\nabla} F(x_0) = \left(\Big(\frac{\partial F}{\partial x_1}(x_0)\Big), \ldots, \Big(\frac{\partial F}{\partial x_n}(x_0)\Big)\right), \tag{2.18}$$

where $\dfrac{\partial F}{\partial x_j}(x_0)$ is the j^e gH-partial derivative of F in x_0.

Definition 2.12 ([43, 44])**.** (Pseudoinvexity)

$$\nabla F(x)^T (y - x) \geq 0 \implies F(y) \geq F(x) \tag{2.19}$$

for all $x, y \in \widetilde{\mathcal{X}}$, where $\nabla F(x)$ denotes the gradient of F at x.

Remark 2.13.
- Relation with Convexity: every convex function is pseudoinvex, but not all pseudoinvex functions are convex. Pseudoinvexity is a generalization of convexity [44].
- Optimality Properties: for a pseudoinvex function F, any stationary point x^* (where $\nabla F(x^*) = 0$) is a global minimum of F over $\mathbb{K}$. This property eliminates the distinction between local and global optima for pseudoinvex functions [45].
- Addition: if F and G are pseudoinvex functions defined on the same domain $\widetilde{\mathcal{X}}$, their sum $F + G$ is also pseudoinvex [46].
- Composition with Strictly Monotone Functions: if F is pseudoinvex and G is a strictly increasing function, then the composition $G \circ F$ is pseudoinvex [47, 48].
- Sufficient Condition for Twice Differentiable Functions: for a twice-differentiable function F, a sufficient condition for pseudoinvexity is: $\nabla^2 F(x)$ is positive semi-definite in the direction $(y - x)$ $\quad \forall x,\ y \in \widetilde{\mathcal{X}}$ [38].

3. Main results

3.1. **Novel method.** Let us consider a fuzzy single objective optimization problem with the following formulation:

$$(FO)\begin{cases} \min F(x) \\ s.t \\ g_j(x) \leq 0,\ j = 1, \cdots, p \text{ and } x\ \in \mathbb{K} \subset \mathbb{R}^n, \end{cases} \tag{3.1}$$

where F and g_j are fuzzy functions. The feasible set of the problem (FO) is given by $\widetilde{\mathcal{X}} = \{x \in \mathbb{K},\ g_j(x) \leq 0,\ j = 1, \cdots, p\}$.
We say that (FO) *is a fuzzy Pseudoinvex II problem*, if F is a gH-differentiable function, g_j are gH-differentiable functions on $\mathbb{K}$ and for all $x,\ x^* \in \widetilde{\mathcal{X}}$, there exists $\eta(x^*, x) \in \mathbb{R}^n$ such that:

$$\begin{aligned} F(x) \preceq F(x^*) &\Rightarrow \tilde{\nabla} F(x^*) \cdot \eta(x^*, x) \preceq 0, \\ -\nabla g_j(x^*) \cdot \eta(x^*, x) &\leq 0 \quad \text{for} \quad j \in I(x^*), \end{aligned} \tag{3.2}$$

where $I(x^*)$ is the set of active constraints. If

$$\begin{aligned} F(x) \prec F(x^*) &\Rightarrow \tilde{\nabla} F(x^*) \cdot \eta(x^*, x) \prec 0, \\ -\nabla g_j(x^*) \cdot \eta(x^*, x) &\leq 0 \quad \text{for} \quad j \in I(x^*), \end{aligned} \tag{3.3}$$

then we say that (FO) *is a fuzzy Pseudoinvex I problem.*
To solve this problem using our method, four steps are necessary.

STEP 1: By using the α-coupe concept, we can rewrite the problem as follows:

$$(P_\alpha)\begin{cases}\min[\underline{F_\alpha}(x),\overline{F_\alpha}(x)]\\ s.t\\ [\underline{g_{j\alpha}}(x),\overline{g_{j\alpha}}(x)]\leq 0\ \forall\ j=1,\cdots,p\\ x\geq 0,\end{cases}\tag{3.4}$$

where $\alpha\in[0,1]$.

Theorem 3.1. *Let x^* be the optimal solution of the problem (P_α), then x^* is also the optimal solution of the problem (FO).*

Proof. Let x^* be the optimal solution of the problem (P_α). Let us assume that x^* is not the optimal solution to the problem (FO). It means that there exists another point $y\in\mathbb{K}$ such that $F(y)\leq F(x^*)$. Then $\forall\alpha\in[0,1]$, $F_\alpha(y)\leq F_\alpha(x^*)$. Otherwise, we have $[\underline{F_\alpha}(y),\overline{F_\alpha}(y)]\leq[\underline{F_\alpha}(x),\overline{F_\alpha}(x)]$. That is equivalent to

$$\begin{cases}\underline{F_\alpha}(y)\leq\underline{F_\alpha}(x^*)\\ \overline{F_\alpha}(y)\leq\overline{F_\alpha}(x^*).\end{cases}$$

That is wrong, according to our assertion. Thus x^* is an optimal solution to the problem (FO). □

STEP 2: By using the arithmetic operations on fuzzy intervals, we can rewrite the problem as follows:

$$\begin{cases}\max\underline{F_\alpha}(x)\\ \min\overline{F_\alpha}(x)\\ s.t\\ [\underline{g_{j\alpha}}(x),\overline{g_{j\alpha}}(x)]\leq 0\ \forall\ j=1,\cdots,p\\ x\geq 0.\end{cases}\tag{3.5}$$

That can be rewritten into the following form:

$$(P_\alpha M)\begin{cases}\min(-\underline{F_\alpha}(x))\\ \min\overline{F_\alpha}(x)\\ s.t\\ [\underline{g_{j\alpha}}(x),\overline{g_{j\alpha}}(x)]\leq 0\ \forall\ j=1,\cdots,p\\ x\geq 0.\end{cases}\tag{3.6}$$

The feasible set of the problem $(P_\alpha M)$ is given by:

$$\widetilde{\mathcal{X}}=\{x\in\mathbb{K}:g_{j\alpha}\leq 0,\ j=1,\cdots,p\}.$$

Theorem 3.2. *Let x^* be a Pareto optimal solution of the problem $(P_\alpha M)$, then x^* is also the global optimal solution of the problem (P_α).*

Proof. Suppose, for the sake of contradiction, that x^* is a Pareto optimal solution of the problem $(P_\alpha M)$, but it is not a global solution of the problem (P_α). This means that there exists a point $y \in \widetilde{\mathcal{X}}$ such that:

$$[\underline{F_\alpha}(y), \overline{F_\alpha}(y)] < [\underline{F_\alpha}(x^*), \overline{F_\alpha}(x^*)], \tag{3.7}$$

which means that simultaneously:

$$-\underline{F_\alpha}(y) \leq -\underline{F_\alpha}(x^*) \quad \text{and} \quad \overline{F_\alpha}(y) \leq \overline{F_\alpha}(x^*), \tag{3.8}$$

with at least one of the inequalities being strict.

This means that y strictly dominates x^* on both objectives, that is,

$$-\underline{F_\alpha}(y) < -\underline{F_\alpha}(x^*) \quad \text{and} \quad \overline{F_\alpha}(y) < \overline{F_\alpha}(x^*). \tag{3.9}$$

Then y simultaneously improves both objectives compared to x^*. However, by the definition of Pareto optimality in $(P_\alpha M)$, no feasible point $y \in \widetilde{\mathcal{X}}$ can simultaneously improve both objectives $-\underline{F_\alpha}(x)$ and $\overline{F_\alpha}(x)$ compared to x^*. This assumption leads to a contradiction with the fact that x^* is a Pareto optimal solution of $(P_\alpha M)$. In conclusion, the assumption that x^* is not a global solution of the problem (P_α), while being a Pareto optimal solution of the problem $(P_\alpha M)$, leads to a contradiction. Thus if x^* is a Pareto optimal solution of $(P_\alpha M)$, then it is also a global solution of the problem (P_α). So the result holds. □

STEP 3: At this step, the weighted sum is used to transform the problem into a single objective optimization problem.

$$(P_\alpha O)\begin{cases} \min[\lambda(\underline{F_\alpha}(x) + \overline{F_\alpha}(x)) - \underline{F_\alpha}(x)] \\ s.t \\ g_j(x) \leq 0 \ \forall j = 1, \cdots, p \\ x \geq 0; \ 0 < \lambda < 1. \end{cases} \tag{3.10}$$

Theorem 3.3. *Let x^* be the global optimal solution of the problem $(P_\alpha O)$, then x^* is also the Pareto optimal solution of the problem $(P_\alpha M)$.*

Proof. As the problem $(P_\alpha O)$ is the weighted sum sub-problem with strictly positive weights of the problem $(P_\alpha M)$. Then by using the Theorem 4.6 of the Chankong works [27], we have obtained that if x^* is is the global optimal solution of $(P_\alpha O)$, then it is also Pareto optimal solution of $(P_\alpha M)$ $\alpha \in [0, 1]$. □

STEP 4: The problem $(P_\alpha O)$ is solved by using KKT conditions. According to the waited solutions to the problems, we have organized the optimality conditions into two groups: non-dominated solutions and weakly non-dominated solutions.

Here are the optimality conditions for obtaining non-dominated solutions.

Theorem 3.4. *Let us assume that the constraint functions g_j, $j = 1, \cdots, p$ are convex and differentiable on $\mathbb{K}$. In addition, we assume that the fuzzy function $F : \mathbb{K} \to \mathcal{F}_\mathcal{C}$ est gH-differentiable and the function $\lambda_1\overline{F_\alpha}(x) + \lambda_2(-\underline{F_\alpha}(x))$ is convex on $\mathbb{K}$ $\forall$ $\alpha \in [0, 1]$ and $\lambda_1 + \lambda_2 = 1$. If there exists a* *real positive numbers $\mu_j(\alpha)$ with $j = 1, \cdots, p$, such as KKT conditions are verified:*

(i) $\nabla\Big(\lambda_1 \overline{F_\alpha}(x) + \lambda_2(-\underline{F_\alpha}(x))\Big)(x^*) + \sum_{j=1}^{p} \mu_j(\alpha)\nabla g_j(x^*) = 0$ *for all* $\alpha \in [0,1]$ *and* $\lambda_1 + \lambda_2 = 1$,

(ii) $\mu_j(\alpha) g_j(x^*) = 0$ *for all* $j = 1, \cdots, p$,

then x^* is the non-dominated solution to the problem (FO).

Proof. From the Proposition 2.10, if the function F is gH-differentiable, then $\lambda_1 \overline{F_\alpha}(x) + \lambda_2(-\underline{F_\alpha}(x))$ is differentiable on $\widetilde{\mathcal{X}}$ and $\alpha \in [0,1]$. The conditions (i) and (ii) involve that x^* is a point of Karush-Kunh-Tucker for the sub-problem $(P_\alpha O)$ $\forall \alpha \in [0,1]$, obtained by using weighted sum function. If g_j are convex function on $\mathbb{K}$, $j = 1, \ldots, p$ and $\lambda_1 \overline{F_\alpha}(x) + \lambda_2(-\underline{F_\alpha}(x))$ is convex on $\mathbb{K}$, then x^* is a Pareto optimal solution for all $\alpha \in [0,1]$. As $(P_\alpha O)$ is the sub-problem of (P_α) and the weights are strictly positive, by using the Chankong's theorem, we obtain x^* is a Pareto optimal solution of (P_α) for all $\alpha \in [0,1]$. From the Lemma 1 in [29], we can deduce that x^* is a non-dominated solution of (FO). □

Corollary 3.5. *Let us assume that the constraint functions g_j, $j = 1, \cdots, p$ are convex and differentiable on $\mathbb{K}$. In addition, we assume that the fuzzy function $F : \mathbb{K} \to \mathcal{F}_\mathcal{C}$ is gH-differentiable and convex on $\mathbb{K}$. If there exists a real positive numbers $\mu_j(\alpha)$ with $j = 1, \cdots, p$, such as KKT conditions are verified:*

(i) $\nabla\Big(\lambda_1 \overline{F_\alpha}(x) + \lambda_2(-\underline{F_\alpha}(x))\Big)(x^*) + \sum_{j=1}^{p} \mu_j(\alpha)\nabla g_j(x^*) = 0$ *for all* $\alpha \in [0,1]$ *and* $\lambda_1 + \lambda_2 = 1$,

(ii) $\mu_j(\alpha) g_j(x^*) = 0$ *for all* $j = 1, \cdots, p$,

then x^* is the non-dominated solution to the problem (FO).

Proof. According to the convexity of the function F, we have the convexity of $\lambda_1 \overline{F_\alpha}$ and also of $(-\lambda_1 \underline{F_\alpha})$ for all $\alpha \in [0,1]$. There remains the proof of Theorem 3.4. □

Theorem 3.6. *Let (FO) be the pseudoinvex optimization problem from II on $\mathbb{K}$. Let us suppose that $\forall \alpha \in [0,1]$ and $\lambda_1 + \lambda_2 = 1$. If there exists a real positive numbers $\mu_j(\alpha)$, $j = 1, \cdots, p$ such as KKT conditions are verified:*

(i) $\nabla\Big(\lambda_1 \overline{F_\alpha}(x) + \lambda_2(-\underline{F_\alpha}(x))\Big)(x^*) + \sum_{j=1}^{p} \mu_j(\alpha)\nabla g_j(x^*) = 0$ *for all* $\alpha \in [0,1]$ *and* $\lambda_1 + \lambda_2 = 1$,

(ii) $\mu_j(\alpha) g_j(x^*) = 0$ *for each* $j = 1, \cdots, p$,

then x^* is the non-dominated solution of the problem (FO).

Proof. For proving the Theorem 3.6, we will proceed by reasoning by absurdity. Let us assume that x^* is not a weakly non-dominated solution. Then there exists a $\hat{x} \in \widetilde{\mathcal{X}}$ such as $F(\hat{x}) \preceq F(x^*)$. If (FO) is a fuzzy Pseudoinvex function defined from II to $\mathbb{K}$, then there exists $\eta(\hat{x}, x^*)$ such as $\tilde{\nabla} F(x^*).\eta(\hat{x}, x^*) \preceq 0$ and for all $\alpha \in [0,1]$, we have:

$$\left(\left[\frac{\partial(-\lambda_2 \underline{F_\alpha})}{\partial x_1}(x^*), \frac{\partial(\lambda_1 \overline{F_\alpha})}{\partial x_1}(x^*)\right], \ldots, \left[\frac{\partial(-\lambda_2 \underline{F_\alpha})}{\partial x_n}(x^*), \frac{\partial(\lambda_1 \overline{F_\alpha})}{\partial x_n}(x^*)\right]\right).\eta(\hat{x}, x^*) \preceq 0. \tag{3.11}$$

By multiplying each interval by $\eta(\hat{x}, x^*)$, we obtain

$$\begin{aligned}\left[\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_1}(x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_1}(x^*)\right].\eta_1(\hat{x}, x^*) + \cdots \\ + \left[\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_n}(x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_n}(x^*)\right].\eta_n(\hat{x}, x^*) \preceq 0.\end{aligned} \tag{3.12}$$

Thus there exists $\alpha^* \in [0, 1]$ such as:

$$\begin{aligned}&\min\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x}, x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x}, x^*)\right\} + \cdots \\ &+ \min\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_n}(x^*).\eta_n(\hat{x}, x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_n}(x^*).\eta_n(\hat{x}, x^*)\right\} \leq 0,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}&\max\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x}, x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x}, x^*)\right\} + \cdots \\ &+ \max\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_n}(x^*).\eta_n(\hat{x}, x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_n}(x^*).\eta_n(\hat{x}, x^*)\right\} \leq 0\end{aligned} \tag{3.14}$$

with strict inequality. By applying the Proposition 2.10, the function $\lambda_2\underline{F_\alpha} + \lambda_1\overline{F_\alpha}$ is differentiable in x^* and we have:

$$\begin{aligned}\frac{\partial\big(-\lambda_2\underline{F_{\alpha^*}} + \lambda_1\overline{F_{\alpha^*}}\big)}{\partial x_1}(x^*).\eta_1(\hat{x}, x^*) + \cdots \\ + \frac{\partial\big(-\lambda_2\underline{F_{\alpha^*}} + \lambda_1\overline{F_{\alpha^*}}\big)}{\partial x_n}(x^*).\eta_n(\hat{x}, x^*) \leq 0\end{aligned} \tag{3.15}$$

and consequently,

$$\nabla\big(-\lambda_2\underline{F_{\alpha^*}} + \lambda_1\overline{F_{\alpha^*}}\big)(x^*)^T.\eta(\hat{x}, x^*) \leq 0 \ \forall\alpha \in [0, 1]. \tag{3.16}$$

From the assumption of the Pseudo-invexity of the problem (FO), we have:

$$\nabla g_j(x^*).\eta(\hat{x}, x^*) \leq 0 \ \forall j \in I(x^*). \tag{3.17}$$

By applying the Motzkin's alternatif theorem, there does not exist a $0 < \epsilon_0 \in \mathbb{R}$ and $0 < \epsilon_j \in \mathbb{R},\ j \in I(x^*)$ such as:

$$\epsilon_0(\alpha)\nabla\big(-\lambda_2\underline{F_{\alpha^*}} + \lambda_1\overline{F_{\alpha^*}}\big)(x^*) + \sum_{j\in I(x^*)} \epsilon_j(\alpha).\nabla g_j(x^*) = 0 \ \forall\alpha \in [0, 1], \tag{3.18}$$

reciprocally, there does not exist parameters $\mu_j(\alpha^*) \in \mathbb{R},\ j \in I(x^*)$ such as:

$$\nabla\big(-\lambda_2\underline{F_\alpha} + \lambda_1\overline{F_\alpha}\big)(x^*) + \sum_{j\in I(x^*)} \mu_j(\alpha^*).\nabla g_j(x^*) = 0, \tag{3.19}$$

where $\mu_j(\alpha^*) = \epsilon_j/\epsilon_0$.

If $I(x^*)$ is the set of active constraints of the problem, then we have $g_j(x^*) < 0,\ j \notin I(x^*)$. That is why, if $j \notin I(x^*)$, then the condition (ii) involves that $\mu_j(\alpha) = 0$ for all $\alpha \in [0, 1]$. Thus from Equation 3.19, there does not exist some parameters $0 \leq \mu_j(\alpha^*) \in \mathbb{R}$ such as the conditions (i) and (ii) are satisfied, which is a contradiction. So the result holds. □

Here are the optimality conditions for obtaining a weakly non-dominated solution.

Theorem 3.7. *Let us assume that the constraint functions $g_j,\ j=1,\cdots,p$ are convex and differentiable on $\mathbb{K}$. In addition, let us assume that the fuzzy function $F:\mathbb{K}\to\mathcal{F}_{\mathcal{C}}$ is gH-differentiable and the function $\lambda_1\overline{F_\alpha}(x)+\lambda_2(-\underline{F_\alpha}(x))$ is convex on $\mathbb{K}\ \forall\ \alpha^*\in[0,1]$ and $\lambda_1+\lambda_2=1$. If there exist some positive real numbers $\mu_j,\ j=1,\cdots,p$ such as KKT conditions are verified:*

(i) $\nabla\Big(\lambda_1\overline{F_{\alpha^*}}(x)+\lambda_2(-\underline{F_{\alpha^*}}(x))\Big)(x^*)+\sum_{j=1}^p\mu_j(\alpha)\nabla g_j(x^*)=0$ *for all* $\alpha\in[0,1]$ *and* $\lambda_1+\lambda_2=1$,

(ii) $\mu_j(\alpha^*)g_j(x^*)=0$, *for each* $j=1,\cdots,p$;,

then x^* is a weakly non-dominated solution of the problem (FO).

Proof. From Proposition 2.10, if F is a gH-differentiable function, then $\lambda_1\overline{F_{\alpha^*}}(x)+\lambda_2(-\underline{F_{\alpha^*}}(x))$ is differentiable on $\overline{\chi}$ and $\alpha^*\in[0,1]$. The conditions (i) and (ii) involve that x^* is a Karush-Kuhn-Tucker's point for the weighted sum problem $(P_\alpha O)$ $\alpha^*\in[0,1]$. If $g_j,\ j=1,\cdots,p$ are convex on $\mathbb{K}$ and $\lambda_1\overline{F_{\alpha^*}}(x)+\lambda_2(-\underline{F_{\alpha^*}}(x))$ is convex on $\mathbb{K}$, then x^* is an optimal solution for all $\alpha\in[0,1]$. As $(P_{\alpha^*}O)$ is a sub-problem of (P_{α^*}) with positive strict weights, by using the Theorem 4.6 from [27], we can deduce that x^* is a Pareto optimal solution of (P_{α^*}) for all $\alpha^*\in[0,1]$. From Lemma 2, taken in [29], x^* is a weakly non-dominated solution of (FO). ☐

Theorem 3.8. *Let (FO) be a pseudoinvex problem defined from I to $\mathbb{K}$. In addition, let us assume that $\forall\alpha\in[0,1]$ and $\lambda_1+\lambda_2=1$, there exist some positive real numbers $\mu_j,\ j=1,\ldots,p$ such as KKT conditions are verified: for $\alpha^*\in[0,1]$,*

(i) $\nabla\Big(\lambda_1\overline{F_{\alpha^*}}(x)+\lambda_2(-\underline{F_{\alpha^*}}(x))\Big)(x^*)+\sum_{j=1}^p\mu_j(\alpha)\nabla g_j(x^*)=0$ *for all* $\alpha\in[0,1]$ *and* $\lambda_1+\lambda_2=1$;

(ii) $\mu_jg_j(x^*)=0$ *for all* $j=1,\cdots,p$,

then x^* is a non-dominated solution of the problem (FO).

Proof. We will proceed with absurd reasoning to prove the Theorem 3.8. Let us assume that x^* is not a weakly non-dominated solution. Then there exists a $\hat{x}\in\widetilde{\mathcal{X}}$ such as $F(\hat{x})\prec F(x^*)$. If (FO) is a fuzzy Pseudoinvex function defined from I to $\mathbb{K}$, then there exists $\eta(x^*,x)$ such as $\tilde{\nabla}F(x^*).\eta(\hat{x},x^*)\prec 0$ and for all $\alpha\in[0,1]$, we have:

$$\left(\left[\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_1}(x^*),\frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_1}(x^*)\right],\ldots,\left[\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_n}(x^*),\frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_n}(x^*)\right]\right).\eta(\hat{x},x^*)\prec 0. \tag{3.20}$$

Thus by multiplying each interval by $\eta(\hat{x},x^*)$, we obtain

$$\begin{aligned}\left[\frac{\partial(-\lambda_2\underline{F_{\alpha^*}})}{\partial x_1}(x^*),\frac{\partial(\lambda_1\overline{F_{\alpha^*}})}{\partial x_1}(x^*)\right].\eta_1(\hat{x},x^*)+\cdots\\+\left[\frac{\partial(-\lambda_2\underline{F_{\alpha^*}})}{\partial x_n}(x^*),\frac{\partial(\lambda_1\overline{F_{\alpha^*}})}{\partial x_n}(x^*)\right].\eta_n(\hat{x},x^*)\prec 0.\end{aligned} \tag{3.21}$$

So there exists $\alpha \in [0,1]$ such as:

$$\begin{aligned}(3.22)\quad &\min\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x},x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x},x^*)\right\}+\cdots\\ &+\min\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_n}(x^*).\eta_n(\hat{x},x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_n}(x^*).\eta_n,(\hat{x},x^*)\right\}<0\end{aligned}$$

where

$$\begin{aligned}(3.23)\quad &\max\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x},x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_1}(x^*).\eta_1(\hat{x},x^*)\right\}+\cdots\\ &+\max\left\{\frac{\partial(-\lambda_2\underline{F_\alpha})}{\partial x_n}(x^*).\eta_n(\hat{x},x^*), \frac{\partial(\lambda_1\overline{F_\alpha})}{\partial x_n}(x^*).\eta_n(\hat{x},x^*)\right\}<0\end{aligned}$$

with strict inequality. By applying Proposition 2.10, the function $\lambda_2\underline{F_\alpha}+\lambda_1\overline{F_\alpha}$ is differentiable in x^* and we have:

$$\frac{\partial\big(-\lambda_2\underline{F_\alpha}+\lambda_1\overline{F_\alpha}\big)}{\partial x_1}(x^*).\eta_1(\hat{x},x^*)+\cdots + \frac{\partial\big(-\lambda_2\underline{F_\alpha}+\lambda_1\overline{F_\alpha}\big)}{\partial x_n}(x^*).\eta_n(\hat{x},x^*)<0$$

and consequently,

$$(3.24)\qquad \nabla\big(-\lambda_2\underline{F_\alpha}+\lambda_1\overline{F_\alpha}\big)(x^*)^T.\eta(\hat{x},x^*)\leq 0\ \forall\alpha\in[0,1].$$

From the assumption that the problem (FO) is Pseudoinvex, we have:

$$(3.25)\qquad \nabla g_j(x^*).\eta(\hat{x},x^*)\leq 0\ \forall j\in I(x^*).$$

By applying Motzkin's alternative theorem, there does not exist a $0<\epsilon_0\in\mathbb{R}$ and $0<\epsilon_j\in\mathbb{R},\ j\in I(x^*)$ such as:

$$(3.26)\qquad \epsilon_0(\alpha)\nabla\big(-\lambda_2\underline{F_\alpha}+\lambda_1\overline{F_\alpha}\big)(x^*)+\sum_{j\in I(x^*)}\epsilon_j(\alpha).\nabla g_j(x^*)=0\ \forall\alpha\in[0,1],$$

reciprocally, there does not exist some parameters $\mu_j(\alpha)\in\mathbb{R},\ \ j\in I(x^*)$ such as:

$$(3.27)\qquad \nabla\big(-\lambda_2\underline{F_\alpha}+\lambda_1\overline{F_\alpha}\big)(x^*)+\sum_{j\in I(x^*)}\mu_j(\alpha).\nabla g_j(x^*)=0,$$

where $\mu_j(\alpha)=\epsilon_j(\alpha)/\epsilon_0(\alpha)$.

If $I(x^*)$ is the set of active constraints, then we have $g_j(x^*)<0,\ j\notin I(x^*)$. That is why, if $j\notin I(x^*)$, then the condition (ii) involves that $\mu_j(\alpha)=0\ \forall\alpha\in[0,1]$. From Equation 3.27, there does not exist some parameters $0\leq\mu_j(\alpha)\in\mathbb{R}$ such as the conditions (i) and (ii) are satisfied and that is a contradiction. Thus the result holds. □

We can summarize the process of the method by using the following pseudo-code.

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(1) Using α-coupe to transform the fuzzy objective function into the real interval objective function.

(2) Using fuzzy arithmetic operations to transform the minimization of an interval function into a minimization of two deterministic functions.
(3) Using the weighted sum function to transform the bi-objective function into a single deterministic function.
(4) Using Karush-Kuhn-Tucker to reach the global optimal solution of the problem.

3.2. **Numerical experiences.** We have dealt with two didactic examples to highlight the steps of our method and prove its numerical performance.

3.2.1. *Problem 1.* [29] Let us consider the following problem

$$\begin{cases} \min \tilde{3}x_1 + \tilde{2}x_2^2 \\ s.t \\ (x_1-2)^2 + x_2^2 \leq 4 \\ x_1, x_2 \geq 0, \end{cases} \tag{3.28}$$

where $\tilde{3} = (2,3,5)$ and $\tilde{2} = (1,2,4)$ are fuzzy triangular numbers.

Let us set $F(x_1,x_2) = \tilde{3}x_1 + \tilde{2}x_2^2$ and $g_1(x_1,x_2) = (x_1-2)^2 + x_2^2 - 4$.
For all $\alpha \in [0,1]$, we have

$$\big[F(x_1,x_2)\big]^\alpha = \begin{cases} \big[(2+\alpha)x_1 + (1+\alpha)x_2^2, (5-2\alpha)x_1 + (4-2\alpha)x_2^2\big], & \text{if } x_1 \geq 0, x_2 \in \mathbb{R} \\ \big[(5-2\alpha)x_1 + (1+\alpha)x_2^2, (2+\alpha)x_1 + (4-2\alpha)x_2^2\big], & \text{if } x_1 < 0, x_2 \in \mathbb{R}. \end{cases} \tag{3.29}$$

As x_1 and $x_2 > 0$ are positive, the α-coupes of F are given by:

$$\big[F(x_1,x_2)\big]^\alpha = \big[(2+\alpha)x_1 + (1+\alpha)x_2^2, (5-2\alpha)x_1 + (4-2\alpha)x_2^2\big] \ \forall\alpha \in [0,1] \tag{3.30}$$

Let us set $\underline{F_\alpha}(x_1,x_2) = (2+\alpha)x_1 + (1+\alpha)x_{2^2}$ and $\overline{F_\alpha}(x_1,x_2) = (5-2\alpha)x_1 + (4-2\alpha)x_2^2$. By replacing $\underline{F_\alpha}(x_1,x_2)$ and $\overline{F_\alpha}(x_1,x_2)$ in Equation 3.10, we obtain problem 3.31. A Pareto optimal solution to this problem is also an optimal solution to the initial problem.

$$\begin{cases} \min \Big[\lambda\big((7-\alpha)x_1 + (5-\alpha)x_2^2\big) - (2+\alpha)x_1 - (1+\alpha)x_2^2\Big] \\ s.t \\ (x_1-2)^2 + x_2^2 - 4 \leq 0. \end{cases} \tag{3.31}$$

(*) Let us verify the first KKT condition

$$\nabla\Big[\lambda\big((7-\alpha)x_1 + (5-\alpha)x_2^2\big) - (2+\alpha)x_1 - (1+\alpha)x_2^2\Big] = \begin{bmatrix} \lambda(7-\alpha) - (2+\alpha) \\ 2\lambda(5-\alpha)x_2 - 2(1-\alpha)x_2 \end{bmatrix}$$

$$\nabla\Big[g(x_1,x_2)\Big] = \begin{bmatrix} 2(x_1-2) \\ 2x_2 \end{bmatrix}.$$

If $x^* = (0,0)$ satisfies the first KKT condition, then we have:

$$\lambda(7-\alpha) - (2+\alpha) - 4\mu = 0 \Rightarrow \lambda = \frac{4\mu + \alpha + 2}{7-\alpha}.$$

From the hypotheses of the theorem 3.8, for all $\alpha \in [0,1]$ $\lambda \in [0,1]$ and $\mu \geq 0$, we get:

$$\begin{aligned} 0 \leq \lambda \leq 1 &\Rightarrow 0 \leq \frac{4\mu + \alpha + 2}{7 - \alpha} \leq 1 \\ &\Rightarrow 0 \leq 4\mu + \alpha + 2 \leq 7 - \alpha \\ &\Rightarrow \frac{-\alpha - 2}{4} \leq \mu \leq \frac{5 - 2\alpha}{4}. \end{aligned}$$

Thus for all $\alpha \in [0,1]$, $\frac{-\alpha-2}{4} < 0$, we have $0 \leq \mu \leq \frac{5-2\alpha}{4}$. By bounding $\frac{5-2\alpha}{4}$, we get:

$$\begin{aligned} 0 \leq \alpha \leq 1 &\Rightarrow \frac{3}{4} \leq \frac{5 - 2\alpha}{4} \leq \frac{5}{4} \\ &\Rightarrow 0 \leq 4\mu + \alpha + 2 \leq 7 - \alpha. \end{aligned}$$

So $0 \leq \mu \leq \frac{3}{4} < 1 < \frac{5}{4}$. For $\mu = 1$, we have $\lambda = \frac{6+\alpha}{7-\alpha}$. In particular, for $\alpha = 1$, we get $\lambda > 1$, but $\lambda \in [0,1]$. Hence $\mu \in [0, \frac{3}{4}]$. We can conclude that the solution $x^* = (0,0)$ satisfies the first KKT condition for $\lambda = \frac{4\mu+\alpha-2}{7-\alpha}$, with $\alpha \in [0,1]$ and $\mu \in [0, \frac{3}{4}]$.

(**) Let us verify the second KKT condition:

$$\mu g(0,0) = 0$$

since $g(0,0) = 0$. Then the second condition (ii) is satisfied for all $\mu \in [0, \frac{3}{4}]$. From (*) and (**), we can conclude that $x^* = (0,0)$ satisfies KKT conditions for $\lambda = \dfrac{4\mu + \alpha - 2}{7 - \alpha}$, with $\alpha \in [0,1]$ and $\mu \in [0, \frac{3}{4}]$.

The following table gives our obtained solutions for Problem 1 and the value of the ranking function of each solution.

Table 1. Numerical solutions for Problem 1

α	0	0.2	0.8
λ	0	0.3	0.8
x	$(3; 2.64)$	$(4.82; 0.0002)$	$(0.1 * 10^{-7}; 0.14 * 10^{-3})$
$F(x)$	$(12.9696; 22.9392; 42.8784)$	$(9.64; 14.46; 24.1)$	$(0; 0; 0)$
$\mathcal{R}_1[F(x)]$	26.2624	16.066	0

The solution $x = (0; 0)$ has the best value ranking function. So it is the optimal solution of the problem. The solution is the same as obtained in the work of Chalco-Cano et al. [29].

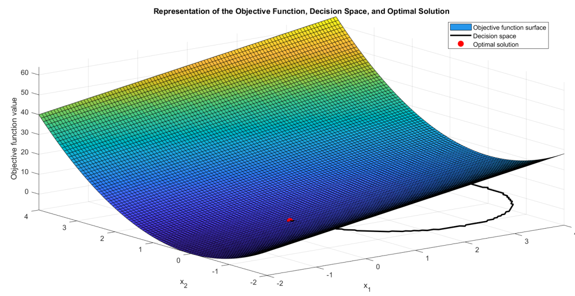

FIGURE 1. The objective function for each α and $\lambda \in [0,1], x_1, x_2 \in \widetilde{\mathcal{X}}$

3.2.2. *Problem 2.* [31] Let us consider the following problem

$$\begin{cases} \min\left\{F(x,y) = \tilde{a}x^2 + \tilde{b}y^2\right\} \\ s.t \\ \tilde{h}(x,y) = \tilde{c}(x-2)^2 + \tilde{d}(y-2)^2 \leq \tilde{k}, \\ x, y \geq 0, \end{cases} \tag{3.32}$$

where $\tilde{a} = (0,2,4)$, $\tilde{b} = (0,2,4)$, $\tilde{c} = (0,2,4)$, $\tilde{d} = (0,2,4)$ and $\tilde{k} = (0,2,4)$ are fuzzy triangular numbers.

By using the α-coupes, we obtain the following formulation of the problem:

$$\begin{cases} \min\left[2\alpha x^2 + 2\alpha y^2; (4-2\alpha)x^2 + (4-2\alpha)y^2\right] \\ s.t \\ (x-2)^2 + (y-2)^2 - 1 \leq 0 \\ x, y > 0. \end{cases} \tag{3.33}$$

Let us set $\underline{F_\alpha}(x,y) = 2\alpha x^2 + 2\alpha y^2$, $\overline{F_\alpha}(x,y) = (4-2\alpha)x^2 + (4-2\alpha)y^2$ and $g(x,y) = (x-2)^2 + (y-2)^2 - 1 \ \forall \alpha \in [0,1]$.

By using the weighted sum function, we transform the problem into the form below:

$$\begin{cases} \min\left[\lambda(4x^2 + 4y^2) - 2\alpha x^2 - 2\alpha y^2\right] \\ s.t \\ (x-2)^2 + (y-2)^2 - 1 \leq 0 \\ x, y > 0. \end{cases} \tag{3.34}$$

(*) Let's verify the first KKT condition:

$$\nabla\left[\lambda(4x^2 + 4y^2) - 2\alpha)x^2 - 2\alpha)y^2\right] = \begin{bmatrix} 8\lambda x - 4\alpha x \\ 8\lambda y - 4\alpha y, \end{bmatrix}$$

$$\nabla\left[g(x,y)\right] = \begin{bmatrix} 2(x-2) \\ 2(y-2) \end{bmatrix}.$$

We have $x^* = (1.2696; 1.2696)$. Let $1.2696 = \beta$ If $x^* = (\beta, \beta)$ satisfies KKT condition (i), then we have:

$$8\lambda x - 4\alpha\beta + 2\mu(\beta - 2) = 0 \Rightarrow \lambda = \frac{4\alpha\beta + 2\mu(2-\beta)}{8\beta} \Rightarrow \lambda = \frac{\alpha}{2} + \mu\Big(\frac{1}{2\beta} - \frac{1}{4}\Big).$$

From the assumptions of the Theorem 3.8, for $\alpha \in [0,1], \lambda \in [0,1]$ and $\mu \geq 0$, we get:

$$\begin{aligned} 0 \leq \lambda \leq 1 &\Rightarrow 0 \leq \frac{\alpha}{2} + \mu\Big(\frac{1}{2\beta} - \frac{1}{4}\Big) \leq 1 \\ &\Rightarrow \frac{-4\alpha\beta}{2-\beta} \leq \mu \leq \frac{2\beta(2-\alpha)}{2-\beta}. \end{aligned}$$

Thus for all $\alpha \in [0,1]$, $\mu \geq 0$, we have $\frac{-4\alpha\beta}{2-\beta} < 0$. So $0 \leq \mu \leq \frac{2\beta(2-\alpha)}{2-\beta}$. By bounding $\frac{2\beta(2-\alpha)}{2-\beta}$, we get:

$$\begin{aligned} 0 \leq \alpha \leq 1 &\Rightarrow 1 \leq 2 - \alpha \leq 2 \\ &\Rightarrow \frac{2\beta}{2-\beta} \leq \frac{2\beta(2-\alpha)}{2-\beta} \leq \frac{4\beta}{2-\beta}. \end{aligned}$$

Hence $0 \leq \mu \leq \frac{2\beta}{2-\beta} < \frac{4\beta}{2-\beta}$. We can finally conclude that $\mu \in [0, \frac{2\beta}{2-\beta}]$. We can conclude that the solution $x^* = (\beta, \beta)$ satisfies the first KKT condition for $\lambda = \frac{\alpha}{2} + \mu\Big(\frac{1}{2\beta} - \frac{1}{4}\Big)$, with $\alpha \in [0,1]$ and $\mu \in [0, \frac{2\beta}{2-\beta}]$.

(**) Let's verify the second KKT condition:

$$\mu g(\beta, \beta) = 0.$$

$g(\beta, \beta) = 0$, because the point x^* is located on the circle with center $(2,2)$ and radius 1. Then KKT condition (ii) is satisfied for all $\mu \in [0, \frac{3}{4}]$. From (*) and (**), we can conclude that $x^* = (\beta, \beta)$ satisfies KKT conditions for $\lambda = \frac{\alpha}{2} + \mu\Big(\frac{1}{2\beta} - \frac{1}{4}\Big)$, with $\alpha \in [0,1]$ and $\mu \in [0, \frac{2\beta}{2-\beta}]$.

Table 2. Numerical solutions for Problem 2

α	0	0.3	0.9
λ	0	0.1	0.9
(x, y)	$(1.45; 1.45)$	$(2.71; 2.71)$	$(1.2928; 1.2928)$
$F(x, y)$	$(0; 8.41; 16.82)$	$(0; 29.376; 58.752)$	$(0; 6.685; 13.370)$
$\mathcal{R}_1[F(x, y)]$	8.41	29.376	6.685

The solution $x = (1.2928; 1.2928)$ gives the best ranking value. This is the same as the solution obtained in the works of Panigrahini, Panda et al. However, this method allows obtaining sufficient solutions depending on the chosen preference threshold.

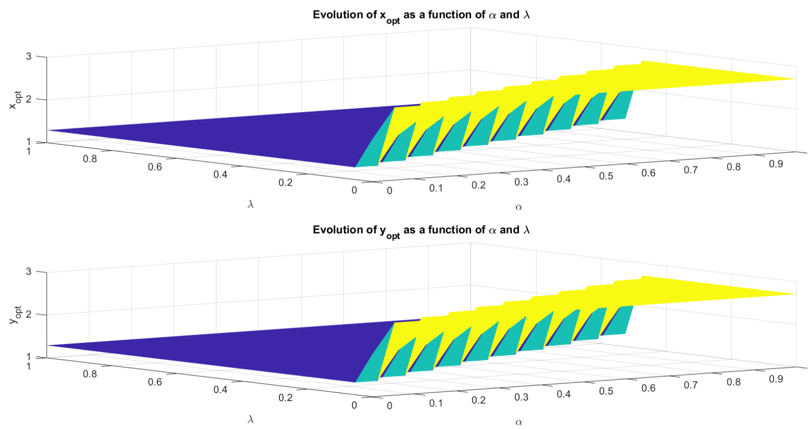

FIGURE 2. Solutions x_{opt} and y_{opt} for each α and $\lambda \in [0, 1]$

3.2.3. *Problem 3.* [50] Let us consider the following problem

$$(3.35)\qquad \begin{cases} \min \left[\widetilde{u}_1(x_1^2) + \widetilde{u}_2 \arctan x_2\right] \\ s.t \\ 1 - x_1 \leq 0, \\ -\arctan x_2 \leq 0, \\ x_1 \arctan x_2 = 0, \end{cases}$$

where $\widetilde{u}_1 = (1, 2, 3)$ and $\widetilde{u}_2 = (2, 5, 6)$.
This problem is equivalent to the following deterministic mono-objective problem:

$$(3.36)\qquad \begin{cases} \min \left[\lambda[4x_1^2 + (8 + 2\alpha)\arctan x_2] - (1 + \alpha)x_1^2 - (2 + 3\alpha)\arctan x_2\right] \\ s.t \\ 1 - x_1 \leq 0, \\ -\arctan x_2 \leq 0; \\ x_1 \arctan x_2 \leq 0, \\ -x_1 \arctan x_2 \leq 0, \end{cases}$$

where $\alpha, \lambda \in [0, 1]$. KKT system is as follows:

$$(3.37)\qquad \begin{cases} 8\lambda x_1 - 2(1 + \alpha)x_1 - \mu_1 + \mu_3 \arctan(x_2) - \mu_4 \arctan(x_2) = 0, \\ \lambda(8 + 2\alpha)\frac{1}{1+x_2^2} - (2 + 3\alpha)\frac{1}{1+x_2^2} - \mu_2 \frac{1}{1+x_2^2} + \mu_3 x_1 \frac{1}{1+x_2^2} - \mu_4 x_1 \frac{1}{1+x_2^2} = 0, \\ \mu_1(1 - x_1) = 0, \\ \mu_2(-\arctan(x_2)) = 0, \\ \mu_3(x_1 \arctan(x_2)) = 0, \\ \mu_4(-x_1 \arctan(x_2)) = 0. \end{cases}$$

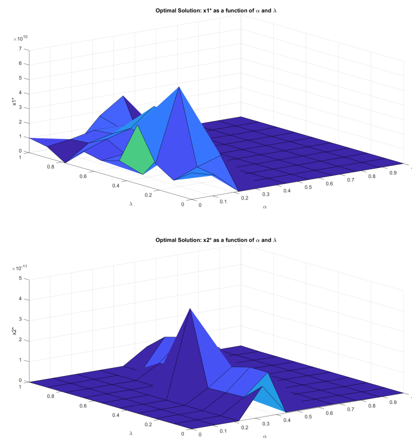

FIGURE 3. Solutions x_1^* and x_2^* for each α and $\lambda \in [0, 1]$

For $\alpha = 0.4$ and $\lambda = 0.35$, we have $x_0 = (1, 0)$, which satisfies KKT conditions for $\mu_1 > 0$ and $\mu_2 = \mu_3 = \mu_4 = 0$.

TABLE 3. Numerical solutions for Problem 3

α	$1/3$	0.89	0.4
λ	$1/3$	0.44	0.35
x	$(1.23; 0)$	$(1.33; 0)$	$(1; 0)$
$F(x)$	$(1.513; 3.03; 4.54)$	$(1.77; 3.54; 5.31)$	$(1; 2; 3)$
$\mathcal{R}_1[F(x)]$	3.07	3.54	2

The solution $x = (1; 0)$ gives the best ranking value. Then it is the optimal solution of the problem. The solution is the same as the one obtained in the works of Antczak.

3.2.4. *Problem 4.* [50] Let us consider the following problem

$$\begin{cases} \min \left[\widetilde{u}_1(0.002x^2 - 1000\ln x + 7500)\right] \\ s.t \\ -x \leq 0, \\ x - 400 \leq 0, \end{cases} \tag{3.38}$$

where $\widetilde{u}_1 = (1, 2, 3)$.

This problem is equivalent to the following deterministic mono-objective problem:
(3.39)

$$\begin{cases} \min \left[\lambda[0.008x^2 - 4000\ln x + 30000] - (1+\alpha)(0.002x^2 - 1000\ln x + 7500)\right] \\ \text{s.t} \\ -x \leq 0, \\ x - 400 \leq 0, \end{cases}$$

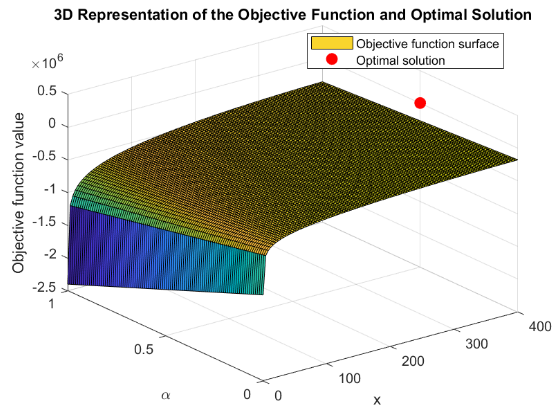

FIGURE 4. Objective function for $\alpha, \lambda \in [0,1]$ and $x_1, x_2 \in \widetilde{\mathcal{X}}$

where $\alpha, \lambda \in [0,1]$. The KKT system is as follows:

$$(3.40) \qquad \begin{cases} \lambda\left[0.016x - \frac{4000}{x}\right] - (1+\alpha)\left[0.004x - \frac{1000}{x}\right] - \mu_1 + \mu_2 = 0 \\ \mu_1 \geq 0 \\ \mu_2 \geq 0 \\ \mu_1(-x) = 0 \\ \mu_2(x-400) = 0. \end{cases}$$

For $\alpha = 0.2$ and $\lambda = 0.5$, we have $x_0 = 400$, which satisfies the KKT conditions for $\mu_1 = 0$ and $\mu_2 = 0.72$.

TABLE 4. Numerical solutions for Problem 4

α	$1/2$	$1/4$	0.2
λ	$1/3$	$1/2$	0.5
x	400	400	400
$F(x)$	$(1828; 3657; 5485)$	$(1828; 3657; 5485)$	$(1828; 3657; 5485)$
$\mathcal{R}_1[F(x)]$	3657.5	3657.5	3657.5

The solution $x = 400$ gives the best ranking value. Then it is the optimal solution of the problem. The solution is the same as the one obtained in the work of Antczak.

3.2.5. *Problem 5.* [51] Let us consider the following problem

$$(3.41) \qquad \begin{cases} \min\left[\widetilde{2}\ln((x^2+|x|+1)e) \ominus_H \widetilde{1}\right] \\ s.t \\ x^2 - 5x \leq 0, \end{cases}$$

where $\widetilde{1} = (0,1,2)$ and $\widetilde{2} = (1,2,4)$.

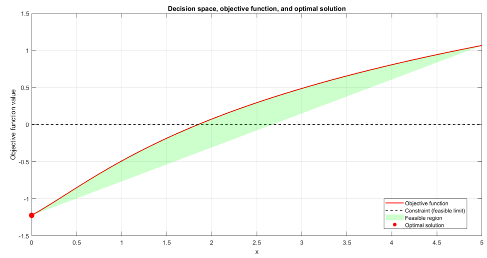

FIGURE 5. Objective function for $\alpha, \lambda \in [0, 1]$ and $x \in \widetilde{\mathcal{X}}$

This problem is equivalent to the following deterministic mono-objective problem:

(3.42)
$$\begin{cases} \min \left[\lambda[4\ln((x^2+|x|+1)e)] - (2\alpha-2) - 2\alpha\ln((x^2+|x|+1)e) + \alpha - 2\right] \\ \text{s.t} \\ x^2 - 5x \le 0, \end{cases}$$

where $\alpha, \lambda \in [0, 1]$. The KKT system is as follows:

(3.43)
$$\left\{ \lambda \cdot \frac{8x}{x^2+|x|+1} - 2\alpha \cdot \frac{8x}{x^2+|x|+1} + 2\mu(x-5) = 0, \mu(x2-5x) = 0, \mu \ge 0, x2 - 5x \le 0. \right.$$

Substitute $x = 0$ into each equation. The first equation becomes :

(3.44)
$$\lambda \cdot \frac{8\cdot 0}{0^2+|0|+1} - 2\alpha \cdot \frac{8\cdot 0}{0^2+|0|+1} + 2\mu(0-5) = 0 \quad \Rightarrow \quad 0-0-10\mu = 0 \quad \Rightarrow \quad \mu = 0.$$

The second equation becomes:

$$\mu(0^2 - 5\cdot 0) = 0 \quad \Rightarrow \quad 0 = 0, \tag{3.45}$$

which is obviously. The third equation becomes:

$$\mu \ge 0 \quad \Rightarrow \quad 0 \ge 0, \tag{3.46}$$

which is obviously. The fourth equation becomes:

$$0^2 - 5\cdot 0 \le 0 \quad \Rightarrow \quad 0 \le 0, \tag{3.47}$$

which is obviously. Then for $x = 0$, we get $\mu = 0$ and the solution is valid for all values of λ and $\alpha \in [0, 1]$. Thus $x_0 = 0$, satisfies KKT conditions.

TABLE 5. Numerical solutions for Problem 5

α	0	0	1
λ	0	1	1
x	0	1.23	1.115
F(x)	$(1; 2; 3)$	$(0; 3.64; 7.28)$	$(0; 3.42; 6.84)$
$\mathcal{R}_1[F(x)]$	2	3.64	3.42

3.3. Discussion.

In order to highlight the performance of our method, we have provided some theoretical and numerical results.

According to the proposed six (6) theorems (Theorems 3.2 through Theorem 3.8), we have proven that our method provides an optimal solution for the fuzzy nonlinear optimization problem with fuzzy triangular numbers.

According to Table 1, Table 2, Table 3, Table 4 and Table 5, we can confirm that our method is the best for the two problems we dealt with.

We note that the proposed method, in addition to obtaining the same optimal solutions as other existing methods, provides to the decision makers a wide range of possible solutions. Furthermore, as shown by some of the graphs, by varying alpha and beta, all possible solutions can be obtained.

4. Conclusion

In this work, we have presented a new method to solve the fuzzy nonlinear optimization problem with fuzzy triangular numbers. We described the main steps of the method and suggested some theorems to show that the solutions will be optimal. In addition, five didactic examples have been solved to show numerical performance and strengthen the theoretical results. Based on numerical results, we can conclude that the proposed method is a good method to solve the nonlinear fuzzy optimization problem when it is fuzzy triangular numbers that are concerned.

In our future work, we will focus on applying our results to solving multi-objective cases, which are many used in the modelling of many real-world problems.

References

[1] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975) 199–249.

[2] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-II, Inform. Sci. 8 (1975) 301–357.

[3] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, Inform. Sci. 8 (1975) 43–80.

[4] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319–330.

[5] M. Delgado, J. Kacprzyk, J. L. Verdegay and M. A. Vila, Fuzzy Optimization: Recent Advances, New York, Physica-Verlag (1994).

[6] W. A. Lodwick and J. Kacprzyk, Fuzzy Optimization: Recent Advances and Applications, Berlin, Springer (2010).

[7] J. C. Sama and K. Some, Solving fuzzy nonlinear optimization problems using null set concept, International Journal of Fuzzy Systems 26 (2023) 674–685.

[8] J. C. Sama, K. Some and A. Compaore, Hybrid approach for solving fuzzy fractional linear optimization problems, Ann. Fuzzy Math. Inform. 25 (2) (2023) 111–123.

[9] O. M. Saad and E. Fathy, On the solution of linear programming problems with rough interval coefficients in a fuzzy environment, Ann. Fuzzy Math. Inform. 18 (2) (2019) 161–171.

[10] H. C. Wu, Duality theory in fuzzy linear programming problems with fuzzy coefficients, Fuzzy Optimization and Decision Making 2 (2003) 61 – 73.

[11] D. Dubois and H. Prade, Fuzzy Sets and Systems : Theory and Applications, Academic Press (1980).
[12] O. Abu Arqub, Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm–Volterra integrodifferential equations, Neural Computing and Applications 28 (7) (2017) 1591–1610.
[13] O. Abu Arqub, R. Mezghiche and B. Maayah, Fuzzy M-fractional integrodifferential models: Theoretical existence and uniqueness results, and approximate solutions utilizing the Hilbert reproducing kernel algorithm, Frontiers in Physics 11 (2023) 1–20.
[14] O. Abu Arqub, J. Singh, B. Maayah and M. Alhodaly, Reproducing kernel approach for numerical solutions of fuzzy fractional initial value problems under the Mittag-Leffler Kernel differential operator, Mathematical Methods in the Applied Sciences 46 (7) (2023) 7965–7986.
[15] O. Arqub, J. Singh and M. Alhodaly, Adaptation of kernel functions-based approach with ABC distributed order derivative for solutions of fuzzy fractional Volterra and Fredholm integrodifferential equations, Mathematical Methods in the Applied Sciences 46 (7) (2021) 7807–7834.
[16] B. Maayah and O. Abu Arqub, Uncertain M-fractional differential problems: Existence, uniqueness, and approximations using Hilbert reproducing technique provisioner with the case application: Series resistor-inductor circuit, Physica Scripta, 99 (2) (2024) 1–25.
[17] M. R. Seikh, P. K. Nayak and M. Pal, Generalized triangular fuzzy numbers in intuitionistic fuzzy environment, International Journal of Engineering Research and Development 5 (1) (2012) 08–13.
[18] M. R. Seikh, P. K. Nayak and M. Pal, Notes on triangular intuitionistic fuzzy numbers, International Journal of Mathematics in Operational Research 5 (4) (2013) 446–465.
[19] M. R. Seikh, P. K. Nayak and M. Pal, Application of intuitionistic fuzzy mathematical programming with exponential membership and quadratic non-membership functions in matrix games, Ann. Fuzzy Math. Inform 9 (2) (2015) 183–195.
[20] M. R. Seikh and S. Dutta, Solution of interval-valued matrix games using intuitionistic fuzzy optimisation technique: An effective approach, International Journal of Mathematics in Operational Research 20 (3) (2021) 297–322.
[21] M. R. Seikh, P. K. Nayak and M. Pal, Matrix games in intuitionistic fuzzy environment, International Journal of Mathematics in Operational Research 5 (6) (2013) 693–708.
[22] K. Sivakumar and A. Saraswathi, Fuzzy mathematical approach for solving multi-objective fuzzy linear fractional programming problem with trapezoidal fuzzy numbers, Mathematical Modelling of Engineering Problem 11 (1) (2024) 255–262.
[23] N. Okumus and D. Kesen, Power aggregation operators on trapezoidal fuzzy multi-numbers and their applications to a zero-waste problem, Ann. Fuzzy Math. Inform. 27 (3) (2024) 169–189.
[24] J. C. Sama, D. P. Traore and K. Some, New approach to solving fuzzy multiobjective linear fractional optimization problems, International Journal of Analysis and Applications 22 (2024) 1–11.
[25] H. Rommelfanger and R. Slowinski, Fuzzy linear programming with single or multiple objective functions. In R. Slowinski (Ed.), Fuzzy sets in decision analysis, operations research and statistics. Handbook fuzzy sets series, Boston, MA: Kluwer Academic Publisher 179–213 (1988).
[26] H. C. Wu, The Karush–Kuhn–Tucker optimality conditions for multi-objective programming problems with fuzzy-valued objective functions, Fuzzy Optimization and Decision Making 8 (2009) 1 – 28.
[27] V. Chankong and Y. Y. Haimes, Multiobjective Decision Making: Theory and Methodology, New York, North-Holland (1983).
[28] N. Yu and D. Qiu, The Karush-Kuhn-Tucker optimality conditions for the fuzzy optimization problems in the quotient space of fuzzy numbers, Complexity 2017 (2017) Article ID 1242841.
[29] Y. Chalco-Cano, W. A. Lodwick, R. Osuna-Gómez and A. Rufian-Lizana, The Karush Kuhn Tucker optimality conditions for fuzzy optimization problems, Fuzzy Optimization and Decision Making 15 (2016) 57 – 73.

[30] H. C. Wu, The Karush–Kuhn–Tucker optimality conditions for the optimization problem with fuzzy-valued objective function, Mathematical Methods of Operations Research 66 (2007) 203 – 224.
[31] M. Panigrahi, G. Panda and S. Nanda, Convex fuzzy napping with differentiability and its application in fuzzy optimization, European Journal of Operational Research 185 (2008) 47–62.
[32] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, J. Math. Anal. Appl. 91 (1983) 552 – 558.
[33] A. Rufián-Lizana, Y. Chalco-Cano, R. Osuna-Gómez and G. Ruiz-Garzón, On invex fuzzy mappings and fuzzy variational-like inequalities, Fuzzy Sets and Systems 2012 (2012) 84–98.
[34] Z. Wu and J. Xu, Generalized convex fuzzy mappings and fuzzy variational-like inequality, Fuzzy Sets and Systems 160 (2009) 1590 –1619.
[35] [L. C. De Barros, R. C. Bassanezi and W. A. Lodwick, A First Course in Fuzzy Logic, Fuzzy Dynamical Systems and Biomathematics: Theory and Applications, Springer-Verlag (2017).
[36] L. Stefanini, A Generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets and Systems 161 (2010) 1564– 1584.
[37] A. Kauffman and M. M. Gupta, Introduction to Fuzzy Arithmetic: Theory and Applications, Van Nostrand Reinhold (1985).
[38] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications 114 (2) (1986) 409–422.
[39] M. Hanss, Applied Fuzzy Arithmetic: An Introduction with Engineering Applications, Springer (2005).
[40] H. J. Zimmermann, Fuzzy Set Theory– and Its Applications, Kluwer Academic Publishers (2001).
[41] R. R. Yager, Defining the fuzzy set operations, Fuzzy Sets and Systems 6 (1981) 185–190.
[42] Y. Chalco-Cano, A. Rufian-Lizana, H. Román-Flores and M. D. Jiménez-Gamero, Calculus for interval-valued functions using generalized Hukuhara derivative and applications, Fuzzy Sets and Systems 219 (2013) 49 – 67.
[43] M. A. Hanson, On sufficiency of Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.
[44] O. L. Mangasarian, Nonlinear Programming, McGraw-Hill (1969).
[45] M. Avriel, W. E. Diewert, S. Schaible and I. Zang, Generalized Concavity, Springer (2010).
[46] B. D. Craven, Mathematical Programming and Control Theory, Chapman and Hall (1981).
[47] M. S. Bazaraa, H. D. Sherali and C. M. Shetty, tNonlinear Programming: Theory and Algorithms, 3rd ed., Wiley–Interscience (2006).
[48] M. S. Bazaraa, H. D. Sherali and C. M. Shetty, Nonlinear Programming: Theory and Algorithms, Wiley-Interscience (2013).
[49] A. Nemirovski and K. Roos, Motzkin Transposition Theorem, C. Floudas and P. Pardalos, (eds) Encyclopedia of Optimization, Springer, Boston, MA. 2345–2348 (2008).
[50] T. Antczak, On optimality for fuzzy optimization problems with granular differentiable fuzzy objective functions, Expert Systems with Applications 240 (2024) 1–16.
[51] T. Antczak, Optimality conditions for invex nonsmooth optimization problems with fuzzy objective functions, Fuzzy Optimization and Decision Making 22 (1) (2023) 1–21.

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