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# Generalized version of $\alpha^{\theta}$-closed set and its applications in fuzzy setting

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Abstract. This paper deals with a new type of generalized version of fuzzy closed set, viz., $f\alpha^{\theta}g$-closed set, the class of which is strictly larger than that of fuzzy $\alpha$-closed set [1]. Using this concept as a basic tool, here we introduce an idempotent operator. Afterwards, $f\alpha^{\theta}g$-open (resp., $f\alpha^{\theta}g$-closed) function is introduced and characterized. It is shown that the class of $f\alpha^{\theta}g$-open (resp., $f\alpha^{\theta}g$-closed) functions is strictly larger than that of fuzzy open [2] (resp., fuzzy closed [2]) function. In the last section a new type of fuzzy separation axiom is introduced and some applications of the function defined here are established.

2020 AMS Classification: 54A40, 03E72

Keywords: Fuzzy semiopen set, Fuzzy $\alpha$-open set, $f\alpha^{\theta}g$-closed set, $f\alpha^{\theta}g$-open function, $f\alpha^{\theta}g$-closed function, $fT_{\alpha^{\theta}g}$-space, $f\alpha^{\theta}g$-$T_2$-space.

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## 1. Introduction

In 1965, Zadeh [3] introduced fuzzy set. In 1968, Chang [4] introduced fuzzy topology. Afterwards, many researchers have engaged themselves to introduce and study several types of fuzzy open-like sets. Fuzzy regular open and fuzzy semiopen sets are introduced by Azad [5]. Fuzzy $\alpha$-open set is introduced by Shahna [1]. In [6, 7], generalized version of fuzzy closed set is introduced and studied. After that different types of fuzzy generalized version of closed sets are investigated. In this regard, $fmg$-closed set [8], $fwg$-closed set [8], $fswg$-closed set [8] and $frwg$-closed set [8] have to be mentioned. In this paper, taking fuzzy $\alpha$-open set as a basic tool, we introduce and study $f\alpha^{\theta}g$-closed set. The mutual relationships of $f\alpha^{\theta}g$-closed sets with $fg$-closed set, $fmg$-closed set, $fwg$-closed set, $fswg$-closed set and $frwg$-closed set are shown here. Fuzzy open function (resp., fuzzy closed

function) is introduced by Wong [2]. Here we introduce $f\alpha^\theta g$-open (resp., $f\alpha^\theta g$-closed) function and characterize in several ways. We also establish the mutual relationships of $f\alpha^\theta g$-open function (resp., $f\alpha^\theta g$-closed function) with $fg$-open (resp., $fg$-closed) function [7], $fmg$-open (resp., $fmg$-closed) function [9], $fwg$-open (resp., $fwg$-closed) function [10], $fswg$-open (resp., $fswg$-closed) function [11] and $frwg$-open (resp., $frwg$-closed) function [12]. Fuzzy $T_2$-space is introduced in [13]. Here we introduce $f\alpha^\theta g$-$T_2$-space which is strictly larger than fuzzy $T_2$-space.
Recently, new types of fuzzy sets, viz., fuzzy soft set and fuzzy octahedron set are introduced and studied. A new branch in fuzzy system is developed using these types of fuzzy sets. In this context, we have to mention [14, 15, 16, 17, 18].

## 2. Preliminaries

Throughout this paper, $(X, \tau)$ or simply by $X$, we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [4]. A fuzzy set $A$ is a function from a non-empty set $X$ into the closed interval $I = [0, 1]$, i.e., $A \in I^X$ [3]. The support of a fuzzy set $A$, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$ [3]. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value $t$ $(0 < t \leq 1)$ will be denoted by $x_t$. $0_X$ and $1_X$ are the constant fuzzy sets taking values 0 and 1 respectively in $X$. The complement of a fuzzy set $A$ in $X$ is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$ for each $x \in X$ [3]. For any two fuzzy sets $A$, $B$ in $X$, the *intersection* and the *union* of $A$ and $B$, denoted by $A \wedge B$ and $A \vee B$, are defined as follows: for each $x \in X$,

$$(A \wedge B) = A(x) \wedge B(x) \text{ and } (A \vee B)(x) = A(x) \vee B(x),$$

where $A(x) \wedge B(x) = min\{A(x), B(x)\}$ and $A(x) \vee B(x) = max\{A(x), B(x)\}$.
For any two fuzzy sets $A$, $B$ in $X$, $A \leq B$ means $A(x) \leq B(x)$ for all $x \in X$ [3] while $AqB$ means $A$ is quasi-coincident (q-coincident, for short) with $B$, if there exists $x \in X$ such that $A(x) + B(x) > 1$ [19]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy point $x_t$ and a fuzzy set $A$, $x_t \in A$ means $A(x) \geq t$, i.e., $x_t \leq A$. For a fuzzy set $A$, $clA$ and $intA$ will stand for fuzzy closure [4] and fuzzy interior [4] of $A$ respectively. A fuzzy set $A$ is called a *fuzzy neighbourhood* (fuzzy nbd, for short) of a fuzzy point $x_t$, if there exists a fuzzy open set $U$ in $X$ such that $x_t \in U \leq A$ [19]. If, in addition, $A$ is fuzzy open, then $A$ is called a *fuzzy open nbd* of $x_t$ [19]. A fuzzy set $A$ is called a *fuzzy quasi neighbourhood* (fuzzy $q$-nbd, for short) [19] of a fuzzy point $x_t$ in an fts $X$, if there is a fuzzy open set $U$ in $X$ such that $x_t qU \leq A$. If, in addition, $A$ is fuzzy open, then $A$ is called a *fuzzy open q-nbd* [19] of $x_t$. A fuzzy set $A$ in $X$ is called a *fuzzy regular open* [5] (*fuzzy semi open* [5], fuzzy $\alpha$-open [1]), if $A = intclA$ (resp., $A \leq cl(intA)$, $A \leq intclintA$). The complement of a fuzzy $\alpha$-open set is said to be *fuzzy $\alpha$-closed* [1]. The intersection (resp., union) of all fuzzy $\alpha$-closed (resp., fuzzy $\alpha$-open) sets containing (resp., contained in) a fuzzy set $A$ is called fuzzy $\alpha$-closure [1] (resp., fuzzy $\alpha$-interior [1]) of $A$, to be denoted by $\alpha clA$ (resp., $\alpha intA$). The collection of all fuzzy regular open (resp. fuzzy semiopen, fuzzy $\alpha$-open) sets in an fts $X$ is denoted by $FRO(X)$ (resp., $FSO(X)$, $F\alpha O(X)$) and that of fuzzy $\alpha$-closed sets is denoted by $F\alpha C(X)$.

## 3. $f\alpha^\theta g$-closed set: Some properties

In this section we first introduce $f\alpha^\theta g$-closed set and establish some of its properties. Then establish the mutual relationships of this newly defined set with the sets defined in [6, 7, 8].

**Definition 3.1.** Let $(X, \tau)$ be an fts and $A \in I^X$. Then $A$ is called an *$f\alpha^\theta g$-closed set* in $X$, if $cl(intA) \leq U$, whenever $A \leq U \in F\alpha O(X)$.

The complement of this set is called an *$f\alpha^\theta g$-open set* in $X$.
The collection of all $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) sets in an fts $X$ is denoted by $F\alpha^\theta GC(X)$ (resp., $F\alpha^\theta GO(X)$).

**Remark 3.2.** Union and intersection of two $f\alpha^\theta g$-closed sets may not be so, as it seen from the following example.

**Example 3.3.** Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$, where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.3, B(b) = 0.2$. Then $(X, \tau)$ is an fts. Here $F\alpha O(X) = \{0_X, 1_X, U\}$, where $B \leq U \leq A$. Now consider the fuzzy sets $C$ and $D$ defined by $C(a) = 0.6, C(b) = 0.2, D(a) = 0.3, D(b) = 0.6$. Clearly $C$ and $D$ are $f\alpha^\theta g$-closed sets in $(X, \tau)$. Let $E = C \wedge D$. Then $E = B \leq B \in F\alpha O(X)$. But $cl(intE) = 1_X \setminus A \not\leq B$. Thus $E$ is not $f\alpha^\theta g$-closed set in $X$.

Again, consider two fuzzy sets $S$ and $T$ defined by $S(a) = 0.3, S(b) = 0, T(a) = 0, T(b) = 0.2$. Then clearly $S, T \in F\alpha^\theta GC(X)$. Let $U = S \vee T$. Then $U = B \leq B \in F\alpha O(X)$. But $cl(intU) = 1_X \setminus A \not\leq B$. Thus $U \notin F\alpha^\theta GC(X)$.

**Note 3.4.** So we can conclude that the set of all $f\alpha^\theta g$-open sets in an fts $(X, \tau)$ does not form a fuzzy topology.

**Theorem 3.5.** *Let $(X, \tau)$ be an fts and $A, B \in I^X$. If $A \leq B \leq cl(intA)$ and $A$ is an $f\alpha^\theta g$-closed set in $X$, then $B$ is also an $f\alpha^\theta g$-closed set in $X$.*

*Proof.* Let $U \in F\alpha O(X)$ such that $B \leq U$. Then by hypothesis, $A \leq B \leq U$. As $A$ is $f\alpha^\theta g$-closed set in $X$, $cl(intA) \leq U$. Thus we have

$$cl(intA) \leq cl(intB) \leq cl(int(cl(intA))) \leq cl(intA) \leq U.$$

So $B$ is an $f\alpha^\theta g$-closed set in $X$. □

**Theorem 3.6.** *Let $(X, \tau)$ be an fts and $A, B \in I^X$. If $int(clA) \leq B \leq A$ and $A$ is an $f\alpha^\theta g$-open set in $X$, then $B$ is also an $f\alpha^\theta g$-open set in $X$.*

*Proof.* Suppose $int(clA) \leq B \leq A$ and $A$ is an $f\alpha^\theta g$-open set in $X$. Then we get

$$int(clA) \leq B \leq A \Rightarrow 1_X \setminus A \leq 1_X \setminus B \leq 1_X \setminus int(clA) = cl(int(1_X \setminus A)),$$

where $1_X \setminus A$ is an $f\alpha^\theta g$-closed set in $X$. Thus by Theorem 3.5, $1_X \setminus B$ is an $f\alpha^\theta g$-closed set in $X$. So $B$ is an $f\alpha^\theta g$-open set in $X$. □

**Theorem 3.7.** *Let $(X, \tau)$ be an fts and $A \in I^X$. Then $A$ is an $f\alpha^\theta g$-open set in $X$ if and only if $K \leq int(clA)$, whenever $K \leq A$ and $K$ is a fuzzy $\alpha$-closed set in $(X, \tau)$.*

*Proof.* Suppose $A$ is an $f\alpha^\theta g$-open set in $X$ and let $A(\in I^X)$ be an $f\alpha^\theta g$-open set in $X$ and $K \leq A$, where $K$ is a fuzzy $\alpha$-closed set in $(X,\tau)$. Then $1_X \setminus A \leq 1_X \setminus K$, where $1_X \setminus A$ is an $f\alpha^\theta g$-closed set in $X$ and $1_X \setminus K$ is a fuzzy $\alpha$-open set in $(X,\tau)$. By hypothesis, $cl(int(1_X \setminus A)) \leq 1_X \setminus K$. Thus $1_X \setminus int(clA) \leq 1_X \setminus K$. So $K \leq int(clA)$.

Conversely, suppose $K \leq int(clA)$, whenever $K \leq A$, $K \in F\alpha C(X)$. Then $1_X \setminus A \leq 1_X \setminus K$, where $1_X \setminus K \in F\alpha O(X)$. By hypothesis, $1_X \setminus int(clA) \leq 1_X \setminus K$. Then $cl(int(1_X \setminus A)) \leq 1_X \setminus K$. Thus $1_X \setminus A$ is an $f\alpha^\theta g$-closed set in $X$. So $A$ is an $f\alpha^\theta g$-open set in $X$. □

**Theorem 3.8.** *Let $(X,\tau)$ be an fts and $A, B \in I^X$. If $A$ is an $f\alpha^\theta g$-closed set in $X$ and $B$ is a fuzzy $\alpha$-closed set in $(X,\tau)$ with $A \not{q} B$, then $cl(intA) \not{q} B$.*

*Proof.* By hypothesis, $A \not{q} B$. Then $A \leq 1_X \setminus B \in F\alpha O(X)$. Thus $cl(intA) \leq 1_X \setminus B$. So $cl(intA) \not{q} B$. □

**Remark 3.9.** The converse of Theorem 3.8 may not be true, in general, as it seen from the following example.

**Example 3.10.** Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$, where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.4, B(b) = 0.5$. Then $(X,\tau)$ is an fts. Here $F\alpha O(X) = \{0_X, 1_X, B, U\}$, where $U \geq A$ and $F\alpha C(X) = \{0_X, 1_X, 1_X \setminus B, 1_X \setminus U\}$, where $1_X \setminus U \leq 1_X \setminus A$. Consider the fuzzy set $D$ defined by $D(a) = D(b) = 0.5$. Then $D \leq A \in F\alpha O(X)$. But $cl(intD) = 1_X \setminus B \not\leq A$. Thus $D$ is not an $f\alpha^\theta g$-closed set in $X$. Again, $D \not{q} E \in F\alpha C(X)$, where $E(a) = E(b) = 0.3$. Also $cl(intD) = (1_X \setminus B) \not{q} E$.

Now we recall the following definitions from [6, 7, 8] for ready references.

**Definition 3.11.** Let $(X,\tau)$ be an fts and $A \in I^X$. Then $A$ is called an:

(i) *$fg$-closed set* [6, 7], if $clA \leq U$, whenever $A \leq U \in \tau$ and the complement of $fg$-closed set is said to be *$fg$-open*,

(ii)*$fwg$-closed set* [8], if $clintA \leq U$, whenever $A \leq U \in \tau$ and the complement of $fwg$-closed set is said to be *$fwg$-open*,

(iii) *$fmg$-closed set* [8], if $clintA \leq U$, whenever $A \leq U$, $U$ is $fg$-open set in $X$ and the complement of $fmg$-closed set is said to be *$fmg$-open*,

(iv) *$fswg$-closed set* [8], if $clintA \leq U$, whenever $A \leq U \in FSO(X)$ and the complement of $fswg$-closed set is said to be *$fswg$-open*,

(v) *$frwg$-closed set* [8], if $clintA \leq U$, whenever $A \leq U \in FRO(X)$ and the complement of $frwg$-closed set is said to be *$frwg$-open*.

**Note 3.12.** It is clear from definitions that every fuzzy $\alpha$-closed set is an $f\alpha^\theta g$-closed. But the converse may not be true, follows from Example 3.3. Here $S$ is $f\alpha^\theta g$-closed, but $clintclS = 1_X \setminus A \not\leq S$.

**Remark 3.13.** (1) $fmg$-closed set, $fswg$-closed set $\Rightarrow$ $f\alpha^\theta g$-closed set $\Rightarrow$ $fwg$-closed set, $frwg$-closed set, but the reverse implications are not necessarily true, follow from the next examples.

(2) $fg$-closed set and $f\alpha^\theta g$-closed are independent concepts, follow from the next examples.

**Example 3.14.** An $f\alpha^\theta g$-closed set does not imply an $fg$-closed set.
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.6$. Then $(X,\tau)$ is

an fts. Thus $F\alpha O(X) = \{0_X, 1_X, U\}$, where $A \leq U$. Consider the fuzzy set $B$ defined by $B(a) = B(b) = 0.5$. Then $B \leq A \in \tau$ (also $B \leq A \in F\alpha O(X)$). Thus $clB = 1_X \not\leq A$. So $B$ is not an $fg$-closed set in $X$. But $cl(intB) = 0_X < A$. Hence $B$ is an $f\alpha^\theta g$-closed set in $X$.

**Example 3.15.** An $fg$-closed set and an $fwg$-closed set do not imply an $f\alpha^\theta g$-closed set.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$, where $A(a) = 0.45, A(b) = 0.55, B(a) = 0.4, B(b) = 0.5$. Then $(X, \tau)$ is an fts. Here $F\alpha O(X) = \{0_X, 1_X, B, U\}$, where $U \geq A$. Consider the fuzzy set $C$ defined by $C(a) = C(b) = 0.5$. Then $C < 1_X (\in \tau)$ only and thus $clC = 1_X \setminus B < 1_X$. So $C$ is an $fg$-closed set in $X$. Also, $clintC = 1_X \setminus B < 1_X$. Hence $C$ is an $fwg$-closed set in $X$. Now $C < D \in F\alpha O(X)$, where $D(a) = 0.5, D(b) = 0.55$. But $clintC = 1_X \setminus B \not\leq D$. Therefore $C$ is not an $f\alpha^\theta g$-closed set in $X$.

**Example 3.16.** An $f\alpha^\theta g$-closed set does not imply an $fswg$-closed set.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.4$. Then $(X, \tau)$ is an fts. Thus $F\alpha O(X) = \tau$ and $FSO(X) = \{0_X, 1_X, U\}$, where $A \leq U \leq 1_X \setminus A$. Consider the fuzzy set $B$ defined by $B(a) = B(b) = 0.5$. So $B < 1_X \in F\alpha O(X)$ only and hence $B$ is an $f\alpha^\theta g$-closed set in $X$. Now $B \leq B \in FSO(X)$. But $clintB = 1_X \setminus A \not\leq B$. Therefore $B$ is not an $fswg$-closed set in $X$.

**Example 3.17.** An $frwg$-closed set does not imply an $f\alpha^\theta g$-closed set.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.6$. Then $(X, \tau)$ is an fts. Thus $F\alpha O(X) = \{0_X, 1_X, U\}$, where $A \leq U$ and $FRO(X) = \{0_X, 1_X\}$. Consider the fuzzy set $B$ defined by $B(a) = B(b) = 0.6$. Clearly $B$ is an $frwg$-closed set in $X$. Now $B \leq B \in F\alpha O(X)$. But $clintB = 1_X \not\leq B$. So $B$ is not an $f\alpha^\theta g$-closed set in $X$.

**Example 3.18.** An $f\alpha^\theta g$-closed set does not imply an $fmg$-closed set.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.4$. Then $(X, \tau)$ is an fts. Thus $F\alpha O(X) = \tau$ and the collection of $fg$-open sets in $X$ is $\{0_X, 1_X, U\}$, where $U \not\geq 1_X \setminus A$. Consider the fuzzy set $B$ defined by $B(a) = B(b) = 0.5$. Then clearly $B$ is an $f\alpha^\theta g$-closed set in $X$. Now $B \leq B$, where $B$ is $fg$-open set in $X$. But $clintB = 1_X \setminus A \not\leq B$. Thus $B$ is not an $fmg$-closed set in $X$.

**Definition 3.19.** A fts $(X, \tau)$ is called an $fT_{\alpha^\theta g}$*-space*, if every $f\alpha^\theta g$-closed set in $X$ is a fuzzy closed set in $X$.

**Note 3.20.** In $fT_{\alpha^\theta g}$-space, an $f\alpha^\theta g$-closed set is $fg$-closed, $fmg$-closed, $fswg$-closed. Indeed, if $A$ is closed, then $A \leq U \Rightarrow clA = A \leq U \Rightarrow clintA \leq clA \leq U$.

Now we introduce a new type of generalized version of neighbourhood system in an fts.

**Definition 3.21.** Let $(X, \tau)$ be an fts and $x_t$, a fuzzy point in $X$. A fuzzy set $A$ is called an $f\alpha^\theta g$*-neighbourhood* ($f\alpha^\theta g$-nbd, for short) of $x_t$, if there exists an $f\alpha^\theta g$-open set $U$ in $X$ such that $x_t \in U \leq A$. If, in addition, $A$ is $f\alpha^\theta g$-open set in $X$, then $A$ is called an $f\alpha^\theta g$*-open nbd* of $x_t$.

**Definition 3.22.** Let $(X, \tau)$ be an fts and $x_t$, a fuzzy point in $X$. A fuzzy set $A$ is called an $f\alpha^\theta g$-*quasi neighbourhood* ($f\alpha^\theta g$-$q$-nbd, for short) of $x_t$, if there is an $f\alpha^\theta g$-open set $U$ in $X$ such that $x_t q U \leq A$. If, in addition, $A$ is $f\alpha^\theta g$-open set in $X$, then $A$ is called an $f\alpha^\theta g$-*open* $q$-*nbd* of $x_t$.

**Note 3.23.** (1) It is clear from definitions that every $f\alpha^\theta g$-open set is an $f\alpha^\theta g$-open nbd of each of its points. But every $f\alpha^\theta g$-nbd of $x_t$ may not be an $f\alpha^\theta g$-open set containing $x_t$ follows from the following example.

(2) Also every fuzzy open nbd (resp., fuzzy open $q$-nbd) of a fuzzy point $x_t$ is an $f\alpha^\theta g$-open nbd (resp., $f\alpha^\theta g$-open $q$-nbd) of $x_t$. But the converses are not necessarily true, in general, as it is seen from the following example.

**Example 3.24.** Consider Example 3.10 and the fuzzy point $a_{0.4}$ and the fuzzy set $D$. Here $D$ is not $f\alpha^\theta g$-closed as well as $f\alpha^\theta g$-open set in $X$. Also $a_{0.4} \in D$. Now consider the fuzzy set $E$ defined by $E(a) = 0.6, E(b) = 0.5$. Then $E \leq U \in F\alpha O(X)$, where $U(a) = U(b) = 0.6$. Now $clintE = 1_X \setminus B \leq U$. Thus $E$ is an $f\alpha^\theta g$-closed set in $X$ and so $1_X \setminus E$ is an $f\alpha^\theta g$-open set in $X$ containing $a_{0.4}$. Now $1_X \setminus E \leq D$. Hence $D$ is an $f\alpha^\theta g$-nbd of $a_{0.4}$ but not an $f\alpha^\theta g$-open nbd of $a_{0.4}$.

Again consider the fuzzy point $b_{0.6}$. Then $b_{0.6} q (1_X \setminus E) \leq D$. Thus $D$ is an $f\alpha^\theta g$-$q$-nbd of $b_{0.6}$, but not an $f\alpha^\theta g$-open $q$-nbd of $b_{0.6}$.

**Example 3.25.** Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.4, A(b) = 0.5$. Then $(X, \tau)$ is an fts. Here $F\alpha O(X) = \tau$. Consider the fuzzy point $a_{0.45}$ and the fuzzy set $B$ defined by $B(a) = B(b) = 0.5$. Then clearly $B$ is an $f\alpha^\theta g$-closed set as well as an $f\alpha^\theta g$-open set in $X$ containing $a_{0.45}$. Thus $B$ is an $f\alpha^\theta g$-open nbd of $a_{0.45}$. But as there is no fuzzy open set in $X$ containing $a_{0.45}$ contained in $B$, $B$ is not a fuzzy open nbd of $a_{0.45}$.

Next consider the fuzzy point $a_{0.6}$. Then $a_{0.6} q B$. Thus $B$ is an $f\alpha^\theta g$-open $q$-nbd of $a_{0.6}$. But there does not exist any fuzzy open set in $X$ $q$-coincident with $a_{0.6}$ contained in $B$. So $B$ is not a fuzzy open $q$-nbd of $a_{0.6}$.

## 4. $f\alpha^\theta g$-open function and $f\alpha^\theta g$-closed function

In this section, we first introduce and study a new type of generalized version of fuzzy closure-like operator which is seen to be an idempotent operator. Then using this operator as a basic tool, two types of functions are introduced and characterized.

**Definition 4.1.** Let $(X, \tau)$ be an fts and $A \in I^X$. Then $f\alpha^\theta g$-*closure* and $f\alpha^\theta g$-*interior* of $A$, denoted by $f\alpha^\theta gcl(A)$ and $f\alpha^\theta gint(A)$, are defined as follows:

$$f\alpha^\theta gcl(A) = \bigwedge\{F : A \leq F, F \text{ is } f\alpha^\theta g\text{-closed set in } X\},$$
$$f\alpha^\theta gint(A) = \bigvee\{G : G \leq A, G \text{ is } f\alpha^\theta g\text{-open set in } X\}.$$

**Remark 4.2.** It is clear from definition that for any $A \in I^X$, $A \leq f\alpha^\theta gcl(A) \leq clA$. If $A$ is an $f\alpha^\theta g$-closed set in an fts $X$, then $A = f\alpha^\theta gcl(A)$. Similarly, $intA \leq f\alpha^\theta gint(A) \leq A$. If $A$ is an $f\alpha^\theta g$-open set in an fts $X$, then $A = f\alpha^\theta gint(A)$. It follows from Remark 3.2 that $f\alpha^\theta gcl(A)$ (resp., $f\alpha^\theta gint(A)$) may not be an $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) set in an fts $X$.

**Theorem 4.3.** *Let $(X, \tau)$ be an fts and $A \in I^X$. Then for a fuzzy point $x_t$ in $X$, $x_t \in f\alpha^\theta gcl(A)$ if and only if every $f\alpha^\theta g$-open $q$-nbd $U$ of $x_t$, $UqA$.*

*Proof.* Let $x_t \in f\alpha^\theta gcl(A)$ for any fuzzy set $A$ in an fts $X$ and $F$ be any $f\alpha^\theta g$-open $q$-nbd of $x_t$. Then $x_t qF$. Thus $x_t \notin 1_X \setminus F$ which is $f\alpha^\theta g$-closed set in $X$. By Definition 4.1, $A \not\leq 1_X \setminus F$. So there exists $y \in X$ such that $A(y) > 1 - F(y)$. Hence $AqF$.

Conversely, let for every $f\alpha^\theta g$-open $q$-nbd $F$ of $x_t$, $FqA$. If possible, let $x_t \notin f\alpha^\theta gcl(A)$. Then by Definition 4.1, there exists an $f\alpha^\theta g$-closed set $U$ in $X$ with $A \leq U$, $x_t \notin U$. Thus $x_t q(1_X \setminus U)$ which being an $f\alpha^\theta g$-open set in $X$ is an $f\alpha^\theta g$-open $q$-nbd of $x_t$. By assumption, $(1_X \setminus U)qA$. So $(1_X \setminus A)qA$ which is absurd. □

**Theorem 4.4.** *Let $(X, \tau)$ be an fts and $A, B \in I^X$. Then the following statements are true:*

(1) $f\alpha^\theta gcl(0_X) = 0_X$,
(2) $f\alpha^\theta gcl(1_X) = 1_X$,
(3) $A \leq B \Rightarrow f\alpha^\theta gcl(A) \leq f\alpha^\theta gcl(B)$,
(4) $f\alpha^\theta gcl(A \vee B) = f\alpha^\theta gcl(A) \vee f\alpha^\theta gcl(B)$,
(5)$f\alpha^\theta gcl(A \wedge B) \leq f\alpha^\theta gcl(A) \wedge f\alpha^\theta gcl(B)$, *equality does not hold, in general, follows from Example 3.3,*
(6) $f\alpha^\theta gcl(f\alpha^\theta gcl(A)) = f\alpha^\theta gcl(A)$.

*Proof.* The proofs of (1), (2) and (3) are obvious.

(4) From (3), $f\alpha^\theta gcl(A) \vee f\alpha^\theta gcl(B) \leq f\alpha^\theta gcl(A \vee B)$.

To prove the converse, let $x_t \in f\alpha^\theta gcl(A \vee B)$. Then by Theorem 4.3, for any $f\alpha^\theta g$-open set $U$ in $X$ with $x_t qU$, $Uq(A \vee B)$. Thus there exists $y \in X$ such that $U(y) + max\{A(y), B(y)\} > 1$. So we have the implications:
either $U(y) + A(y) > 1$ or $U(y) + B(y) > 1$
$\Rightarrow$ either $UqA$ or $UqB$
$\Rightarrow$ either $x_t \in f\alpha^\theta gcl(A)$ or $x_t \in f\alpha^\theta gcl(B)$
$\Rightarrow x_t \in f\alpha^\theta gcl(A) \vee f\alpha^\theta gcl(B)$.
So $f\alpha^\theta gcl(A \vee B) \leq f\alpha^\theta gcl(A) \vee f\alpha^\theta gcl(B)$. Hence we get

$$f\alpha^\theta gcl(A \vee B) = f\alpha^\theta gcl(A) \vee f\alpha^\theta gcl(B).$$

(5) The proof follows from (3).

(6) As $A \leq f\alpha^\theta gcl(A)$ for any $A \in I^X$, by (3), $f\alpha^\theta gcl(A) \leq f\alpha^\theta gcl(f\alpha^\theta gcl(A))$.

Conversely, let $x_t \in f\alpha^\theta gcl(f\alpha^\theta gcl(A)) = f\alpha^\theta gcl(B)$, where $B = f\alpha^\theta gcl(A)$. Let $U$ be any $f\alpha^\theta g$-open set in $X$ with $x_t qU$. Then $UqB$ implies that there exists $y \in X$ such that $U(y) + B(y) > 1$. Let $B(y) = s$. Then $y_s qU$ and $y_s \in B = f\alpha^\theta gcl(A)$. Thus $UqA$ implies that $x_t \in f\alpha^\theta gcl(A)$. So $f\alpha^\theta gcl(f\alpha^\theta gcl(A)) \leq f\alpha^\theta gcl(A)$. Hence $f\alpha^\theta gcl(f\alpha^\theta gcl(A)) = f\alpha^\theta gcl(A)$. □

**Theorem 4.5.** *Let $(X, \tau)$ be an fts and $A \in I^X$. Then the following statements hold:*

(1) $f\alpha^\theta gcl(1_X \setminus A) = 1_X \setminus f\alpha^\theta gint(A)$,
(2) $f\alpha^\theta gint(1_X \setminus A) = 1_X \setminus f\alpha^\theta gcl(A)$.

*Proof.* (1) Let $x_t \in f\alpha^\theta gcl(1_X \setminus A)$ for a fuzzy set $A$ in an fts $(X, \tau)$. Assume that $x_t \notin 1_X \setminus f\alpha^\theta gint(A)$. Then we have the following implications:
$1 - (f\alpha^\theta gint(A))(x) < t$
$\Rightarrow [f\alpha^\theta gint(A)](x) + t > 1$

$\Rightarrow f\alpha^\theta gint(A)qx_t$

$\Rightarrow$ there exists at least one $f\alpha^\theta g$-open set $F \leq A$ with $x_tqF$.

Thus $x_tqA$. As $x_t \in f\alpha^\theta gcl(1_X \setminus A), Fq(1_X \setminus A)$. So $Aq(1_X \setminus A)$, which is absurd. Hence we get

$$(4.1) \qquad f\alpha^\theta gcl(1_X \setminus A) \leq 1_X \setminus f\alpha^\theta gint(A).$$

Conversely, let $x_t \in 1_X \setminus f\alpha^\theta gint(A)$. Then $1 - [(f\alpha^\theta gint(A)](x) \geq t$. Thus $x_t \not{q}(f\alpha^\theta gint(A))$. So we have

$$(4.2) \qquad x_t \not{q}F \text{ for every } f\alpha^\theta g-\text{open set } F \text{ contained in } A.$$

Let $U$ be any $f\alpha^\theta g$-closed set in $X$ such that $1_X \setminus A \leq U$. Then $1_X \setminus U \leq A$. Now $1_X \setminus U$ is $f\alpha^\theta g$-open set in $X$ contained in $A$. By (4.2), $x_t \not{q}(1_X \setminus U)$ implies that $x_t \in U$. Thus $x_t \in f\alpha^\theta gcl(1_X \setminus A)$. So we have

$$(4.3) \qquad 1_X \setminus f\alpha^\theta gint(A) \leq f\alpha^\theta gcl(1_X \setminus A).$$

Hence by (4.1) and (4.3), (1) holds.

(2) Putting $1_X \setminus A$ for $A$ in (1). Then we get $f\alpha^\theta gcl(A) = 1_X \setminus f\alpha^\theta gint(1_X \setminus A)$. Thus $f\alpha^\theta gint(1_X \setminus A) = 1_X \setminus f\alpha^\theta gcl(A)$. □

Let us now introduce a new type of generalized version of fuzzy open-like function.

**Definition 4.6.** A function $h : X \to Y$ is called an *$f\alpha^\theta g$-open function*, if $h(U)$ is $f\alpha^\theta g$-open set in $Y$ for every fuzzy open set $U$ in $X$.

**Theorem 4.7.** *For a bijective function $h : X \to Y$, the following statements are equivalent:*

(1) *$h$ is $f\alpha^\theta g$-open,*

(2) *$h(intA) \leq f\alpha^\theta gint(h(A))$ for all $A \in I^X$,*

(3) *for each fuzzy point $x_t$ in $X$ and each fuzzy open set $U$ in $X$ containing $x_t$, there exists an $f\alpha^\theta g$-open set $V$ in $Y$ containing $h(x_t)$ such that $V \leq h(U)$.*

*Proof.* (1) $\Rightarrow$ (2) Let $A \in I^X$. Then $intA$ is a fuzzy open set in $X$. By (1), $h(intA)$ is an $f\alpha^\theta g$-open set in $Y$. Since $h(intA) \leq h(A)$ and $f\alpha^\theta gint(h(A))$ is the union of all $f\alpha^\theta g$-open sets contained in $h(A)$, we have $h(intA) \leq f\alpha^\theta gint(h(A))$.

(2) $\Rightarrow$ (1) Let $U$ be any fuzzy open set in $X$. Then by (2), we have

$$h(U) = h(intU) \leq f\alpha^\theta gint(h(U)).$$

Thus $h(U)$ is an $f\alpha^\theta g$-open set in $Y$. So $h$ is an $f\alpha^\theta g$-open function.

(2) $\Rightarrow$ (3) Let $x_t$ be a fuzzy point in $X$ and $U$ a fuzzy open set in $X$ such that $x_t \in U$. Then by (2), $h(x_t) \in h(U) = h(intU) \leq f\alpha^\theta gint(h(U))$. Thus $h(U)$ is an $f\alpha^\theta g$-open set in $Y$. Let $V = h(U)$. Then $h(x_t) \in V$ and $V \leq h(U)$.

(3) $\Rightarrow$ (1) Let $U$ be any fuzzy open set in $X$ and $y_t$ any fuzzy point in $h(U)$, i.e., $y_t \in h(U)$. Since $h$ is bijective, there exists unique $x \in X$ such that $h(x) = y$. Then $[h(U)](y) \geq t \Rightarrow U(h^{-1}(y)) \geq t \Rightarrow U(x) \geq t \Rightarrow x_t \in U$. By (3), there exists an $f\alpha^\theta g$-open set $V$ in $Y$ such that $h(x_t) \in V$ and $V \leq h(U)$. Thus $h(x_t) \in V = f\alpha^\theta gint(V) \leq f\alpha^\theta gint(h(U))$. Since $y_t$ is taken arbitrarily and $h(U)$ is the union of all fuzzy points in $h(U)$, $h(U) \leq f\alpha^\theta gint(h(U))$. So $h(U)$ is an $f\alpha^\theta g$-open set in $Y$. Hence $h$ is an $f\alpha^\theta g$-open function. □

**Theorem 4.8.** *If $h : X \to Y$ is an $f\alpha^\theta g$-open, injective function, then the following statements are true:*

(1) *for each fuzzy point $x_t$ in $X$ and each fuzzy open $q$-nbd $U$ of $x_t$ in $X$, there exists an $f\alpha^\theta g$-open $q$-nbd $V$ of $h(x_t)$ in $Y$ such that $V \leq h(U)$,*

(2) *$h^{-1}(f\alpha^\theta gcl(B)) \leq cl(h^{-1}(B))$ for all $B \in I^Y$.*

*Proof.* (1) Let $x_t$ be a fuzzy point in $X$ and $U$ any fuzzy open $q$-nbd of $x_t$ in $X$. Then $x_t qU = intU$. Thus by Theorem 4.7 (1)⇒(2), $h(x_t)qh(intU) \leq f\alpha^\theta gint(h(U))$. So there exists at least one $f\alpha^\theta g$-open $q$-nbd $V$ of $h(x_t)$ in $Y$ with $V \leq h(U)$.

(2) Let $x_t$ be any fuzzy point in $X$ such that $x_t \notin cl(h^{-1}(B))$ for any $B \in I^Y$. Then there exists a fuzzy open $q$-nbd $U$ of $x_t$ in $X$ such that $U \not q h^{-1}(B)$. Now

$$h(x_t)qh(U), \tag{4.4}$$

where $h(U)$ is an $f\alpha^\theta g$-open set in $Y$. Now $h^{-1}(B) \leq 1_X \setminus U$ which is a fuzzy closed set in $X$. Since $h$ is injective, $B \leq h(1_X \setminus U) \leq 1_Y \setminus h(U)$. Thus $B \not q h(U)$. Let $V = 1_Y \setminus h(U)$. Then $B \leq V$ which is an $f\alpha^\theta g$-closed set in $Y$. We claim that $h(x_t) \notin V$. Assume that $h(x_t) \in V = 1_Y \setminus h(U)$. Then $1 - [h(U)](h(x)) \geq t$. Thus $h(U) \not q h(x_t)$, contradicting (4.4). So $h(x_t) \notin V \Rightarrow h(x_t) \notin f\alpha^\theta gcl(B) \Rightarrow x_t \notin h^{-1}(f\alpha^\theta gcl(B)) \Rightarrow h^{-1}(f\alpha^\theta gcl(B)) \leq cl(h^{-1}(B))$. □

**Theorem 4.9.** *An injective function $h : X \to Y$ is $f\alpha^\theta g$-open if and only if for each $B \in I^Y$ and $F$, a fuzzy closed set in $X$ with $h^{-1}(B) \leq F$, there exists an $f\alpha^\theta g$-closed set $V$ in $Y$ such that $B \leq V$ and $h^{-1}(V) \leq F$.*

*Proof.* Suppose $h : X \to Y$ is $f\alpha^\theta g$-open and injective, and let $B \in I^Y$ and $F$, a fuzzy closed set in $X$ with $h^{-1}(B) \leq F$. Then $1_X \setminus h^{-1}(B) \geq 1_X \setminus F$, where $1_X \setminus F$ is a fuzzy open set in $X$. Since $h$ is injective, $h(1_X \setminus F) \leq h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus B$, where $h(1_X \setminus F)$ is an $f\alpha^\theta g$-open set in $Y$. Let $V = 1_Y \setminus h(1_X \setminus F)$. Then $V$ is an $f\alpha^\theta g$-closed set in $Y$ such that $B \leq V$. Thus $h^{-1}(V) = h^{-1}(1_Y \setminus h(1_X \setminus F)) = 1_X \setminus h^{-1}(h(1_X \setminus F)) \leq F$.

Conversely, suppose the necessary condition holds and let $F$ be a fuzzy open set in $X$. Then $1_X \setminus F$ is a fuzzy closed set in $X$. We have to show that $h(F)$ is an $f\alpha^\theta g$-open set in $Y$. As $h$ is injective, $h^{-1}(1_Y \setminus h(F)) \leq 1_X \setminus F$. Then by the hypothesis, there exists an $f\alpha^\theta g$-closed set $V$ in $Y$ such that

$$1_Y \setminus h(F) \leq V \tag{4.5}$$

and $h^{-1}(V) \leq 1_X \setminus F$. As $h$ is injective, $F \leq 1_X \setminus h^{-1}(V)$ implies that

$$h(F) \leq h(1_X \setminus h^{-1}(V)) \leq 1_Y \setminus V. \tag{4.6}$$

Thus by (4.5) and (4.6), $h(F) = 1_Y \setminus V$ which is an $f\alpha^\theta g$-open set in $Y$. So $h$ is an $f\alpha^\theta g$-open function. □

**Definition 4.10.** A function $h : X \to Y$ is called an *$f\alpha^\theta g$-closed function*, if $h(A)$ is an $f\alpha^\theta g$-closed set in $Y$ for each fuzzy closed set $A$ in $X$.

**Theorem 4.11.** *A function $h : X \to Y$ is an $f\alpha^\theta g$-closed function if and only if $f\alpha^\theta gcl(h(A)) \leq h(clA)$ for all $A \in I^X$.*

*Proof.* Suppose $h$ is an $f\alpha^\theta g$-closed function and let $A \in I^X$. Then $h(cl(A))$ is an $f\alpha^\theta g$-closed set in $Y$. Since $h(A) \leq h(clA)$ and $f\alpha^\theta gcl(h(A))$ is the intersection of all $f\alpha^\theta g$-closed sets in $Y$ containing $h(A)$, $f\alpha^\theta gcl(h(A)) \leq h(clA)$.

Conversely, suppose the necessary condition holds and let for any $A \in I^X$, $f\alpha^\theta gcl(h(A)) \leq h(clA)$. Let $U$ be any fuzzy closed set in $X$. Then $h(U) = h(clU) \geq f\alpha^\theta gcl(h(U))$. Thus $h(U)$ is an $f\alpha^\theta g$-closed set in $Y$. So $h$ is an $f\alpha^\theta g$-closed function. □

**Theorem 4.12.** *If $h : X \to Y$ is an $f\alpha^\theta g$-closed bijective function, then the following statements hold:*

(1) *for each fuzzy point $x_t$ in $X$ and each fuzzy closed set $U$ in $X$ with $x_t \not{q} U$, there exists an $f\alpha^\theta g$-closed set $V$ in $Y$ with $h(x_t) \not{q} V$ such that $V \geq h(U)$,*

(2) *$h^{-1}(f\alpha^\theta gint(B)) \geq int(h^{-1}(B))$, for all $B \in I^Y$.*

*Proof.* (1) Let $x_t$ be a fuzzy point in $X$ and $U$ any fuzzy closed set in $X$ with $x_t \not{q} U = clU$. Then by Theorem 4.11, $h(x_t) \not{q} h(clU) \geq f\alpha^\theta gcl(h(U))$. Thus $h(x_t) \not{q} V$ for some $f\alpha^\theta g$-closed set $V$ in $Y$ with $V \geq h(U)$.

(2) Let $B \in I^Y$ and $x_t$ be any fuzzy point in $X$ such that $x_t \in int(h^{-1}(B))$. Then there exists a fuzzy open set $U$ in $X$ with $U \leq h^{-1}(B)$ such that $x_t \in U$. Thus $1_X \setminus U \geq 1_X \setminus h^{-1}(B)$. So $h(1_X \setminus U) \geq h(1_X \setminus h^{-1}(B))$, where $h(1_X \setminus U)$ is an $f\alpha^\theta g$-closed set in $Y$. Let $V = 1_Y \setminus h(1_X \setminus U)$. Then $V$ is an $f\alpha^\theta g$-open set in $Y$. As $h$ is bijective, $V = 1_Y \setminus h(1_X \setminus U) \leq 1_Y \setminus h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus (1_Y \setminus B) = B$. Now $U(x) \geq t \Rightarrow x_t \not{q}(1_X \setminus U) \Rightarrow h(x_t) \not{q} h(1_X \setminus U) \Rightarrow h(x_t) \leq 1_Y \setminus h(1_X \setminus U) = V \Rightarrow h(x_t) \in V = f\alpha^\theta gint(V) \leq f\alpha^\theta gint(B) \Rightarrow x_t \in h^{-1}(f\alpha^\theta gint(B))$. Since $x_t$ is taken arbitrarily, $int(h^{-1}(B)) \leq h^{-1}(f\alpha^\theta gint(B))$ for all $B \in I^Y$. □

**Note 4.13.** Composition of two $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) functions need not be so, as it seen from the following example.

**Example 4.14.** Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, C\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, A, B\}$ where $A(a) = 0.45, A(b) = 0.55$, $B(a) = 0.4, B(b) = 0.5, C(a) = C(b) = 0.5$. Then $(X, \tau_1)$, $(X, \tau_2)$ and $(X, \tau_3)$ are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly $i_1$ and $i_2$ are $f\alpha^\theta g$-closed functions. Let $i_3 = i_2 \circ i_1 : (X, \tau_1) \to (X, \tau_3)$. We claim that $i_3$ is not an $f\alpha^\theta g$-closed function. Now $C \in \tau_1^c, i_3(C) = C < D \in F\alpha O(X, \tau_3)$, where $D(a) = 0.5, D(b) = 0.55$. But $cl_{\tau_3} int_{\tau_3} C = 1_X \setminus B \not\leq D$. Then $C$ is not an $f\alpha^\theta g$-closed set in $(X, \tau_3)$. Thus $i_3$ is not an $f\alpha^\theta g$-closed function. Similarly, we can show that the composition of two $f\alpha^\theta g$-open functions may not be so.

Now we recall some definitions from [2, 7, 9, 10, 11, 12] for ready references.

**Definition 4.15.** Let $h : (X, \tau_1) \to (Y, \tau_2)$ be a function. Then $h$ is called:

(i) a *fuzzy closed* (resp., *fuzzy open*) *function* [2], if $h(U)$ is a fuzzy closed (resp., fuzzy open) set in $Y$ for every fuzzy closed (resp., fuzzy open) set $U$ in $X$,

(ii) an *$fg$-closed* (resp., *$fg$-open*) *function* [7], if $h(U)$ is an $fg$-closed set in $Y$ for every fuzzy closed (resp., fuzzy open) set $U$ in $X$,

(iii) an *$fwg$-closed* (resp., *$fwg$-open*) *function* [10], if $h(U)$ is an $fwg$-closed (resp., $fwg$-open) set in $Y$ for every fuzzy closed (resp., fuzzy open) set $U$ in $X$,

(iv) an *$fmg$-closed* (resp., *$fmg$-open*) *function* [9], if $h(U)$ is an $fmg$-closed (resp.,$fmg$-open) set in $Y$ for every fuzzy closed (resp., fuzzy open) set $U$ in $X$,

(v) an $fswg$-*closed* (resp., $fswg$-*open*) $function$ [11], if $h(U)$ is an $fswg$-closed (resp., $fswg$-open) set in $Y$ for every fuzzy closed (resp., fuzzy open) set $U$ in $X$,

(vi) an $frwg$-*closed* (resp., $frwg$-*open*) $function$ [12], if $h(U)$ is an $frwg$-closed (resp., $frwg$-open) set in $Y$ for every fuzzy closed (resp., fuzzy open) set $U$ in $X$.

**Remark 4.16.** It is clear from definitions that

(1) fuzzy closed (resp., fuzzy open), $fmg$-closed (resp., $fmg$-open), $fswg$-closed (resp., $fswg$-open) functions are an $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) function, but the converse is not true (See Example 4.17),

(2) an $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) function is $fwg$-closed (resp., $fwg$-open) and $frwg$-closed (resp., $frwg$-open) functions, but the converses are not true (See Example 4.18),

(3) an $fg$-closed (resp., $fg$-open) function and an $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) function are independent concepts (See Examples 4.19, 4.20 and 4.21).

**Example 4.17.** An $f\alpha^\theta g$-closed function does not imply a fuzzy closed function and an $fg$-closed function.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5 = B(b)$. Then $(X, \tau_1)$ and $(X, \tau_2)$ are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly, $B \in \tau_1^c$, $i(B) = B \leq A \in F\alpha O(X, \tau_2)$. Then $cl_{\tau_2} int_{\tau_2} B = 0_X < A$. Thus $B$ is an $f\alpha^\theta g$-closed set in $(X, \tau_2)$. So $i$ is an $f\alpha^\theta g$-closed function. But as $B \notin \tau_2^c$, $i$ is not a fuzzy closed function. On the other hand, $B \leq A \in \tau_2$ but $cl_{\tau_2} B = 1_X \not\leq A$. Then $B$ is not an $fg$-closed set in $(X, \tau_2)$. Thus $i$ is not an $fg$-closed function.

**Example 4.18.** An $fg$-closed function and an $fwg$-closed function do not imply an $f\alpha^\theta g$-closed function.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, C\}$, $\tau_2 = \{0_X, 1_X, A, B\}$, where $A(a) = 0.45, A(b) = 0.55, B(a) = 0.4, B(b) = 0.5, C(a) = C(b) = 0.5$. Then $(X, \tau_1)$ and $(X, \tau_2)$ are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Then clearly, $i$ is an $fg$-closed as well as an $fwg$-closed function. Now $C \in \tau_1^c$, $i(C) = C \leq D \in F\alpha O(X, \tau_2)$, where $D(a) = 0.5, D(b) = 0.55$. But $cl_{\tau_2}(int_{\tau_2} C) = 1_X \setminus B \not\leq D$. Thus $C$ is not an $f\alpha^\theta g$-closed set in $(X, \tau_2)$. So $i$ is not an $f\alpha^\theta g$-closed function.

**Example 4.19.** An $f\alpha^\theta g$-closed function does not imply an $fswg$-closed function.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.5$. Then $(X, \tau_1)$ and $(X, \tau_2)$ are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. It is obvious that $B \in \tau_1^c$, $i(B) = B < 1_X (\in F\alpha O(X, \tau_2))$ only and thus $B$ is an $f\alpha^\theta g$-closed set in $(X, \tau_2)$. So $i$ is an $f\alpha^\theta g$-closed function. On the other hand, $B \leq B \in FSO(X, \tau_2)$. Then $cl_{\tau_2} int_{\tau_2} B = 1_X \setminus A \not\leq B$. Thus $B$ is not an $fswg$-closed set in $(X, \tau_2)$. So $i$ is not an $fswg$-closed function.

**Example 4.20.** An $frwg$-closed function does not imply an $f\alpha^\theta g$-closed function.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = B(b) = 0.4$. Then $(X, \tau_1)$ and $(X, \tau_2)$ are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Then $1_X \setminus B \in \tau_1^c$, $i(1_X \setminus B) = 1_X \setminus B < 1_X (\in FRO(X, \tau_2))$ only and thus $1_X \setminus B$ is an $frwg$-closed set in $(X, \tau_2)$. So $i$ is an $frwg$-closed function. On the other hand, $1_X \setminus B \leq 1_X \setminus B \in F\alpha O(X, \tau_2)$. But

$cl_{\tau_2} int_{\tau_2}(1_X \setminus B) = 1_X \not\leq 1_X \setminus B$. Then $1_X \setminus B$ is not an $f\alpha^\theta g$-closed set in $(X, \tau_2)$. Thus $i$ is not an $f\alpha^\theta g$-closed function.

**Example 4.21.** An $f\alpha^\theta g$-closed function does not imply an $fmg$-closed function.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.5$. Then $(X, \tau_1)$ and $(X, \tau_2)$ are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Then $B \in \tau_1^c, i(B) = B < 1_X (\in F\alpha O(X, \tau_2))$ only and thus $B$ is $f\alpha^\theta g$-closed set in $(X, \tau_2)$. So $i$ is an $f\alpha^\theta g$-closed function. On the other hand, $B \leq B$, where $B$ is an $fg$-open set in $(X, \tau_2)$. But $cl_{\tau_2} int_{\tau_2} B = 1_X \setminus A \not\leq B$. Then $B$ is not an $fmg$-closed set in $(X, \tau_2)$. Thus $i$ is not an $fmg$-closed function.

In a similar manner we can cite counter examples for fuzzy generalized version of open-like functions.

**Theorem 4.22.** *If $h_1 : X \to Y$ is a fuzzy closed (resp., fuzzy open) function and $h_2 : Y \to Z$ is an $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) function, then $h_2 \circ h_1 : X \to Z$ is an $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) function.*

*Proof.* Obvious. □

## 5. Applications of $f\alpha^\theta g$-open function

In this section we first introduce a new type of separation axiom, viz., $f\alpha^\theta g$-$T_2$-space and then establish some applications of $f\alpha^\theta g$-open function.

We first recall the definition and theorem from [13, 20] for ready references.

**Definition 5.1** ([13])**.** An fts $(X, \tau)$ is called a *fuzzy $T_2$-space*, if for any two distinct fuzzy points $x_\alpha$ and $y_\beta$; when $x \neq y$, there exist fuzzy open sets $U_1, U_2, V_1, V_2$ such that $x_\alpha \in U_1, y_\beta q V_1, U_1 \not{q} V_1$ and $x_\alpha q U_2, y_\beta \in V_2, U_2 \not{q} V_2$; when $x = y$ and $\alpha < \beta$ (say), there exist fuzzy open sets $U$ and $V$ in $X$ such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

**Theorem 5.2** ([20])**.** *An fts $(X, \tau)$ is fuzzy $T_2$-space if and only if for any two distinct fuzzy points $x_\alpha$ and $y_\beta$ in $X$; when $x \neq y$, there exist fuzzy open sets $U, V$ in $X$ such that $x_\alpha q U$, $y_\beta q V$ and $U \not{q} V$; when $x = y$ and $\alpha < \beta$ (say), $x_\alpha$ has a fuzzy open nbd $U$ and $y_\beta$ has a fuzzy open $q$-nbd $V$ such that $U \not{q} V$.*

Now we introduce the following concept.

**Definition 5.3.** An fts $(X, \tau)$ is called an *$f\alpha^\theta g$-$T_2$-space*, if for any two distinct fuzzy points $x_t$ and $y_s$ in $X$; when $x \neq y$, there exist $f\alpha^\theta g$-open sets $U, V$ in $X$ such that $x_t q U$, $y_s q V$ and $U \not{q} V$; when $x = y$ and $t < s$ (say), $x_t$ has an $f\alpha^\theta g$-open nbd $U$ and $y_s$ has an $f\alpha^\theta g$-open $q$-nbd $V$ such that $U \not{q} V$.

**Remark 5.4.** Clearly, a fuzzy $T_2$-space is an $f\alpha^\theta g$-$T_2$-space, but the converse is not necessarily true, follows from the following example.

**Example 5.5.** Let $X = \{a, b\}$, $\tau = \{0_X, 1_X\}$. Then $(X, \tau)$ is an fts. Clearly, $(X, \tau)$ is not a fuzzy $T_2$-space. Here every fuzzy set in $(X, \tau)$ is an $f\alpha^\theta g$-open set in $(X, \tau)$. Clearly, it is an $f\alpha^\theta g$-$T_2$-space.

**Theorem 5.6.** *If a bijective function $h : X \to Y$ is an $f\alpha^\theta g$-open function from a fuzzy $T_2$-space $X$ onto an fts $Y$, then $Y$ is an $f\alpha^\theta g$-$T_2$-space.*

*Proof.* Let $z_t$ and $w_s$ be two fuzzy points in $Y$. Since $h$ is bijective, there exist unique $x, y$ in $X$ such that $h(x) = z, h(y) = w$, i.e., $h(x_t) = z_t, h(y_s) = w_s$.

Case I. Suppose $z \neq w$. Then $x \neq y$. Since $X$ is a fuzzy $T_2$-space, there exist fuzzy open sets $U, V$ in $X$ such that $x_t qU, y_s qV$ and $U \not{q} V$. Thus $h(x_t)(= z_t)qh(U), h(y_s)(= w_s)qV$ and $h(U) \not{q} h(V)$, where $h(U)$ and $h(V)$ are $f\alpha^\theta g$-open sets in $Y$ as $h$ is an $f\alpha^\theta g$-open function [Indeed, $h(U)qh(V) \Rightarrow$ there exists $p \in Y$ such that $[h(U)](p) + [h(V)](p) > 1 \Rightarrow U(h^{-1}(p)) + V(h^{-1}(p)) > 1$ where $h^{-1}(p) \in X \Rightarrow UqV$, a contradiction].

Case II. Suppose $z = w$ and $t < s$ (say). Then $x = y$ and $t < s$. Since $X$ is a fuzzy $T_2$-space, there exist a fuzzy open nbd $U$ of $x_t$ and a fuzzy open $q$-nbd $V$ of $y_s$ such that $U \not{q} V$. Thus $h(x_t) \in h(U), h(y_s)qh(V)$ and $h(U) \not{q} h(V)$, where $h(U), h(V)$ are $f\alpha^\theta g$-open sets in $Y$, i.e., $h(U)$ is an $f\alpha^\theta g$-open nbd of $z_t$, $h(V)$ is an $f\alpha^\theta g$-open $q$-nbd of $w_s$ and $h(U) \not{q} h(V)$. Consequently, $Y$ is $f\alpha^\theta g$-$T_2$-space. □

In a similarly manner, we can prove the following theorem easily.

**Theorem 5.7.** *If a bijective function $h : X \to Y$ is an $f\alpha^\theta g$-open function from a fuzzy $T_2$-space $X$ onto an $fT_{\alpha^\theta g}$-space $Y$, then $Y$ is a fuzzy $T_2$-space.*

## 6. Conclusions

Here introducing a new type of generalized version of fuzzy closed set, we introduce and discuss about new types of generalized version of fuzzy open and fuzzy closed functions. Some applications of these functions are established here. Next our aim is to introduce generalized version of fuzzy continuous-like functions using $f\alpha^\theta g$-closed set as a basic tool. Also new types of generalized version of fuzzy separation axioms and compactness are to be introduced and the applications of the generalized version of fuzzy continuous-like functions are to be discussed.

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