Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 29, No. 2, (April 2025) pp. 107–127
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2025.29.2.107

@FMI

Octahedron KU-ideals of KU-algebras

J. I. Baek, Samy M. Mostafa, S. H. Han, M. Cheong, K. Hur

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 29, No. 2, April 2025

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 29, No. 2, (April 2025) pp. 107–127
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2025.29.2.107

@FMI

Octahedron KU-ideals of KU-algebras

J. I. Baek, Samy M. Mostafa, S. H. Han, M. Cheong, K. Hur

Received 26 September 2024; Revised 23 October 2024; Accepted 26 November 2024

Abstract. In this paper, we introduce the notion of octahedron KU-subalgebras and octahedron KU-ideals of a KU-algebra, and study there properties. Also, we define a level set of an octahedron set and deal with relationships between octahedron KU-subalgebras [resp. octahedron KU-ideals] and KU-subalebras [resp. KU-ideals] of a KU-algebra. Next, we discuss some properties of the image [resp. preimage] of octahedron KU-ideals under a KU-homomorphism. Finally, the Cartesian product of two octahedron KU-ideals of the Cartesian product of two KU-algebras is studied..

2020 AMS Classification: 06F35, 03G25, 08A72

Keywords: Octahedron set, KU-algebra, Homomorphism of KU-algebras, Octahedron KU-subalgebra, Octahedron KU-ideal, Lebel set of an octahedron set, The image and preimage of octahedron KU-ideal, The Cartesian product of ctahedron KU-ideals.

Corresponding Author: Samy M. Mostafa (samymostafa@edu.asu.edu.eg)
S. H. Han (shhan235@wku.ac.kr)

1. Introduction

Logical algebra, commonly known as Boolean algebra, is a foundational concept in digital circuit design and computer science. Boolean algebra plays a crucial role in various fields of computer science and electrical engineering. Understanding and applying it is fundamental to modern technological advancements, enabling the design and optimization of complex systems. In 1978, Iséki and Tanaka [1] defined a BCK-algebra as a generalization of Boolean algebras and deal with its various properties. After then, many researchers [2, 3, 4, 5, 6, 7] proposed various types of logical algebra, which are subclasses of Boleanalgebra, and studied their properties. In particular, Prabpayak and Leerawat [8] defined a new algebraic structure which is called KU-*algebra* and investigated some properties of the images and the preimages of a homomorphism of KU-algebras (See [9, 10, 11, 12, 13, 14] for the further research). In 2023, Almuhaimeed [15] defined an interior KU-algebra and studied some of its structures. In 2024, Beak et al. [16] introduce the notion of Γ-KU-algebras as a generalization of KU-algebras and discussed its various structures. Also, Ansari and Koam [17] dealt with some properties of modules over KU-algebras.

In 2020, Lee et al. [18] introduced the notion of octahedron sets combined by fuzzy sets (See [19]), intuitionistic fuzzy sets (See [20]) and interval-valued fuzzy sets (See [21, 22]) as a tool expressing mathematically uncertainty problems. Also, Lee et al. [23] introduced the concepts of octahedron subdroups and octahedron subrings, and dealt with some of their properties. Şenel et al. [24] discussed decision-making problems defining distance and similarity measures for octahedron sets. Lee et al. [25] applied octahedron sets to topological structures. Mostafa et al. [26] introduced the concept of crossing octahedron sets and studied some properties of crossing octahedron Q-ideals of a Q-algebra.

In logical algebra, the concept of an ideal is important because it helps in understanding the structure and properties of logical algebras. It aids in the simplification and optimization of digital circuits, enhances structural analysis, ensures logical consistency, and supports various applications in computer science and cryptography. So, we would like to study the structures of the ideal in KU-algebra based on the octahedron set.

We intend to conduct research as follows. In Section 2, we list some basic concepts needed in the next sections. In Section 3, we define the octahedron KU-ideal of a KU-algebra and discuss its various properties. In particular, by an level set of an octahedron set, we obtain a relationship between octahedron KU-ideals and KU-ideals of a KU-algebra. In Section 4, we deal with some results on the images and preimages of octahedron KU-ideals of a KU-algebra under a homomorphism of KU-algebras. In Section 5, we study the Cartesian product of two octahedron KU-ideals of the Cartesian product of two KU-algebras in the viewpoint of Section 3.

2. Preliminaries

In this section, we recall some notions related to KU-algebras and octahedron sets. Throughout this paper, we denote $I = [0, 1]$.

Definition 2.1 ([8])**.** An algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ is called a *KU-algebra*, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(KU_1) $(x * y) * [(y * z) * (x * z)] = 0$,
(KU_2) $x * 0 = 0$,
(KU_3) $0 * x = x$,
(KU_4) $x * y = 0 = y * x$ implies $x = y$.

From now on, unless otherwise specified, X represents KU-algebra. We define a binary relation $\leq$ on X by: for any $x,\ y \in X$,

$$x \leq y \text{ if and only if } y * x = 0.$$

Result 2.2 (See [27]). *An algebra $(X, *, 0)$ is a KU-algebra if and only if it satisfies the following conditions: $x,\ y,\ z \in X$,*
($KU_{1'}$) $(y * z) * (x * z) \leq x * y$,
($KU_{2'}$) $0 \leq x * 0$,
($KU_{3'}$) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
($KU_{4'}$) $x \leq y$ *if and only if* $y * x = 0$.

Result 2.3 (See [27]). *In X, the followings hold: $x,\ y,\ z \in X$,*
(1) $x * x = 0$,
(2) $z * (x * z) = 0$,
(3) $x \leq y$ *implies* $y * z \leq x * z$,
(4) $z * (y * x) = y * (z * x)$,
(5) $y * [(y * x) * x] = 0$.

Definition 2.4 ([28]). A subset S of X is called a *subalgebra* of X, if $x * y \in S$ for any $x,\ y \in S$.

Definition 2.5 ([28]). Let A be a nonempty subset of X. Then A is called a *KU-ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(KUI_1) $0 \in A$,
(KUI_2) $x * (y * z),\ y \in A$ implies $x * z \in A$.

For a nonempty set X, a mapping $A : X \to I$ is called a *fuzzy set* in X (See [19]). The empty fuzzy set and the whole fuzzy set, denoted by $\mathbf{0}$ and $\mathbf{1}$, are fuzzy sets in X defined by, respectively: for each $x \in X$,

$$\mathbf{0}(x) = 0, \text{ and } \mathbf{1}(x) = 1.$$

We will denote the set of all fuzzy sets in X by I^X.

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$, where $I = [0, 1]$. Then each member $\bar{a}$ of $I \oplus I$ is called an *intuitionistic point* or *intuitionistic number* (See [29]). In particular, we denote $(0, 1)$ and $(1, 0)$ as $\overline{0}$ and $\overline{1}$, respectively. Refer to [29] for the definitions of the order ($\leq$) and the equality ($=$) of two intuitionistic numbers, and the infimum and the supremum of any intuitionistic numbers.

For a nonempty set X, a mapping $\overline{A} = (A^{\in}, A^{\notin}) : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IFS) in X, where $A(x) = (A^{\in}(x), A^{\notin}(x))$ for each $x \in X$ (See [20]). The *empty intuitionistic fuzzy set* and the *whole intuitionistic fuzzy set*, denoted by $\overline{\mathbf{0}}$ and $\overline{\mathbf{1}}$, are defined by, respectively: for each $x \in X$,

$$\overline{\mathbf{0}}(x) = \overline{0} \text{ and } \overline{\mathbf{1}}(x) = \overline{1}.$$

We will denote the set of all IFSs in X by $IFS(X)$.

For the definitions of the inclusion, the equality, the union and the intersection of two IFSs, the complement of an IFS, two operations [ ] and $\diamond$ on $(I \oplus I)^X$, refer to [20].

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called it interval numbers and are denoted by $\widetilde{a},\ \widetilde{b},\ \widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$ (See [30]).

For the definitions of the order and the equality of two interval numbers, and the infimum and the supremum of any interval numbers, refer to [18, 31, 32].

For a nonempty set X, a mapping $\widetilde{A} = [A^-, A^+] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVFS) in X, where for $x \in X$, $\widetilde{A}(x) = [A^-(x), A^+(x)]$ is called the *degree of membership* of an element x to A, and A^- and $A^+ \in I^X$ are called the *lower fuzzy set* and the *upper fuzzy set* in X of $\widetilde{A}(x)$, respectively (See [21, 22]). $\widetilde{0}$ and $\widetilde{1}$ denote the *empty interval-valued fuzzy set* and the *whole interval-valued fuzzy set* in X defined by, respectively: for each $x \in X$,

$$\widetilde{0}(x) = \mathbf{0} \text{ and } \widetilde{1}(x) = \mathbf{1}.$$

We will denote the set of all IVSs in X by $IVFS(X)$.

For the definitions of the inclusion, the equality, the union, the intersection of two IVSs and the complement of an IVS, refer to [21, 22].

For a nonempty set X, a mapping $\mathcal{A} = \left\langle \widetilde{A}, A \right\rangle : X \to [I] \times I$ is called a *cubic set* in X. We will denote the set of all cubic sets in X by $\mathcal{C}(X)$ (See [33]).

Now members of $[I] \times (I \oplus I) \times I$ are written as $\widetilde{\widetilde{a}} = \langle \widetilde{a}, \bar{a}, a \rangle = \langle [a^-, a^-], (a^{\in}, a^{\notin}), a \rangle$, $\widetilde{\widetilde{b}} = \langle \widetilde{b}, \bar{b}, b \rangle = \langle [b^-, b^-], (b^{\in}, b^{\notin}), b \rangle$, etc. and are called *octahedron numbers*. See [18] for the definition of octahedron numbers.

Definition 2.6 ([18])**.** For a nonempty set X, a mapping $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle : X \to [I] \times (I \oplus I) \times I$ is called an *octahedron* set in X.

We can consider following special octahedron sets in X:

$\left\langle \widetilde{0}, \bar{\mathbf{0}}, \mathbf{0} \right\rangle = \ddot{0}$,

$\left\langle \widetilde{0}, \bar{\mathbf{0}}, \mathbf{1} \right\rangle$, $\left\langle \widetilde{0}, \bar{\mathbf{1}}, \mathbf{0} \right\rangle$, $\left\langle \widetilde{1}, \bar{\mathbf{0}}, \mathbf{0} \right\rangle$,

$\left\langle \widetilde{0}, \bar{\mathbf{1}}, \mathbf{1} \right\rangle$, $\left\langle \widetilde{1}, \bar{\mathbf{0}}, \mathbf{1} \right\rangle$, $\left\langle \widetilde{1}, \bar{\mathbf{1}}, \mathbf{0} \right\rangle$,

$\left\langle \widetilde{1}, \bar{\mathbf{1}}, \mathbf{1} \right\rangle = \ddot{1}$.

In this case, $\ddot{0}$ [resp., $\ddot{1}$] is called the *empty* [resp. *whole*] *octahedron set* in X. We denote the set of all octahedron sets as $\mathcal{O}(X)$. See [18] for some operations on $\mathcal{O}(X)$.

We can consider octahedron sets as a generalization of cubic sets proposed by Jun et al. [33]. Moreover, we can see that octahedron sets as a generalization of classical subsets of a set.

3. Octahedron KU-ideals of KU-algebras

In this section, a concept of octahedron KU-ideals is introduced and some of its basic properties are discussed. Unless otherwise specified, X represents KU-algebra.

Definition 3.1. $A \in I^X$ is called a *fuzzy KU-subalgebra of* X, if the following inequality holds:

$$A(x * y) \geq A(x) \wedge A(y) \text{ for any } x,\ y \in X.$$

Definition 3.2 ([34])**.** $\overline{A} \in IFS(X)$ is called an *intuitionistic fuzzy KU-subalgebra* of X, if the following inequality holds:

$$\overline{A}(x * y) \geq \overline{A}(x) \wedge \overline{A}(y) \text{ for any } x,\ y \in X.$$

Definition 3.3 ([35]). $\widetilde{A} \in IVFS(X)$ is called an *interval-valued fuzzy KU-subalgebra of X*, if the following inequality holds:

$$\widetilde{A}(x * y) \geq \widetilde{A}(x) \wedge \widetilde{A}(y) \text{ for any } x,\ y \in X.$$

Definition 3.4. $\mathcal{A} = \left\langle \widetilde{A}, A \right\rangle \in \mathcal{C}(X)$ is called a *cubic KU-subalgebra* of X, if it satisfies the following conditions:

(CKUSA_1) $\widetilde{A}$ is an interval-valued fuzzy KU-subalgebra of X

(CKUSA_2) $A(x * y) \leq A(x) \vee A(y)$ for any $x,\ y \in X$.

Definition 3.5. $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle \in \mathcal{O}(X)$ is called an *octahedron KU-subalgebra* of X, if the following conditions hold:

(OKUSA_1) $\widetilde{A}$ is an interval-valued fuzzy KU-subalgebra of X,

(OKUSA_2) $\overline{A}(x * y) \leq \overline{A}(x) \vee \overline{A}(y)$ for any $x,\ y \in X$,

(CKUSA_2) $A(x * y) \leq A(x) \vee A(y)$ for any $x,\ y \in X$.

Example 3.6. Let $X = \{0, 1, 2, 3, 4\}$ be the KU-algebra in which defined by the following Cayley table:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	1
2	0	3	0	3	4
3	0	1	2	0	1
4	0	0	0	0	0

Table 3.1

Consider the octahedron set $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle$ in X defined as follows: for each $x \in X$,

$$\widetilde{A}(x) = \begin{cases} [0.3, 0.9] & \text{if } x \in \{0, 1, 2\} \\ [0.1, 0.6] & \text{if } x \in \{2, 4\}, \end{cases}$$

$$\overline{A}(x) = \begin{cases} (0.1, 0.3) & \text{if } x \in \{0, 1, 2\} \\ (0.4, 0.1) & \text{if } x \in \{2, 4\}, \end{cases}$$

$$A(x) = \begin{cases} 0.1 & \text{if } x \in \{0, 1, 2\} \\ 0.4 & \text{if } x \in \{2, 4\}. \end{cases}$$

Then by routine calculations, we can see that $\mathcal{A}$ is an octahedron KU-subalgebra of X.

Remark 3.7. (1) If $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle$ is an octahedron KU-subalgebra of X, then A^-, A^+, $A^{\in^c}$, $A^{\notin}$ and A^c are fuzzy subalgebras of X, respectively. Furthermore, $\left\langle \widetilde{A}, A \right\rangle$ and $\langle [A^{\notin}, (A^{\in})^c], A \rangle$ are cubic KU-subalgebras of X, respectively.

(2) If $\mathcal{A} = \left\langle \widetilde{A}, A \right\rangle$ is a cubic KU-subalgebra of X, then

$$\left\langle \widetilde{A}, (A, A^c), A \right\rangle$$

is an octahedron KU-subalgebra of X.

(3) If $\widetilde{A} = [A^-, A^+]$ is an interval-valued KU-subalgebra of X, then

$$\left\langle \widetilde{A}, ({A^+}^c, A^-), {A^+}^c \right\rangle$$

is an octahedron KU-subalgebra of X.

(4) If $\overline{A} = (A^\in, A^\notin)$ is an intuitionistic fuzzy KU-subalgebra of X, then

$$\left\langle [A^\in, {A^\notin}^c], \overline{A}, {A^\in}^c \right\rangle \text{ and } \left\langle [A^\in, {A^\notin}^c], \overline{A}, A^\notin \right\rangle$$

are octahedron KU-subalgebras of X.

Lemma 3.8. *If $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle$ is an octahedron KU-subalgebra of X, then the following inequalities hold: for each $x \in X$,*

$$\widetilde{A}(0) \geq \widetilde{A}(x), \ \overline{A}^c(0) \leq \overline{A}^c(x), \ A^c(0) \leq A^c(x).$$

Proof. Suppose $\mathcal{A}$ is an octahedron KU-subalgebra of X and let $x \in X$. Then we get

$\widetilde{A}(0) = \widetilde{A}(x * x)$ [By Result 2.3 (1)]

$\geq \widetilde{A}(x) \wedge \widetilde{A}(x)$ [By Definition 3.3]

$= \widetilde{A}(x)$.

Similarly, $A(0) \geq A(x)$. Thus $A^c(0) \leq A^c(x)$. On the other hand, by [Lemma 4.4, [34]], $A^\in(0) \geq A^\in(x)$ and $A^\notin(0) \leq A^\notin(x)$. So we have

$$\overline{A}^c(0) = (A^\notin(0), A^\in(0)) \leq (A^\notin(x), A^\in(x)) = \overline{A}^c(x).$$

This completes the proof. □

Definition 3.9. $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle \in \mathcal{O}(X)$ is called an *octahedron KU-ideal* of X, if it satisfies the following conditions: for any $x, \ y \in X$,

(OKUI_1) $\widetilde{A}(0) \geq \widetilde{A}(x)$, $\overline{A}(0) \leq \overline{A}(x)$, $A(0) \leq A(x)$,

(OKUI_2) $\widetilde{A}(x * z) \geq \widetilde{A}(x * (y * z)) \wedge \widetilde{A}(y)$, $\overline{A}(x * z) \leq \overline{A}(x * (y * z)) \vee \overline{A}(y)$, $A(x * z) \leq A(x * (y * z)) \vee A(y)$.

Example 3.10. Let $X = \{0, 1, 2, 3, 4\}$ be the KU-algebra given in Example 3.6. Consider the octahedron set $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle$ in X defined as follows: for each $x \in X$,

$$\widetilde{A}(x) = \begin{cases} [0.6, 0.7] & \text{if } x = 0 \\ [0.4, 0.5] & \text{if } x \in \{1, 3\} \\ [0.1, 0.3] & \text{if } x \in \{2, 4\}, \end{cases}$$

$$\overline{A}(x) = \begin{cases} (0.1, 0.6) & \text{if } x = 0 \\ (0.3, 0.4) & \text{if } x \in \{1, 3\} \\ (0.6, 0.3) & \text{if } x \in \{2, 4\}, \end{cases}$$

$$A(x) = \begin{cases} 0.1 & \text{if } x = 0 \\ 0.3 & \text{if } x \in \{1, 3\} \\ 0.6 & \text{if } x \in \{2, 4\}. \end{cases}$$

Then by routine calculations, we can check that $\mathcal{A}$ is an octahedron KU-ideal of X.

Remark 3.11. (1) If $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle$ is an octahedron KU-ideal of X, then A^-, A^+, $A^{\in c}$, $A^{\notin}$ and A^c are fuzzy KU-ideals of X, respectively in the sense of Mostafa et al. [27]. Furthermore, $\left\langle \widetilde{A}, A \right\rangle$ and $\langle [A^{\notin}, (A^{\in})^c], A \rangle$ are cubic KU-deals of X, respectively in the viewpoint of Yaqoob et al. [36].

(2) If $\mathcal{A} = \left\langle \widetilde{A}, A \right\rangle$ is a cubic KU-ideal of X, then $\left\langle \widetilde{A}, (A, A^c), A \right\rangle$ is an octahedron KU-ideal of X.

(3) If $\widetilde{A} = [A^-, A^+]$ is an interval-valued KU-ideal of X in the sense of Mostafa et al. [35], then $\left\langle \widetilde{A}, ({A^+}^c, A^-), {A^+}^c \right\rangle$ is an octahedron KU-ideal of X.

(4) If $\overline{A} = (A^{\in}, A^{\notin})$ is an intuitionistic fuzzy KU-ideal of X in the viewpoint of Mostafa et al. [34], then $\left\langle [A^{\in}, {A^{\notin}}^c], \overline{A}, A^{\in c} \right\rangle$ and $\left\langle [A^{\in}, {A^{\notin}}^c], \overline{A}, A^{\notin} \right\rangle$ are octahedron KU-ideals of X.

Lemma 3.12. *Let $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle$ be an octahedron KU-ideal of X. If $x \leq y$ in X, then the following inequalities hold:*

$$\widetilde{A}(x) \geq \widetilde{A}(y),\ \overline{A}(x) \leq \overline{A}(y),\ A(x) \leq A(y).$$

Proof. Suppose $x \leq y$ in X. Then clearly, $y * x = 0$. Thus we have
$\widetilde{A}(x) = \widetilde{A}(0 * x)$ [By (KU_3)]
$\geq \widetilde{A}(0 * (y * x)) \wedge \widetilde{A}(y)$ [By ($OKUI_2$)]
$= \widetilde{A}(0 * 0) \wedge \widetilde{A}(y)$ [Since $y * x = 0$]
$= \widetilde{A}(0) \wedge \widetilde{A}(y)$ [By Result 2.3 (1)]
$= \widetilde{A}(y)$ [By ($OKUI_1$)],
$\overline{A}(x) = \overline{A}(0 * x)$
$\leq \overline{A}(0 * (y * x)) \vee \overline{A}(y)$
$= \overline{A}(0 * 0) \vee \overline{A}(y)$
$= \overline{A}(0) \vee \overline{A}(y)$
$= \overline{A}(y)$.
Similarly, we get $A(x) \leq A(y)$. The proof is complete. $\square$

Lemma 3.13. *Let $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle$ be an octahedron KU-ideal of X. If $x * y \leq z$ in X, then the following inequalities hold:*

(1) $\widetilde{A}(y) \geq \widetilde{A}(x) \wedge \widetilde{A}(z)$,
(2) $\overline{A}(y) \leq \overline{A}(x) \vee \overline{A}(z)$,
(3) $A(y) \leq A(x) \vee A(z)$.

Proof. Suppose $x * y \leq z$ in X. It is clear that $z * (x * y) = 0$. Then we get
$\widetilde{A}(y) = \widetilde{A}(0 * y)$ [By (KU$_3$)]
$\geq \widetilde{A}(0 * (x * y)) \wedge \widetilde{A}(x)$ [By (OKUI$_2$)]
$= \widetilde{A}(x * y) \wedge \widetilde{A}(x)$ [By (KU$_3$)]
$\geq (\widetilde{A}(x * (z * y)) \wedge \widetilde{A}(z)) \wedge \widetilde{A}(x)$ [By (OKUI$_2$)]
$= (\widetilde{A}(z * (x * y)) \wedge \widetilde{A}(z)) \wedge \widetilde{A}(x)$ [By Result 2.4 (4)]
$= (\widetilde{A}(0) \wedge \widetilde{A}(z)) \wedge \widetilde{A}(x)$ [Since $z * (x * y) = 0$]
$= \widetilde{A}(z) \wedge \widetilde{A}(x)$ [By (OKUI$_1$)],
$\overline{A}(y) = \overline{A}(0 * y)$
$\leq \overline{A}(0 * (x * y)) \vee \overline{A}(x)$
$= \overline{A}(x * y) \vee \overline{A}(x)$
$\leq (\overline{A}(x * (z * y)) \vee \overline{A}(z)) \vee \overline{A}(x)$
$= (\overline{A}(z * (x * y)) \vee \overline{A}(z)) \vee \overline{A}(x)$
$= (\overline{A}(0) \vee \overline{A}(z)) \vee \overline{A}(x)$
$= \overline{A}(z) \vee \overline{A}(x)$.
Similarly, we have $A(y) \leq A(z) \vee A(x)$. Thus (1), (2) and (3) hold. □

Definition 3.14 ([18]). Let X be a nonempty set and let $(\mathcal{A}_j)_{j \in J} = \left(\left\langle \widetilde{A}_j, \overline{A}_j, A_j \right\rangle\right)_{j \in J}$ be a family of octahedron sets in X, where J denotes an index set. Then

(i) the *union* $\bigcup_{j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \widetilde{A}_j, \bigcap_{j \in J} \overline{A}_j, \bigcap_{j \in J} A_j \right\rangle$ of $(\mathcal{A}_j)_{j \in J}$ is an octahedron set in X defined as follows: for each $x \in X$,

$$\left(\bigcup_{j \in J} \mathcal{A}_j\right)(x) = \left\langle \bigvee_{j \in J} \widetilde{A}_j(x), \bigwedge_{j \in J} \overline{A}_j(x), \bigwedge_{j \in J} A_j(x) \right\rangle,$$

(ii) the *intersection* $\bigcap_{j \in J} \mathcal{A}_j = \left\langle \bigcap_{j \in J} \widetilde{A}_j, \bigcup_{j \in J} \overline{A}_j, \bigcup_{j \in J} A_j \right\rangle$ of $(\mathcal{A}_j)_{j \in J}$ is an octahedron set in X defined as follows: for each $x \in X$,

$$\left(\bigcap_{j \in J} \mathcal{A}_j\right)(x) = \left\langle \bigwedge_{j \in J} \widetilde{A}_j(x), \bigvee_{j \in J} \overline{A}_j(x), \bigvee_{j \in J} A_j(x) \right\rangle.$$

Proposition 3.15. *Let $(\mathcal{A}_j)_{j \in J} = \left(\left\langle \widetilde{A}_j, \overline{A}_j, A_j \right\rangle\right)_{j \in J}$ be a family of octahedron KU-ideals of X. Then $\bigcap_{j \in J} \mathcal{A}_j$ is an octahedron KU-ideal of X.*

Proof. Let $\mathcal{A} = \bigcap_{j \in J} \mathcal{A}_j$ and $x \in X$. Then by (OKUI$_1$), we have

$$\left(\bigcap_{j \in J} \widetilde{A}_j\right)(x) = \bigwedge_{j \in J} \widetilde{A}_j(x) \leq \bigwedge_{j \in J} \widetilde{A}_j(0) = (\bigcap_{j \in J} \widetilde{A}_j)(0),$$

$$\left(\bigcup_{j\in J}\overline{A}_j\right)(x)=\bigvee_{j\in J}\overline{A}_j(x)\geq\bigvee_{j\in J}\overline{A}_j(0)=(\bigcup_{j\in J}\overline{A}_j)(0),$$
$$\left(\bigcup_{j\in J}A_j\right)(x)=\bigvee_{j\in J}A_j(x)\geq\bigvee_{j\in J}A_j(0)=(\bigcup_{j\in J}A_j)(0).$$

Thus $\mathcal{A}$ satisfies the condition (OKUI_1).

To prove that $\mathcal{A}$ satisfies the condition (OKUI_2), let us take $x,\ y,\ z\in X$. Then by Definition 3.14 and the hypothesis, we get

$\left(\bigcap_{j\in J}\widetilde{A}_j\right)(x*z)=\bigwedge_{j\in J}\widetilde{A}_j(x*z)$
$\geq\bigwedge_{j\in J}[\widetilde{A}_j(x*(y*z))\wedge\widetilde{A}_j(y)]$
$=\left(\bigwedge_{j\in J}\widetilde{A}_j(x*(y*z))\right)\wedge\left(\bigwedge_{j\in J}\widetilde{A}_j(y)\right)$
$=\left(\bigcap_{j\in J}\widetilde{A}_j\right)(x*(y*z))\wedge\left(\bigcap_{j\in J}\widetilde{A}_j\right)(y),$
$\left(\bigcup_{j\in J}\overline{A}_j\right)(x*z)=\bigvee_{j\in J}\overline{A}_j(x*z)$
$\leq\bigvee_{j\in J}[\overline{A}_j(x*(y*z))\vee\overline{A}_j(y)]$
$=\left(\bigvee_{j\in J}\overline{A}_j(x*(y*z))\right)\vee\left(\bigvee_{j\in J}\overline{A}_j(y)\right)$
$=\left(\bigcup_{j\in J}\overline{A}_j\right)(x*(y*z))\vee\left(\bigcup_{j\in J}\overline{A}_j\right)(y).$

Similarly, we have $\left(\bigcup_{j\in J}A_j\right)(x*z)\leq\left(\bigcup_{j\in J}A_j\right)(x*(y*z))\vee\left(\bigcup_{j\in J}A_j\right)(y)$. Thus $\mathcal{A}$ satisfies the condition (OKUI_2). So $\mathcal{A}$ is an octahedron KU-ideal of X. □

Proposition 3.16. *If $\mathcal{A}=\left\langle\widetilde{A},\overline{A},A\right\rangle$ is an octahedron KU-ideal of X, then the following inequalities hold: for any $x,\ y\in X$,*

$$\widetilde{A}(x*(x*y))\geq\widetilde{A}(y),\ \overline{A}(x*(x*y))\leq\overline{A}(y),\ A(x*(x*y))\leq A(y).$$

Proof. Suppose $\mathcal{A}$ is an octahedron KU-ideal of X and let $x,\ y\in X$. Then we have

$\widetilde{A}(x*(x*y))\geq\widetilde{A}(x*(y*(x*y)))\wedge\widetilde{A}(y)$ [By (OKUI_2)]
$=\widetilde{A}(x*(x*(y*y)))\wedge\widetilde{A}(y)$ [By Result 2.3 (4)]
$=\widetilde{A}(x*(x*0))\wedge\widetilde{A}(y)$ [By Result 2.3 (1)]
$=\widetilde{A}(0)\wedge\widetilde{A}(y)$ [By (KU_1)]
$=\widetilde{A}(y)$ [By (OKUI_1)],
$\overline{A}(x*(x*y))\leq\overline{A}(x*(y*(x*y)))\vee\overline{A}(y)$
$=\overline{A}(x*(x*(y*y)))\vee\overline{A}(y)$
$=\overline{A}(x*(x*0))\vee\overline{A}(y)$
$=\overline{A}(0)\vee\overline{A}(y)$
$=\overline{A}(y).$

Similarly, $A(x*(x*y))\leq A(y)$. Thus the inequalities hold. □

Definition 3.17. Let X be a nonempty set, let $\widetilde{\widetilde{a}}=\langle\widetilde{a},\bar{a},a\rangle$ be an octahedron number and let $\mathcal{A}=\left\langle\widetilde{A},\overline{A},A\right\rangle\in\mathcal{O}(X)$. Then two subsets $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ and $[\mathcal{A}]^*_{\widetilde{\widetilde{a}}}$ of X are defined as follows:

$$[\mathcal{A}]_{\widetilde{\widetilde{a}}}=\{x\in X:\widetilde{A}(x)\geq\widetilde{a},\ \overline{A}(x)\leq\bar{a},\ A(x)\leq a\},$$

$$[\mathcal{A}]^*_{\widetilde{\bar{a}}} = \{x \in X : \widetilde{A}(x) > \widetilde{a},\ \overline{A}(x) > \bar{a},\ A(x) > a\}.$$

In this case, $[\mathcal{A}]_{\widetilde{\bar{a}}}$ is called an $\widetilde{\bar{a}}$*-level set* of $\mathcal{A}$ and $[\mathcal{A}]^*_{\widetilde{\bar{a}}}$ is called a *strong* $\widetilde{\bar{a}}$*-level set* of $\mathcal{A}$.

We have a relationship between octahedron KU-subalebras and KU-subalgebras of a KU-algebra X.

Theorem 3.18. *$\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle \in \mathcal{O}(X)$ is an octahedron KU-sualgebra of X if and only if either $[\mathcal{A}]_{\widetilde{\bar{a}}} = \varnothing$ or $[\mathcal{A}]_{\widetilde{\bar{a}}}$ is a KU-subalgebra of X for each octahedron number $\widetilde{\bar{a}} = \langle \widetilde{a}, \bar{a}, a\rangle$.*

In this case, $[\mathcal{A}]_{\widetilde{\bar{a}}}$ is called an $\widetilde{\bar{a}}$*-level KU-subalgebra* of $\mathcal{A}$.

Proof. The proof follows from Definition 3.4 and 3.17. □

The following give a relationship between octahedron KU-ideal and KU-ideal of a KU-algebra X.

Theorem 3.19. *$\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle \in \mathcal{O}(X)$ is an octahedron KU-ideal of X if and only if either $[\mathcal{A}]_{\widetilde{\bar{a}}} = \varnothing$ or $[\mathcal{A}]_{\widetilde{\bar{a}}}$ is a KU-ideal of X for each octahedron number $\widetilde{\bar{a}} = \langle \widetilde{a}, \bar{a}, a\rangle$.*

In this case, $[\mathcal{A}]_{\widetilde{\bar{a}}}$ is called an $\widetilde{\bar{a}}$*-level KU-ideal* of $\mathcal{A}$.

Proof. Suppose $\mathcal{A}$ is an octahedron KU-ideal of X, let $\widetilde{\bar{a}}$ be any octahedron number such that $[\mathcal{A}]_{\widetilde{\bar{a}}} \neq \varnothing$ and $x \in [\mathcal{A}]_{\widetilde{\bar{a}}}$. Then by ($\text{OKUI}_1$), we have

$$\widetilde{A}(0) \geq \widetilde{A}(x) \geq \widetilde{a},\ \overline{A}(0) \leq \overline{A}(x) \leq \bar{a},\ A(0) \leq A(x) \leq a.$$

Thus $0 \in [\mathcal{A}]_{\widetilde{\bar{a}}}$. So the condition ($\text{KUI}_1$) holds.

Suppose $x * (y * z),\ y \in [\mathcal{A}]_{\widetilde{\bar{a}}}$ for any $x,\ y,\ z \in X$. Then clearly,

$$\widetilde{A}(x * (y * z)) \geq \widetilde{a},\ \overline{A}(x * (y * z)) \leq \bar{a},\ A(x * (y * z)) \leq a,$$

$$\widetilde{A}(y) \geq \widetilde{a},\ \overline{A}(y) \leq \bar{a},\ A(y)) \leq a.$$

Thus by (OKUI_2), we get

$$\widetilde{A}(x * z) \geq \widetilde{A}(x * (y * z) \wedge \widetilde{A}(y) \geq \widetilde{a},$$

$$\overline{A}(x * z) \leq \overline{A}(x * (y * z) \vee \overline{A}(y) \leq \bar{a},$$

$$A(x * z) \leq A(x * (y * z) \vee A(y) \leq a.$$

So $x * z \in [\mathcal{A}]_{\widetilde{\bar{a}}}$, i.e., the condition ($\text{KUI}_2$) holds. Hence $[\mathcal{A}]_{\widetilde{\bar{a}}}$ is a KU-ideal of X.

Conversely, suppose $[\mathcal{A}]_{\widetilde{\bar{a}}}$ is a KU-ideal of X. Assume that the condition (OKUI_1) does not hold, i.e., there is $x \in X$ such that

$$\widetilde{A}(0) < \widetilde{A}(x),\ \overline{A}(0) > \overline{A}(x),\ A(0) > A(x). \tag{3.1}$$

Let us take an octahedron number $\widetilde{\bar{a}} = \langle \widetilde{a}, \bar{a}, a\rangle$ as follows:

$$\widetilde{a} = \frac{1}{2}(\widetilde{A}(0) + \widetilde{A}(x)),\ \bar{a} = \frac{1}{2}(\overline{A}(0) + \overline{A}(x)),\ a = \frac{1}{2}(A(0) + A(x)). \tag{3.2}$$

Then by (3.1) and (3.2), we have the following inequalities:

$$\widetilde{A}(0) < \widetilde{a} < \widetilde{A}(x),\ \overline{A}(0) > \bar{a} > \overline{A}(x),\ A(0) > a > A(x). \tag{3.3}$$

Thus by (3.3), $x \in [\mathcal{A}]_{\widetilde{\overline{a}}}$ but $0 \notin [\mathcal{A}]_{\widetilde{\overline{a}}}$. This is a contradiction from the fact that $0 \in [\mathcal{A}]_{\widetilde{\overline{a}}}$. So the condition ($OKUI_1$) holds.

Now assume that the condition ($OKUI_2$) does not hold, i.e., there are $x,\ y,\ z \in X$ such that

(3.4)
$$\widetilde{A}(x*z) < \widetilde{A}(x*(y*z)) \wedge \widetilde{A}(y),\ \overline{A}(x*z) > \overline{A}(x*(y*z)) \vee \overline{A}(y),\ A(x*z) > A(x*(y*z)) \vee A(y).$$

Let us take an octahedron number $\widetilde{\overline{a}} = \langle \widetilde{a}, \bar{a}, a \rangle$ as follows:

$$\widetilde{a} = \frac{1}{2}[\widetilde{A}(x*z) + (\widetilde{A}(x*(y*z)) \wedge \widetilde{A}(y))], \tag{3.5}$$

$$\bar{a} = \frac{1}{2}[\overline{A}(x*z) + (\bar{(}x*(y*z)) \vee \overline{A}(y))], \tag{3.6}$$

$$a = \frac{1}{2}[A(x*z) + (A(x*(y*z)) \vee A(y))]. \tag{3.7}$$

Then from (3.4), (3.5), (3.6) and (3.7), we get the following inequalities:

$$\widetilde{A}(x*z) < \widetilde{a} < \widetilde{A}(x*(y*z)) \wedge \widetilde{A}(y), \tag{3.8}$$

$$\overline{A}(x*z) > \bar{a} > \overline{A}(x*(y*z)) \vee \overline{A}(y), \tag{3.9}$$

$$A(x*z) > a > A(x*(y*z)) \vee A(y). \tag{3.10}$$

Thus by (3.8), (3.9) and (3.10), $x*(y*z) \in [\mathcal{A}]_{\widetilde{\overline{a}}}$ but $x*y \notin [\mathcal{A}]_{\widetilde{\overline{a}}}$. This is a contradiction. So the condition ($OKUI_2$) holds. Hence $\mathcal{A}$ is an octahedron KU-ideal of X. □

4. The images and the preimages of octahedron KU-ideals

In this section we obtain some results on the images and the preimages of octahedron KU-ideals of a KU-algebra under a KU-homomorphism.

Definition 4.1 ([18])**.** Let $X,\ Y$ be two sets, $f : X \to Y$ be a mapping and $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle \in \mathcal{O}(X)$, $\mathcal{B} = \left\langle \widetilde{B}, \overline{B}, B \right\rangle \in \mathcal{O}(Y)$.

(i) The *preimage of* $\mathcal{B}$ *under* f, denoted by $f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\widetilde{B}), f^{-1}(\overline{B}), f^{-1}(B) \right\rangle$, is the octahedron set in X defined as follows: for each $x \in X$,

$$f^{-1}(\mathcal{B})(x) = \left\langle [B^-(f(x)), B^+(f(x))], (B^{\in}(f(x)), B^{\notin}(f(x))), (B(f(x)) \right\rangle.$$

We will denote $[B^-(f(x)), B^+(f(x))]$ and $(B^{\in}(f(x)), B^{\notin}(f(x)))$ by $\widetilde{B}(f(x))$ and $\overline{B}(f(x))$, respectively.

(ii) The *image of* $\mathcal{A}$ *under* f, denoted by $f(\mathcal{A}) = \left\langle f(\widetilde{A}), f(\overline{A}), f(A) \right\rangle$, is the octahedron set in Y defined as follows: for each $y \in Y$,

$$f(\widetilde{A})(y) = \begin{cases} [\bigvee_{x \in f^{-1}(y)} A^-(x), \bigvee_{x \in f^{-1}(y)} A^+(x)] = \bigvee_{x \in f^{-1}(y)} \widetilde{A}(x) & \text{if } f^{-1}(y) \neq \varnothing \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$f(\overline{A})(y) = \begin{cases} (\bigvee_{x \in f^{-1}(y)} A^{\in}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin}(x)) = \bigvee_{x \in f^{-1}(y)} \overline{A}(x) & \text{if } f^{-1}(y) \neq \phi \\ \bar{1} & \text{otherwise,} \end{cases}$$

$$f(A)(y)=\begin{cases}\bigvee_{x\in f^{-1}(y)}A(x) & \text{if } f^{-1}(y)\neq\phi\\ 1 & \text{otherwise.}\end{cases}$$

It is obvious that $f(x_{\widetilde{a}})=[f(x)]_{\widetilde{a}}$ for each $x_{\widetilde{a}}\in\mathcal{O}_P(X)$.

Proposition 4.2. *Let $(X,*,0)$ and $(Y,*^{'},0^{'})$ be two KU-algebras, let $f:X\to Y$ be a homomorphism and let $\mathcal{B}=\left\langle\widetilde{B},\overline{B},B\right\rangle\in\mathcal{O}(Y)$. If $\mathcal{B}$ is an octahedron KU-ideal of Y, then $f^{-1}(\mathcal{B})$ is an octahedron KU-ideal of X.*

Proof. Suppose $\mathcal{B}=\left\langle\widetilde{B},\overline{B},B\right\rangle$ is an octahedron KU-ideal of Y, let $\mathcal{A}=f^{-1}(\mathcal{B})$ and let $x\in X$. Then we get
$\widetilde{A}(x)=\widetilde{B}(f(x))$ [By Definition 4.1 (i)]
$\leq\widetilde{B}(0^{'})$ [By (OKUI$_1$)]
$=\widetilde{B}(f(0))$ [Since f is a homomorphism]
$=\widetilde{A}(0)$,
$\overline{A}(x)=\overline{B}(f(x))\geq\overline{B}(0^{'})=\overline{B}(f(0))=\overline{A}(0)$.
Similarly, $A(x)\geq A(0)$. Thus $\mathcal{A}$ satisfies the condition (OKUI$_1$).

To show that $\mathcal{A}$ satisfies the condition (OKUI$_2$), pick $x,\ y,\ z\in X$. Then we have
$\widetilde{A}(x*y)=\widetilde{B}(f(x*y))$ [By Definition 4.1 (i)]
$=\widetilde{B}(f(x)*^{'}f(y))$ [Since f is a homomorphism]
$\geq\widetilde{B}(f(x)*^{(}f(y)*^{'}f(z)))\wedge\widetilde{B}(f(y))$ [By (OKUI$_2$)]
$=\widetilde{B}(f(x*(y*z)))\wedge\widetilde{B}(f(y))$ [Since f is a homomorphism]
$=\widetilde{A}(x*(y*z))\wedge\widetilde{A}(y)$ [By Definition 4.1 (i)],
$\overline{A}(x*y)=\overline{B}(f(x*y))$
$=\overline{B}(f(x)*^{'}f(y))$
$\leq B(f(x)*^{(}f(y)*^{'}f(z)))\vee\overline{B}(f(y))$
$=\overline{B}(f(x*(y*z)))\vee\overline{B}(f(y))$
$=\overline{A}(x*(y*z))\vee\overline{A}(y)$.
Similarly, $A(x*y)\leq A(x*(y*z))\vee A(y)$. Thus $\mathcal{A}$ satisfies the condition (OKUI$_2$). So $\mathcal{A}=f^{-1}(\mathcal{A})$ is an octahedron KU-ideal of Y. □

Proposition 4.3. *Let $(X,*,0)$ and $(Y,*^{'},0^{'})$ be two KU-algebras, let $\mathcal{B}=\left\langle\widetilde{B},\overline{B},B\right\rangle\in\mathcal{O}(X)$ and let $f:X\to Y$ be an epimorphism. If $f^{-1}(\mathcal{B})$ is an octahedron KU-ideal of X, then $\mathcal{B}$ is an octahedron KU-ideal of Y.*

Proof. Suppose $f^{-1}(\mathcal{B})$ is an octahedron KU-ideal of X and let $y\in Y$. Since f is surjective, there is $a\in X$ such that $y=f(a)$. Then we obtain
$\widetilde{B}(y)=\widetilde{B}(f(a))$ [Since $y=f(a)$]
$=f^{-1}(\widetilde{B})(a)$ [By Definition 4.1 (i)]
$\leq f^{-1}(\widetilde{B})(0)$ [By (OKUI$_1$)]
$=\widetilde{B}(f(0))$ [By Definition 4.1 (i)]
$=\widetilde{B}(0^{'})$ [Since f is a homomorphism],
$\overline{B}(y)=\overline{B}(f(x))=f^{-1}(\widetilde{B})(a)\geq f^{-1}(\overline{B})(0)=\overline{B}(f(0))=\overline{B}(0^{'})$.
Similarly, $B(y)\geq B(0^{'})$. Thus $\mathcal{B}$ satisfies the condition (OKUI$_1$).

To prove that the condition (OKUI$_1$) holds, let $x,\ y,\ z \in X$. Since f is sujective, there are $a,\ b,\ c \in X$ such that $x = f(a)$, $y = f(b)$, $z = f(c)$. Since f is a homomorphism, $x * y = f(a * b)$ and $x * (y * z) = f(a * (b * c))$. Then we have
$\widetilde{B}(x * y) = \widetilde{B}(f(a * b))$ [By Definition 4.1 (i)]
$= f^{-1}(\widetilde{B})(a * b)$ [By Definition 4.1 (i)]
$\geq f^{-1}(\widetilde{B})(a * (b * c)))) \wedge f^{-1}(\widetilde{B})(c)$ [By (OKUI$_2$)]
$= \widetilde{B}(f(a * (b * c))) \wedge \widetilde{B}(f(c))$ [By Definition 4.1 (i)]
$= \widetilde{B}(f(a) *^{'} (f(b) *^{'} f(c))) \wedge \widetilde{B}(f(c))$ [Since f is a homomorphism]
$= \widetilde{B}(x *^{'} (y *^{'} z)) \wedge \widetilde{B}(y)$,
$\overline{B}(x * y) = \overline{B}(f(a * b))$
$= f^{-1}(\overline{B})(a * b)$
$\leq f^{-1}(\overline{B})(a * (b * c)))) \vee f^{-1}(\overline{B})(c)$
$= \overline{B}(f(a * (b * c))) \vee \overline{B}(f(c))$
$= \overline{(}f(a) *^{'} (f(b) *^{'} f(c))) \vee \overline{B}(f(c))$
$= \overline{B}(x *^{'} (y *^{'} z)) \vee \overline{B}(y)$.
Similarly, $B(x *^{'} y) \leq B(x *^{'} (y *^{'} z)) \vee B(y)$. Thus $\mathcal{B}$ satisfies the condition (OKUI$_2$). So $\mathcal{B}$ is an octahedron KU-ideal of X. □

Definition 4.4 (See [23])**.** Let X be a nonempty set and $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle \in \mathcal{O}(X)$. Then we say that $\mathcal{A}$ *has the sup-property*, if for each $T \in 2^X$, there is $t_0 \in T$ such that

$$\mathcal{A}(t_0) = \left\langle \bigvee_{t \in T} \widetilde{A}(t), \bigwedge_{t \in T} \overline{A}(t), \bigwedge_{t \in T} A(t) \right\rangle.$$

It is obvious that if $\mathcal{A}$ takes on only finitely many values, then it has the sup-property. In fact, $\mathcal{A} \in \mathcal{O}(X)$ has the sup-property if and only if $\widetilde{A}$ has the sup-property, $\overline{A}$ and A have the inf-property.

Proposition 4.5. *Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be two KU-algebras, let $f : X \to Y$ be a homomorphism and let $\mathcal{A} = \left\langle \widetilde{A}, \overline{A}, A \right\rangle \in \mathcal{O}(X)$ have the sup-property. If $\mathcal{A}$ an octahedron KU-ideal of X, then $f(\mathcal{A})$ is an octahedron KU-ideal of Y.*

Proof. Suppose $\mathcal{A}$ an octahedron KU-ideal of X and let $\mathcal{B} = f(\mathcal{A})$, $y \in Y$. If $f^{-1}(y) = \varnothing$, then by Definition 4.1 (i), $\widetilde{B}(y) = \mathbf{0}$, $\overline{B}(y) = \overline{1}$ and $B(y) = 1$. It is clear that $\mathcal{B} = f(\mathcal{A})$ satisfies the condition (OKUI$_1$). Without loss of generality, we can suppose $f^{-1}(y) \neq \varnothing$ for each $y \in Y$. Then we get
$\widetilde{B}(0^{'}) = \bigvee_{t \in f^{-1}(0^{'})} \widetilde{A}(t)$ [By Definition 4.1 (ii)]
$= \widetilde{A}(0)$ [Since f is a homomorphism, $0^{'} = f(0)$.]
$\geq \widetilde{A}(a)$ for each $a \in X$ [By (OKUI$_1$)],
$\overline{B}(0^{'}) = \bigwedge_{t \in f^{-1}(0^{'})} \overline{A}(t) = \overline{A}(0) \leq \overline{A}(a)$ for each $a \in X$
Similarly, $B(0^{'}) \leq A(a)$ for each $a \in X$. Thus

$$\widetilde{B}(0^{'}) \geq \widetilde{A}(a),\ \overline{B}(0^{'}) \leq \overline{A}(a),\ B(0^{'}) \leq A(a) \text{ for each } a \in X. \tag{4.1}$$

By (4.1) and Definition 4.1 (ii), we have

$$\widetilde{B}(0^{'}) \geq \widetilde{B}(y),\ \overline{B}(0^{'}) \leq \overline{B}(y),\ B(0^{'}) \leq B(y). \tag{4.2}$$

So $\mathcal{B}$ satisfies the condition (OKUI$_1$).

Take $x,\ y,\ z \in Y$ such that $f^{-1}(x) \neq \varnothing$, $f^{-1}(y) \neq \varnothing$ and $f^{-1}(z) \neq \varnothing$. Since $\mathcal{A}$ has the sup-property, there are $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$ and $z_0 \in f^{-1}(z)$ such that the following identities hold:

$$\widetilde{A}(x_0 * z_0) = \bigvee_{t \in f^{-1}(x *' z)} \widetilde{A}(t),\ \widetilde{A}(y_0) = \bigvee_{t \in f^{-1}(y)} \widetilde{A}(t), \tag{4.3}$$

$$\overline{A}(x_0 * z_0) = \bigwedge_{t \in f^{-1}(x *' z)} \overline{A}(t),\ \overline{A}(y_0) = \bigwedge_{t \in f^{-1}(y)} \overline{A}(t), \tag{4.4}$$

$$A(x_0 * z_0) = \bigwedge_{t \in f^{-1}(x *' z)} A(t),\ A(y_0) = \bigwedge_{t \in f^{-1}(y)} A(t), \tag{4.5}$$

$$\widetilde{A}(x_0 * (y_0 * z_0)) = \bigvee_{t \in f^{-1}(x *' (y *' z))} \widetilde{A}(t), \tag{4.6}$$

$$\overline{A}(x_0 * (y_0 * z_0)) = \bigwedge_{t \in f^{-1}(x *' (y *' z))} \overline{A}(t), \tag{4.7}$$

$$A(x_0 * (y_0 * z_0)) = \bigwedge_{t \in f^{-1}(x *' (y *' z))} A(t). \tag{4.8}$$

Then we have

$\widetilde{B}(x *' z) = \bigvee_{t \in f^{-1}(x * y)} \widetilde{A}(t)$ [By Definition 4.1 (ii)]
$= \widetilde{A}(x_0 * z_0)$ [By (4.3)]
$\geq \widetilde{A}(x_0 * (y_0 * z_0)) \wedge \widetilde{A}(y_0)$ [By (OKUI$_2$)]
$= \left(\bigvee_{t \in f^{-1}(x *' (y *' z))} \widetilde{A}(t)\right) \wedge \left(\bigvee_{t \in f^{-1}(y)} \widetilde{A}(t)\right)$
[By (4.3) and (4.6)]
$= \widetilde{B}(x *' (y *' z)) \wedge \widetilde{B}(y)$ [By Definition 4.1 (ii)],
$\overline{B}(x *' z) = \bigwedge_{t \in f^{-1}(x * y)} \overline{A}(t)$
$= \overline{A}(x_0 * z_0)$ [By (4.4)]
$\leq \overline{A}(x_0 * (y_0 * z_0)) \vee \overline{A}(y_0)$
$= \left(\bigwedge_{t \in f^{-1}(x *' (y *' z))} \overline{A}(t)\right) \vee \left(\bigwedge_{t \in f^{-1}(y)} \overline{A}(t)\right)$
[By (4.4) and (4.7)]
$= \overline{B}(x *' (y *' z)) \vee \overline{B}(y)$,
$B(x *' z) = \bigwedge_{t \in f^{-1}(x * y)} A(t)$
$= A(x_0 * z_0)$ [By (4.5)]
$\leq A(x_0 * (y_0 * z_0)) \vee A(y_0)$
$= \left(\bigwedge_{t \in f^{-1}(x *' (y *' z))} A(t)\right) \vee \left(\bigwedge_{t \in f^{-1}(y)} A(t)\right)$
[By (4.5) and (4.8)]
$= B(x *' (y *' z)) \vee B(y)$.

Thus $\mathcal{B} = f(\mathcal{A})$ satisfies the condition (OKUI$_2$). So $f(\mathcal{A})$ is an octahedron KU-ideal of Y. □

5. The Cartesian product of octahedron KU-ideals

In this section, we define the Cartesian product of two octahedron sets and study some of its octahedron ideals of the Cartesian product of two KU-algebras.

Definition 5.1. Let $X,\ Y$ be two nonempty sets and $\mathcal{A} \in \mathcal{O}(X),\ \mathcal{B} \in \mathcal{O}(Y)$. Then the *Cartesian product* $\mathcal{A}\times\mathcal{B} = \left\langle \widetilde{A}\times\widetilde{B}, \overline{A}\times\overline{B}, A\times B\right\rangle$ of $\mathcal{A}$ and $\mathcal{B}$ is an octahedron set in $X \times Y$ defined as follows: for each $(x,y) \in X \times Y$,
(i) $(\widetilde{A}\times\widetilde{B})(x,y) = \widetilde{A}(x) \wedge \widetilde{B}(y)$,
(ii) $(\overline{A}\times\overline{B})(x,y) = \overline{A}(x) \vee \overline{B}(y)$,
(iii) $(A\times B)(x,y) = A(x) \vee B(y)$.

Remark 5.2. Let $(X,*,0)$ and $(Y,*,0)$ be two KU-algebras. We define the operation $*^{'}$ on $X \times Y$ by: for any $(x,y),\ (u,v) \in X \times Y$,

$$(x,y) *^{'} (u,v) = (x*u),(y*v).$$

Then it is obvious that $(X\times Y, *^{'}, (0,0))$ is a KU-algebra.

Proposition 5.3. *Let $(X,*,0)$ and $(Y,*,0)$ be two KU-algebras and $\mathcal{A} \in \mathcal{O}(X),\ \mathcal{B} \in \mathcal{O}(Y)$. If $\mathcal{A}$ and $\mathcal{B}$ are octahedron KU-subalgebras of X and Y,respectively, then $\mathcal{A}\times\mathcal{B}$ is an octahedron KU-subalgebra of $X \times Y$.*

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are octahedron KU-subalgebras of X and Y, respectively and let $(x,y),\ (u,v) \in X \times Y$. Then we get
$(\widetilde{A}\times\widetilde{B})((x,y) *^{'} (u,v)) = (\widetilde{A}\times\widetilde{B})(x*u, y*v)$ [By Remark 5.2]
$= \widetilde{A}(x*u) \wedge \widetilde{B}(y*v)$ [Definition 5.1 (i)]
$\geq (\widetilde{A}(x) \wedge \widetilde{A}(u)) \wedge (\widetilde{B}(y) \wedge \widetilde{B}(v))$
[By (OKUSA$_1$)]
$= (\widetilde{A}(x) \wedge \widetilde{B}(y)) \wedge (\widetilde{A}(u) \wedge \widetilde{B}(v))$
$= (\widetilde{A}\times\widetilde{B})(x,y) \wedge (\widetilde{A}\times\widetilde{B})(u,v)$,
$(\overline{A}\times\overline{B})((x,y) *^{'} (u,v)) = (\overline{A}\times\overline{B})(x*u, y*v)$
$= \overline{A}(x*u) \vee \overline{B}(y*v)$ [Definition 5.1 (ii)]
$\leq (\overline{A}(x) \vee \overline{A}(u)) \vee (\overline{B}(y) \vee \overline{B}(v))$
[By (OKUSA$_2$)]
$= (\overline{A}(x) \vee \overline{B}(y)) \vee (\overline{A}(u) \vee \overline{B}(v))$
$= (\overline{A}\times\overline{B})(x,y) \vee (\overline{A}\times\overline{B})(u,v)$,
$(A\times B)((x,y) *^{'} (u,v)) = (A\times B)(x*u, y*v)$
$= A(x*u) \vee B(y*v)$ [Definition 5.1 (iii)]
$\leq (A(x) \vee A(u)) \vee (B(y) \vee B(v))$ [By (OKUSA$_3$)]
$= (A(x) \vee B(y)) \vee (A(u) \vee B(v))$
$= (A\times B)(x,y) \vee (A\times B)(u,v)$.
Thus $\mathcal{A}\times\mathcal{B}$ is an octahedron KU-subalgebra of $X \times Y$. □

Proposition 5.4. *Let $(X,*,0)$ and $(Y,*,0)$ be two KU-algebras and $\mathcal{A} \in \mathcal{O}(X),\ \mathcal{B} \in \mathcal{O}(Y)$. If $\mathcal{A}$ and $\mathcal{B}$ are octahedron KU-ideals of X and Y, respectively, then $\mathcal{A}\times\mathcal{B}$ is an octahedron KU-ideal of $X \times Y$.*

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are octahedron KU-ideals of X and Y, respectively and let $(x,y) \in X \times Y$. Then we have

$(\widetilde{A} \times \widetilde{B})(x, y) = \widetilde{A}(x) \wedge \widetilde{B}(y)$
$\geq \widetilde{A}(0) \wedge \widetilde{B}(0)$ [By (OKUI$_1$)]
$= (\widetilde{A} \times \widetilde{B})(0, 0),$
$(\overline{A} \times \overline{B})(x, y) = \overline{A}(x) \vee \overline{B}(y)$
$\leq \overline{A}(0) \vee \overline{B}(0)$
$= (\overline{A} \times \overline{B})(0, 0),$
$(A \times B)(x, y) = A(x) \vee B(y)$
$\leq A(0) \vee B(0)$
$= (A \times B)(0, 0).$
Thus $\mathcal{A} \times \mathcal{B}$ satisfies the condition (OKUI$_1$).

To show that the condition (OKUI$_1$) holds, let $(x_1, x_2),\ (y_1, y_2),\ (z_1, z_2) \in X \times Y$. Then we obtain
$(\widetilde{A} \times \widetilde{B})((x_1, x_2) *^{'} (z_1, z_2))$
$= (\widetilde{A} \times \widetilde{B})(x_1 * z_1, x_2 * z_2)$
$= \widetilde{A}(x_1 * z_1) \wedge \widetilde{B}(x_2 * z_2)$
$\geq (\widetilde{A}(x_1 * (y_1 * z_1)) \wedge \widetilde{A}(y_1)) \wedge (\widetilde{B}(x_2 * (y_2 * z_2)) \wedge \widetilde{B}(y_2))$ [By (OKUI$_2$)]
$= (\widetilde{A}(x_1 * (y_1 * z_1)) \wedge \widetilde{B}(x_2 * (y_2 * z_2))) \wedge (\widetilde{A}(y_1) \wedge \widetilde{B}(y_2))$
$= (\widetilde{A} \times \widetilde{B})((x_1, x_2) *^{'} ((y_1, y_2) *^{'} (z_1, z_2))) \wedge (\widetilde{A} \times \widetilde{B})(y_1, y_2),$
$(\overline{A} \times \overline{B})((x_1, x_2) *^{'} (z_1, z_2))$
$= (\overline{A} \times \overline{B})(x_1 * z_1, x_2 * z_2)$
$= \overline{A}(x_1 * z_1) \vee \overline{B}(x_2 * z_2)$
$\leq (\overline{A}(x_1 * (y_1 * z_1)) \vee \overline{A}(y_1)) \vee (\overline{B}(x_2 * (y_2 * z_2)) \vee \overline{B}(y_2))$
$= (\overline{A}(x_1 * (y_1 * z_1)) \vee \overline{B}(x_2 * (y_2 * z_2))) \vee (\overline{A}(y_1) \vee \overline{B}(y_2))$
$= (\overline{A} \times \overline{B})((x_1, x_2) *^{'} ((y_1, y_2) *^{'} (z_1, z_2))) \vee (\overline{A} \times \overline{B})(y_1, y_2).$
Similarly, we get

$$(A \times B)((x_1, x_2) *^{'} (z_1, z_2)) \leq (A \times B)((x_1, x_2) *^{'} ((y_1, y_2) *^{'} (z_1, z_2))) \vee (A \times B)(y_1, y_2).$$

Thus $\mathcal{A} \times \mathcal{B}$ satisfies the condition (OKUI$_2$). So $\mathcal{A} \times \mathcal{B}$ is an octahedron KU-ideal of $X \times Y$. □

From Definitions 3.17 and 5.1, we obtain the following.

Lemma 5.5. *Let $X,\ Y$ be two nonempty sets, $\widetilde{\widetilde{a}} = \langle \widetilde{a}, \bar{a}, a \rangle$ an octahedron number and $\mathcal{A} \in \mathcal{O}(X),\ \mathcal{B} \in \mathcal{O}(Y)$. Then $[\mathcal{A} \times \mathcal{B}]_{\widetilde{\widetilde{a}}} = [\mathcal{A}]_{\widetilde{\widetilde{a}}} \times [\mathcal{B}]_{\widetilde{\widetilde{a}}}$.*

The following is a similar consequence to Theorem 3.18.

Theorem 5.6. *Let $(X, *, 0)$ and $(Y, *, 0)$ be two KU-algebras and $\mathcal{A} \in \mathcal{O}(X),\ \mathcal{B} \in \mathcal{O}(Y)$. Then $\mathcal{A} \times \mathcal{B}$ is an octahedron KU-subalgebra of $X \times Y$ if and only if for each octahedron number $\widetilde{\widetilde{a}} = \langle \widetilde{a}, \bar{a}, a \rangle$, either $[\mathcal{A} \times \mathcal{B}]_{\widetilde{\widetilde{a}}} = \varnothing$ or $[\mathcal{A} \times \mathcal{B}]_{\widetilde{\widetilde{a}}}$ is a KU-subalgebra of $X \times Y$.*

Proof. The proof is straightforward. □

Also, we have a similar consequence to Theorem 3.19.

Theorem 5.7. *Let $(X, *, 0)$ and $(Y, *, 0)$ be two KU-algebras and $\mathcal{A} \in \mathcal{O}(X),\ \mathcal{B} \in \mathcal{O}(Y)$. Then $\mathcal{A} \times \mathcal{B}$ is an octahedron KU-ideal of $X \times Y$ if and only if for each* *octahedron number* $\widetilde{\widetilde{a}} = \langle \widetilde{a}, \bar{a}, a\rangle$, *either* $[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}} = \varnothing$ *or* $[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$ *is a* KU*-ideal of* $X\times Y$.

Proof. Suppose $\mathcal{A}\times\mathcal{B}$ is an octahedron KU-ideal of $X\times Y$. Without loss of generality, assume that $[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}} \neq \varnothing$ and let $(x,y)\in[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$. Then by Lemma 5.5 and (OKUI$_1$), we have

$$\widetilde{a} \leq (\widetilde{A}\times\widetilde{B})(x,y) = \widetilde{A}(x)\wedge\widetilde{B}(y) \leq \widetilde{A}(0)\wedge\widetilde{B}(0) = (\widetilde{A}\times\widetilde{B})(0,0),$$
$$\bar{a} \geq (\overline{A}\times\overline{B})(x,y) = \overline{A}(x)\vee\overline{B}(y) \geq \overline{A}(0)\vee\overline{B}(0) = (\overline{A}\times\overline{B})(0,0),$$
$$a \geq (A\times B)(x,y) = A(x)\vee B(y) \geq A(0)\vee B(0) = (A\times B)(0,0).$$

Thus $(0,0)\in[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$. So $\mathcal{A}\times\mathcal{B}$ satisfies the condition (KUI$_1$).

Now suppose $(x_1,x_2)*^{'}((y_1,y_2)*^{'}(z_1,z_2)),\ (y_1,y_2)\in[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$ for any (x_1,x_2), $(y_1,y_2),\ (z_1,z_2)\in X\times Y$. Then clearly,

$$(\widetilde{A}\times\widetilde{B})((x_1,x_2)*^{'}((y_1,y_2)*^{'}(z_1,z_2))) \geq \widetilde{a},\ (\widetilde{A}\times\widetilde{B})(y_1,y_2)\geq\widetilde{a}, \tag{5.1}$$

$$(\overline{A}\times\overline{B})((x_1,x_2)*^{'}((y_1,y_2)*^{'}(z_1,z_2))) \leq \bar{a},\ (\overline{A}\times\overline{B})(y_1,y_2)\leq\bar{a}, \tag{5.2}$$

$$(A\times B)((x_1,x_2)*^{'}((y_1,y_2)*^{'}(z_1,z_2))) \leq a,\ (A\times B)(y_1,y_2)\leq a. \tag{5.3}$$

Thus we have

$(\widetilde{A}\times\widetilde{B})((x_1,x_2)*^{'}(z_1,z_2))$
$\geq (\widetilde{A}\times\widetilde{B})((x_1,x_2)*^{'}((y_1,y_2)*^{'}(z_1,z_2)))\wedge(\widetilde{A}\times\widetilde{B})(y_1,y_2)$ [By (OKUI$_1$)
$\geq \widetilde{a}$ [By (5.1)],
$(\overline{A}\times\overline{B})((x_1,x_2)*^{'}(z_1,z_2))$
$\leq (\overline{A}\times\overline{B})((x_1,x_2)*^{'}((y_1,y_2)*^{'}(z_1,z_2)))\vee(\overline{A}\times\overline{B})(y_1,y_2)$
$\leq \bar{a}$ [By (5.2)],
$(A\times B)((x_1,x_2)*^{'}(z_1,z_2))$
$\leq (A\times B)((x_1,x_2)*^{'}((y_1,y_2)*^{'}(z_1,z_2)))\vee(A\times B)(y_1,y_2)$
$\leq a$ [By (5.3)].

Thus $(x_1,x_2)*^{'}(z_1,z_2)\in[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$. So $[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$ satisfies the condition (KUI$_2$). Hence $[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$ is a KU-ideal of $X\times Y$.

Conversely, suppose the necessary condition holds, say $[\mathcal{A}\times\mathcal{B}]_{\widetilde{\widetilde{a}}}$ is a KU-ideal of $X\times Y$ for each octahedron number $\widetilde{\widetilde{a}}$. Assume that the condition (OKUI$_2$) does not hold, i.e there is $(x,y)\in X\times Y$ such that

$$(\widetilde{A}\times\widetilde{B})(0,0) < (\widetilde{A}\times\widetilde{B})(x,y), \tag{5.4}$$

$$(\overline{A}\times\overline{B})(0,0) > (\overline{A}\times\overline{B})(x,y), \tag{5.5}$$

$$(A\times B)(0,0) > (A\times B)(x,y). \tag{5.6}$$

Let us take an octahedron number $\widetilde{\widetilde{a}}$ such that

$$\widetilde{a} = \frac{1}{2}\left((\widetilde{A}\times\widetilde{B})(0,0)+(\widetilde{A}\times\widetilde{B})(x,y)\right), \tag{5.7}$$

$$\bar{a} = \frac{1}{2}\left((\overline{A}\times\overline{B})(0,0)+(\overline{A}\times\overline{B})(x,y)\right), \tag{5.8}$$

$$a = \frac{1}{2}\left((A \times B)(0,0) + (A \times B)(x,y)\right). \tag{5.9}$$

Then we obtain the following inequalities:

$$(\widetilde{A} \times \widetilde{B})(0,0) < \widetilde{a} < (\widetilde{A} \times \widetilde{B})(x,y), \tag{5.10}$$

$$(\overline{A} \times \overline{B})(0,0) > \bar{a} > (\overline{A} \times \overline{B})(x,y), \tag{5.11}$$

$$(A \times B)(0,0) > a > (\widetilde{A} \times \widetilde{B})(x,y). \tag{5.12}$$

Thus $(x,y) \in [\mathcal{A} \times \mathcal{B}]_{\widetilde{\widetilde{a}}}$ but $(0,0) \notin [\mathcal{A} \times \mathcal{B}]_{\widetilde{\widetilde{a}}}$. This is a contradiction. So $\mathcal{A} \times \mathcal{B}$ satisfies the condition (OKUI_1).

Now assume that the condition (OKUI_2) does not hold, i.e., there are (x_1,x_2), (y_1,y_2), $(z_1,z_2) \in X \times Y$ such that

$$(\widetilde{A} \times \widetilde{B})((x_1,x_2) *^{'} (z_1,z_2))$$
$$< (\widetilde{A} \times \widetilde{B})((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \wedge (\widetilde{A} \times \widetilde{B})(y_1,y_2),$$

$$(\overline{A} \times \overline{B})((x_1,x_2) *^{'} (z_1,z_2))$$
$$> (\overline{A} \times \overline{B})((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \vee (\overline{A} \times \overline{B})(y_1,y_2),$$

$$(A \times B)((x_1,x_2) *^{'} (z_1,z_2))$$
$$> (A \times B)((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \vee (A \times B)(y_1,y_2).$$

Let us take an octahedron number $\widetilde{\widetilde{a}}$ such that

$$\widetilde{a} = \tfrac{1}{2}[(\widetilde{A} \times \widetilde{B})((x_1,x_2) *^{'} (z_1,z_2))$$
$$+(\widetilde{A} \times \widetilde{B})((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \wedge (\widetilde{A} \times \widetilde{B})(y_1,y_2)],$$
$$\bar{a} = \tfrac{1}{2}[(\overline{A} \times \overline{B})((x_1,x_2) *^{'} (z_1,z_2))$$
$$+(\overline{A} \times \overline{B})((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \vee (\overline{A} \times \overline{B})(y_1,y_2)],$$
$$a = \tfrac{1}{2}[(A \times B)((x_1,x_2) *^{'} (z_1,z_2))$$
$$+(A \times B)((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \vee (A \times B)(y_1,y_2)].$$

Then we obtain the following inequalities:

$$(\widetilde{A} \times \widetilde{B})((x_1,x_2) *^{'} (z_1,z_2)$$
$$< \widetilde{a} < (\widetilde{A} \times \widetilde{B})((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \wedge (\widetilde{A} \times \widetilde{B})(y_1,y_2),$$
$$(\overline{A} \times \overline{B})((x_1,x_2) *^{'} (z_1,z_2)$$
$$> \bar{a} > (\overline{A} \times \overline{B})((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \vee (\overline{A} \times \overline{B})(y_1,y_2),$$
$$(A \times B)((x_1,x_2) *^{'} (z_1,z_2)$$
$$> a > (A \times B)((x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))) \vee (A \times B)(y_1,y_2).$$

Thus $(x_1,x_2) *^{'} ((y_1,y_2) *^{'} (z_1,z_2))$, $(y_1,y_2) \in [\mathcal{A} \times \mathcal{B}]_{\widetilde{\widetilde{a}}}$ but $(x_1,x_2) *^{'} (z_1,z_2) \notin [\mathcal{A} \times \mathcal{B}]_{\widetilde{\widetilde{a}}}$. This is a contradiction. So $\mathcal{A} \times \mathcal{B}$ satisfies the condition (OKUI_2). Hence $\mathcal{A} \times \mathcal{B}$ is an octahedron KU-ideal of $X \times Y$. □

For two KU-algebras $(X,*,0)$ and $(Y,*,0)$, we define binary operation $\leq^{'}$ on $X \times Y$ as follows: for any (x,y), $(u,v) \in X \times Y$,

$$(x,y) \leq^{'} (u,v) \text{ if and only if } x \leq_X u,\ y \leq_Y v,$$

where $\leq_X$ and $\leq_Y$ denote the partial orders on X and Y, respectively.

It is obvious that $\leq^{'}$ is a partial order on $X \times Y$.

The following is the similar consequence of Lemma 3.12.

Proposition 5.8. *Let $(X,*,0)$ and $(Y,*,0)$ be two KU-algebras, $\mathcal{A}\in\mathcal{O}(X)$, $\mathcal{B}\in\mathcal{O}(Y)$ and $\mathcal{A}\times\mathcal{B}$ be an octahedron KU-ideal of $X\times Y$. If $(x_1,x_2)\leq^{'}(y_1,y_2)$ for any $(x_1,x_2),\ (y_1,y_2)\in X\times Y$, then the following inequalities hold:*

$$(\widetilde{A}\times\widetilde{B})(y_1,y_2)\geq^{'}(\widetilde{A}\times\widetilde{B})(x_1,x_2),$$

$$(\overline{A}\times\overline{B})(y_1,y_2)\leq^{'}(\overline{A}\times\overline{B})(x_1,x_2),$$

$$(A\times B)(y_1,y_2)\leq^{'}(A\times B)(x_1,x_2).$$

Proof. The proof follows from Lemma 3.12 and the definition of the partial order $\leq^{'}$. □

Also, we have the similar consequence of Lemma 3.13.

Proposition 5.9. *Let $(X,*,0)$ and $(Y,*,0)$ be two KU-algebras, $\mathcal{A}\in\mathcal{O}(X)$, $\mathcal{B}\in\mathcal{O}(Y)$ and $\mathcal{A}\times\mathcal{B}$ is an octahedron KU-ideal of $X\times Y$. If $(x_1,x_2)*^{'}(y_1,y_2)\leq^{'}(z_1,z_2)$ for any $(x_1,x_2),\ (y_1,y_2),\ (z_1,z_2)\in X\times Y$, then the following inequalities hold:*

$$(\widetilde{A}\times\widetilde{B})(y_1,y_2)\geq^{'}(\widetilde{A}\times\widetilde{B})(x_1,x_2)\wedge(\widetilde{A}\times\widetilde{B})(z_1,z_2),$$

$$(\overline{A}\times\overline{B})(y_1,y_2)\leq^{'}(\overline{A}\times\overline{B})(x_1,x_2)\vee(\overline{A}\times\overline{B})(z_1,z_2),$$

$$(A\times B)(y_1,y_2)\leq^{'}(A\times B)(x_1,x_2)\vee(A\times B)(z_1,z_2).$$

Proof. The proof follows from Lemma 3.13 and the definition of the partial order $\leq^{'}$. □

6. Conclusions

By defining the notions of octahedron KU-subalgeras and octahedron KU-ideals of a KU-algebra, we obtained some of their properties. In particular, we dealt with a relationship between octahedron KU-subalgebras [resp. KU-ideals] and classical KU-subalgebras [resp. KU-ideals] of a KU-algebra. Also, we discusses some properties of the images and the preimages of octahedron KU-ideals of a KU-algebra under a homomorphism of KU-algebras. Furthermore, we studied that the Cartesian products of two octahedron KU-ideals [resp. KU-subalgebras] of the Cartesian products of two KU-algebras have similar properties as discussed in Section 3.

In the future, we expect to define a new octahedron set that combines the octahedron set with soft set, bipolar fuzzy set, hesitant fuzzy set, Pythagorean fuzzy set, and neutrosophic set, and apply each to topology, abstract algebra, logical algebra, decision making problems, etc.

Acknowledgments: This paper was supported by Wonkwang University in 2025. We are very grateful to the reviewers for their careful reading and their meaningful suggestions.

References

[1] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. 23 (1) (1978) 1– 26.

[2] K. Iséki, On BCI-algebras, Math. Seminar Notes 8 (1980) 125–130.

[3] Q. P. Hu and X. Li, On BCH-algebras, Math. Seminar Notes Kobe Univ. 11 (2) (1983) 313–320.

[4] Wieslaw A. Dudek, On BCC-algebras, Logique et Analyse 33 (129/130) (1990) 103–111.

[5] Sun Shin Ahn and Hee Sik Kim, On QS-algebras, Journal of The Chungcheong Mathematical Society 12 (1999) 33–41.

[6] Joseph Neggers, Sun Shin Ahn and Hee Sik Kim, On Q-algebras, IJMMS 27 (12) (2001) 749–757.

[7] Hee Sik Kim and Young Hee Kim, On BE-algebras, Scientiae Mathematicae Japonicae Online 1 (2006) 1299–1302.

[8] C. Prabpayak and U. Leerawat, On ideals and congruences in KU-algebras, Scientia Magna 5 (1) (2009) 54–57.

[9] S. M. Mostafa, R. A. K. Omar and O. W. Abd El-Baseer, Sub implicative ideals of KU-algebras, International Journal of Modern Science and Technology 2 (5) (2017) 223–227.

[10] S. M. Mostafa, Intuitionistic fuzzy dot hyper-(subalgebra) ideals on hyper KU-algebras, Journal of Universal Mathematics 1 (2) (2018) 230–244.

[11] Ali N. A. Koam, Azeem Haider and Moin A. Ansari, On an extension of KU-algebras, AIMS Mathematics 6 (2) (2020) 1249–1267.

[12] Qasim Mohsin Luhaiba and Fatema Faisal Kareemb, Study of the dual ideal of KU-algebra, Int. J. Nonlinear Anal. Appl. 12 (2) (2021) 1099–1107.

[13] Samy M. Mostafa, Ehab f. Adb-elfattah, Mostafa A. Hassan, Y. B. Jun and K. Hur, Crossing intuitionistic KU-ideals on KU-algebras as an extension of bipolar fuzzy sets, Ann. Fuzzy Math. Inform. 22 (3) (2021) 283–295.

[14] Javad Golzarpoor and Saeed Mehrshad, State KU-algebras, Curr. Appl. Sci. 2 (1) (2022) 91–100.

[15] Areej Almuhaimeed, On the structure of interior KU-algebras and KU-ideals, Communications in Mathematics and Applications 14 (1) (2023) 471–479.

[16] J. I. Baek, Samy M. Mostafa, Fatema F. Kareem, S. H. Han and K. Hur, Γ-KU-algebras, Ann. Fuzzy Math. Inform. 27 (1) (2024) 1–27.

[17] Moin A. Ansari and Ali N. A. Koam, On KU-modules over KU-algebras, AInt. J. Anal. Appl. 22 (57) (2024) 1–13.

[18] J. G. Lee, G. Şenelnel, P. K. Lim, J. Kim, K. Hur, Octahedron sets, Ann. Fuzzy Math. Inform. 19 (3) (2020) 211–238.

[19] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[20] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[21] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975) 199–249.

[22] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy sets and Systems 21 (1987) 1–17.

[23] Jeong-Gon Lee, Young Bae Jun and Kul Hur, Octahedron subgroups and subrings, Mathematics 2020, 8, 1444; doi:10.3390/math8091444.

[24] Güzide Şenel, Jeong-Gon Lee and Kul Hur, Distance and similarity measures for octahedron sets and their application to MCGDM problems, Mathematics 2020, 8, 1690; doi:10.3390/math8101690.

[25] J. G. Lee, G. Şenel, K. Hur, J. Kim and J. I. Baek, Octahedron topological spaces, Ann. Fuzzy Math. Inform. 22 (1) (2021) 77–101.

[26] Samy M. Mostafa, S. H. Han, E. F. Adb-elfattah, M. A. Hassan, K. Hur and J. I. Baek, Crossing octahedron sets and their applications to Q-algebras, Ann. Fuzzy Math. Inform. 23 (2) (2022) 161–180.

[27] Samy M. Mostafa, Mokhtar A. Abd- Elnaby and Moustafa M. M. Yousef, Fuzzy ideals of KU-algebras, International Mathematical Forum 6 (63) (2011) 3139–3149.

[28] C. Prabpayak and U. Leerawat, On isomorphisms of KU-algebras, Scientia Magna journal 5 (3) (2009) 25–31.
[29] Minseok Cheong and K. Hur, Intuitionistic interval-valued fuzzy sets, J. Korean Institute of Intelligent Systems 20 (6) (2010) 864–874.
[30] T. K. Mondal and S. K. Samanta, Topology of interval-valued fuzzy sets, Indian J. Pure Appl. Math. 30 (1) (1999) 133–189.
[31] J. Kim, P. K. Lim, J. G. Lee, K. Hur, Interval-valued intuitionistic cubic structures of medial ideals on BCI-algebras, Ann. Fuzzy Math. Inform. 19 (1) (2020) 21–43.
[32] J. Kim, Minseok Cheong, J. G. Lee, K. Hur, S. M. Mostafa, Interval-valued intuitionistic cubic structures of medial ideals on BCI-algebras, Ann. Fuzzy Math. Inform. 19 (2) (2020) 109–125.
[33] Y. B. Jun, C. S. Kim and K. O. Yang, Cubic sets, Ann. Fuzzy Math. Inform. 4 (1) (2012) 83–98.
[34] Samy M. Mostafa, Mokhtar A. Abd-Elnaby and Osama R. Elgendy, Intuitionistic fuzzy KU-ideals in KU-algebras, Int. J. of Mathematical Sciences and Applications 1 (3) (2011) 1379–1384.
[35] Samy M. Mostafa, Mokhtar A. Abd-Elnaby and Osama R. Elgendy, Interval-valued fuzzy KU-ideals in KU-algebras, International Mathematical Forum 6 (64) (2011) 3151–3159.
[36] Naveed Yaqoob, Samy M. Mostafa and Moin A. Ansari, On cubic KU-ideals of KU-algebras, ISRN Algebra 2013 (2013) Article ID 935905 10 pages.http://dx.doi.org/10.1155/2013/935905.

J. I. Baek (jibaek@wku.ac.kr)
School of Big Data and Financial Statistics, Wonkwang University, Korea

Samy M. Mostafa (samymostafa@edu.asu.edu.eg)
Department of pure mathematics Ain Shams University , Roxy , Cairo , Egypt

S. H. Han (shhan235@wku.ac.kr)
Department of Applied Mathematics, Wonkwang University, Korea

M. Cheong (poset@kau.ac.kr)
School of Liberal Arts and Sciences, Korea Aerospace University, Korea

K. Hur (kulhur@wku.ac.kr)
Division of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea