Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 29, No. 2, (April 2025) pp. 143–169
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2025.29.2.143

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 29, No. 2, April 2025

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Received 1 November 2024; Revised 14 November 2024; Accepted 11 December 2024

Abstract. This paper introduces states on pseudo L-algebras. We first define pseudo KL-algebras and pseudo CL-algebras, then explore their properties and relationships between pseudo L-algebras and pseudo BCK-algebras. Next, we define Bosbach states on pseudo L-algebras and discuss their properties. At the same time, we also introduce the fantastic ideal and the pseudo MV-ideal, also use these ideals to study the existence of states on pseudo L-algebras. Moreover, we introduce the state-morphism on pseudo L-algebras, discuss the relationship between Bosbach states and state-morphisms, and give an equivalent description of state-morphisms. Further, we introduce the concept of Riečan states and study their properties, as well as the relationship between Bosbach states and Riečan states. We have shown that any Bosbach state can be considered a Riečan state in bounded pseudo L-algebras, but not the other way around. Finally, we prove that Riečan states on special pseudo L-algebras are Bosbach states.

2020 AMS Classification: 03E72, 08A72

Keywords: Pseudo L-algebra, Bosbach state, Riečan state, State-morphism, Ideal.

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1. Introduction

In 2008, Rump [1] introduced the concept of L-algebras, a quantum structure closely associated with non-classical logical algebras and the decryption of quantum Yang-Baxter equations. In 2021, Ciungu [2] presented L-algebras arising from other structures, such as BCK-algebras, pseudo MV-algebras, pseudo BL-algebras, pseudo hoops, bounded Rℓ-monoids, BE-algebras and Hilbert algebras.

As a result, the study of L-algebras has increasingly captured the attention of scholars. In 2022, Mao et al. [3] presented the EL-algebra, a novel algebraic structure that extends from the realm of L-algebras. In 2023, Kologani [4] investigated the connection between L-algebras and other logical algebras. Furthermore, in the same year, Kologani [5] investigated different types of L-algebras and introduced the concept of ideals in L-algebras.

In 2022, Xin et al. [6] introduced pseudo L-algebras, which were the multiplication reducts of pseudo hoops and structures that combined two L-algebras with one compatible order. It was proven that every pseudo hoop gave rise to a pseudo L-algebra, and every pseudo effect algebra led to a pseudo L-algebra. Moreover, the extension of pseudo L-algebras was studied, which included their pseudo self-similar closures and structured groups.

The notion of states is an analogue to probability measure, and plays a very important role in the theory of quantum structures. In 1995, Mundici [7] introduced states on MV-algebras as averaging the truth value in Łukasiewicz logic. States constitute measures on their associated MV-algebras, which generalize the usual probability measures on Boolean algebras. Then, the notion of state has been extended to other logic algebras such as BL-algebras, residuated lattices, EQ-algebras, and their non-commutative cases. Different approaches to the generalization mainly give rise to two different notions, namely Riečan states and Bosbach states. In 2001, Dvurečenskij [8] proved a state on MV-algebras always exists. In 2004, Georgescu [9] defined Bosbach states and Riečan states on pseudo BL-algebras, and for a good pseudo BL-algebra, he proved that any Bosbach state is also a Rie čan state. He asked to find an example of Riečan state on a good pseudo BL-algebra which is not a Bosbach state. In 2017, Xin et al. [10]) studied states on pseudo BCI-algebras. In 2020, Xin et al. [11]studied the notions of fantastic filters and investigated the existence of Bosbach states and Riečan states on EQ-algebras by using of fantastic filters. In 2021, Hua [12] studied states on L-algebras and derivations of L-algebras. In 2022, Shi et al. [13] investigated states on pseudo EQ-algebras and proved that any Bosbach state is a Riečan state in normal pseudo EQ-algebras, but the inverse is not true in general.

The first motivation, with the progress of the times, the adaptability of classical logic in people's reasoning and thinking activities is gradually insufficient, which gives birth to non-classical logic, in which fuzzy logic is particularly important. As the corresponding algebraic structure of fuzzy logic, the study of pseudo L-algebra can effectively promote the development of fuzzy logic. Moreover, the pseudo L-algebra has a quantum background, which is helpful to the cross study of fuzzy logic and quantum logic. The second motivation, as a probability measure, states are of great significance in measuring the average truth value of propositions in Łukasiewicz logic and are crucial in quantum structure theory. The study of state theory on pseudo L-algebra is to extend the concept of state to a broader fuzzy structure, which can lay a solid algebraic foundation for fuzzy event probabilistic reasoning. The third motivation, the existence of states has always attracted much attention. Studying the existence of states in logic systems is equivalent to studying the testability problem, which is of great value to the whole research.

The organization of this article is as follows: In Section 2, we review several basic definitions and properties used in this paper. In Section 3, we introduce and study ideals and congruence relations on pseudo L-algebras, and we provide the concepts of special ideals. In Section 4, we provide the concepts of Bosbach states and state-morphisms on pseudo L-algebras and explore some fundamental properties. We also introduce fantastic ideals and pseudo MV-ideals, also use these ideals to study the existence of states. In Section 5, we introduce the concept of Riečan states and investigate their properties as well as the relationship between Bosbach states and Riečan states.

2. Preliminaries

This section recalls fundamental definitions and properties of algebras pertinent to this paper.

Definition 2.1 ([6]). A *pseudo L-algebra* is an algebra $(L, \to, \rightsquigarrow, 1)$ with two binary operations $\to$, $\rightsquigarrow$ and one constant 1 such that: for all $x, y, z \in L$,
$(PL1)$ $1 \to x = x = 1 \rightsquigarrow x$ and $x \to 1 = 1$,
$(PL2)$ $x \to x = 1$,
$(PL3)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,
$(PL4)$ $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z)$,
$(PL5)$ $x \to y = y \to x = 1$ implies $x = y$,
$(PL6)$ $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$.

Remark 2.2 ([6]). Let L be a pseudo L-algebra. Define a binary relation $\leq$ as follows:

$$x \leq y \Leftrightarrow x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1.$$

Then $\leq$ is a partial order on L.

Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra. We can see that the reducts $(L, \to, 1)$ and $(L, \rightsquigarrow, 1)$ of $(L, \to, \rightsquigarrow, 1)$ are both L-algebras.

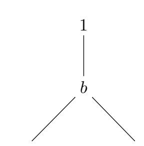
For an element x of an L-algebra with 0, we define $x^- := x \to 0$.

We say that an L-algebra L has a negation if L admits a smallest element 0 such that the map $x \mapsto x^-$ is bijective. The inverse map will then be denoted by $x \mapsto x^\sim$.

Example 2.3. Let $L = \{a, b, c, 1\}$ such that $a, c \leq b \leq 1$, and b and c are incomparable. Define the operations $\to$ and $\rightsquigarrow$ using the following two tables.

$\to$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	a	1	1	1
1	a	b	c	1

$\rightsquigarrow$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	b	1	1	1
1	a	b	c	1



In this example we can see that a pseudo L-algebra can have no 0, but an L-algebra with a negation must have 0.

Example 2.4. Let $L = \{0, a, b, 1\}$ be a lattice such that $0 < a < b < 1$. Define the operations $\to$ and $\rightsquigarrow$ using the following two tables.

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then we can verify that $(L, \rightarrow, \rightsquigarrow, 1)$ is a pseudo L-algebra, but it is not an L-algebra with a negation. Since the map $x \mapsto x^-$ is not bijective.

Example 2.5. Let $L = \{0, a, b, c, 1\}$ and define the operation $\rightarrow$ on L as follows:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	a	a	1
b	b	b	1	b	1
c	c	c	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	b	c	1
b	b	a	1	c	1
c	c	a	b	1	1
1	0	a	b	c	1

Clearly, $(L, \rightarrow, 0, 1)$ is an L-algebra with a negation. Now, we show that $(L, \rightsquigarrow, 1)$ is not an L-algebra, which means $(L, \rightarrow, \rightsquigarrow, 1)$ is not a pseudo L-algebra, where $x \rightsquigarrow y = y^{\sim} \rightarrow x^{\sim}$. Since $(c \rightsquigarrow b) \rightsquigarrow (c \rightsquigarrow 0) = b \rightsquigarrow c = c \neq (b \rightsquigarrow c) \rightsquigarrow (b \rightsquigarrow 0) = c \rightsquigarrow b = b$, which implies that $(PL4)$ is not true. Therefore, $(L, \rightarrow, \rightsquigarrow, 1)$ is not a pseudo L-algebra. Besides, we also can prove that $(L, \rightarrow, 1)$ is not a semi-regular L-algebra. Since $((a \rightarrow 1) \rightarrow b) \rightarrow ((1 \rightarrow a) \rightarrow b) = b \rightarrow a = b \neq 1 = b \rightarrow b = ((a \rightarrow 1) \rightarrow b) \rightarrow b$.

Proposition 2.6 ([6])**.** *Let L be a pseudo L-algebra. Then the following are equivalent: for all $x, y, z \in L$,*
(1) *$y \leq z$,*
(2) *$x \rightarrow y \leq x \rightarrow z$,*
(3) *$x \rightsquigarrow y \leq x \rightsquigarrow z$.*

Definition 2.7 ([14])**.** A pseudo L-algebra L is called a *bounded pseudo L-algebra*, if there is an element $0 \in L$ such that $0 \leq x$ for all $x \in L$.

Let $(L, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded pseudo L-algebra. We define two negations, denoted as $-$ and $\sim$, as follows: $x^- = x \rightarrow 0$ and $x^{\sim} = x \rightsquigarrow 0$ for all $x \in L$.

A bounded pseudo L-algebra L is said to be *good*, if $x^{-\sim} = x^{\sim-}$ for all $x \in L$.

A bounded pseudo L-algebra L is said to *have the pseudo-double negation property* (pDN for short), if $x^{-\sim} = x^{\sim-} = x$ for all $x \in L$.

Definition 2.8 ([14])**.** A pseudo L-algebra L is called a *pseudo KL-algebra*, if it satisfies (K) $x \leq y \rightarrow x$ and $x \leq y \rightsquigarrow x$ for all $x, y, z \in L$.

Proposition 2.9 ([14])**.** *Let L be a pseudo KL-algebra. Then for all $x, y, z \in L$, $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$.*

Definition 2.10 ([14])**.** A pseudo L-algebra L is called a *pseudo CL-algebra*, if it satisfies the condition (C): $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$ for all $x, y, z \in L$.

Proposition 2.11. *Let L be a pseudo CL-algebra. Then, we have the following: for all $x, y, z \in L$,*
(1) $x \leq (x \to y) \rightsquigarrow y$ *and* $x \leq (x \rightsquigarrow y) \to y$,
(2) $(((x \to y) \rightsquigarrow y) \rightsquigarrow x) \rightsquigarrow (((x \to y) \rightsquigarrow y) \rightsquigarrow z) = x \rightsquigarrow z$ *and*
$(((x \rightsquigarrow y) \to y) \rightsquigarrow x) \rightsquigarrow (((x \rightsquigarrow y) \to y) \rightsquigarrow z) = x \rightsquigarrow z$,
(3) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$ *and* $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$,
(4) $x \leq y \to z$ *if and only if* $y \leq x \rightsquigarrow z$,
(5) *if* $x \leq y$, *then* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$,
(6) $((x \to y) \rightsquigarrow y) \to y = x \to y$ *and* $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$,
(7) $y \leq (x \to y) \rightsquigarrow y$ *and* $y \leq (x \rightsquigarrow y) \to y$,
if L is bounded, then
(8) $x \leq x^{-\sim}$ *and* $x \leq x^{\sim -}$,
(9) $x \to y^{-\sim} = y^- \rightsquigarrow x^-$ *and* $x \rightsquigarrow y^{\sim -} = y^\sim \to x^\sim$,
(10) $x \to y \leq y^- \rightsquigarrow x^-$ *and* $x \rightsquigarrow y \leq y^\sim \to x^\sim$,
(11) $x^{-\sim -} = x^-$ *and* $x^{\sim - \sim} = x^\sim$.

Proof. (1) By (C), we have $x \to ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow (x \to y) = 1$. Then $x \leq ((x \to y) \rightsquigarrow y)$. Similarly, we can get $x \leq ((x \rightsquigarrow y) \to y)$.

(2) By (PL4) and (1), we get

$$\begin{aligned} &(x \to y) \rightsquigarrow y) \rightsquigarrow x) \rightsquigarrow ((x \to y) \rightsquigarrow y) \rightsquigarrow z) \\ =&(x \rightsquigarrow ((x \to y) \rightsquigarrow y)) \rightsquigarrow (x \rightsquigarrow z) \\ =&(1 \rightsquigarrow (x \rightsquigarrow z)) \\ =&(x \rightsquigarrow z) \end{aligned}$$

Similarly, we have $(((x \rightsquigarrow y) \to y) \rightsquigarrow x) \rightsquigarrow (((x \rightsquigarrow y) \to y) \rightsquigarrow z) = x \rightsquigarrow z$.

(3) By (PL1), (PL3) and (C), we have

$$\begin{aligned} &(x \to y) \to ((y \to z) \rightsquigarrow (x \to z)) \\ =&(y \to z) \rightsquigarrow ((x \to y) \to (x \to z)) \\ =&(y \to z) \rightsquigarrow ((y \to x) \to (y \to z)) \\ =&(y \to x) \to ((y \to z) \rightsquigarrow (y \to z)) \\ =&(y \to x) \to 1 \\ =&1 \end{aligned}$$

Then $(x \to y) \leq ((y \to z) \rightsquigarrow (x \to z))$. Similarly, we prove

$$x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z).$$

(4) Since $x \leq y \to z$, $x \rightsquigarrow (y \to z) = 1$. Then by (C), $y \to (x \rightsquigarrow z) = 1$. Thus $y \leq x \rightsquigarrow z$. Similarly, we get the inverse implication.

(5) Since $y \to z \leq y \to z$, $y \leq (y \to z) \rightsquigarrow z$ by (4). From $x \leq y$, we get $x \leq y \leq (y \to z) \rightsquigarrow z$. Then, $y \to z \leq x \to z$. Similarly, we prove $y \rightsquigarrow z \leq x \rightsquigarrow z$.

(6) By (1) and (5), we can get $((x \to y) \rightsquigarrow y) \to y \geq x \to y$. Now, we prove

$$((x \to y) \rightsquigarrow y) \to y \leq x \to y.$$

It is obvious that

$$\begin{aligned} &((x \to y) \rightsquigarrow y) \to ((x \to y) \rightsquigarrow y) = 1 \\ iff&(x \to y) \rightsquigarrow (((x \to y) \rightsquigarrow y) \to y) = 1 \\ iff&x \to y \leq ((x \to y) \rightsquigarrow y) \to y. \end{aligned}$$

Then $((x \to y) \rightsquigarrow y) \to y = x \to y$. Similarly, we get $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$.

(7) $y \to (x \to y) \rightsquigarrow y = (x \to y) \rightsquigarrow (y \to y) = (x \to y) \rightsquigarrow 1 = 1$. Then $y \leq (x \to y) \rightsquigarrow y$. Similarly, we get $y \leq (x \rightsquigarrow y) \to y$.

(8) Setting $y = 0$ in (1).

(9) By (C), we get

$$x \to y^{-\sim} = x \to ((y \to 0) \rightsquigarrow 0) = (y \to 0) \rightsquigarrow (x \to 0) = y^{-} \rightsquigarrow x^{-}.$$

Similarly, we have

$$x \to y^{\sim -} = x \rightsquigarrow ((y \rightsquigarrow 0) \to 0) = (y \rightsquigarrow 0) \to (x \rightsquigarrow 0) = y^{\sim} \to x^{\sim}.$$

(10)-(11) The proofs are obvious. □

Definition 2.12. A pseudo L-algebra L is called a *commutative pseudo L-algebra*, if it satisfies the following conditions: for all $x, y \in L$,

(C1) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$,

(C2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

Example 2.13. Consider the bounded pseudo L-algebra L from Example 2.3. Now, when $x = a, y = c$, we have

$$\begin{aligned} &(a \to c) \rightsquigarrow c = a \rightsquigarrow c = a;\ (c \to a) \rightsquigarrow a = a \rightsquigarrow a = 1, \\ &(a \rightsquigarrow c) \to c = a \to c = a;\ (c \rightsquigarrow a) \to a = b \to a = a. \end{aligned}$$

Then the condition (C1) does not hold, while the condition (C2) holds. This shows that the condition (C1) and the condition (C2) are independent of each other, i.e., they are not related or dependent on each other.

For all $x, y \in L$, define:

$$x \vee_1 y = (x \to y) \rightsquigarrow y, x \vee_2 y = (x \rightsquigarrow y) \to y.$$

Proposition 2.14. *Let L be a pseudo L-algebra. Then we have the following: for all $x, y, z \in L$,*

(1) $1 \vee_1 x = x \vee_1 1 = 1 = 1 \vee_2 x = x \vee_2 1$,

(2) $x \leq y$ *implies* $x \vee_1 y = y$ *and* $x \vee_2 y = y$,

(3) $x \vee_1 x = x \vee_2 x = x$,

if L is bounded and satisfies the condition (C), then

(4) *if* $x_1 \leq x_2$, *then* $x_1 \vee_1 y \leq x_2 \vee_1 y$ *and* $x_1 \vee_2 y \leq x_2 \vee_2 y$,

(5) $x \vee_1 y^{-\sim} = x^{-\sim} \vee_1 y^{-\sim}$ *and* $x \vee_2 y^{\sim -} = x^{\sim -} \vee_2 y^{\sim -}$, (6) $x \vee_1 y^{\sim} = x^{-\sim} \vee_1 y^{\sim}$ *and* $x \vee_2 y^{\sim} = x^{\sim -} \vee_2 y^{-}$,

(7) $(x \vee_1 y) \to y = x \to y$ *and* $(x \vee_2 y) \rightsquigarrow y = x \rightsquigarrow y$.

Proof. (1)-(3) The proofs are obvious.

(4) Suppose $x_1 \leq x_2$. Then by Proposition 2.11(5), we have

$$x_2 \to y \leq x_1 \to y \text{ and } (x_1 \to y) \rightsquigarrow y \leq (x_2 \to y) \rightsquigarrow y.$$

Thus $x_1 \vee_1 y \leq x_2 \vee_1 y$. Analogously, we prove $x_1 \vee_2 y \leq x_2 \vee_2 y$.

(5) Note that

$$x \vee_1 y^{-\sim} = (x \to y^{-\sim}) \rightsquigarrow y^{-\sim} \text{ and } x^{-\sim} \vee_1 y^{-\sim} = (x^{-\sim} \to y^{-\sim}) \rightsquigarrow y^{-\sim}.$$

Then it suffices to show that $x \to y^{-\sim} = x^{-\sim} \to y^{-\sim}$. By Proposition 2.11(11), we know that $x^{-\sim} \to 0 = x \to 0$. Thus we have

$$\begin{aligned} x \to y^{-\sim} &= x \to [(y \to 0) \rightsquigarrow 0] \\ &= (y \to 0) \rightsquigarrow (x \to 0) \\ &= (y \to 0) \rightsquigarrow (x^{-\sim} \to 0) \\ &= x^{-\sim} \to [(y \to 0) \rightsquigarrow 0] \\ &= x^{-\sim} \to y^{-\sim}. \end{aligned}$$

So $x \vee_1 y^{-\sim} = x^{-\sim} \vee_1 y^{-\sim}$. Analogously, $x \vee_2 y^{\sim -} = x^{\sim -} \vee_2 y^{\sim -}$ holds.

(6) By taking $y = y^{-}$ and $y = y^{\sim}$ in (5), we immediately get the conclusions.

(7) By proposition 2.11(6), we immediately get the conclusions. □

Definition 2.15 ([14])**.** Let L be a pseudo L-algebra.

(i) If $x \vee_1 y = y \vee_1 x$ for all $x, y \in L$, then L is said to be $\vee_1 -$ *commutative*.

(ii) If $x \vee_2 y = y \vee_2 x$ for all $x, y \in L$, then L is said to be $\vee_2 -$ *commutative*.

Proposition 2.16. *Let L be a pseudo L-algebra and L satisfies the condition (C).*

(1) If L is $\vee_1 -$ commutative, then $(L, \vee_1)$ forms a joint semi-lattice (under $\leq$).

(2) If L is $\vee_2 -$ commutative, then $(L, \vee_2)$ forms a joint semi-lattice (under $\leq$).

Proof. (1)We need to prove that $x \vee_1 y$ is the least upper bound of $\{x, y\}$ for all $x, y \in L$. Assume that $z \in L$ is an upper bound of $\{x, y\}$. Then we have

$$\begin{aligned} (x \vee_1 y) \to z &= (x \vee_1 y) \to (y \vee_1 z) \\ &= (x \vee_1 y) \to (z \vee_1 y) \\ &= (x \vee_1 y) \to ((z \to y) \rightsquigarrow y) \\ &= (z \to y) \rightsquigarrow ((x \vee_1 y) \to y) \\ &= (z \to y) \rightsquigarrow (x \to y) \\ &= x \to ((z \to y) \rightsquigarrow y) \\ &= x \to (z \vee_1 y) \\ &= x \to z \\ &= 1. \end{aligned}$$

It follows that $x \vee_1 y \leq z$.

(2) Similarly as (1). □

Definition 2.17 ([15])**.** A pseudo BCK-algebra (more precisely, reversed left-pseudo BCK-algebra) is a structure $X = (X, \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on X, $\rightarrow$ and $\rightsquigarrow$ are binary operations on X, and 1 is an element of X satisfies the following conditions: for all $x, y, z \in X$,

$(X1)$ $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,
$(X2)$ $x \leq (x \rightarrow y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \rightarrow y$,
$(X3)$ $x \leq x$,
$(X4)$ $x \leq 1$,
$(X5)$ if $x \leq y$ and $y \leq x$, then $x = y$,
$(X6)$ $x \leq y$ if and only if $x \rightarrow y = 1$ if and only if $x \rightsquigarrow y = 1$.

Proposition 2.18 ([14])**.** *Any pseudo CL-algebra is a pseudo KL-algebra.*

Conversely, a pseudo KL-algebra may not be a pseudo CL-algebra, as the following example.

Example 2.19. Let $L = \{a, b, c, 1\}$ be a lattice such that $a < b < c < 1$. Define the operations $\rightarrow$ and $\rightsquigarrow$ using the following two tables:

$\rightarrow$	a	b	c	1
a	1	1	1	1
b	c	1	1	1
c	a	b	1	1
1	a	b	c	1

$\rightsquigarrow$	a	b	c	1
a	1	1	1	1
b	b	1	1	1
c	a	b	1	1
1	a	b	c	1

Then, $(L, \rightarrow, \rightsquigarrow, 1)$([14]) is a pseudo KL-algebra which is not a pseudo CL-algebra, because

$$b \rightarrow (c \rightsquigarrow a) = b \rightarrow a = c \neq 1 = c \rightsquigarrow c = c \rightsquigarrow (b \rightarrow a).$$

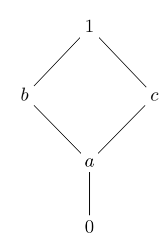
Proposition 2.20 ([14])**.** *Any pseudo CL-algebra is a pseudo BCK-algebra.*

Note: The reverse of Proposition 2.20 is not true. We will provide an example.

Example 2.21 ([16])**.** Let $L = \{0, a, b, c, 1\}$ be a lattice. Define the operations $\rightarrow$ and $\rightsquigarrow$ by the following two tables.

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	a	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1



Then $(L, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo BCK-algebra, but it is not a pseudo CL-algebra, because

$$b = c \rightarrow a = (b \rightarrow c) \rightarrow (b \rightarrow a) \neq (c \rightarrow b) \rightarrow (c \rightarrow a) = b \rightarrow b = 1.$$

Proposition 2.22. *Any commutative pseudo KL-algebra is a pseudo BCK-algebra.*

Proof. Clearly, (A3)-(A6) hold. We need to prove (A1) and (A2). By (K), we have

$$x \to ((x \to y) \rightsquigarrow y) = x \to ((y \to x) \rightsquigarrow x) = 1$$

and

$$x \to ((x \rightsquigarrow y) \to y) = x \to ((y \rightsquigarrow x) \to x) = 1.$$

Then $x \leq ((x \to y) \rightsquigarrow y)$ and $x \leq ((x \rightsquigarrow y) \to y)$. Thus (A2) holds.

Now by (PL3) and (K), we have

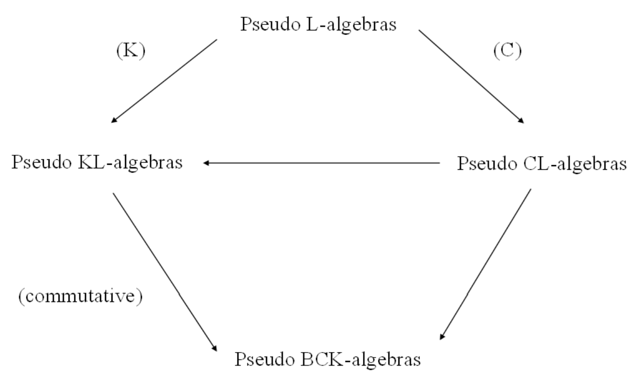
$$y \to z \leq (y \to x) \to (y \to z) = (x \to y) \to (x \to z).$$

Then by Proposition 2.11(5), we get

$$(y \to z) \rightsquigarrow (x \to z) \geq ((x \to y) \to (x \to z)) \rightsquigarrow (x \to z) \geq x \to y.$$

Similarly, we have $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$. □

Next, we illustrate the relationships among pseudo L-algebras, pseudo KL-algebras, pseudo CL-algebras, and pseudo BCK-algebras through a diagram.



3. Structure on pseudo L-algebras

Definition 3.1. Let L be a pseudo L-algebra. A nonempty subset I of L is called an *ideal*, if it satisfies the following conditions: for any $x, y \in L$,

(I1) $1 \in I$,

(I2) $x, x \to y \in I$ implies $y \in I$,

(I3) $x, x \rightsquigarrow y \in I$ implies $y \in I$,

(I4) $x \in I$ implies $(x \to y) \rightsquigarrow y \in I$ and $(x \rightsquigarrow y) \to y \in I$,

(I5) $x \in I$ implies $y \to x, y \rightsquigarrow x \in I$,

(I6) $x \in I$ implies $y \rightsquigarrow (x \to y), y \to (x \rightsquigarrow y) \in I$.

Remark 3.2. (1)If L satisfies condition (K), then (I5) and (I6) can be omitted.

(2)If L satisfies condition (C), then (I4), (I5) and (I6) can be omitted.

We will denote by $I(L)$ the set of all ideals of the pseudo L-algebra L. It is clear that $\{1\} \in I(L)$ and $L \in I(L)$. An ideal I is said to be proper if $I \neq L$.

Example 3.3. Consider the pseudo L-algebra L from Example 2.3. We can check that all ideals of L are $I_1 = \{1\}, I_2 = \{b, 1\}$ and L.

Let $(L, \rightarrow, \rightsquigarrow, 1)$ be a pseudo L-algebra. We define two binary relations $\sim_1$ and $\sim_2$ on L as follows: for all $x, y \in L$,

$$x \sim_1 y \text{ iff } x \rightarrow y, y \rightarrow x \in I \text{ and } x \sim_2 y \text{ iff } x \rightsquigarrow y, y \rightsquigarrow x \in I.$$

Theorem 3.4. *Let L be a pseudo L-algebra and $I \in I(L)$. Then the relations $\sim_1$ and $\sim_2$ are equivalence relations on L.*

Proof. Reflexivity and symmetry are evident. Now, we proceed to prove transitivity. Assuming $x \sim_1 y \sim_1 z$, we have $x \rightarrow y, y \rightarrow x \in I$ and $y \rightarrow z, z \rightarrow y \in I$. To establish $x \rightarrow z, z \rightarrow x \in I$, we utilize (I5) to obtain

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z) \in I.$$

Moverover, by (I2), we deduce that $x \rightarrow z \in I$. Similarly, we establish $z \rightarrow x \in I$, confirming the transitivity of $\sim_1$.

Similarly, we deduce that $\sim_2$ is an equivalence relation on L. □

Definition 3.5. An ideal I of a pseudo L-algebra L is said to be *normal*, if it satisfies the condition: for any $x, y \in L$,

$$x \rightarrow y \in I \text{ iff } x \rightsquigarrow y \in I.$$

We denote by $I_o(L)$ the set of all normal ideals of L.

Proposition 3.6. *Let L be a pseudo L-algebra and $I \in I(L)$. If $I \in I_o(L)$, then $\sim_1 = \sim_2$.*

Proof. Clear. □

If I is a normal ideal of L, then $\sim_{1(I)}$ and $\sim_{2(I)}$ coincide. Thus we use the notation $\sim_I$ to denote $\sim_{1(I)}$ and $\sim_{2(I)}$.

Lemma 3.7. *Let $I \in I(L)$. Then the following hold: for any $x, y, z \in L$,*
(1) *if $x \in I$ and $(x \rightsquigarrow y) \rightarrow z \in I$, then $y \rightarrow z \in I$,*
(2) *if $x \in I$ and $(x \rightarrow y) \rightsquigarrow z \in I$, then $y \rightsquigarrow z \in I$.*

Proof. (1) Since $I \in I(L)$, by (I5) and assumption,

$$((x \rightsquigarrow y) \rightarrow y) \rightarrow ((x \rightsquigarrow y) \rightarrow z) \in I.$$

Then, by (L2), we get $(y \rightarrow (x \rightsquigarrow y)) \rightarrow (y \rightarrow z) \in I$. Since $x \in I$, by (I6), we have $y \rightarrow (x \rightsquigarrow y)$. Thus by (I2), we obtain $y \rightarrow z \in I$.
(2) It is similar to (1). □

Theorem 3.8. *Let L be a pseudo L-algebra and I be a normal ideal of L. Then the binary relation $\sim_I$ is a congruence relation on L. Conversely, every congruence relation $\sim_I$ defines an ideal $I = \{x \in L \mid x \sim_I 1\}$.*

Proof. By Theorem 3.4, the binary relation $\sim_I$ is an equivalence relation on L. We need to show that $\sim_I$ is a congruence relation, meaning that $x \sim_I y$ implies

$$(z \rightarrow x) \sim_I (z \rightarrow y), (x \rightarrow z) \sim_I (y \rightarrow z).$$

Now, we prove that $(z \rightarrow x) \sim_I (z \rightarrow y)$ for any $x, y, z \in L$. Assume that $x \sim_I y$. Then we have $x \rightarrow y$, $y \rightarrow x \in I$. Thus we have

$$(z \to x) \to (z \to y) = (x \to z) \to (x \to y) \in I.$$

Similarly, we have $(z \to y) \to (z \to x) \in I$. So $(z \to x) \sim_I (z \to y)$.

Next, we prove that $(x \to z) \sim_I (y \to z)$ for any $x, y, z \in L$. By Lemma 3.7, we get $((x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)) \to (y \rightsquigarrow z) = ((y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z)) \to (y \rightsquigarrow z) \in I$, which implies $(x \rightsquigarrow z) \to (y \rightsquigarrow z) \in I$. Similarly, we get $(y \rightsquigarrow z) \to (x \rightsquigarrow z) \in I$. By symmetry, we also obtain $(x \to z) \rightsquigarrow (y \to z) \in I$ and $(y \to z) \rightsquigarrow (x \to z) \in I$. Then $(x \to z) \sim_I (y \to z)$ and $(x \rightsquigarrow z) \sim_I (y \rightsquigarrow z)$. Since I be a normal ideal of L, $(x \to z) \sim_I (y \to z)$. Thus $\sim_I$ is a congruence.

Conversely, consider $I = \{x \in L \mid x \sim_I 1\}$. We need to prove $I \in I(L)$. Since the relation $\sim_I$ is a congruence relation on L, by reflexivity, we have $1 \sim_I 1$, and then $I \neq \varnothing$. Thus (I1) holds. Suppose x, $x \to y \in I$. Then $x \sim_I 1$ and $(x \to y) \sim_I 1$. Clearly, $y \sim_I (1 \to y) \sim_I (x \to y) \sim_I 1$. Thus $y \in I$. So (I2) holds. Similarly, we can show that (I3) holds. From $x \in I$, we have $x \sim_I 1$. Then $(x \to y) \sim_I y$ and $(x \rightsquigarrow y) \sim_I y$. Thus $((x \to y) \rightsquigarrow y) \sim_I (y \rightsquigarrow y)$ and $((x \rightsquigarrow y) \to y) \sim_I (y \to y)$. Hence $(x \to y) \rightsquigarrow y \in I$ and $(x \rightsquigarrow y) \to y \in I$. So (I4) holds. By similar reasoning, we can show that (I5) and (I6) hold. Therefore $I \in I(L)$. □

Let us denote by $L/I = \{[x]_{\sim_I} \mid x \in L\}$, where $[x]_{\sim_I} = \{y \in L \mid x \sim_I y\}$ and I is a normal ideal in $I_0(L)$. The binary relation $\leq$ on L/I is defined as follows:

$$[x]_{\sim_I} \leq [y]_{\sim_I} \text{ if and only if } x \to y \in I.$$

Lemma 3.9. *Let L be a pseudo L-algebra and $I \in I_0(L)$. Then $[1]_{\sim_I} = I$.*

Proof. Suppose that $I \in I_0(L)$. Then by Proposition 3.6, we have $\sim_1 = \sim_2$. Thus

$$[1]_{\sim_I} = \{x \in L | x_{\sim_I} 1\} = \{x \in L | x \in I\} = I.$$

□

Theorem 3.10. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra and I be a normal ideal of L. Then $(L/I, \to, \rightsquigarrow, [1]_{\sim_I})$ is a pseudo L-algebra, where $[x]_{\sim_I} \to [y]_{\sim_I} = [x \to y]_{\sim_I}, [x]_{\sim_I} \rightsquigarrow [y]_{\sim_I} = [x \rightsquigarrow y]_{\sim_I}$, for any $x, y \in L$.*

Proof. First, we need to prove $\to$ and $\rightsquigarrow$ are well-defined in L/I. Suppose $[x]_{\sim_I} = [x_1]_{\sim_I}$ and $[y]_{\sim_I} = [y_1]_{\sim_I}$. Then $x \sim_I x_1$ and $y \sim_I y_1$. Thus we have

$$x \to y \sim_I x_1 \to y_1 \text{ and } x \rightsquigarrow y \sim_I x_1 \rightsquigarrow y_1.$$

So we get

$$[x]_{\sim_I} \to [y]_{\sim_I} = [x \to y]_{\sim_I} = [x_1 \to y_1]_{\sim_I} = [x_1]_{\sim_I} \to [y_1]_{\sim_I}$$

and

$$[x]_{\sim_I} \rightsquigarrow [y]_{\sim_I} = [x \rightsquigarrow y]_{\sim_I} = [x_1 \rightsquigarrow y_1]_{\sim_I} = [x_1]_{\sim_I} \rightsquigarrow [y_1]_{\sim_I}.$$

Hence $\to$ and $\rightsquigarrow$ are well-defined in L/I.

It is now proved to be a pseudo L-algebra. Let $[x]_{\sim_I} \in L/I$. Then we have

$$1 \to [x]_{\sim_I} = [1]_{\sim_I} \to [x]_{\sim_I} = [1 \to x]_{\sim_I} = [x]_{\sim_I},$$

$$[1 \rightsquigarrow x]_{\sim_I} = [x]_{\sim_I},$$

$$[x]_{\sim_I} \to [1]_{\sim_I} = [x \to 1]_{\sim_I} = [1]_{\sim_I} = 1$$

and

$$[x]_{\sim_I} \to [x]_{\sim_I} = [x \to x]_{\sim_I} = [1]_{\sim_I}.$$

Thus (PL1) and (PL2) hold. Let $[x]_{\sim_I}, [y]_{\sim_I} \in L/I$. Then we have

$$\begin{aligned}([x]_{\sim_I} \to [y]_{\sim_I}) \to ([x]_{\sim_I} \to [z]_{\sim_I}) &= ([x \to y]_{\sim_I}) \to ([x \to z]_{\sim_I}) \\ &= [(x \to y) \to (x \to z)]_{\sim_I} \\ &= [(y \to x) \to (y \to z)]_{\sim_I} \\ &= ([y \to x]_{\sim_I}) \to ([y \to z]_{\sim_I}) \\ &= ([y]_{\sim_I} \to [x]_{\sim_I}) \to ([y]_{\sim_I} \to [z]_{\sim_I}).\end{aligned}$$

Similarly, $([x]_{\sim_I} \rightsquigarrow [y]_{\sim_I}) \rightsquigarrow ([x]_{\sim_I} \rightsquigarrow [z]_{\sim_I}) = ([y]_{\sim_I} \rightsquigarrow [x]_{\sim_I}) \rightsquigarrow ([y]_{\sim_I} \rightsquigarrow [z]_{\sim_I})$. Thus (PL3) and (PL4) hold. Now let $[x]_{\sim_I}, [y]_{\sim_I} \in L/I$ such that

$$[x]_{\sim_I} \to [y]_{\sim_I} = [y]_{\sim_I} \to [x]_{\sim_I} = [1]_{\sim_I}.$$

Since $[x]_{\sim_I} \to [y]_{\sim_I} = [x \to y]_{\sim_I}$ and $[y]_{\sim_I} \to [x]_{\sim_I} = [y \to x]_{\sim_I}$, by lemma 3.9, we get $x \to y \in [1]_{\sim_I} = I$ and $y \to x \in [1]_{\sim_I} = I$. Since $I \in I_0(L)$, $x \rightsquigarrow y, y \rightsquigarrow x \in I$. Thus $x \sim_I y$, ie.,$[x]_{\sim_I} = [y]_{\sim_I}$. So (PL5) holds. Finally let $[x]_{\sim_I}, [y]_{\sim_I} \in L/I$ such that $[x]_{\sim_I} \to [y]_{\sim_I} = [1]_{\sim_I}$, ie., $[x \to y]_{\sim_I} = [1]_{\sim_I} = I$. Then $x \to y \in I$. Since $I \in I_0(L)$, we get $x \rightsquigarrow y \in I$. Thus $[x \rightsquigarrow y]_{\sim_I} = [1]_{\sim_I}$, which means that $[x]_{\sim_I} \rightsquigarrow [y]_{\sim_I} = [1]_{\sim_I}$. Conversely, let $[x]_{\sim_I} \rightsquigarrow [y]_{\sim_I} = [1]_{\sim_I}$. Then we can get $[x]_{\sim_I} \to [y]_{\sim_I} = [1]_{\sim_I}$. Thus (PL6) holds. So $(L/I, \to, \rightsquigarrow, [1]_{\sim_I})$ is a pseudo L-algebra. □

Example 3.11. Consider the pseudo L-algebra L from Example 2.3. Then we can check that $I = \{b, 1\}$ is an ideal on L. Thus $L/I = \{[1]_{\sim_I}, [a]_{\sim_I}, [c]_{\sim_I}\}$, where $[1]_{\sim_I} = [b]_{\sim_I} = \{1, b\}, [a]_{\sim_I} = \{a\}, [c]_{\sim_I} = \{c\}$.

Definition 3.12. Let L be a pseudo L-algebra. I is called a *fantastic ideal* of L, if it satisfies the following conditions: for any $x, y \in L$,

$(F1)$ $y \to x \in I$ implies $((x \to y) \rightsquigarrow y) \to x \in I$,

$(F2)$ $y \rightsquigarrow x \in I$ implies $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in I$.

Note: We will denote by $F_I(L)$ the set of all fantastic ideals of a pseudo L-algebra L.

Example 3.13. Consider the pseudo L-algebra L from Example 2.4. Then we can verify that $F_I = \{\{b, 1\}, \{a, b, c, 1\}\}$.

Theorem 3.14. *Let L be a pseudo CL-algebra and I be a subset of L. I is a fantastic ideal if and only if it satisfies the following conditions: for all $x, y, z \in L$,*

$(FI1)$ $1 \in I$,

$(FI2)$ $z \to (x \to y) \in I$ *and* $z \in I$ *imply* $((y \to x) \rightsquigarrow x) \to y \in I$,

$(FI3)$ $z \rightsquigarrow (x \rightsquigarrow y) \in I$ *and* $z \in I$ *imply* $((y \rightsquigarrow x) \to x) \rightsquigarrow y \in I$.

Proof. Suppose I is a fantastic ideal. Then clearly, $1 \in I$. Suppose for any $x, y, z \in L$, $z \to (x \to y) \in I$ and $z \in I$. Then by (I2), $x \to y \in I$. Thus $((y \to x) \rightsquigarrow x) \to y \in I$. Likewise, $y \rightsquigarrow x \in I$ implies $((y \rightsquigarrow x) \to x) \rightsquigarrow y \in I$. So the conditions (FI1), (FI2) and (FI3) hold.

Conversely, suppose the necessary condition hold. Then we aim to demonstrate that I is a fantastic ideal. To begin, let's prove that I is an ideal. As L is a bounded pseudo CL-algebra, validating conditions (1)-(3) suffices. Clearly, $1 \in I$. Now, if x and $x \to y \in I$, then $x \to (1 \to y) = x \to y \in I$. Applying (FI2), we obtain $((y \to 1) \rightsquigarrow 1) \to y \in I$. Thus leading to $y \in I$. Applying a similar reasoning, when x and $x \rightsquigarrow y \in I$, we conclude that $y \in I$.

Next, let's establish that I is a fantastic ideal. Suppose $x, y \in L$ such that $y \to x \in I$. Since $1 \to (y \to x) = y \to x \in I$ and $1 \in I$, by (FI2), it follows that $((x \to y) \rightsquigarrow y) \to x \in I$. Similarly, if $y \rightsquigarrow x \in I$, then $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in I$. Thus I is a fantastic ideal. □

Proposition 3.15. *Let F and G be ideals of a pseudo CL-algebra L such that $F \subseteq G$. If F is a fantastic ideal of L, then so is G.*

Proof. Let $y \to x \in G$, for any $x, y \in L$. Since L is a pseudo CL-algebra, we have

$$y \to ((y \to x) \rightsquigarrow x) = (y \to x) \rightsquigarrow (y \to x) = 1 \in I.$$

Since I is a fantastic ideal, we get

$$((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to ((y \to x) \rightsquigarrow x) \in F \subseteq G.$$

Thus by (C) , $(y \to x) \rightsquigarrow (((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x) \in G$. Since G is an ideal and $y \to x \in G$, $((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x \in G$. So by Proposition 2.11(5), $((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x \leq ((x \to y) \rightsquigarrow y) \to x$. Since G is an ideal, $((x \to y) \rightsquigarrow y) \to x \in G$. Similarly, suppose that $x, y \in L$ such that $y \rightsquigarrow x \in G$. Then we have $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in G$. Hence G is a fantastic ideal. □

Definition 3.16. Let L be a pseudo L-algebra and I be an ideal of L. If for any $x, y \in L$, it satisfies $((y \to x) \rightsquigarrow x) \to ((x \to y) \rightsquigarrow y) \in I$ and $((y \rightsquigarrow x) \to x) \rightsquigarrow ((x \rightsquigarrow y) \to y) \in I$, then we call I a *pseudo MV-ideal*.

Example 3.17. Consider the pseudo L-algebra L from Example 2.6. Then we can verify that $\{1\}, \{b, 1\}, \{a, b, c, 1\}$ are pseudo MV-ideals.

Theorem 3.18. *Let L be a pseudo CL-algebra and I be an ideal of L. Then I be a pseudo MV-ideal if and only if I is a fantastic ideal.*

Proof. Suppose I is a pseudo MV-ideal and let $x, y, z \in L$ satisfying $z, z \to (x \to y) \in I$. Then by (I2), we get $x \to y \in I$. Since $(x \to y) \rightsquigarrow (((y \to x) \rightsquigarrow x) \to y) = ((y \to x) \rightsquigarrow x) \to ((x \to y) \rightsquigarrow y) \in I$, by (I3), we obtain that $((y \to x) \rightsquigarrow x) \to y \in I$. Analogously, we can show that (FI3) holds. Thus I is a fantastic ideal.

Conversely, suppose I is a fantastic ideal and let $x, y \in L$. We first prove $x \to y \in I$ and $y \to z \in I$ imply $x \to z \in I$. By Proposition 2.11(3), we have $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$. Then $(y \to z) \rightsquigarrow (x \to z) \in I$. By (I3), we obtain $x \to z \in I$. Since I is a fantastic ideal of L and $x, y \in L$, by Proposition 2.11(1), we get $x \to ((x \to y) \rightsquigarrow y) = 1 \in I$. Thus we obtain

$$((((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x) \to ((x \to y) \rightsquigarrow y) \in I.$$

On the other hand, since $y \leq (x \to y) \rightsquigarrow y$, it follows that

$$(y \to x) \rightsquigarrow x \leq (((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x.$$

So we have

$$((y \to x) \rightsquigarrow x) \to ((((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x) = 1 \in I.$$

Hence $((y \to x) \rightsquigarrow x) \to ((x \to y) \rightsquigarrow y) \in I$. □

4. Bosbach States on pseudo L-algebras

In this section, we introduce Bosbach states and morphisms on pseudo L-algebras, study their properties, and discuss their relationships. Also, we introduce two specific ideals(i.e. fantastic ideals and Pseudo MV-ideals) to discuss the existence of Bosbach states.

Definition 4.1. Let $(L, \to, \rightsquigarrow, a, 1)$ be a pseudo L-algebra and $s : L \to [0, 1]$ be a function satisfying the following conditions: for all $x, y \in L$,

$(BS1)$ $s(a) = 0, s(1) = 1$, for some $a \in L\backslash\{1\}$,

$(BS2)$ $s(x) + s(x \to y) = s(y) + s(y \to x)$,

$(BS3)$ $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$.

Then we call s a *Bosbach state* on L.

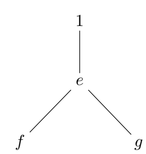
Denote by $BS(L)$ the set of all Bosbach states on L.

Next, we give an example of a Bosbach state on a pseudo L-algebra.

Example 4.2. Let $L = \{1, e, f, g\}$, where $f, g < e < 1$, and f and g are incomparable. Define the operations $\to$ and $\rightsquigarrow$ using the following tables. Then we can verify

$\to$	1	e	f	g
1	1	e	f	g
e	1	1	f	g
f	1	1	1	g
g	1	1	f	1

$\rightsquigarrow$	1	e	f	g
1	1	e	f	g
e	1	1	f	g
f	1	1	1	g
g	1	1	g	1



that $(L, \to, \rightsquigarrow, 1)$ forms a pseudo L-algebra. Define the function $s : L \to [0, 1]$ by $s(1) = 1$, $s(e) = 1$, $s(f) = 0$, and $s(g) = 0$. Then s is a Bosbach state on L.

Proposition 4.3. *Let L be a pseudo L-algebra and s be a Bosbach state on L. Then the following properties hold: for all $x, y \in L$,*

(1) $x \leq y$ implies $s(x) \leq s(y)$,

(2) $x \leq y$ implies $s(y \to x) = s(y \rightsquigarrow x) = 1 - s(y) + s(x)$,

if L is bounded, then

(3) $s(0) = 0$,

(4) $s(x^-) = s(x^\sim) = 1 - s(x)$,

(5) $s(x^{\sim\sim}) = s(x^{--}) = s(x^{-\sim}) = s(x^{\sim-}) = s(x)$.

Proof. (1) Suppose $x \leq y$. Then $x \to y = 1$. Thus we have

$$s(x) + 1 = s(x) + s(x \to y) = s(y) + s(y \to x).$$

So $s(x) - s(y) = s(y \to x) - 1 \leq 0$. Hence $s(x) \leq s(y)$.

(2) Suppose $x \leq y$. Then $x \to y = 1$. Thus we have

$$s(x) + 1 = s(x) + s(x \to y) = s(y) + s(y \to x).$$

So $s(y \to x) = 1 - s(y) + s(x)$. Similarly, $s(y \rightsquigarrow x) = 1 - s(y) + s(x)$. Hence $s(y \to x) = s(y \rightsquigarrow x) = 1 - s(y) + s(x)$.

(3) Obvious.

(4) By (BS2), $s(x) + s(x \to 0) = s(0) + s(0 \to x)$. Then we deduce that $s(x^-) = 1 - s(x)$. Similarly, $s(x^\sim) = 1 - s(x)$. Thus $s(x^-) = s(x^\sim) = 1 - s(x)$.

(5) The proof follows from (4). ☐

Proposition 4.4. *Let L be a pseudo CL-algebra and s be a Bosbach state on L. Then the following properties hold: for all $x, y \in L$,*

(1) $s(x \to y) = 1 - s(x \vee_1 y) + s(y)$ *and* $s(x \rightsquigarrow y) = 1 - s(x \vee_2 y) + s(y)$,

(2) $s(x \vee_1 y) = s(y \vee_1 x)$ *and* $s(x \vee_2 y) = s(y \vee_2 x)$,

(3) $s(x \vee_1 y) = s(x \vee_2 y)$,

(4) $s(x \to y) = s(x \rightsquigarrow y)$.

Proof. (1) Let $x, y \in L$. Then clearly, $y \leq x \vee_1 y$. Thus by Propositions 2.11(6) and 4.3(2), we have

$$s(x \to y) = s(x \vee_1 y \to y) = 1 - s(x \vee_1 y) + s(y).$$

Similarly, the second equation holds.

(2) From (1), $s(x \to y) = 1 - s(x \vee_1 y) + s(y)$ and $s(y \to x) = 1 - s(y \vee_1 x) + s(x)$. Then $s(x \vee_1 y) - s(y \vee_1 x) = s(y) + s(y \to x) - (s(x) + s(x \to y)) = 0$. Thus $s(x \vee_1 y) = s(y \vee_1 x)$. Analogously, $s(x \vee_2 y) = s(y \vee_2 x)$.

(3) Initially proving the equality for $y \leq x$, we deduce from (2) that

$$s(x \vee_1 y) = s(y \vee_1 x) = s(x) \text{ and } s(x \vee_2 y) = s(y \vee_2 x) = s(x).$$

Then, $s(x \vee_1 y) = s(x \vee_2 y)$. For any $x, y \in L$, since $x \leq (x \vee_1 y)$, we have

$$\begin{aligned} s(x \vee_1 y) &= s(x \vee_1 (x \vee_1 y)) = s((x \vee_1 y) \vee_1 x) = s((x \vee_1 y) \vee_2 x) \\ &\geq s(y \vee_2 x) = s(x \vee_2 y) = s((x \vee_2 y) \vee_2 y) = s((x \vee_2 y) \vee_1 y) \\ &\geq s(x \vee_1 y). \end{aligned}$$

Thus $s(x \vee_1 y) = s(x \vee_2 y)$.

(4) Combining (1) and (3), we conclude $s(x \to y) = s(x \rightsquigarrow y)$. ☐

Now, let's give examples of the Bosbach state of a bounded pseudo L-algebra.

Example 4.5. Let $L = \{0, a, b, c, 1\}$ be a lattice, where $0 < a < b$, $c < 1$, and b and c are incomparable. Define the operations $\to$ and $\rightsquigarrow$ using the following tables.

Then we can verify that $(L, \to, \rightsquigarrow, 0, 1)$ forms a bounded pseudo L-algebra. Define

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	0	a	1	c	1
c	0	a	c	1	1
1	0	a	b	c	1

a function $s : L \to [0,1]$ by $s(0) = 0$, $s(a) = 0$, $s(b) = 1$, $s(c) = 1$ and $s(1) = 1$. Then s is a Bosbach state on L.

Example 4.6. Let $L = \{0, m, n, x, y, 1\}$. Define the operations $\to$ and $\rightsquigarrow$ using the following tables. Then we can verify that $(L, \to, \rightsquigarrow, 0, 1)$([14]) forms a bounded

$\to$	0	m	n	x	y	1
0	1	1	1	1	1	1
m	y	1	y	1	y	1
n	m	m	1	1	1	1
x	0	m	y	1	y	1
y	m	m	x	x	1	1
1	0	m	n	x	y	1

$\rightsquigarrow$	0	m	n	x	y	1
0	1	1	1	1	1	1
m	n	1	n	1	n	1
n	m	m	1	1	1	1
x	0	m	y	1	y	1
y	m	m	x	x	1	1
1	0	m	n	x	y	1

pseudo L-algebra. Define a function $s : L \to [0,1]$ by $s(0) = 0$, $s(m) = 0.5$, $s(n) = 0.5$, $s(x) = 1$, $s(y) = 1$ and $s(1) = 1$. Then s is a Bosbach state on L.

Example 4.7. Let $(L, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo L-algebra L. Define the binary operations $\to$ and $\rightsquigarrow$ as following:

$$x \to y = \begin{cases} 1, & x \leq y \\ \frac{y}{x}, & x > y \end{cases} \quad \textit{and} \qquad x \rightsquigarrow y = \begin{cases} 1, & x \leq y \\ y, & x > y \end{cases}$$

Define a function $s : L \to [0,1]$ by

$$s(a) = \begin{cases} 0, & a = 0 \\ 1, & a \neq 0 \end{cases}$$

Then one can easily verify that s is a Bosbach state on L.

The following example indicates that there exist a bounded pseudo L-algebra which have no Bosbach states.

Example 4.8. Consider the bounded pseudo L-algebra L from Example 2.3. One can check that L has no Bosbach states. Indeed, if s is a Bosbach state on L, then $s(0) = 0$ and $s(1) = 1$. From $s(a) + s(a \to 0) = s(0) + s(0 \to a)$ and $s(b) + s(b \to 0) = s(0) + s(0 \to b)$, we can obtain $s(a) = 1, s(b) = 1$. Now, by $s(a) + s(a \rightsquigarrow 0) = s(0) + s(0 \rightsquigarrow a)$, we have $s(a) + s(b) = s(0) + s(1)$, which leads to $1 + 1 = 0 + 1$, a contradiction. Therefore, there is a bounded pseudo L-algebra that has no Bosbach state.

In the following, we give some equivalent characterizations of the Bosbach states on bounded pseudo L-algebras.

Proposition 4.9. *Let L be a pseudo L-algebra and $s : L \to [0,1]$ be a function. Then the following properties are equivalent: for all $x, y \in L$,*

(1) *$s \in BS(L)$,*

(2) *if $x \leq y$, then $s(y \to x) = s(y \rightsquigarrow x) = 1 - s(y) + s(x)$,*

(3) *if L satisfies (C), then $s(x \to y) = 1 - s(x \vee_1 y) + s(y)$ and $s(x \rightsquigarrow y) = 1 - s(x \vee_2 y) + s(y)$.*

Proof. (1) $\Rightarrow$ (2): It follows from Proposition 4.3(2).

(2) $\Rightarrow$ (3): It follows from Proposition 4.4(1).

(3) $\Rightarrow$ (1): By Proposition4.4(2), we obtain $s(x)+s(x \to y) = s(x)+1-s(x \vee_1 y)+s(y) = 1-s(y \vee_1 x)+s(x)+s(y) = s(y)+s(y \to x)$. Similarly, we have $s(x)+s(x \rightsquigarrow y) = s(x)+1-s(x \vee_2 y)+s(y) = 1-s(y \vee_2 x)+s(x)+s(y) = s(y)+s(y \rightsquigarrow x)$. Furthermore, by (3), we conclude $s(1) = s(x \to x) = 1-s(x)+s(x) = 1$. Then we can assert that s is a Bosbach state on L. □

Proposition 4.10. *Let L be a bounded pseudo L-algebra and s a Bosbach state on it. Let $m = 1-s$. Then the following conditions hold: for all $x, y \in L$,*

(1) $m(0) = 1$,

(2) if $x \leq y$, then $m(y \to x) = m(y \rightsquigarrow x) = m(x) - m(y)$.

Conversely, if $m : L \to [0,1]$ is a map satisfying conditions (1) and (2), then $s = 1-m \in BS(L)$.

Proof. (1) $m(0) = 1-s(0) = 1$.

(2) Suppose $x \leq y$. Then $x \to y = 1$ and $x \rightsquigarrow y = 1$. By Proposition 4.3(2), we have

$$s(y \to x) = 1-s(y)+s(x) \text{ and } s(y \rightsquigarrow x) = 1-s(y)+s(x).$$

Thus we get

$$1-m(y \to x) = 1-(1-m(y))+(1-m(x))$$

and

$$1-m(y \rightsquigarrow x) = 1-(1-m(y))+(1-m(x)).$$

So $m(y \to x) = m(y \rightsquigarrow x) = m(x)-m(y)$.

Conversely, suppose m satisfies conditions (1) and (2). Then $m(0) = 1$ and $m(1) = m(x \to x) = m(x)-m(x) = 0$. By the condition (2), we have

$$m(x)+m(x \to y) = m(x)+m(1) = m(x)+0 = m(x)$$

and

$$m(y)+m(y \to x) = m(y)+(m(x)-m(y)) = m(x).$$

Thus we get

$$m(x \to y)+m(x) = m(y \to x)+m(y) \text{ or } s(x \to y)+s(x) = s(y \to x)+s(y).$$

Similarly $s(x \rightsquigarrow y)+s(x) = s(y \rightsquigarrow x)+s(y)$. So we can conclude that $s \in BS(L)$. □

The set

$$Ker(s) = \{x \in L \mid s(x) = 1\}$$

is called the *kernel* of a Bosbach state s on L.

Proposition 4.11. *Let L be a commutative pseudo L-algebra and s a Bosbach state on L. Then $Ker(s) = \{x \in L \mid s(x) = 1\}$ is an ideal of L.*

Proof. Since s is a Bosbach state, $s(1) = 1$, i.e., $1 \in Ker(s)$. Let $x, x \to y \in Ker(s)$ for any $x, y \in L$. Then $s(x) = 1$ and $s(x \to y) = 1$. Since $x \leq y \to x$, by Proposition 4.3(1), $s(x) \leq s(y \to x)$. Thus, $s(y \to x) = 1$. Since $s(x) + s(x \to y) = s(y) + s(y \to x)$, we can get $s(y) = 1$. So $y \in Ker(s)$. Analogously, we can prove that $x, x \rightsquigarrow y \in Ker(s)$ implies $y \in Ker(s)$. Let $x \in Ker(s)$, then $s(x) = 1$. $x \to ((x \to y) \rightsquigarrow y) = x \to ((y \to x) \rightsquigarrow x) = 1$. Thus $x \leq ((x \to y) \rightsquigarrow y)$, i.e., $s(x) \leq s(((x \to y) \rightsquigarrow y))$. So $s(((x \to y) \rightsquigarrow y)) = 1$. Hence $((x \to y) \rightsquigarrow y) \in Ker(s)$. Analogously, $((x \rightsquigarrow y) \to y) \in Ker(s)$. Therefore $Ker(s) = \{x \in L \mid s(x) = 1\}$ is an ideal. □

Corollary 4.12. *Let L be a pseudo CL-algebra and s a Bosbach state on L. Then $Ker(s) = \{x \in L \mid s(x) = 1\}$ is an ideal of L.*

Proposition 4.13. *Let s be a Bosbach state on a pseudo CL-algebra L and $K = Ker(s)$. In the quotient pseudo L-algebra $(L/K, \leq, \to, 1/K)$ we have:*

(1) $a/K \leq b/K$ if and only if $s(a \to b) = 1$ if and only if $s(a \vee_1 b) = s(b)$ if and only if $s(a \vee_2 b) = s(b)$,

(2) $a/K = b/K$ if and only if $s(a \to b) = s(b \to a) = 1$ if and only if $s(a) = s(b) = s(a \vee_1 b)$ if and only if $s(a \rightsquigarrow b) = s(b \rightsquigarrow a) = 1$ if and only if $s(a) = s(b) = s(a \vee_2 b)$.

Moreover, the mapping $\hat{s} : L/K \to [0, 1]$ defined by $\hat{s}(a/K) := s(a)$ for $a \in A$ is a Bosbach state on L/K.

Proof. (1) It follows easily: $a/K \leq b/K$ if and only if $(a \to b)/K = a/K \to b/K = 1/K = K$ if and only if $a \to b \in K$ if and only if $s(a \to b) = 1$. As $s(a \to b) = 1 - s(a \vee_1 b) + s(b)$, we get $a/K \leq b/K$ if and only if $s(a \vee_1 b) = s(b)$. Similarly, $a/K \leq b/K$ if and only if $(a \rightsquigarrow b)/K = a/K \rightsquigarrow b/K = 1/K = K$ if and only if $a \rightsquigarrow b \in K$ if and only if $s(a \rightsquigarrow b) = 1$. As $s(a \rightsquigarrow b) = 1 - s(a \vee_2 b) + s(b)$, we get $a/K \leq b/K$ if and only if $s(a \vee_2 b) = s(b)$.

(2) It follows easily from (1).

The fact that $\hat{s}$ is a well-defined Bosbach state on L/K is now straightforward. □

Proposition 4.14. *Let s be a Bosbach state on a bounded pseudo CL-algebra L, and let $K = Ker(s)$. For every element $x \in L$, we have $x^{\sim -}/K = x/K = x^{-\sim}/K$, i.e., L/K satisfies the (pDN) condition.*

Proof. By proposition 2.11(8), we have $x \leq x^{-\sim}$. From the definition of a Bosbach state and Proposition 4.3(5), we have $s(x^{-\sim} \to x) = s(x) + s(x \to x^{-\sim}) - s(x^{-\sim}) = s(x \to x^{-\sim}) = s(1) = 1$. Then $x^{-\sim}/K = x/K$. In a similar manner, we can prove the second identity. □

Let s be a Bosbach state on a pseudo CL-algebra L. According to the proof of Proposition 4.13, we have $s(x^{-\sim} \to x) = 1 = s(x^{\sim -} \to x)$ and $s(x^{-\sim} \rightsquigarrow x) = 1 = s(x^{\sim -} \rightsquigarrow x)$.

Proposition 4.15. *Let s be a Bosbach state on a pseudo CL-algebra L. Then L/K is $\vee_1 -$commutative, where $K = Ker(s)$. In addition, L/K is a $\vee_1 -$semilattice .*

Proof. From Theorem 3.10, L/K is a pseudo L-algebra. For any $x \in L$, denote by $\bar{x} := x/Ker(s)$ and $\hat{s}(\bar{x}) = s(x)$. Then from proposition 4.13, $\hat{s}$ is a Bosbach state on L/K.

(1) First, we verify that if $\bar{x} \leq \bar{y}$, then

$$\bar{x} \vee_1 \bar{y} = \bar{y} = \bar{y} \vee_1 \bar{x}. \tag{4.1}$$

(i) From Proposition 2.14(2), we have $\bar{x} \vee_1 \bar{y} = \bar{y}$.

(ii) Since L is a pseudo CL-algebra, we have $\bar{y} \leq (y \to x) \rightsquigarrow x = y \vee_1 x$. Then by Proposition 4.3(2), we get

$$\begin{aligned} s(y \vee_1 x) = s((y \to x) \rightsquigarrow x) &= \hat{s}((\bar{y} \to \bar{x}) \rightsquigarrow \bar{x}) \\ &= 1 - \hat{s}(\bar{y} \to \bar{x}) + \hat{s}(\bar{x}) \\ &= 1 + \hat{s}(\bar{x}) - (1 - \hat{s}(\bar{y}) + \hat{s}(\bar{x}) \\ &= \hat{s}(\bar{y}) = s(y). \end{aligned}$$

Thus we have

$$\begin{aligned} s((y \vee_1 x) \to y) = \hat{s}((\bar{y} \vee_1 \bar{x}) \to \bar{y}) &= 1 + \hat{s}(\bar{y}) - \hat{s}(\bar{y} \vee_1 \bar{x}) \\ &= 1 + \hat{s}(\bar{y}) - \hat{s}(\bar{x} \vee_1 \bar{y}) = 1 + \hat{s}(\bar{y}) - \hat{s}(\bar{y}) = 1. \end{aligned}$$

So (4.1) holds for $\bar{x} \leq \bar{y}$.

(2) Now we show that (4.1) holds for all $x, y \in L$. By (1), we have

$$\begin{aligned} x \vee_1 \bar{y} &= \bar{x} \vee_1 (\bar{x} \vee_1 \bar{y}) = (\bar{x} \vee_1 \bar{y}) \vee_1 \bar{x} \\ &\geq \bar{y} \vee_1 \bar{x} = \bar{y} \vee_1 (\bar{y} \vee_1 \bar{x}) = (\bar{y} \vee_1 x) \vee_1 \bar{y} \\ &\geq \bar{x} \vee_1 \bar{y}. \end{aligned}$$

Thus L/K is $\vee_1-$*commutative*. So by Proposition 2.16, L/K is a $\vee_1$-semilattice. □

Corollary 4.16. *Let s be a Bosbach state on a pseudo CL-algebra L. Then L/K is $\vee_2 -$ commutative, where $K = Ker(s)$. In addition, L/K is a $\vee_2$-semilattice.*

Proposition 4.17. *Let L be a pseudo CL-algebra and s be a Bosbach state on it. Then, $Ker(s) = \{x \in L \mid s(x) = 1\}$ is a fantastic ideal of L.*

Proof. It follows from Corollary 4.12 that $Ker(s)$ is an ideal of L. We need to demonstrate that $y \to x \in Ker(s)$ implies $((x \to y) \rightsquigarrow y) \to x \in Ker(s)$ and $y \rightsquigarrow x \in Ker(s)$ implies $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in Ker(s)$.

suppose $y \to x \in Ker(s)$. Then $s(y \to x) = 1$. Since $s \in BS(L)$, we have

$$s(x) + s(x \to y) = s(y) + s(y \to x) = s(y) + 1.$$

Thus $s(x \to y) + s((x \to y) \rightsquigarrow y) = s(y) + s(y \rightsquigarrow (x \to y)) = s(y) + 1$. So $s(x) = s((x \to y) \rightsquigarrow y)$. As $s((x \to y) \rightsquigarrow y) + s(((x \to y) \rightsquigarrow y) \to x) = s(x) + s(x \to ((x \to y) \rightsquigarrow y)) = s(x) + 1$, we get $s(((x \to y) \rightsquigarrow y) \to x) = 1$. Hence $((x \to y) \rightsquigarrow y) \to x \in Ker(s)$.

Analogously, we can show that $y \rightsquigarrow x \in Ker(s)$ implies $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in Ker(s)$. Therefore the conclusion holds. □

Theorem 4.18. *Let L be a bounded pseudo CL-algebra and satisfies (C1) and (C2). Then $(L, \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra, where $x^- := x \to 0, x^\sim := x \rightsquigarrow 0$ and $x \oplus y := x^\sim \to y^{\sim -} = y^- \rightsquigarrow x^{\sim -}$ for all $x, y \in L$.*

Proof. By Proposition 2.20, we know that a pseudo CL-algebra is a pseudo BCK-algebra. And, by ([17]), we know that pseudo MV-algebras are term equivalent with bounded commutative pseudo BCK-algebras. Then we obtain that $(L, \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra. □

Theorem 4.19. *Let L be a bounded pseudo CL-algebra and satisfies (C1), (C2) and I be a normal ideal. If I is a pseudo MV-ideal, then L/I is a pseudo MV-algebra.*

Proof. For any $[x], [y] \in L/I$, since $(([y] \to [x]) \rightsquigarrow [x]) \to (([x] \to [y]) \rightsquigarrow [y]) = [((y \to x) \rightsquigarrow x) \to ((x \to y) \rightsquigarrow y)] = [1]$, $(([y] \to [x]) \rightsquigarrow [x]) \leq (([x] \to [y]) \rightsquigarrow [y])$. Similarly, we can prove that $(([x] \to [y]) \rightsquigarrow [y]) \leq (([y] \to [x]) \rightsquigarrow [x])$. That is, (C1) holds. Analogously, the (C2) holds. Furthermore, by Theorem 4.18, we obtain that L/I is a pseudo MV-algebra. □

We shall say that a pseudo L-algebra L is representable if it can be represented as (i.e. it is) a subdirect product of totally ordered pseudo L-algebras.

Theorem 4.20. *Let L be a bounded, representable pseudo CL-algebra and satisfies (C1), (C2) and I be a normal ideal. If I is a pseudo MV-ideal, then L/I possesses at least one state.*

Proof. By Theorem 4.19 and ([8]: Theorem 5.8), the proof is clear. □

Theorem 4.21. *Let L be a pseudo L-algebra and I be an ideal on L. If L/I has a Bosbach state s, then there exists a Bosbach state on L.*

Proof. Define a function $s^{'}$ from L to $[0, 1]$ satisfying $s^{'}(x) = s([x])$ for each $x \in L$. Then we show that s is a Bosbach state on L. Indeed, for any $x, y \in L$, we have $s^{'}(x) + s^{'}(x \to y) = s([x]) + s([x \to y]) = s([x]) + s([x] \to [y]) = s([y]) + s([y] \to [x]) = s^{'}(y) + s^{'}(y \to x)$. Similarly, $s^{'}(x) + s^{'}(x \rightsquigarrow y) = s^{'}(y) + s^{'}(y \rightsquigarrow x)$. Moreover, $s^{'}(1) = s([1]) = 1$ and $s^{'}(0) = s([0]) = 0$. Hence, the conclusion holds. □

Theorem 4.22. *Let L be a bounded, representable pseudo CL-algebra and satisfies (C1), (C2) and I be a normal ideal. Then the following statements are equivalent:*
(1) *L has a fantastic ideal,*
(2) *L has a pseudo MV-ideal,*
(3) *L has a Bosbach state.*

Proof. (1) ⇔ (2) It follows from Theorem 3.18.
(2) ⇒ (3) By Theorems 4.20 and 4.21, the proof is clear.
(3) ⇒ (1) By Proposition 4.17, $Ker(s)$ is a fantastic ideal. □

Consider the real interval [0, 1] of reals equipped with the Lukasiewicz implication $\to_{\text{Ł}}$ defined by: for any $x, y \in [0, 1]$,

$$x \to_{\text{Ł}} y = x^{-} \oplus y = min\{1 - m(x) + m(y), 1\}.$$

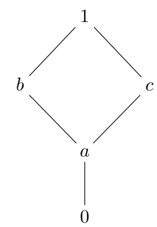
Definition 4.23. Let L be a bounded pseudo L-algebra. A *state-morphism* on L is a function $m : L \to [0, 1]$ satisfying the following conditions: for all $x, y \in L$,
$(SM1)$ $m(0) = 0$,
$(SM2)$ $m(x \to y) = m(x \rightsquigarrow y) = m(x) \to_{\text{Ł}} m(y) = min\{1 - m(x) + m(y), 1\}$.

Example 4.24. Let $(L, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo L-algebra. Then
(1) the $1_{L(x)} = 1$ for all $x \in L$ is a state-morphism on L,
(2) the identity map $Id_L : L \to L$ serves as a state-morphism on L.

Example 4.25 ([14])**.** Let $L = \{0, a, b, c, 1\}$ be a bounded pseudo L-algebra such that $0 < a < b$, $c < 1$, and b and c are incomparable. Define the operations $\to$ and $\rightsquigarrow$ using the following two tables.
Then we assert that the map $s : L \to [0, 1]$, defined as follows:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	a	a	1	1
1	0	a	b	c	1



$$s(0) = 0,\ s(a) = 1,\ s(b) = 1,\ s(c) = 1,\ s(1) = 1$$

is indeed a state-morphism on L. We also can prove that s is a Bosbach state of L.

Proposition 4.26. *Let L be a bounded pseudo L-algebra. Then every state-morphism on L is a Bosbach state on L.*

Proof. By (SM1), we get $m(0) = 0$. For each $x \in L, m(1) = m(x \to x) = min\{1, 1 - m(x) + m(x)\} = 1$. Moreover, $m(x) + m(x \to y) = m(x) + min\{1, 1 - m(x) + m(y)\} = min\{1 + m(x), 1 + m(y)\} = m(y) + min\{1 - m(y) + m(x), 1\} = m(y) + m(y \to x)$. Analogously, $m(x) + m(x \rightsquigarrow y) = m(y) + m(y \rightsquigarrow x)$ holds. Thus $m \in BS(L)$. □

Proposition 4.27. *Let L be a pseudo CL-algebra. A Bosbach state m on L is a state-morphism if and only if*

$$m(x \vee_1 y) = m(x \vee_2 y) = max\{m(x), m(y)\},\ \textit{for all } x, y \in L.$$

Proof. Suppose m is a state-morphism on L and let $x, y \in L$. By Proposition 4.4(3), $m(x \vee_1 y) = m(x \vee_2 y)$. Since m is a state-morphism on L, by Proposition 4.26, m is a Bosbach state. Using the relation: $m(x \to y) = 1 - m(x \vee_1 y) + m(y)$, we obtain $m(x \vee_1 y) = 1 + m(y) - m(x \to y) = 1 + m(y) - (m(x) \to_Ł m(y)) = 1 + m(y) - min\{1 - m(x) + m(y), 1\} = 1 + m(y) + max\{-1 + m(x) - m(y), -1\} = max\{m(x), m(y)\}$. Then the necessary condition holds.

Conversely, suppose m is a Bosbach state on L such that $m(x \vee_1 y) = max\{m(x), m(y)\}$ for all $x, y \in L$. Then using again the relation: $m(x \to y) = 1 - m(x \vee_1 y) + m(y)$, we have $m(x \to y) = 1 + m(y) - max\{m(x), m(y)\} = 1 + m(y) + min\{-m(x), -m(y)\} = min\{1 - m(x) + m(y), 1\} = m(x) \to_Ł m(y)$. Similarly, $m(x \rightsquigarrow y) = m(x) \to_Ł m(y)$. Thus m is a state-morphism on L. □

Proposition 4.28. *Let L be a bounded pseudo CL-algebra and m be a state-morphism on L. Then we have the following: for all $x, y \in L$,*
(1) $m(x^{\sim} \to y^{-\sim}) = min\{m(x) + m(y), 1\}$,
(2) $m(y^{-} \rightsquigarrow x^{\sim -}) = min\{m(x) + m(y), 1\}$.

Proof. (1) Let $x, y \in L$. Since m is a state-morphism on L, m is a Bosbach state on L. Thus by Propositions 4.3(4), $m(x^{\sim} \to y^{-\sim}) = m(x^{\sim}) \to_{\text{L}} m(y^{-\sim}) = min\{1 - m(x^{\sim}) + m(y^{-\sim}), 1\} = min\{1 - (1 - m(x)) + m(y), 1\} = min\{m(x) + m(y), 1\}$.

(2)Similarly, we can prove $m(y^{-} \rightsquigarrow x^{\sim -}) = min\{m(x) + m(y), 1\}$ for all $x, y \in L$. □

5. Riečan States on pseudo L-algebras

We introduce the notion of Riečan states in bounded pseudo L-algebras and investigate the relationships between Bosbach states and Riečan states.

Definition 5.1. Let $(L, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo L-algebra. Two elements $x, y \in L$ are said to be *orthogonal*, if $x^{-\sim} \leq y^{\sim}$, denoted as $x \perp y$.

Let $(L, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo L-algebra. For two orthogonal elements $x, y \in L$, we define a partial operation $+$ on L by $x + y := y^{\sim} \to x^{-\sim}$.

Proposition 5.2. *Let L be a bounded pseudo L-algebra. Then the following properties hold: for any $x, y \in L$,*

(1) *$x \perp x^{-}$ and $x + x^{-} = 1$,*
(2) *$x \perp 0$ and $x + 0 = x^{-\sim}$,*
(3) *$0 \perp x$ and $0 + x = x^{\sim -}$,*
if L has (pDN) property, then
(4) *$x^{\sim} \perp x$ and $x^{\sim} + x = 1$,*
let L satisfy (C) and with (pDN) property:
(5) *if $x \perp y$, then $y^{-\sim} \leq x^{-}$,*
(6) *if $x \leq y$, then $x \perp y^{-}, y^{\sim} \perp x$ and $x + y^{-} = y \to x^{-\sim}, y^{\sim} + x = y \rightsquigarrow x^{\sim -}$.*

Proof. (1) Since $x^{-\sim} \leq x^{-\sim} = (x^{-})^{\sim}$, $x \perp x^{-}$ and $x + x^{-} = x^{-\sim} \to x^{-\sim} = 1$.

(2)Since $x^{-\sim} \leq 1 = 0^{\sim}$, $x \perp 0$ and $x + 0 = 0^{\sim} \to x^{-\sim} = 1 \to x^{-\sim} = x^{-\sim}$.

(3)Since $x^{-\sim} \leq 1 = 0^{-}$, $0 \perp x$ and $0 + x = x^{\sim} \to 0^{\sim -} = x^{\sim} \to 0 = x^{\sim -}$.

(4) Since $x^{-\sim} \leq x^{-\sim} = (x^{\sim})^{-}$, we get

$$x^{\sim} \perp x \text{ and } x^{\sim} + x = x^{\sim} \to x^{\sim -\sim} = x^{\sim} \to x^{\sim} = 1.$$

(5) Suppose $x^{-\sim} \leq y^{\sim}$. Then $y^{\sim -} \leq x^{-\sim -} = x^{-}$.

(6) Suppose $x \leq y$. Then by Proposition 2.11(1), we have $x^{-\sim} \leq y^{-\sim}$. Thus $x \perp y^{-}$. $x + y^{-} = y^{-\sim} \to x^{-\sim} = y \to x^{-\sim}$. Similarly, we have $y^{\sim} \leq x^{\sim}$. So $(y^{\sim})^{-\sim} \leq x^{\sim}$. Hence $y^{\sim} \perp x$ and $y^{\sim} + x = x^{\sim} \to y^{\sim -\sim} = x^{\sim} \to y^{\sim} = y \rightsquigarrow x^{\sim -}$. □

Definition 5.3. Let $(L, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo L-algebra and let $s : L \to [0, 1]$ be a function. Then s is said to be a *Riečan state*, if it satisfies the following conditions: for each $x, y \in L$,

$(RI1)$ $s(1) = 1$,
$(RI2)$ $s(x + y) = s(x) + s(y)$ whenever $x \perp y$.

Denote by $RI(L)$ the set of all Riečan states on L.

Example 5.4. Consider the bounded pseudo L-algebra L from Example 4.5. Let $s : L \to [0, 1]$, defined as $s(0) = 0$, $s(a) = 0$, $s(b) = 1$, $s(c) = 1$, and $s(1) = 1$, is a Riečan state on L. We consider pairs (x, y) of orthogonal elements of L as given in the following table:

x	y	$x^{-\sim}$	$y^{\sim}$	$x+y$
0	0	0	1	0
0	a	0	c	0
0	b	0	0	1
0	c	0	0	1
0	1	0	0	1
a	0	0	1	0
a	a	0	c	0
a	b	0	0	1
a	c	0	0	1
a	1	0	0	1
b	0	1	1	1
c	0	1	1	1
1	0	1	1	1

Then one can easily verify that s is a Riečan state.

Example 5.5. Consider the bounded pseudo L-algebra L from Example 4.24. Let $s : L \to [0,1]$, defined as $s(0) = 0$, $s(a) = 1$, $s(b) = 1$, $s(c) = 1$, and $s(1) = 1$, is a Riečan state on L. We consider pairs (x, y) of orthogonal elements of L as given in the following table:

x	y	$x^{-\sim}$	$y^{\sim}$	$x+y$
0	0	0	1	0
0	a	0	0	1
0	b	0	0	1
0	c	0	0	1
0	1	0	0	1
a	0	1	1	1
b	0	1	1	1
c	0	1	1	1
1	0	1	1	1

Then one can easily verify that s is a Riečan state.

Example 5.6. Let $(L, \to, \rightsquigarrow, 0, 1)$ be a pseudo L-algebra with the greatest element 1 and $L \setminus \{1\}$ be discrete. Define the binary operations $\to$ and $\rightsquigarrow$ as following:

$$x \to y = \begin{cases} y, & x, y \in L \setminus \{1\} \\ y, & x = 1 \\ 1, & y = 1, \end{cases} \qquad \textit{and} \qquad x \rightsquigarrow y = \begin{cases} x, & x, y \in L \setminus \{1\} \\ y, & x = 1 \\ 1, & y = 1, \end{cases}$$

Define a function $s : L \to [0,1]$, $s(a) = 0, a \in L$. Then one can easily verify that s is a Riečan state.

Proposition 5.7. *If s is a Riečan state on a bounded pseudo L-algebra L, then the following properties hold: for any $x, y \in L$,*

(1) $s(x^-) = s(x^\sim) = 1 - s(x)$,
(2) $s(0) = 0$,
(3) $s(x^{-\sim}) = s(x^{\sim-}) = s(x^{--}) = s(x^{\sim\sim}) = s(x)$,
let L satisfies the (K) and have (pDN) property:
(4) *if* $x \le y$, *then* $s(x) \le s(y)$ *and* $s(y \to x^{-\sim}) = s(y \rightsquigarrow x^{\sim-}) = 1 + s(x) - s(y)$,
if L satisfies the (C) and has (pDN) property, then
(5) $s((x \vee_1 y) \to x^{-\sim}) = s((x \vee_1 y) \rightsquigarrow x^{-\sim}) = 1 - s(x \vee_1 y) + s(x)$ *and*

$s((x \vee_2 y) \to x^{-\sim}) = s((x \vee_2 y) \rightsquigarrow x^{-\sim}) = 1 - s(x \vee_2 y) + s(x)$,
(6) $s((x \vee_1 y) \to y^{-\sim}) = s((x \vee_1 y) \rightsquigarrow y^{-\sim}) = 1 - s(x \vee_1 y) + s(y)$ *and*
$s((x \vee_2 y) \to y^{-\sim}) = s((x \vee_2 y) \rightsquigarrow y^{-\sim}) = 1 - s(x \vee_2 y) + s(y)$.

Proof. (1) By Proposition 5.2(1), we know that $x \perp x^-$ and $x + x^- = 1$. Then $s(x) + s(x^-) = s(1) = 1$. Thus $s(x^-) = 1 - s(x)$. Similarly, $s(x^\sim) = 1 - s(x)$.

(2) It follows from the fact that $0 \perp 0$ and $0 + 0 = 0$.

(3) Since $x \perp 0$ and $x + 0 = x^{-\sim}$, $s(x^{-\sim}) = s(x + 0) = s(x) + s(0) = s(x)$. Then $s(x^{-\sim}) = s(x)$. Similarly, $s(x^{\sim-}) = s(x)$. Thus by (1), we have

$$s(x^{--}) = 1 - s(x^-) = 1 - (1 - s(x)) = s(x)$$

and

$$s(x^{\sim\sim}) = 1 - s(x^\sim) = 1 - (1 - s(x)) = s(x).$$

(4) Suppose $x \leq y$. Then by Proposition 5.2(6), we have $x \perp y^-$ and $x + y^- = y \to x^{-\sim}$. Thus, $s(x) + s(y^-) = s(y \to x^{-\sim})$. So $s(x) - s(y) = s(y \to x^{-\sim}) - 1 \leq 0$, i.e., $s(x) \leq s(y)$. We also have $s(y \to x^{-\sim}) = 1 + s(x) - s(y)$. Similarly, since $x \leq y$, $y^\sim \perp x$ and $y^\sim + x = y \rightsquigarrow x^{\sim-}$. Hence $s(y \rightsquigarrow x^{\sim-}) = 1 + s(x) - s(y)$.

(5) Since $x \leq x \vee_1 y$, by Proposition 5.2(6), it follows that $x \perp (x \vee_1 y)^-$ and $x + (x \vee_1 y)^- = (x \vee_1 y) \to x^{-\sim}$. Then $s(x + (x \vee_1 y)^-) = s((x \vee_1 y) \to x^{-\sim})$. Thus $s((x \vee_1 y) \to x^{-\sim}) = 1 - s(x \vee_1 y) + s(x)$. Similarly, from $x \leq x \vee_1 y$, we get $(x \vee_1 y)^\sim \perp x$ and $(x \vee_1 y)^\sim + x = (x \vee_1 y) \rightsquigarrow x^{-\sim}$. So, $s((x \vee_1 y)^\sim + x) = s((x \vee_1 y) \rightsquigarrow x^{-\sim})$ and we get $1 - s(x \vee_1 y) + s(x) = s(x \vee_1 y \rightsquigarrow x^{-\sim})$. Hence, $s((x \vee_1 y) \to x^{-\sim}) = s((x \vee_1 y) \rightsquigarrow x^{-\sim}) = 1 - s(x \vee_1 y) + s(x)$. Similarly, $s((x \vee_2 y) \to x^{-\sim}) = s((x \vee_2 y) \rightsquigarrow x^{-\sim}) = 1 - s(x \vee_2 y) + s(x)$.

(6) It follows similarly as (5). □

Next, we discuss the relationships between Bosbach states and Riečan states.

Theorem 5.8. *Let L be a bounded pseudo L-algebra. Then any Bosbach state on L is a Riečan state.*

Proof. Let s be a Bosbach state on L. Assume $x \perp y$, i.e., $x^{-\sim} \leq y^\sim$. Then we have

$$1 + s(x^{-\sim}) = s(y^\sim) + s(y^\sim \to x^{-\sim}).$$

Thus we get

$$1 + s(x) = 1 - s(y) + s(y^\sim \to x^{-\sim}).$$

So $s(y^\sim \to x^{-\sim}) = s(x) + s(y)$. Hence $s(x + y) = s(x) + s(y)$. By BS(1), $s(1) = 1$. Therefore s is a Riečan state on L. □

In the next example we show that there exists a Riečan state which is not a Bosbach state.

Example 5.9. Let $L = \{0, a, b, c, 1\}$ be a lattice, where $0 < a < b$, $c < 1$, and b and c are incomparable. Define the operations $\to$ and $\rightsquigarrow$ using the following tables.

Then we can verify that $(L, \to, \rightsquigarrow, 1)$ forms a bounded pseudo L-algebra. Define the function $s : L \to [0, 1]$ by

$$s(0) = 0,\ s(a) = 1/2,\ s(b) = 1,\ s(c) = 1 \text{ and } s(1) = 1.$$

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	c	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

is a Riečan state. The orthogonal elements of L are the pairs (x, y) in the table.

x	y	$x^{-\sim}$	$y^{\sim}$	$x+y$
0	0	0	1	0
0	b	0	a	b
0	c	0	0	1
0	1	0	0	1
a	0	a	1	a
b	0	c	1	c
c	0	1	1	1
1	0	1	1	1

One can easily check that s is a Riečan state, but the function s defined above is not a Bosbach state. Indeed, checking condition (BS3), we obtain

$$s(a) + s(a \rightsquigarrow 0) = s(a) + s(c) = 1/2 + 1 = 3/2,$$
$$s(0) + s(0 \rightsquigarrow a) = s(0) + s(1) = 0 + 1 = 1,$$

so condition (BS3) does not hold. We conclude that s is not a Bosbach state.

Remark 5.10. Theorem 5.7 and Example 5.8 show that the concept of Riečan states is more general than that of Bosbach states.

Theorem 5.11. *Let L be a $\vee_1$-commutative and $\vee_2$-commutative pseudo CL-algebra with (PDN) property. Then every Riečan state on L is a Bosbach state.*

Proof. Let s be a Riečan state on L. Then s(1)=1. By Propositions 5.7(2), s(0) = 0. Since $x \leq x \vee_1 y$, by Proposition 2.11(5), $x^{-\sim} \leq y^{-\sim}$, i.e. $x \perp (x \vee_1 y)^-$. Thus $s(x+(x\vee_1 y)^-) = s((x\vee_1 y)^{-\sim}) \rightarrow x^{-\sim}) = s((x\vee_1 y)) \rightarrow x^{-\sim}) = s((x\vee_1 y) \rightarrow x) = s(x)+s((x\vee_1 y)^-) = s(x)+1-s(x\vee_1 y)$ and $s(x+(x\vee_2 y)^-) = s((x\vee_2 y)^{-\sim}) \rightarrow x^{-\sim}) = s(x \vee_2 y) \rightarrow x^{-\sim}) = s((x \vee_2 y) \rightarrow x) = s(x) + s((x \vee_2 y)^-) = s(x) + 1 - s(x \vee_2 y)$. Applying Proposition 2.14(7) and 5.7, we have $s(x \rightarrow y) = 1 - s(x \vee_1 y) + s(y)$ and $s(x \rightsquigarrow y) = 1 - s(x \vee_2 y) + s(y)$. Finally, by Proposition 4.8, it follows that s is a Bosbach state on L. □

6. Conclusion

In this paper, we primarily present states on pseudo L-algebras and investigate their properties, existence, and relationships. Firstly, We discuss the relationship between pseudo KL-algebras, pseudo CL-algebras and pseudo BCK-algebras. Additionally, we introduce ideals and congruence relations on pseudo L-algebras. Subsequently, we define the concept of Bosbach states and morphisms on pseudo L-algebras, then explore their properties and relationships. We prove that each morphism is a Bosbach state. Furthermore, we introduce the notion of fantastic ideals and pseudo MV-ideals, also use these ideals to study the existence of states. Moreover, we introduce the concept of Riečan states and investigate their properties, as well as the connection between Bosbach states and Riečan states.

In our future work, we will study internal states on pseudo L-algebras and explore the relationship between states and internal states on pseudo L-algebras. Additionally, we aim to establish internal states connections between pseudo L-algebras and other logical algebras.

Acknowledgement This research is supported by Research on the Construction of Evaluation Index System for Teacher-Student Interaction in Classrooms during the 14th Five-Year Plan of Shaanxi Province (Project Number: SGH24Y2791), a grant of Foreign Expert Program of China (DL20230410021) and Yulin City Industry University Research Project of China (No.CXY-2022-59).

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