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Abstract. In this paper, as basic tools for analyzing information systems, we introduce the two types (right, left) of interior (closure) operators, interior (closure, extensional) systems and distance functions on a generalized co-residuated lattice as a noncommutative structure. We investigate their relations and give examples.

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1. Introduction

Ward et al. [1] introduced a complete residuated lattice which is an algebraic structure for many valued logic as an extension of left continuous t-norms. Bělohlávek [2, 3, 4, 5] investigated the properties of fuzzy closure operators and fuzzy closure systems on a complete residuated lattice. By using their concepts, topological structures, logic, formal concept, information systems and decision rules are investigated on complete residuated lattices [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. As a non-commutative algebraic structure, Turunen [10] introduced a generalized residuated lattice as an generalization of weak-pseudo-BL-algebras and left continuous pseudo-t-norm [12, 13]. Ko and Kim [14, 15] introduced the notions of right (resp. left) closure operators and right (resp. left) closure systems on a generalized residuated lattice.

Qiao and Hu. [16] introduced fuzzy rough sets based on residuated and co-residuated lattices as an extension of right continuous t-conorms [12, 13, 10]. Employing distance spaces over fuzzy partially ordered spaces, topological structures and formal concepts on complete co-residuated lattices were investigated [9, 17, 18, 19].

The goal of this paper, we introduce the two types (right, left) of interior (closure) operators, interior (closure, extensional) systems and distance functions on a generalized co-residuated lattice defined in [16, 19] as a noncommutative structure. They are basic tools for analyzing information systems.

This paper is organized as follows. In Section 2, we recall some definitions and basic properties of the generalized co-residuated lattice.

In Section 3, we investigate the relations among two types (right, left) of interior (closure) operators, interior (closure, extensional) systems and distance functions.

Moreover, we show that a right interior induces a right (resp. left) interior system and right (resp. left) closure operators induces a right (resp. left) system. We study that a right interior (resp. closure) system induces a right interior (resp. closure) operator and a right distance function, and conversely a right interior (resp. closure) operator induces a right distance function. We show that a right distance function induces a right extensional system, a right interior operator and a right closure operator. We investigate that a right interior (resp. closure) operator induces a right distance function. Left structures, with similar results to those of right structures, are depicted in corollaries.

2. Preliminaries

As an extension of co-residuated lattices [15, 17, 18, 19, 20], we define generalized co-residuated lattices as an non-commutative algebraic structure.

Definition 2.1 ([19])**.** A structure $(L, \vee, \wedge, \oplus, \ominus, \oslash, \bot, \top)$ is called a *generalized co-residuated lattice*, if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \bot, \top)$ is lattice with the least element $\bot$ and the greatest element $\top$,

(GR2) $\bot \oplus x = x \oplus \bot = x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for all $x,\ y,\ z \in L$,

(GR3) it satisfies a co-residuation , i.e., for any $x,\ y,\ z \in X$,

$$x \oplus y \geq z \text{ iff } x \geq z \ominus y \text{ iff } y \geq z \oslash x.$$

A generalized co-residuated lattice is called *co-residuated lattice* if $x \oplus y = y \oplus x$ for any $x,\ y \in L$.

For $\alpha \in L,\ A \in L^X,\ x \in X$, we denote $(A \ominus \alpha),\ (\alpha \oplus A) \in L^X$ as $(A \ominus \alpha)(x) = A(x) \ominus \alpha,\ (\alpha \oplus A)(x) = \alpha \oplus A(x)\ \forall x \in X$ and $\bot_x \in L^X$ defined as

$$\bot_x(y) = \begin{cases} \bot & \text{if } y = x \\ \top & \text{if } y \neq x. \end{cases}$$

Put $n_1(x) = \top \ominus x$ and $n_2(x) = \top \oslash x$ for each $x \in L$. The condition $n_1(n_2(x)) = n_2(n_1(x)) = x$ for each $x \in L$ is called a *double negative law*.

In this paper, we assume $(L, \vee, \wedge, \oplus, \ominus, \oslash, \bot, \top)$ is a generalized co-residuated lattice with a double negative law and if the family supremum or infumum exists, we denote $\bigvee$ and $\bigwedge$.

Lemma 2.2 ([19])**.** *For each $x,\ y,\ z,\ x_i,\ y_i \in L$, we have the following properties.*

(1) $y \oplus (x \oslash y) \geq x$, $(x \ominus y) \oplus y \geq x$, $x \ominus (x \oslash y) \leq y$ *and* $x \oslash (x \ominus y) \leq y$.

(2) *If* $y \leq z$, *then* $(x \oplus y) \leq (x \oplus z)$, $(y \oplus x) \leq (z \oplus x)$, $x \ominus y \leq x \ominus z$ *and* $z \ominus x \leq y \ominus x$ *for* $\ominus \in \{\ominus, \oslash\}$.

(3) $x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ *for* $\ominus \in \{\ominus, \oslash\}$.
(4) $x \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \ominus y_i)$ *and* $(\bigwedge_{i\in\Gamma} x_i) \ominus y \leq \bigwedge_{i\in\Gamma}(x_i \ominus y)$ *for* $\ominus \in \{\ominus, \oslash\}$.
(5) $x \oplus (y \ominus z) \geq (x \oplus y) \ominus z$ *and* $(x \oslash y) \oplus z \geq (x \oplus z) \oslash y$.
(6) $x \ominus (y \oplus z) = (x \ominus z) \ominus y$ *and* $(x \oslash y) \oslash z = x \oslash (y \oplus z)$.
(7) $(x \ominus y) \oslash z = (x \oslash z) \ominus y$.
(8) $(y \oslash z) \oplus (x \oslash y) \geq x \oslash z$ *and* $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.
(9) $(x \oslash z) \geq (y \oplus x) \oslash (y \oplus z)$ *and* $(x \ominus z) \geq (x \oplus y) \ominus (z \oplus y)$.
(10) $y \oslash z \geq (x \oslash z) \ominus (x \oslash y)$ *and* $x \oslash y \geq (x \oslash z) \oslash (y \oslash z)$.
(11) $x \ominus y \geq (x \ominus z) \ominus (y \ominus z)$ *and* $y \ominus z \geq (x \ominus z) \oslash (x \ominus y)$.
(12) $x \ominus x = x \oslash x = \perp$.
(13) $x \ominus y = \perp$ *iff* $x \leq y$ *iff* $x \oslash y = \perp$.
(14) $(\bigvee_{i\in\Gamma} y_i) \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigvee_{i\in\Gamma}(x_i \ominus y_i)$ *and* $(\bigwedge_{i\in\Gamma} x_i) \ominus (\bigwedge_{i\in\Gamma} y_i) \leq \bigvee_{i\in\Gamma}(x_i \ominus y_i)$ *for* $\ominus \in \{\ominus, \oslash\}$.
(15) $x \ominus y = n_1(y) \oslash n_1(x)$ *and* $x \oslash y = n_2(y) \ominus n_2(x)$.
(16) $n_1(y \oplus z) = n_1(z) \ominus y$ *and* $n_2(y \oplus z) = n_2(y) \oslash z$. *Moreover,* $n_2(x \ominus y) = y \oplus n_2(x)$ *and* $n_1(x \oslash y) = n_1(x) \oplus y$.
(17) $x \ominus \perp = x \oslash \perp = x$.
(18) *For any* $k = 1,\ 2$, $n_k(\bigwedge_{i\in\Gamma} x_i) = \bigvee_{i\in\Gamma} n_k(x_i)$ *and* $n_k(\bigvee_{i\in\Gamma} x_i) = n_k(\bigwedge_{i\in\Gamma} x_i)$.

3. Various systems, operators and distance functions

Definition 3.1. Let X be a set. A function $d^r_X : X \times X \to L$ is called a *right distance function*, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(D1) $d^r_X(x,x) = \perp$,
(D2) if $d^r_X(x,y) = d^r_X(y,x) = \perp$, then $x = y$,
(R) $d^r_X(x,y) \oplus d^r_X(y,z) \geq d^r_X(x,z)$.

A function $d^l_X : X \times X \to L$ is called a *left distance function*, if it satisfies (D1), (D2) and
(L) $d^l_X(y,z) \oplus d^l_X(x,y) \geq d^l_X(x,z)$ for all $x,\ y,\ z \in X$.

Remark 3.2. (1) Let d^r_X (resp. d^l_X) be a right (resp. left) distance function on X. Define functions $d^{-r}_X, d^{-l}_X : X \times X \to L$ as $d^{-r}_X(x,y) = d^r_X(y,x), d^{-l}_X(x,y) = d^l_X(y,x)$. Then d^{-r}_X (resp. d^{-l}_X) is a left (resp. right) distance function on X.
(2) Define functions $d^r_L, d^l_L : L \times L \to L$ as $d^r_L(x,y) = x \ominus y,\ d^l_L(x,y) = x \oslash y$. By Lemma 2.2 (8), d^r_L (resp. d^l_L) is a right (resp. left) distance function on L.
(3) Define functions $d^r_{L^X}, d^l_{L^X} : L^X \times L^X \to L$ as

$$d^r_{L^X}(A,B) = \bigvee_{x\in X}(A(x) \ominus B(x)),\ \ d^l_{L^X}(A,B) = \bigvee_{x\in X}(A(x) \oslash B(x)).$$

By Lemma 2.2 (8), $d^r_{L^X}$ (resp. $d^l_{L^X}$) is a right (resp. left) distance function on L^X.

Definition 3.3. An operator $I^r : L^X \to L^X$ is called a *right interior operator* on X, if it satisfies the following conditions: for any $A,\ B \in L^X$,
(I1) $I^r(A) \leq A$ and $I^r(A) \leq I^r(B)$ for $A \leq B$,
(I2) $I^r(I^r(A)) = I^r(A)$,
(IR) $I^r(A \oslash \alpha) \geq I^r(A) \oslash \alpha$ for each $\alpha \in L$.

An operator $I^l : L^X \to L^X$ is called a *left interior operator* on X, if it satisfies the conditions (I1), (I2) and
(IL) $I^l(A \ominus \alpha) \geq I^l(A) \ominus \alpha$ for each $\alpha \in L,\ A \in L^X$.

Definition 3.4. An operator $C^r : L^X \to L^X$ is called a *right closure operator* on X if it satisfies the following conditions: for any $A,\ B \in L^X$,
(C1) $C^r(A) \geq A$ and $C^r(A) \leq C^r(B)$ for $A \leq B$,
(C2) $C^r(C^r(A)) = C^r(A)$,
(CR) $\alpha \oplus C^r(A) \geq C^r(\alpha \oplus A)$ for each $\alpha \in L$.

An operator $C^l : L^X \to L^X$ is called a *left closure operator* on X if it satisfies the conditions (C1), (C2) and for any $A,\ B \in L^X$,
(CL) $C^l(A) \oplus \alpha \geq C^l(A \oplus \alpha)$.

Definition 3.5. (i) A family G^r is called a *right closure system* on X, if $(\alpha \oplus A_i) \in G^r, \bigwedge_{i\in\Gamma} A_i \in G^r$ for all $A_i \in G^r$ and $\alpha \in L$.

(ii) A family G^l is called a *l-closure system* on X, if $(A_i \oplus \alpha), \bigwedge_{i\in\Gamma} A_i \in G^l$ for all $A_i \in G^l$ and $\alpha \in L$.

(iii) A family H^r is called a *r-interior system* on X, if $(A_i \oslash \alpha) \in H^r, \bigvee_{i\in\Gamma} A_i$ for all $A_i \in H^r$ and $\alpha \in L$.

(iv) A family H^l is called a *l-interior system* on X, if $(A_i \ominus \alpha) \in H^l, \bigvee_{i\in\Gamma} A_i$ for all $A_i \in H^l$ and $\alpha \in L$.

(v) A family K^r is called a *right extensional system* on X, if K^r is both a right interior system and a right closure system.

(vi) A family K^l is called a *left extensional system* on X, if K^l is both a left interior system and a left closure system.

Theorem 3.6. (1) *Let $I^r : L^X \to L^X$ be a right interior operator on X. Then $H^r_{I^r} = \{A \mid A = I^r(A)\}$ is a right interior system on X.*

(2) *Let $I^l : L^X \to L^X$ be a left interior operator on X. Then $H^l_{I^l} = \{A \mid A = C^l(A)\}$ is a left interior system on X.*

(3) *Let $C^r : L^X \to L^X$ be a right closure operator on X. Then $G^r_{C^r} = \{A \mid A = C^r(A)\}$ is a right closure system on X.*

(4) *Let $C^l : L^X \to L^X$ be a left closure operator on X. Then $G^l_{C^l} = \{A \mid A = C^l(A)\}$ is a left closure system on X.*

Proof. (1) Let $A \in H^r_{I^r}$. By (IR) and (I1), since $A \oslash \alpha \geq I^r(A \oslash \alpha) \geq I^r(A) \oslash \alpha = A \oslash \alpha$ for each $\alpha \in L,\ A \in L^X$, $I^r(A \oslash \alpha) = A \oslash \alpha$, i.e, $A \oslash \alpha \in H^r_{I^r}$.

For all $A_i \in H^r_{I^r}$, $\bigvee_{i\in\Gamma} I^r(A_i) \leq I^r(\bigvee_{i\in\Gamma} A_i) \leq \bigvee_{i\in\Gamma} A_i = \bigvee_{i\in\Gamma} I^r(A_i)$. Then $\bigvee_{i\in\Gamma} A_i \in H^r_{I^r}$. Thus $H^r_{I^r}$ is a right interior system on X.

(2) It is similarly proved as (1).

(3) Let $B \in G^r_{C^r}$. By (CR) and (C1), since $\alpha \oplus B = \alpha \oplus C^r(B) \leq C^r(\alpha \oplus B) \leq \alpha \oplus C^r(B)$, $\alpha \oplus B \in G^r_{C^r}$.

For all $A_i \in G^r_{C^r}$, $\bigwedge_{i\in\Gamma} A_i = \bigwedge_{i\in\Gamma} C^r(A_i) \leq C^r(\bigwedge_{i\in\Gamma} A_i) \leq \bigwedge_{i\in\Gamma} C^r(A_i)$. Then $\bigwedge_{i\in\Gamma} A_i \in G^r_{C^r}$. Thus $G^r_{C^r}$ is a right closure system on X.

(4) It is similarly proved as (3). □

Theorem 3.7. *Let H^r be a right interior system on X. Then the following properties hold.*

(1) *$I^r_{H^r}(A) = \bigvee_{C\in H^r}(C \oslash d^r_{L^X}(C, A)) = \bigvee_{i\in\Gamma}\{A_i \mid A_i \le A, A_i \in H^r\}$.*
(2) *$I^r_{H^r}$ is a right interior operator on X.*
(3) *$H^r_{I^r_{H^r}} = H^r$ where $H^r_{I^r_{H^r}} = \{A \in L^X \mid A = I^r_{H^r}(A)\}$.*
(4) *Define $e^r_{H^r}(x, y) = \bigvee_{A\in H^r}(A(y) \oslash A(x))$. Then $e^r_{H^r}$ is a right distance function.*
(5) *If I^r is a right interior operator on X, then $I^r_{H^r_{I^r}} = I^r$ and $e^r_{H^r_{I^r}}(x, y) = I^r(\perp_x)(y)$ for any $x,\ y \in X$.*

Proof. (1) Let $I(A) = \bigvee\{A_i \in H^r \mid A_i \le A\}$. Since $d^r_{L^X}(A_i, A) \oplus A \ge A_i$ iff $A \ge A_i \oslash d^r_{L^X}(A_i, A)$, $A \ge \bigvee_{A_i\in H^r}(A_i \oslash d^r_{L^X}(A_i, A)) \in H^r$ for $A_i \in H^r$. Then $I^r_{H^r}(A) \le I(A)$. Since $I(A) \in H^r$, by Lemma 2.2 (13) and (17), we have

$$I^r_{H^r}(A) \ge I(A) \oslash d^r_{L^X}(I(A), A) = I(A) \oslash \perp_X = I(A).$$

Thus $I^r_{H^r}(A) = \bigvee\{A_i \in H^r \mid A_i \le A\}$.
(2) (I1) By the definition of $I^r_{H^r}$, it can be proved easily.
(I2) For each $A \in L^X$, we have

$$\begin{aligned} I^r_{H^r}(A) &= \bigvee_{i\in\Gamma}\{A_i \mid A_i \le A, A_i \in H^r\} \\ &\le \bigvee_{i\in\Gamma}\{A_i \mid A_i = I^r_{H^r}(A_i) \le I^r_{H^r}(A), A_i \in H^r\} \\ &\le \bigvee_{i\in\Gamma}\{B_i \mid B_i \le I^r_{H^r}(A), B_i \in H^r\} \\ &= I^r_{H^r}(I^r_{H^r}(A)). \end{aligned}$$

By (I1), $I^r_{H^r}(A) = I^r_{H^r}(I^r_{H^r}(A))$.
(IR) For each $A \in L^X,\ \alpha \in L$, we have

$$\begin{aligned} I^r_{H^r}(A) \oslash \alpha &= \bigvee_{i\in\Gamma}\{A_i \mid A_i \le A, A_i \in H^r\} \oslash \alpha \\ &\le \bigvee_{i\in\Gamma}\{A_i \oslash \alpha \mid A_i \oslash \alpha \le A \oslash \alpha, A_i \oslash \alpha \in H^r\} \\ &\le \bigvee_{i\in\Gamma}\{B_i \mid B_i \le A \oslash \alpha, B_i \in H^r\} \\ &= I^r_{H^r}(A \oslash \alpha). \end{aligned}$$

(3) $H^r_{I^r_{H^r}} = H^r$,where $H^r_{I^r_{H^r}} = \{A \in L^X \mid A = I^r_{H^r}(A)\}$.
If $A \in H^r$, then $A = I^r_{H^r}(A)$. Thus $A \in H^r_{I^r_{H^r}}$.
If $A \in H^r_{I^r_{H^r}}$, then $A = I^r_{H^r}(A) \in H^r$. Thus $A \in H^r$.
(4) It is easily proved from $e^r_{H^r}(x, y) \oplus e^r_{H^r}(y, z) = \bigvee_{A\in H^r}(A(y) \oslash A(x)) \oplus \bigvee_{A\in H^r}(A(z) \oslash A(y)) \ge \bigvee_{A\in H^r}((A(y) \oslash A(x)) \oplus (A(z) \oslash A(y))) \ge \bigvee_{A\in H^r}(A(z) \oslash A(x)) = e^r_{H^r}(y, z)$.
(5) For any $A,\ B \in L^X$, since $d^r_{L^X}(C, A) \ge d^r_{L^X}(I^r(C), I^r(A))$, $I^r_{H^r_{I^r}} = I^r$ from:

$$\begin{aligned} I^r_{H^r_{I^r}}(A) &= \bigvee_{i\in\Gamma}\{A_i \mid A_i \le A, A_i \in H^r_{I^r}\} \\ &\le \bigvee_{i\in\Gamma}\{A_i \mid A_i = I^r(A_i) \le I^r(A), A_i \in H^r_{I^r}\} \\ &\le I^r(A), \end{aligned}$$

$$\begin{aligned} I^r_{H^r_{I^r}}(A)(x) &= \bigvee_{C\in L^X}(C(x) \oslash d^r_{L^X}(C, A)) \\ &\ge d_{L^X}(I^r(A)(x) \oslash d^r_{L^X}(I^r(A), A)) = I^r(A)(x). \end{aligned}$$

For any $x,\ y \in X$, we get

$$\begin{aligned} e^r_{H^r_{I^r}}(x, y) &= \bigvee_{A\in H^r_{I^r}}(A(y) \oslash A(x)) \\ &= \bigvee_{A\in H^r_{I^r}}(A(y) \oslash d^r_{L^X}(A, \perp_x)) \\ &= I^r_{H^r_{I^r}}(\perp_x)(y) = I^r(\perp_x)(y). \end{aligned}$$

□

Corollary 3.8. *Let H^l be a left interior system on X. For each $A \in L^X$. Then the following properties hold.*

(1) $I^l_{H^l}(A) = \bigvee_{C\in H^l}(C \ominus d^l_{L^X}(C,A)) = \bigvee_{i\in\Gamma}\{A_i \mid A_i \leq A, A_i \in H^l\}$.

(2) $I^l_{H^l}$ *is a left interior operator on X.*

(3) $H^l_{I^l_{H^l}} = H^l$, *where* $H^l_{I^l_{H^l}} = \{A \in L^X \mid A = I^l_{H^l}(A)\}$.

(4) *Define* $e^l_{H^l}(x,y) = \bigvee_{A\in H^l}(A(y) \ominus A(x))$. *Then* $e^l_{H^l}$ *is a left distance function.*

(5) *If I^l is a left interior operator on X, then* $I^l_{H^l_{I^l}} = I^l$ *and* $e^l_{H^l_{I^l}}(x,y) = I^l(\perp_x)(y)$ *for any $x,\ y \in X$.*

Theorem 3.9. *Let G^r be a right closure system on X. Then the following properties hold.*

(1) $C^r_{G^r}(A) = \bigwedge_{B\in G^r}(d^r_{L^X}(A,B) \oplus B) = \bigwedge\{B \in G^r \mid A \leq B\}$.

(2) $C^r_{G^r}$ *is a right closure operator on X.*

(3) $G^r_{C^r_{G^r}} = G^r$ *where* $G^r_{C^r_{G^r}} = \{A \in L^X \mid A = C^r_{G^r}(A)\}$.

(4) *Define* $e^r_{G^r}(x,y) = \bigvee_{A\in G^r}(A(y) \oslash A(x))$. *Then* e_{G^r} *is a right distance function.*

(5) *If C^r is a right closure operator on X, then* $C^r_{G^r_{C^r}} = C^r$.

(6) *Define* $H^l_{G^r} = \{n_1(A) \mid A \in G^r\}$. *Then* $H^l_{G^r}$ *is a l interior system. Moreover, define* $H^r_{G^l} = \{n_2(B) \mid B \in G^l\}$. *Then* $H^r_{G^l}$ *is a right interior system. Moreover, for any $x,\ y \in X$,*

$$\begin{array}{ll} e^l_{H^l_{G^r}}(x,y) & = \bigvee_{B\in H^l_{G^r}}(B(y) \ominus B(x)) = \bigvee_{A\in G^r}(n_1(A)(y) \ominus n_1(A)(x)) \\ & = n_1(C^r_{G^r}(n_2(\perp_x))(y)), \end{array}$$

$$\begin{array}{ll} e^r_{H^r_{G^l}}(x,y) & = \bigvee_{B\in H^r_{G^l}}(B(y) \oslash B(x)) = \bigvee_{A\in G^l}(n_2(A)(y) \oslash n_2(A)(x)) \\ & = n_2(C^l_{G^l}(n_1(\perp_x))(y)). \end{array}$$

(7) *For each* $A \in L^X$, $n_1(C^r_{G^r}(A)) = I^l_{H^l_{G^r}}(n_1(A))$ *Moreover,* $A \in G^r_{C^r_{G^r}}$ *iff* $n_1(A) \in H^l_{I^l_{H^l_{G^r}}}$.

Proof. (1) Put $C_1(A) = \bigwedge_{i\in\Gamma}\{A_i \mid A \leq A_i, A_i \in G^r\}$. Since $A \leq \bigwedge_{C\in G^r}(d^r_{L^X}(A,C) \oplus C)$ and $\bigwedge_{C\in G^r}(d^r_{L^X}(A,C) \oplus C) \in G^r$, $C_1(A) \leq C_{G^r}(A)$. Since $C_1(A) \in G^r$, $C^r_{G^r}(A) \leq d^r_{L^X}(A, C_1(A)) \oplus C_1(A) = C_1(A)$. Then we get

$$C^r_{G^r}(A) = \bigwedge_{i\in\Gamma}\{A_i \mid A \leq A_i, A_i \in G^r\}.$$

(2) (C1) By the definition of $C^r_{G^r}$, it easily proved.

(C2) For each $A \in L^X$, we have

$$\begin{array}{ll} C^r_{G^r}(A) & = \bigwedge_{i\in\Gamma}\{A_i \mid A \leq A_i, A_i \in G^r\} \\ & \geq \bigwedge_{i\in\Gamma}\{A_i \mid C^r_{G^r}(A) \leq A_i = C^r_{G^r}(A_i), A_i \in G^r\} \\ & \geq \bigwedge_{i\in\Gamma}\{B_i \mid C^r_{G^r}(A) \leq B_i, B_i \in G^r\} \\ & = C^r_{G^r}(C^r_{G^r}(A)). \end{array}$$

By (C1), $C^r_{G^r}(A) = C^r_{G^r}(C^r_{G^r}(A))$.

(RI) Since $x \oplus \bigwedge_{i\in\Gamma} y_i \leq \bigwedge_{i\in\Gamma}(x \oplus y_i)$ and $\bigwedge_{i\in\Gamma}(x \oplus y_i) \ominus \bigwedge_{i\in\Gamma} y_i \leq \bigvee_{i\in\Gamma}((x \oplus y_i) \ominus y_i) \leq x$ iff $\bigwedge_{i\in\Gamma}(x \oplus y_i) \leq x \oplus \bigwedge_{i\in\Gamma} y_i$, we have $x \oplus \bigwedge_{i\in\Gamma} y_i = \bigwedge_{i\in\Gamma}(x \oplus y_i)$.

For each $A \in L^X$, $\alpha \in L$, we have

$$\begin{aligned}\alpha \oplus C^r_{G^r}(A) &= \alpha \oplus \textstyle\bigwedge_{i\in\Gamma}\{A_i \mid A \leq A_i, A_i \in G^r\}\\ &\geq \textstyle\bigwedge_{i\in\Gamma}\{\alpha \oplus A_i \mid \alpha \oplus A \leq \alpha \oplus A_i, \alpha \oplus A_i \in G^r\}\\ &\geq \textstyle\bigwedge_{i\in\Gamma}\{B_i \mid \alpha \oplus A \leq B_i, B_i \in G^r\}\\ &= C^r_{G^r}(\alpha \oplus A).\end{aligned}$$

Since $C^r_{G^r}(A) \in G^r$, $C^r_{G^r}(A) = C^r_{G^r}(C^r_{G^r}(A))$ from:

$$C^r_{G^r}(C^r_{G^r}(A)) \quad \leq d^r_{L^X}(C^r_{G^r}(A), C^r_{G^r}(A)) \oplus C^r_{G^r}(A) = C^r_{G^r}(A).$$

Then $C^r_{G^r}$ is a right closure operator on X.

(3), (4) The proofs are similar to Theorem 3.7 (3) and (4), respectively.

(5) For any $A,\ B \in L^X$, since $d^r_{L^X}(A, B) \geq d^r_{L^X}(C^r(A), C^r(B))$,

$$\begin{aligned}C^r_{G^r_{C^r}}(A) &= \textstyle\bigwedge_{i\in\Gamma}\{A_i \mid A \leq A_i, A_i \in G^r_{C^r}\}\\ &\geq \textstyle\bigwedge_{i\in\Gamma}\{A_i \mid C^r(A) \leq C^r(A_i) = A_i, A_i \in G^r_{C^r}\}\\ &\geq C^r(A).\end{aligned}$$

$$\begin{aligned}C^r_{G^r_{C^r}}(B)(x) &= \textstyle\bigwedge_{A\in G^r_{C^r}}(d^r_{L^X}(B, A) \oplus A(x))\\ &\leq d^r_{L^X}(B, C^r(B)) \oplus C^r(B)(x) = C^r(B)(x).\end{aligned}$$

(6) If $A = n_1(n_2(A)) \in H^l_{G^r}$, then $n_2(A) \in G^r$ and $k \oplus n_2(A) \in G^r$. Thus $n_1(k \oplus n_2(A)) = A \ominus k \in H^l_{G^r}$.

If $A_i = n_1(n_2(A_i)) \in H^l_{G^r}$ for each $i \in \Gamma$, then $n_2(A_i) \in G^r$ and $\bigwedge_{i\in\Gamma} n_2(A_i) \in G^r$. Thus $n_1(\bigwedge_{i\in\Gamma} n_2(A_i)) = n_1(n_2(\bigvee_{i\in\Gamma} A_i)) = \bigvee_{i\in\Gamma} A_i \in H^l_{G^r}$. So $H^l_{G^r}$ is a left interior system. Similarly, $H^r_{G^l}$ is a right interior system.

Moreover, for any $x,\ y \in X$, by Lemma 2.2,

$$\begin{aligned}e^l_{H^l_{G^r}}(x, y) &= \textstyle\bigvee_{B\in H^l_{G^r}}(B(y) \ominus B(x)) = \bigvee_{A\in G^r}(n_1(A)(y) \ominus n_1(A)(x))\\ &= \textstyle\bigvee_{A\in G^r}(n_1(A)(y) \ominus d^l_{L^X}(n_1(A), \perp_x))\\ &= n_1(\textstyle\bigwedge_{A\in G^r}(d^r_{L^X}(n_2(\perp_x), A) \oplus A(y)))\\ &= n_1(C^r_{G^r}(n_2(\perp_x))(y)),\end{aligned}$$

$$\begin{aligned}e^r_{H^r_{G^l}}(x, y) &= \textstyle\bigvee_{B\in H^r_{G^l}}(B(y) \oslash B(x)) = \bigvee_{A\in G^l}(n_2(A)(y) \oslash n_2(A)(x))\\ &= \textstyle\bigvee_{A\in G^l}(n_2(A)(y) \oslash d^r_{L^X}(n_2(A), \perp_x))\\ &= n_2(\textstyle\bigvee_{A\in G^l}(A(y) \oplus d^l_{L^X}(n_1(\perp_x), A)))\\ &= n_2(C^l_{G^l}(n_1(\perp_x))(y)).\end{aligned}$$

(7) For each $A \in L^X$,

$$\begin{aligned}n_1(C^r_{G^r}(A)) &= n_1\Big(\textstyle\bigwedge_{B\in G^r} d^r_{L^X}(A, B) \oplus B\Big) = \bigvee_{B\in G^r}\Big(n_1(B) \ominus d^r_{L^X}(A, B)\Big)\\ &= \textstyle\bigvee_{B\in G^r}\Big(n_1(B) \ominus d^l_{L^X}(n_1(B), n_1(A))\Big)\\ &= \textstyle\bigvee_{C\in H^l_{G^r}}\Big(C \ominus d^l_{L^X}(C, n_1(A))\Big)\\ &= I^l_{H^l_{G^r}}(n_1(A)).\end{aligned}$$

Moreover, $A \in G^r_{C_{G^r}}$ iff $n_1(A) = n_1 C^r_{G^r}(A) = I^l_{H^l_{G^r}}(n_1(A))$ iff $n_1(A) \in H^l_{I^l_{H^l_{G^r}}}$. □

Corollary 3.10. *Let G^l be a left closure system on X. Then the following properties hold.*

(1) $C^l_{G^l}(A) = \bigwedge_{B\in G^l}(B \oplus d^l_{L^X}(A,B)) = \bigwedge\{B \in G^l \mid A \le B\}$.

(2) $C^l_{G^l}$ *is a left closure operator on X.*

(3) $G^l_{C^l_{G^l}} = G^l$ *where* $G^l_{C^l_{G^l}} = \{A \in L^X \mid A = C^l_{G^l}(A)\}$.

(4) *If C^l is a left closure operator on X, then* $C^l_{G^l_{C^l}} = C^l$.

(5) *Define* $e^l_{G^l}(x,y) = \bigvee_{A\in G^l}(A(y) \ominus A(x))$. *Then $e^l_{G^l}$ is a left distance function.*

(6) *If C^l is a left closure operator on X, then* $C^l_{G^l_{C^l}} = C^l$.

(7) *For each $A \in L^X$,* $n_2(C^l_{G^l}(A)) = I^r_{H^r_{G^l}}(n_1(A))$ *Moreover,* $A \in G^l_{C^l_{G^l}}$ *iff* $n_1(A) \in H^r_{I^r_{H^r_{G^l}}}$.

Corollary 3.11. *Let K^r be a right extensional system on X. Then the following properties hold.*

(1) $I^r_{K^r}$ *is a right interior operator on X such that* $I^r_{K^r}(A) = \bigvee_{C\in K^r}(C \oslash d^r_{L^X}(C,A)) = \bigvee_{i\in\Gamma}\{A_i \mid A_i \le A, A_i \in K^r\}$.

(2) $C^r_{K^r}$ *is a right closure operator on X such that* $C^r_{K^r}(A) = \bigwedge_{B\in K^r}(d^r_{L^X}(A,B)\oplus B) = \bigwedge\{B \in K^r \mid A \le B\}$.

(3) *For* $I^r \in \{I^r, C^r\}$, $K^r_{I^r_{K^r}} = K^r$, *where* $K^r_{I^r_{K^r}} = \{A \in L^X \mid A = I^r_{K^r}(A)\}$.

Corollary 3.12. *Let K^l be a left extensional system on X. Then the following properties hold.*

(1) $I^l_{K^l}$ *is a left interior operator on X such that* $I^l_{K^l}(A) = \bigvee_{C\in K^l}(C\ominus d^l_{L^X}(C,A)) = \bigvee_{i\in\Gamma}\{A_i \mid A_i \le A, A_i \in K^l\}$.

(2) $C^l_{K^l}$ *is a left closure operator on X such that* $C^l_{K^l}(A) = \bigwedge_{B\in K^l}(B\oplus d^l_{L^X}(A,B)) = \bigwedge\{B \in K^l \mid A \le B\}$.

(3) *For* $I^l \in \{I^l, C^l\}$, $K^l_{I^l_{K^l}} = K^l$, *where* $K^l_{I^l_{K^l}} = \{A \in L^X \mid A = I^l_{K^l}(A)\}$.

Theorem 3.13. *Let d^r_X be a right distance function on X and* $K^r_{d^r_X} = \{A \in L^X \mid A(x) \oplus d^r_X(x,y) \ge A(y)\}$ *be a family on X. Then the following properties hold.*

(1) $K^r_{d^r_X}$ *is a right extensional system on X.*

(2) $K^r_{d^r_X} = \{\bigvee_{x\in X}(A(x) \ominus d^r_X(-,x) \mid A \in L^X\} = \{\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-) \mid A \in L^X\}$.

(3) $I^r_{K^r_{d^r_X}}(A) = \bigvee\{B \in K^r_{d^r_X} \mid B \le A\} = \bigwedge_{y\in X}(A(y) \oplus d^r_X(y,-))$.

(4) $C^r_{K^r_{d^r_X}}(A) = \bigwedge\{B \in K^r_{d^r_X} \mid A \le B\} = \bigvee_{y\in X}(A(y) \ominus d^r_X(-,y))$.

(5) *Define* $e^r_{K^r_{d^r_X}}(x,y) = \bigvee_{A\in K^r_{d^r_X}}(A(y)\oslash A(x))$ *for each $x, y \in X$ with* $e^r_{K^r_{d^r_X}} = d^r_X$.

(6) *Let K^r be a right extensional system on X. Define* $e^r_{K^r}(x,y) = \bigvee_{A\in K^r}(A(y)\oslash A(x))$. *Then $e^r_{K^r}$ is a right distance function with* $K^r = K^r_{e^r_{K^r}}$, $I^r_{K^r_{d^r_X}}(A) = I^r_{K^r}(A)$ *and* $C^r_{K^r_{d^r_X}}(A) = C^r_{K^r}(A)$.

(7) *Let H^r be a right interior system on X. Then* $K^r_{e^r_{H^r}} = \{\bigvee_{x\in X}(A(x) \ominus e^r_{H^r}(-,x) \mid A \in L^X\} = \{\bigwedge_{x\in X}(A(x) \oplus e^r_{H^r}(x,-) \mid A \in L^X\}$ *is the smallest right extensional system on X containing H^r. Moreover,* $e^r_{K^r_{e^r_{H^r}}} = e^r_{H^r}$.

(8) *Let G^r be a right closure system on X. Then* $K^r_{e^r_{G^r}} = \{\bigvee_{x\in X}(A(x) \ominus e^r_{G^r}(-,x) \mid A \in L^X\} = \{\bigwedge_{x\in X}(A(x) \oplus e^r_{G^r}(x,-) \mid A \in L^X\}$ *is the smallest right extensional system on X containing G^r. Moreover,* $e^r_{K^r_{e^r_{G^r}}} = e^r_{G^r}$.

Proof. (1) For each $A \in K^r_{d^r_X}$ and $\alpha \in L$, $(A \oslash \alpha)$, $(\alpha \oplus A) \in K^r_{d^r_X}$ from: $(\alpha \oplus (A(x) \oslash \alpha)) \oplus d^r_X(x,y) \geq A(x) \oplus d^r_X(x,y) \geq A(y)$ iff $(A(x) \oslash \alpha) \oplus d^r_X(x,y) \geq (A(y) \oslash \alpha)$ iff $(\alpha \oplus A(x)) \oplus d^r_X(x,y) \geq \alpha \oplus A(y)$.

(2) Put $H = \{\bigvee_{x\in X}(A(x) \ominus d^r_X(-,x) \mid A \in L^X\}$, $G = \{\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-) \mid A \in L^X\}$. Let $\bigvee_{x\in X}(A(x) \ominus d^r_X(-,x) \in H$. Then

$$(A(x) \ominus d^r_X(z,x)) \oplus d^r_X(z,y) \oplus d^r_X(y,x) \geq (A(x) \ominus d^r_X(z,x)) \oplus d^r_X(z,x) \geq A(x).$$

Thus $\bigvee_{x\in X}(A(x) \ominus d^r_X(z,x)) \oplus d^r_X(z,y) \geq \bigvee_{x\in X}(A(x) \ominus d^r_X(y,x))$. So $\bigvee_{x\in X}(A(x) \ominus d^r_X(-,x) \in K^r_{d^r_X}$. Hence $H \subset K^r_{d^r_X}$.

Let $A \in K^r_{d^r_X}$. Then $A(x) \oplus d^r_X(x,y) \geq A(y)$. Thus $A(x) \geq \bigvee_{y\in X}(A(y) \ominus d^r_X(x,y)) \geq A(x)$. So $A = \bigvee_{y\in X}(A(y) \ominus d^r_X(-,y)) \in H$. Hence $K^r_{d^r_X} \subset H$.

Let $\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-) \in G$. Then

$$\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,y)) \oplus d^r_X(y,z) \geq \bigwedge_{x\in X}(A(x) \oplus d^r_X(x,z)).$$

Thus $\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-) \in K^r_{d^r_X}$. So $G \subset K^r_{d^r_X}$.

Let $A \in K^r_{d^r_X}$. Then $\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,y)) = A(y)$. Thus $A = \bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-)) \in G$. So $K^r_{d^r_X} \subset G$.

(3) By Corollary 3.11 (1), $I^r_{K^r_{d^r_X}}(A) = \bigvee\{B \in K^r_{d^r_X} \mid B \leq A\}$. Since $\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-)) \in K^r_{d^r_X}$ and $\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-)) \leq A$, $\bigwedge_{x\in X}(A(x) \oplus d^r_X(x,-)) \leq I^r_{K^r_{d^r_X}}(A)$. Since $I^r_{K^r_{d^r_X}}(A) \leq A$ and $I^r_{K^r_{d^r_X}}(A) \in K^r_{d^r_X}$, by (2),

$$I^r_{K^r_{d^r_X}}(A) = \bigwedge_{y\in X}(I^r_{K^r_{d^r_X}}(A)(y) \oplus d^r_X(y,-)) \leq \bigwedge_{y\in X}(A(y) \oplus d^r_X(y,-)).$$

Then $I^r_{K^r_{d^r_X}}(A) = \bigwedge_{y\in X}(A(y) \oplus d^r_X(y,-))$.

(4) By Corollary 3.11 (2), $C^r_{K^r_{d^r_X}}(A) = \bigwedge\{B \in K^r_{d^r_X} \mid A \leq B\}$. Since $(A(y) \ominus d^r_X(x,y)) \oplus d^r_X(x,z) \oplus d^r_X(z,y) \geq A(y)$, $\bigvee_{y\in X}(A(y) \ominus d^r_X(-,y))) \in K^r_{d^r_X}$. Since $A \leq \bigvee_{y\in X}(A(y) \ominus d^r_X(-,y)))$, $C^r_{K^r_{d^r_X}}(A) \leq \bigvee_{y\in X}(A(y) \ominus d^r_X(-,y))$. Since $A \leq C^r_{K^r_{d^r_X}}(A)$ and $C^r_{K^r_{d^r_X}}(A) \in K^r_{d^r_X}$, by (2),

$$C^r_{K^r_{d^r_X}}(A) = \bigvee_{y\in X}(C^r_{K^r_{d^r_X}}(A)(y) \ominus d^r_X(-,y)) \geq \bigvee_{y\in X}(A(y) \ominus d^r_X(-,y)).$$

Then $C^r_{K^r_{d^r_X}}(A) = \bigvee_{y\in X}(A(y) \ominus d^r_X(-,y))$.

(5) For all $x,\ y \in X$,

$$\begin{aligned} e^r_{K^r_{d^r_X}}(x,y) &= \bigvee_{A\in L^X}(\bigvee_{z\in X}(A(z) \ominus d^r_X(y,z)) \oslash \bigvee_{w\in X}(A(w) \ominus d^r_X(x,w))) \\ &\geq \bigvee_{z\in X}(d^r_X(x,z) \ominus d^r_X(y,z)) \text{ (put } A = d^r_X(x,-)) \\ &\oslash \bigvee_{w\in X}(d^r_X(x,w) \ominus d^r_X(x,w))) \\ &= \bigvee_{z\in X}(d^r_X(x,z) \ominus d^r_X(y,z)) = d^r_X(x,y). \end{aligned}$$

By Lemma 2.2 (11) and (14),

$$\begin{aligned} e^r_{K^r_{d^r_X}}(x,y) &= \bigvee_{A\in L^X}(\bigvee_{z\in X}(A(z)\ominus d^r_X(y,z))\oslash \bigvee_{w\in X}(A(w)\ominus d^r_X(x,w))) \\ &\leq \bigvee_{A\in L^X}(\bigvee_{z\in X}(A(z)\ominus d^r_X(y,z))\oslash (A(z)\ominus d^r_X(x,z))) \\ &\leq \bigvee_{z\in X}(d^r_X(x,z)\ominus d^r_X(y,z)) = d^r_X(x,y). \end{aligned}$$

(6) Let $A\in K^r$. Then we get

$$A(x)\oplus \bigvee_{B\in K^r}(B(y)\oslash B(x)) \geq A(x)\oplus (A(y)\oslash A(x)) \geq A(y).$$

Thus $A\in K^r_{e^r_{K^r}}$.

Let $A\in K^r_{e^r_{K^r}}$. Then we have

$$\bigwedge_{x\in X}(A(x)\oplus e^r_{K^r}(x,y)) \geq A(y)$$

and

$$\bigwedge_{x\in X}(A(x)\oplus e^r_{K^r}(x,y)) \leq A(y)\oplus e^r_{K^r}(y,y) = A(y).$$

Thus $A=\bigwedge_{x\in X}(A(x)\oplus e^r_{K^r}(x,-)) = \bigwedge_{x\in X}(A(x)\oplus \bigvee_{B\in K^r}(B(-)\oslash B(x)))$. Since $\bigvee_{B\in K^r}(B(-)\oslash B(x))\in K^r$, $A\in K^r$. So $K^r=K^r_{e^r_{K^r}}$. Moreover,

$$\begin{aligned} I^r_{K^r}(A) &= \bigvee_{i\in\Gamma}\{A_i \mid A_i\leq A, A_i\in K^r\} = I^r_{K^r_{d^r_X}}(A), \\ C^r_{K^r}(A) &= \bigwedge_{i\in\Gamma}\{A_i \mid A\leq A_i, A_i\in K^r\} = C^r_{K^r_{d^r_X}}(A). \end{aligned}$$

(7) It is obvious that $K^r_{e^r_{H^r}}=\{B\in L^X \mid B(x)\oplus e^r_{H^r}(x,y)\geq B(y)\}$ is a right extensional system such that $H^r\subset K^r_{e^r_{H^r}}$. By (2), we get

$$K^r_{e^r_{H^r}} = \{\bigvee_{x\in X}(B(x)\ominus e^r_{H^r}(-,x) \mid B\in L^X\} = \{\bigwedge_{x\in X}(B(x)\oplus e^r_{H^r}(x,-) \mid B\in L^X\}.$$

Let K^r be a right extensional system such that $H^r\subset K^r$. For each $\bigwedge_{x\in X}(B(x)\oplus e^r_{H^r}(x,-)\in K^r_{e^r_{H^r}}$, $e^r_{H^r}(x,-)=\bigvee_{A\in H^r}(A(-)\oslash A(x))\in K^r$ and $B(x)\oplus e^r_{H^r}(x,-)\in K^r$. Then $\bigwedge_{x\in X}(B(x)\oplus e^r_{H^r}(x,-)\in K^r$. Thus $K^r_{e^r_{H^r}}\subset K^r$. So $K^r_{e^r_{H^r}}$ is the smallest right extensional system such that $H^r\subset K^r$. Hence by (5), $e^r_{K^r_{e^r_{H^r}}}=e^r_{H^r}$.

(8) It is similarly proved as in that of (7). □

Corollary 3.14. *Let d^l_X be a left distance function on X and $K^l_{d^l_X}=\{A\in L^X \mid d_X(x,y)\oplus A(x)\geq A(y)\}$ be a family on X. Then the following properties hold.*

(1) *$K^l_{d^l_X}$ is a left extensional system.*

(2) *$K^l_{d^l_X}=\{\bigvee_{x\in X}(A(x)\oslash d^l_X(-,x) \mid A\in L^X\}=\{\bigwedge_{x\in X}(d^l_X(x,-)\oplus A(x)) \mid A\in L^X\}$.*

(3) *$I^l_{K^l_{d^l_X}}(A)=\bigvee\{B\in K^l_{d^l_X} \mid B\leq A\}=\bigwedge_{x\in X}(d^l_X(x,-)\oplus A(x))$.*

(4) *$C^l_{K^l_{d^l_X}}(A)=\bigwedge\{B\in K^l_{d^l_X} \mid A\leq B\}=\bigvee_{y\in X}(A(y)\oslash d^l_X(-,y))$.*

(5) *Define $e^l_{K^l_{d^l_X}}(x,y)=\bigvee_{A\in K^l_{d^l_X}}(A(y)\ominus A(x))$ for each $x,y\in X$ with $e^l_{K^l_{d^l_X}}=d^l_X$.*

(6) *Let* K^l *be a left extensional system on* X*. Define* $e^l_{K^l}(x,y) = \bigvee_{A\in K^l}(A(y)\ominus A(x))$*. Then* $e^l_{K^l}$ *is a left distance function with* $K^l = K^l_{e^l_{K^l}}$*. Moreover,* $I^l_{K^l_{d^l_X}}(A) = I^l_{K^l}(A)$ *and* $C^l_{K^l_{d^l_X}}(A) = C^l_{K^l}(A)$*.*

(7) *Let* H^l *be a left interior system on* X*. Then* $K^l_{e^l_{H^l}} = \{\bigvee_{x\in X}(A(x)\oslash e^l_{H^l}(-,x) \mid A \in L^X\} = \{\bigwedge_{x\in X}(e^l_{H^l}(x,-)\oplus A(x)) \mid A\in L^X\}$ *is the smallest left extensional system on* X *containing* H^l*.*

(8) *Let* (X,G^l) *be a left closure system. Then* $K^l_{e^l_{G^l}} = \{\bigvee_{x\in X}(A(x)\oslash e^l_{G^l}(-,x) \mid A \in L^X\} = \{\bigwedge_{x\in X}(e^l_{G^l}(x,-)\oplus A(x)) \mid A\in L^X\}$ *is the smallest left extensional system on* X *containing* G^l*.*

Theorem 3.15. *Let* I^r *be a right interior operator on* X*. Then the following properties hold.*

(1) *Define* $e^r_{I^r}(x,y) = \bigvee_{A\in L^X}(I^r(A)(y)\oslash I^r(A)(x))$*. Then* $e^r_{I^r}$ *is a right distance function with* $I^r_{K^r_{e^r_{I^r}}}(A) \geq I^r(A)$*.*

(2) *Define* $d^r_{I^r}(x,y) = I^r(\perp_x)(y)$ *for all* $x,\ y\in X$*. Then* $d^r_{I^r}$ *is a right distance function with* $d^r_{I^r} = e^r_{H^r_{I^r}} = e^r_{I^r}$ *and* $I^r_{K^r_{d^r_{I^r}}}(A) \geq I^r(A)$*. Moreover, if* $I^r(\bigwedge_{i\in I}A_i) = \bigwedge_{i\in I}I^r(A_i)$ *and* $I^r(\alpha\oplus A) = \alpha\oplus I^r(A)$ *for all* $A,\ A_i\in L^X, \alpha\in L$*, then* $I^r_{K^r_{d^r_{I^r}}}(A) = I^r(A)$ *for each* $A\in L^X$*.*

(3) *Define* $C^l_{I^r}(A) = n_1(I^r(n_2(A)))$ *for each* $A\in L^X$*. Then* $C^l_{I^r}$ *is a left closure operator such that* $d^l_{C^l_{I^r}}(x,y) = I^r(\perp_y)(x) = d^r_{I^r}(y,x)$ *for any* $x,\ y\in X$*.*

Proof. (1) Since $I^r = I^r_{H^r_{I^r}}$ from Theorem 3.7 (5),

$$e^r_{I^r}(x,y) \quad = \bigvee_{A\in L^X}(I^r(A)(y)\oslash I^r(A)(x)) = \bigvee_{B\in H^r_{I^r}}(B(y)\oslash B(x)) = e^r_{H^r_{I^r}}(x,y).$$

By Theorem 3.7 (4), $e^r_{I^r}$ is a right distance function on X. By Theorem 3.13 (3),

$$\begin{aligned} I^r_{K^r_{e^r_{I^r}}}(A)(y) = \bigwedge_{x\in X}(A(x)\oplus e^r_{I^r}(x,y)) \quad &\geq \bigwedge_{x\in X}(I^r(A)(x)\oplus(I^r(A)(y)\oslash I^r(A)(x)) \\ &\geq I^r(A)(y). \end{aligned}$$

(2) Since $A = \bigwedge_{y\in X}(A(y)\oplus\perp_y)$, $I^r(\perp_x) = \bigwedge_{y\in X}(I^r(\perp_x)(y)\oplus\perp_y)$. For any $x,\ y,\ z\in X$,

$$\begin{aligned} d^r_{I^r}(x,z) = I^r(\perp_x)(z) = I^r(I^r(\perp_x))(z) \quad &= I^r(\bigwedge_{y\in X}I^r(\perp_x)(y)\oplus\perp_y)(z) \\ &\leq \bigwedge_{y\in X}I^r(\perp_x)(y)\oplus I^r(\perp_y)(z) \\ &= \bigwedge_{y\in X}(d^r_{I^r}(x,y)\oplus d^r_{I^r}(y,z)). \end{aligned}$$

Since $I^r = I^r_{H^r_{I^r}}$ from Theorem 3.7 (5),

$$\begin{aligned} d^r_{I^r}(x,y) \quad &= I^r(\perp_x)(y) = I^r_{H^r_{I^r}}(\perp_x)(y) = \bigvee_{C\in H^r_{I^r}}(C(y)\oslash d^r_{L^X}(C,\perp_x)) \\ &= \bigvee_{C\in H^r_{I^r}}(C(y)\oslash C(x)) = e^r_{H^r_{I^r}}(x,y) \\ &= \bigvee_{A\in L^X}(I^r(A)(y)\oslash I^r(A)(x)) = e^r_{I^r}(x,y). \end{aligned}$$

Since $\alpha\oplus I^r(A)\geq I^r(\alpha\oplus A)$,

$$\begin{aligned} I^r_{K^r_{d_{I^r}}}(A)(y) \quad &= \bigwedge_{x\in X}(A(x)\oplus d^r_{I^r}(x,y)) = \bigwedge_{x\in X}(A(x)\oplus I^r(\perp_x)(y)) \\ &\geq I^r(\bigwedge_{x\in X}(A(x)\oplus\perp_x))(y) = I^r(A)(y). \end{aligned}$$

Assume that $I^r(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} I^r(A_i)$ and $I^r(\alpha \oplus A) = \alpha \oplus I^r(A)$ for all $A_i,\ A \in L^X$. Since $A = \bigwedge_{x\in X}(A(x) \oplus \bot_x)$,

$$\begin{aligned} I^r_{K^r_{d_{I^r}}}(A)(y) &= \bigwedge_{x\in X}(A(x) \oplus d^r_{I^r}(x,y)) = \bigwedge_{x\in X}(A(x) \oplus I^r(\bot_x)(y)) \\ &= I^r(\bigwedge_{x\in X}(A(x) \oplus \bot_x))(y) = I^r(A)(y). \end{aligned}$$

(3) The operator $C^l_{I^r}$ is a left closure operator from:

$$\begin{aligned} C^l_{I^r}(A) &= n_1(I^r(n_2(A))) \geq n_1(n_2(A)) = A, \\ C^l_{I^r}(C^l_{I^r}(A)) &= C^l_{I^r}(n_1(I^r(n_2(A)))) = n_1(I^r(I^r(n_2(A)))) = n_1(I^r(n_2(A))) \\ &= C^l_{I^r}(A), \end{aligned}$$

$$\begin{aligned} C^l_{I^r}(A \oplus \alpha) = n_1(I^r(n_2(A \oplus \alpha))) &= n_1(I^r(n_2(A) \oslash \alpha)) \\ &\leq n_1(I^r(n_2(A)) \oslash \alpha) \\ &= n_1(I^r(n_2(A))) \oplus \alpha = C^l_{I^r}(A) \oplus \alpha. \end{aligned}$$

Moreover, for any $x,\ y \in X$,

$$\begin{aligned} d^l_{C^l_{I^r}}(x,y) &= n_2(C^l_{I^r}(n_1(\bot_y))(x) \\ &= n_2(n_1(I^r(n_2(n_1(\bot_y))))(x) \\ &= I^r(\bot_y)(x) \\ &= d^r_{I^r}(y,x). \end{aligned}$$

□

Theorem 3.16. *Let C^r be a right closure operator on X. Then the following properties hold.*

(1) *Define $e^r_{C^r}(x,y) = \bigvee_{A\in L^X}(C^r(A)(y) \oslash C^r(A)(x))$. Then $e^r_{C^r}$ is a right distance function on X with $C^r_{K^r_{e^r_{C^r}}}(A) \leq C^r(A)$.*

(2) *Define $d^r_{C^r}(x,z) = n_1(C^r(n_2(\bot_z))(x))$ for all $x, y \in X$. Then $d^r_{C^r}$ is a right distance function on X with $d^r_{C^r}(x,y) = e^l_{H^l_{G^r_{C^r}}}(y,x)$ for any $x,\ y \in X$, where $H^l_{G^r_{C^r}} = \{n_1(A) \mid A \in G^r_{C^r}\}$.*

(3) *Define $I^l_{C^r}(A) = n_1(C^r(n_2(A)))$ for each $A \in L^X$. Then $I^l_{C^r}$ is a left interior operator with $d^l_{I^l_{C^r}}(x,y) = d^r_{C^r}(y,x) = e^l_{H^l_{G^r_{C^r}}}(x,y) = e^l_{I^l_{C^r}}(x,y)$ for any $x,\ y \in X$.*

(4) *$C^r_{K^r_{d^r_{C^r}}}(A) \leq n_1(I^l_{C^r}(n_2(A)))$.*

Moreover, if $C^r(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} C^r(A_i)$ and $C^r(A \oslash \alpha) = C^r(A) \oslash \alpha$ for all $A_i,\ A \in L^X$, then $C^r_{K^r_{d^r_{C^r}}}(A) = n_1(I^l_{C^r}(n_2(A)))$ for each $A \in L^X$.

(5) *Let $d^{-r}_X(x,y) = d_X(y,x)$ for each $r \in \{r,l\}, x, y \in X$. Then $B \in K^r_{d^r_X}$ iff $n_2(B) \in K^l_{d^{-r}_X}$. Similarly, $B \in K^l_{d^l_X}$ iff $n_1(B) \in K^r_{d^{-l}_X}$.*

Proof. (1) Since $C^r = C^r_{G^r_{C^r}}$ from Theorem 3.9 (5),

$$\begin{aligned} e^r_{C^r}(x,y) &= \bigvee_{A\in L^X}(C^r(A)(y) \oslash C^r(A)(x)) \\ &= \bigvee_{B\in G^r_{C^r}}(B(y) \oslash B(x)(x)) = e^r_{G^r_{C^r}}(x,y). \end{aligned}$$

By Theorem 3.9 (4), $e^r_{C^r}$ is a right distance function on X. Moreover,

$$\begin{aligned} C^r_{K^r_{e^r_{C^r}}}(A)(y) &= \bigvee_{x\in X}(A(x)\ominus e^r_{C^r}(y,x)) \\ &\leq \bigvee_{x\in X}(C^r(A)(x)\ominus (C^r(A)(x)\oslash C^r(A)(y))) \\ &\leq C^r(A)(y). \end{aligned}$$

(2) Since $A=\bigwedge_{x\in X}(\perp_x\oplus A(x))$, $n_2(A)=\bigvee_{x\in X}(n_2(\perp_x)\oslash A(x))$. Then we get

$$n_2(n_1(A)) = A = \bigvee_{x\in X}(n_2(\perp_x)\oslash n_1(A)(x)).$$

Thus $C^r(n_2(\perp_x))=\bigvee_{y\in X}(n_2(\perp_y)\oslash n_1(C^r(n_2(\perp_x)))(y))$. Since C^r is a right closure operator on X and $\alpha\oplus(B\oslash\alpha)\geq B$, $C^r(B)\leq C^r(\alpha\oplus(B\oslash\alpha))\leq \alpha\oplus C^r(B\oslash\alpha)$. Thus $C^r(A)\oslash\alpha\leq C^r(A\oslash\alpha)$. It follows

$$\begin{aligned} C^r(n_2(\perp_z))(x)=C^r(C^r(n_2(\perp_z)))(x) &= C^r(\bigvee_{y\in X}(n_2(\perp_y)\oslash n_1(C^r(n_2(\perp_z)))(y)))(x) \\ &\geq \bigvee_{y\in X}(C^r(n_2(\perp_y))(x)\oslash n_1(C^r(n_2(\perp_z)))(y)). \end{aligned}$$

So we have

$$\begin{aligned} d^r_{C^r}(x,z)=n_1(C^r(n_2(\perp_z))(x)) &\leq n_1(\bigvee_{y\in X}(C^r(n_2(\perp_y))(x)\oslash n_1(C^r(n_2(\perp_z)))(y))) \\ &= \bigwedge_{y\in X}(n_1(C^r(n_2(\perp_y)))(x)\oplus n_1(C^r(n_2(\perp_z)))(y))) \\ &= \bigwedge_{y\in X}(d^r_{C^r}(x,y)\oplus d^r_{C^r}(y,z)). \end{aligned}$$

Hence $d^r_{C^r}$ is a right distance function on X. Moreover, for any $x,\ y\in X$,

$$\begin{aligned} d^r_{C^r}(x,y)=d^r_{C^r_{G^r_{C^r}}}(x,y) &= n_1(C^r_{G^r_{C^r}}(n_2(\perp_y)))(x) \\ &= n_1(\bigwedge_{D\in G^r_{C^r}}(d^r_{L^X}(n_2(\perp_y),D)\oplus D(x)) \\ &= \bigvee_{D\in G^r_{C^r}}(n_1(D)(x)\ominus d^r_{L^X}(n_2(\perp_y),D)) \\ &= \bigvee_{D\in G^r_{C^r}}(n_1(D)(x)\ominus d^l_{L^X}(n_1(D),\perp_y)) \\ &= \bigvee_{D\in G^r_{C^r}}(n_1(D)(x)\ominus n_1(D)(y)) \\ &= \bigvee_{B\in H^l_{G^r_{C^r}}}(B(x)\ominus B(y)) \\ &= e^l_{H^l_{G^r_{C^r}}}(y,x). \end{aligned}$$

(3) As a similar method in Theorem 3.13 (3), $I^l_{C^r}$ is a left interior operator on X. Moreover, for any $x,\ y\in X$,

$$d^l_{I^l_{C^r}}(x,y)=I^l_{C^r}(\perp_x)(y)=n_1(C^r(n_2(\perp_x))(y)=d^r_{C^r}(y,x),$$

$$\begin{aligned} e^l_{H^l_{G^r_{C^r}}}(x,y) &= \bigvee_{D\in G^r_{C^r}}(n_1(D)(y)\ominus n_1(D)(x)) \\ &= \bigvee_{B\in L^X}(n_1(C^r(B))(y)\ominus n_1(C^r(B))(x)) \\ &= \bigvee_{B\in L^X}(I^l_{C^r}(n_1(B))(y)\ominus I^l_{C^r}(n_1(B))(x)) \\ &= \bigvee_{A\in L^X}(I^l_{C^r}(A)(y)\ominus I^l_{C^r}(A)(x)) \\ &= e^l_{I^l_{C^r}}(x,y). \end{aligned}$$

(4) For each $A \in L^X,\ y \in X$,

$$\begin{aligned} C^r_{K^r_{d^r_{C^r}}}(A)(y) &= \bigvee_{x\in X}(A(x)\ominus d^r_{C^r}(y,x)) \\ &= \bigvee_{x\in X}(A(x)\ominus n_1(C^r(n_2(\perp_x))(y))) \\ &= \bigvee_{x\in X}(A(x)\ominus I^l_{C^r}(\perp_x)(y)) \\ &= \bigvee_{x\in X} n_1(I^l_{C^r}(\perp_x)(y)\oplus n_2(A)(x)) \\ &\le \bigvee_{x\in X} n_1(I^l_{C^r}(\perp_x\oplus n_2(A)(x))(y)) \\ &= n_1(I^l_{C^r}(\bigwedge_{x\in X}(\perp_x\oplus n_2(A)(x))(y)) \\ &= n_1(I^l_{C^r}(n_2(A))(y). \end{aligned}$$

Assume that $C^r(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} C^r(A_i)$ and $C^r(A\oslash\alpha) = C^r(A)\oslash\alpha$ for all $A_i,\ A\in L^X$. Since $I^l_{C^r}(A) = n_1(C^r(n_2(A)))$ for each $A\in L^X$, we have

$$\begin{aligned} I^l_{C^r}(\bigwedge_{i\in\Gamma} A_i) &= n_1(C^r(n_2(\bigwedge_{i\in\Gamma} A_i))) = n_1(C^r(\bigvee_{i\in\Gamma} n_2(A_i))) \\ &= n_1(\bigvee_{i\in\Gamma} C^r(n_2(A_i))) = \bigwedge_{i\in\Gamma} n_1(C^r(n_2(A_i))) = \bigwedge_{i\in\Gamma} I^l_{C^r}(A_i), \end{aligned}$$

$$\begin{aligned} I^l_{C^r}(A\oplus\alpha) &= n_1(C^r(n_2(A\oplus\alpha))) = n_1(C^r(n_2(A))\oslash\alpha))) \\ &= n_1(C^r(n_2(A)))\oplus\alpha = I^l_{C^r}(A)\oplus\alpha. \end{aligned}$$

Then

$$\begin{aligned} C^r_{K^r_{d^r_{C^r}}}(A)(y) &= \bigvee_{x\in X} n_1(I^l_{C^r}(\perp_x)(y)\oplus n_2(A)(x)) \\ &= \bigvee_{x\in X} n_1(I^l_{C^r}(\perp_x\oplus n_2(A)(x))(y)) \\ &= n_1(I^l_{C^r}(\bigwedge_{x\in X}(\perp_x\oplus n_2(A)(x))(y)) \\ &= n_1(I^l_{C^r}(n_2(A))(y). \end{aligned}$$

(5) By Lemma 2.2, $B\in K^r_{d^r_X}$ iff $n_2(B(x))\oslash d^r_X(x,y)\le n_2(B(y))$ iff $d^{-r}_X(y,x)\oplus n_2(B(y))\ge n_2(B(x))$ iff $n_2(B)\in K^l_{d^{-r}_X}$.

Other case is similarly proved. □

Corollary 3.17. *Let I^l be a left interior operator on X. Then the following properties hold.*

(1) *Define $e^l_{I^l}(x,y) = \bigvee_{A\in L^X}(I^l(A)(y)\ominus I^l(A)(x))$. Then $e^l_{I^l}$ is a left distance function with $I^l_{K^l_{e^l_{I^l}}}(A)\ge I^l(A)$.*

(2) *Define $d^l_{I^l}(x,y) = I^l(\perp_x)(y)$ for all $x,\ y\in X$. Then $d^l_{I^l}$ is a left distance function with $d^l_{I^l} = e^l_{H^l_{I^l}} = e^l_{I^l}$ and $I^l_{K^l_{d^l_{I^l}}}(A)\ge I^l(A)$. Moreover, if $I^l(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} I^l(A_i)$ and $I^l(A\oplus\alpha) = I^l(A)\oplus\alpha$, for all $A,\ A_i\in L^X,\alpha\in L$, then $I^l_{K^l_{d^l_{I^l}}}(A) = I^l(A)$ for each $A\in L^X$.*

(3) *Define $C^r_{I^l}(A) = n_2(I^l(n_1(A)))$ for each $A\in L^X$. Then $C^r_{I^l}$ is a left closure operator such that $d^r_{C^r_{I^l}}(x,y) = I^l(\perp_y)(x) = d^l_{I^l}(y,x)$ for any $x,\ y\in X$.*

Corollary 3.18. *Let C^l be a left closure operator on X. Then the following properties hold.*

(1) *Define $e^l_{C^l}(x,y) = \bigvee_{A\in L^X}(C^l(A)(y)\ominus C^l(A)(x))$. Then $e^l_{C^l}$ is a left distance function on X with $C^l_{K^l_{e^l_{C^l}}}(A)\le C^l(A)$.*

(2) *Define* $d^l_{C^l}(x,y) = n_2(C^l(n_1(\bot_y))(x))$ *for all* $x,\ y \in X$. *Then* $d^l_{C^l}$ *is a left distance function on* X *with* $d^l_{C^l}(x,y) = e^r_{H^r_{G^l_{C^l}}}(y,x)$ *for each* $x,y \in X$, *where* $H^r_{G^l_{C^l}} = \{n_2(A) \mid A \in G^l_{C^l}\}$.

(3) *Define* $I^r_{C^l}(A) = n_2(C^l(n_1(A)))$ *for each* $A \in L^X$. *Then* $I^r_{C^l}$ *is a right interior operator with* $d^r_{I^r_{C^l}}(x,y) = d^l_{C^l}(y,x) = e^r_{H^r_{G^l_{C^l}}}(x,y) = e^r_{I^r_{C^l}}(x,y)$, *for any* $x,\ y \in X$.

(4) $C^l_{K^l_{d^l_{C^l}}}(A) \le n_2(I^r_{C^l}(n_1(A))$.

Moreover, if $C^l(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} C^l(A_i)$ *and* $C^l(A \ominus \alpha) = C^l(A) \ominus \alpha$ *for all* $A_i,\ A \in L^X$, *then* $C^l_{K^l_{d^l_{C^l}}}(A) = n_2(I^r_{C^l}(n_1(A))$ *for each* $A \in L^X$.

Example 3.19. Let $K = \{(x,y) \in R^2 \mid x > 0\}$ be a set and we define an operator $\oplus : K \times K \to K$ as follows: for any $(x_1,y_1),\ (x_2,y_2) \in K$,

$$(x_1,y_1) \oplus (x_2,y_2) = (2x_1x_2, 2x_2y_1 + y_2 - 2x_2).$$

Then $(K,\oplus)$ is a group with $e = (\frac{1}{2},1),\ (x,y)^{-1} = (\frac{1}{4x}, \frac{1-y}{2x} + 1)$.

We define the order $\le$ on K as follows: for $(x_1,y_1),(x_2,y_2) \in K$,

$$(x_1,y_1) \le (x_2,y_2) \quad \Leftrightarrow x_1 < x_2 \ \text{ or } \ x_1 = x_2, y_1 \le y_2.$$

Let $L \subseteq K = \{(x,y) \in R^2 \mid x > 0\}$ be a set. The structure

$$(L, \vee, \wedge, \oplus, \ominus, \oslash, (\tfrac{1}{2},1), (1,0))$$

is a generalized co-residuated lattice with a double negative law where $\bot = (\frac{1}{2},1)$ is the least element and $\top = (1,0)$ is the greatest element from the following statements:

$$\begin{aligned}
(x_1,y_1) \oplus (x_2,y_2) &= (2x_1x_2, 2x_2y_1 + y_2 - 2x_2) \wedge (1,0),\\
(x_1,y_1) \ominus (x_2,y_2) &= ((x_1,y_1) \oplus (x_2,y_2)^{-1}) \vee (\tfrac{1}{2},1)\\
&= ((x_1,y_1) \oplus (\tfrac{1}{4x_2}, \tfrac{1-y_2}{2x_2} + 1)) \vee (\tfrac{1}{2},1)\\
&= (\tfrac{x_1}{2x_2}, 1 + \tfrac{y_1-y_2}{2x_2}) \vee (\tfrac{1}{2},1),\\
(x_1,y_1) \oslash (x_2,y_2) &= ((x_2,y_2)^{-1} \oplus (x_1,y_1)) \vee (\tfrac{1}{2},1)\\
&= (\tfrac{x_1}{2x_2}, y_1 + \tfrac{x_1}{x_2}(1-y_2)) \vee (\tfrac{1}{2},1).
\end{aligned}$$

Then $(x_1,y_1) \oplus (x_2,y_2) \ge (x_3,y_3)$ iff $(x_1,y_1) \ge (x_3,y_3) \ominus (x_2,y_2)$ iff $(x_2,y_2) \ge (x_3,y_3) \oslash (x_2,y_2)$. Furthermore, we have $(x,y) = n_2(n_1(x,y)) = n_1(n_2(x,y))$ from:

$$\begin{aligned}
n_1(x,y) &= (1,0) \ominus (x,y) = (\tfrac{1}{2x}, 1 - \tfrac{y}{2x}),\\
n_2(x,y) &= (1,0) \oslash (x,y) = (\tfrac{1}{2x}, \tfrac{1}{x}(1-y)),\\
n_2(n_1(x,y)) &= (1,0) \oslash (\tfrac{1}{2x}, 1 - \tfrac{y}{2x}) = (x,y),\\
n_1(n_2(x,y)) &= (1,0) \ominus (\tfrac{1}{2x}, \tfrac{1}{x}(1-y)) = (x,y).
\end{aligned}$$

Let $A = \{(\frac{2}{3},y) \mid y \in R\} \subset L$ be given. Then $\bigvee A$ and $\bigwedge A$ do not exist. Thus L is not complete.

Let $A = ((\frac{3}{5},2),(\frac{3}{4},-1),(\frac{1}{2},5)) \in L^X$ on $X = \{a,b,c\}$. Define functions $d^r_A, d^l_A : X \times X \to L$ as $d^r_A(x,y) = A(y) \oslash A(x),\ d^l_A(x,y) = A(y) \ominus A(x)$. Then d^r_A (resp.

d^l_A) is a right (resp. left) distance function such that

$$d^r_A(x,y) = \begin{pmatrix} (\frac{1}{2},1) & (\frac{5}{8},-\frac{9}{4}) & (\frac{1}{2},1) \\ (\frac{1}{2},1) & (\frac{1}{2},1) & (\frac{1}{2},1) \\ (\frac{3}{5},-\frac{14}{5}) & (\frac{3}{4},-7) & (\frac{1}{2},1) \end{pmatrix}$$

$$d^l_A(x,y) = \begin{pmatrix} (\frac{1}{2},1) & (\frac{5}{8},-\frac{3}{2}) & (\frac{1}{2},1) \\ (\frac{1}{2},1) & (\frac{1}{2},1) & (\frac{1}{2},1) \\ (\frac{3}{5},-2) & (\frac{3}{4},-3) & (\frac{1}{2},1) \end{pmatrix}$$

(1) $H^r_A = \{A \oslash \alpha \mid \alpha \in L\}$ is a right interior system because $(A \oslash \alpha) \oslash \beta = A \oslash (\alpha \oplus \beta)$ and $\bigvee_{i\in\Gamma}(A \oslash \alpha_i) = A \oslash \bigwedge_{i\in\Gamma} \alpha_i$ from Lemma 2.2. Since $A(y) \oslash A(x) \geq (A(y) \oslash \alpha) \oslash (A(x) \oslash \alpha)$,

$$\begin{aligned} e^r_{H^r_A}(x,y) &= \bigvee_{B\in H^r_A}(B(y) \oslash B(x)) = A(y) \oslash A(x) = d^r_A(x,y) \\ &= \bigvee_{D\in H^r_A}(D(y) \oslash d^r_{L^X}(D,\perp_x)) \\ &= I^r_{H^r_A}(\perp_x)(y) = d^r_{I^r_{H^r_A}}(x,y) \text{ [By Theorem 3.15 (2)]}, \\ e^r_{I^r_{H^r_A}}(x,y) &= \bigvee_{C\in L^X}(I^r_{H^r_A}(C)(y) \oslash I^r_{H^r_A}(C)(x)) \\ &= e^r_{H_{I^r_{H^r_A}}}(x,y) = e^r_{H^r_A}(x,y) \ \text{(by } H_{I^r_{H^r_A}} = H^r_A). \end{aligned}$$

Define $I^r_{d^r_A} : L^X \to L^X$ as $I^r_{d^r_A}(D)(y) = \bigwedge_{x\in X}(D(x) \oplus d^r_A(x,y))$. Since $I^r_{d^r_A}(D \oslash \alpha)(y) = \bigwedge_{x\in X}((D(x)\oslash\alpha) \oplus d^r_A(x,y)) \geq \bigwedge_{x\in X}(D(x) \oplus d^r_A(x,y)) \oslash \alpha = I^r_{d^r_A}(D)(y) \oslash \alpha$ from Lemma 2.2 (5), $I^r_{d^r_A}$ is a right interior operator with $I^r_{d^r_A}(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} I^r_{d^r_A}(A_i)$ and $I^r_{d^r_A}(\alpha \oplus A) = \alpha \oplus I^r_{d^r_A}(A)$ for all $A_i,\ A \in L^X$. For any $x,\ y \in X$,

$$\begin{aligned} d^r_{I^r_{d^r_A}}(x,y) &= I^r_{d^r_A}(\perp_x)(y) = d^r_A(x,y), \\ e^r_{I^r_{d^r_A}}(x,y) &= \bigvee_{A\in L^X}(I^r_{d^r_A}(A)(y) \oslash I^r_{d^r_A}(A)(x)) \\ &\geq \bigvee_{z\in X}(I^r_{d^r_A}(\perp_z)(y) \oslash I^r_{d^r_A}(\perp_z)(x)) \\ &= \bigvee_{z\in X}(d^r_A(z,y) \oslash d^r_A(z,x)) = d^r_A(x,y), \\ e^r_{I^r_{d^r_A}}(x,y) &= \bigvee_{D\in L^X}(\bigwedge_{z\in X}(D(z) \oplus d^r_A(z,y)) \oslash \bigwedge_{z\in X}(D(z) \oplus d^r_A(z,x))) \\ &\leq \bigvee_{D\in L^X}(\bigvee_{z\in X}(D(z) \oplus d^r_A(z,y)) \oslash (D(z) \oplus d^r_A(z,x))) \\ &\leq \bigvee_{z\in X}(d^r_A(z,y)) \oslash d^r_A(z,x)) = d^r_A(x,y). \end{aligned}$$

Then $d^r_{I^r_{d^r_A}} = e^r_{I^r_{d^r_A}} = d^r_A$.

For $B = ((\frac{4}{5},1),(\frac{2}{3},3),(\frac{3}{4},1))$, $I^r_{H^r_A}(B) \neq I^r_{d^r_A}(B)$ from

$$\begin{aligned} I^r_{H^r_A}(B) &= \bigvee_{\alpha\in L}\{A \oslash \alpha \in H^r_A \mid A \oslash \alpha \leq B\} \\ &= \bigvee_{D\in H^r_A}(D \oslash d^r_{L^X}(D,B)) \\ &= A \oslash (\tfrac{9}{16},-2)_X \\ &= ((\tfrac{8}{15},\tfrac{26}{5}),(\tfrac{2}{3},3),(\tfrac{1}{2},1)), \\ I^r_{d^r_A}(B) &= \bigwedge_{y\in X}(B(y) \oplus (d^r_A)_y) \\ &= ((\tfrac{2}{3},3),(\tfrac{2}{3},3),(\tfrac{2}{3},3)). \end{aligned}$$

For each $C \in L^X$, since $I^r_{H^r}(C) \in H^r_A$,

$$\begin{array}{ll} I^r_{d^r_A}(I^r_{H^r}(C))(x) & = \bigwedge_{y\in X}(I^r_{H^r}(C)(y) \oplus d^r_A(y,x)) \\ & \geq \bigwedge_{y\in X}(I^r_{H^r}(C)(y) \oplus (I^r_{H^r}(C)(x) \oslash I^r_{H^r}(C)(y)) \\ & \geq I^r_{H^r}(C)(x), \\ I^r_{d^r_A}(I^r_{H^r}(C))(x) & = \bigwedge_{y\in X}(I^r_{H^r}(C)(y) \oplus d^r_A(y,x)) \\ & \leq I^r_{H^r}(C)(x) \oplus d^r_A(x,x) = I^r_{H^r}(C)(x). \end{array}$$

Thus $I^r_{H^r}(C) = I^r_{d^r_A}(I^r_{H^r}(C)) \leq I^r_{d^r_A}(C)$ for each $C \in L^X$. For each $C \in L^X,\ y \in X$,

$$\begin{array}{ll} C^l_{I^r_{d^r_A}}(D)(y) & = n_1(I^r_{d^r_A}(n_2(D)))(y) \\ & = n_1(\bigwedge_{x\in X}(n_2(D)(x) \oplus d^r_A(x,y))) \\ & = \bigvee_{x\in X}(n_1(d^r_A(x,y)) \ominus n_2(D)(x)) \\ & = \bigvee_{x\in X}(D(x) \oslash n_1(n_1(d^r_A(x,y)))) \end{array}$$

For $n_1(A) = (\frac{5}{6}, -\frac{2}{3}), (\frac{2}{3}, \frac{5}{3}), (1, -4)),$

$$\begin{array}{ll} n_1(n_1(d^r_A(x,y))) & = n_1(n_1(A(y) \oslash A(x))) \\ & = n_1(n_1(A(y)) \oplus A(x)) = n_1(A(x)) \ominus n_1(A(y)) \\ & = d^l_{n_1(A)}(y,x) \text{ [By Lemma 2.2 (16)]}, \end{array}$$

$$n_1(n_1(d^r_A)) = \begin{pmatrix} (\frac{1}{2}, 1) & (\frac{5}{8}, -\frac{3}{4}) & (\frac{1}{2}, 1) \\ (\frac{1}{2}, 1) & (\frac{1}{2}, 1) & (\frac{1}{2}, 1) \\ (\frac{3}{5}, -1) & (\frac{3}{4}, -\frac{13}{4}) & (\frac{1}{2}, 1) \end{pmatrix}$$

So $C^l_{I^r_{d^r_A}}(D)(y) = \bigvee_{x\in X}(D(x) \oslash d^l_{n_1(A)}(y,x))$.

(2) $H^l_A = \{A \ominus \alpha \mid \alpha \in L\}$ is a left interior system because $\bigvee_{i\in I}(A \ominus \alpha_i) = A \ominus \bigwedge_{i\in I} \alpha_i$ and $(A \ominus \alpha) \ominus \beta = A \ominus (\beta \oplus \alpha)$ from Lemma 2.2. Since $A(y) \ominus A(x) \geq (A(y) \ominus \alpha) \ominus (A(x) \ominus \alpha)$,

$$\begin{array}{ll} e^l_{H^l_A}(x,y) & = \bigvee_{B\in H^l_A}(B(y) \ominus B(x)) = A(y) \ominus A(x) = d^l_A(x,y), \\ & = \bigvee_{D\in H^l_A}(D(y) \ominus d^l_{L^X}(D, \perp_x)) \\ & = I^l_{H^l_A}(\perp_x)(y) = d^l_{I^l_{H^l_A}}(x,y) \text{ [By Corollary 3.17 (2)]}, \\ e^l_{I^l_{H^l_A}}(x,y) & = \bigvee_{C\in L^X}(I^l_{H^l_A}(C)(y) \ominus I^l_{H^l_A}(C)(x)) \\ & = e^l_{H^l_{I^l_{H^l_A}}}(x,y) = e^l_{H^l_A}(x,y). \end{array}$$

Define $I^l_{d^l_A} : L^X \to L^X$ as $I^l_{d^l_A}(D)(y) = \bigwedge_{x\in X}(d^l_A(x,y) \oplus D(x))$. Since $I^l_{d^l_A}(D \ominus \alpha)(y) = \bigwedge_{x\in X}(d^l_A(x,y) \oplus (D(x) \ominus \alpha)) \geq \bigwedge_{x\in X}(d^l_A(x,y) \oplus D(x)) \ominus \alpha = I^l_{d^l_A}(D)(y) \ominus \alpha$ from Lemma 2.2 (5), $I^l_{d^l_A}$ is a left interior operator with $I^l_{d^l_A}(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} I^l_{d^l_A}(A_i)$

and $I^l_{d^l_A}(A \oplus \alpha) = I^r_{d^r_A}(A) \oplus \alpha$ for all $A_i,\ A \in L^X,\ \alpha \in L$.

$$\begin{array}{ll} e^l_{I^l_{d^l_A}}(x,y) & = I^l_{d^l_A}(\perp_x)(y) = d^l_A(x,y), \\ e^l_{I^l_{d^l_A}}(x,y) & = \bigvee_{A\in L^X}(I^l_{d^l_A}(A)(y) \ominus I^l_{d^l_A}(A)(x)) \\ & \geq \bigvee_{z\in X}(I^l_{d^l_A}(\perp_z)(y) \ominus I^l_{d^l_A}(\perp_z)(x)) \\ & = \bigvee_{z\in X}(d^l_A(z,y) \ominus d^l_A(z,x)) = d^l_A(x,y), \\ e^l_{I^l_{d^l_A}}(x,y) & = \bigvee_{D\in L^X}(\bigwedge_{z\in X}(d^l_A(z,y) \oplus D(z)) \ominus \bigwedge_{z\in X}(d^l_A(z,x) \oplus D(z))) \\ & \leq \bigvee_{D\in L^X}(\bigvee_{z\in X}(d^l_A(z,y) \oplus D(z)) \ominus (d^l_A(z,x) \oplus D(z))) \\ & \leq \bigvee_{z\in X}(d^l_A(z,y)) \ominus d^l_A(z,x)) = d^l_A(x,y). \end{array}$$

Hence $d^l_{I^l_{d^l_A}} = e^l_{I^l_{d^l_A}} = d^l_A$.

For $B = ((\frac{4}{5}, 1), (\frac{2}{3}, 3), (\frac{3}{4}, 1))$,

$$\begin{array}{ll} I^l_{H^l}(B) & = \bigvee_{\alpha\in L}\{A \ominus \alpha \in H^l_A \mid A \ominus \alpha \leq B\} \\ & = \bigvee_{D\in H^l_A}(D \ominus d^l_{L^X}(D,B)) = A \ominus (\frac{9}{16}, -\frac{13}{4})_X \\ & = ((\frac{8}{15}, \frac{17}{3}), (\frac{2}{3}, 3), (\frac{1}{2}, 1)), \\ I^l_{d^l_A}(B) & = \bigwedge_{y\in X}((d^l_A)_y \oplus B(y)) = ((\frac{2}{3}, 3), (\frac{2}{3}, 3).(\frac{2}{3}, 3)), \end{array}$$

For each $D \in L^X$, since $I^l_{H^l}(D) \in H^l_A$,

$$\begin{array}{ll} I^l_{d^l_A}(I^l_{H^l}(D))(x) & = \bigwedge_{y\in X}(d^l_A(y,x) \oplus I^l_{H^l}(D)(y)) \\ & \geq \bigwedge_{y\in X}((I^l_{H^l}(D)(x) \ominus I^l_{H^l}(D)(y) \oplus I^l_{H^l}(D)(y)) \\ & \geq I^l_{H^l}(D)(x), \\ I^l_{d^l_A}(I^l_{H^l}(D))(x) & = \bigwedge_{y\in X}(d^l_A(y,x) \oplus I^l_{H^l}(D)(y)) \\ & \leq d^l_A(y,x) \oplus I^l_{H^l}(D)(x) = I^l_{H^l}(D)(x). \end{array}$$

Then $I^l_{H^l}(D) = I^l_{d^l_A}(I^l_{H^l}(D)) \leq I^l_{d^l_A}(D)$ for each $D \in L^X$. By Corollary 3.17 (3),

$$\begin{array}{ll} C^r_{I^l_{d^l_A}}(D)(y) & = n_2(I^l_{d^l_A}(n_1(D)))(y) = n_2(\bigwedge_{x\in X}(d^l_A(x,y) \oplus n_1(D)(x))) \\ & = \bigvee_{x\in X}(n_2(d^l_A(x,y)) \oslash n_1(D)(x)) = \bigvee_{x\in X}(D(x) \ominus n_2(n_2(d^l_A(x,y)))). \end{array}$$

For $n_2(A) = ((\frac{5}{6}, -\frac{5}{3}), (\frac{2}{3}, \frac{8}{3}), (1, -8))$,

$$\begin{array}{ll} n_2(n_2(d^l_A(x,y))) & = n_2(n_2(A(y) \ominus A(x))) = n_2(A(x) \oplus n_2(A(y))) \\ & = n_2(A(x)) \oslash n_2(A(y)) = d^r_{n_2(A)}(y,x), \end{array}$$

$$n_2(n_2(d^l_A)) = \begin{pmatrix} (\frac{1}{2}, 1) & (\frac{5}{8}, -\frac{15}{4}) & (\frac{1}{2}, 1) \\ (\frac{1}{2}, 1) & (\frac{1}{2}, 1) & (\frac{1}{2}, 1) \\ (\frac{3}{5}, -\frac{24}{5}) & (\frac{3}{4}, -\frac{21}{2}) & (\frac{1}{2}, 1) \end{pmatrix}.$$

Thus $C^r_{I^l_{d^l_A}}(D)(y) = \bigvee_{x\in X}(D(x) \ominus d^r_{n_2(A)}(y,x))$.

(3) $G^r_A = \{\alpha \oplus A \mid \alpha \in L\}$ is a right closure system.

Since $A(y) \oslash A(x) \geq (\alpha \oplus A(y)) \oslash (\alpha \oplus A(x))$,

$$e^r_{G^r_A}(x,y) = \bigvee_{B\in G^r_A}(B(y) \oslash B(x)) = A(y) \oslash A(x) = d^r_A(x,y).$$

Define $C^r_{d^r_A}: L^X \to L^X$ as $C^r_{d^r_A}(D)(y) = \bigvee_{x\in X}(D(x)\ominus d^r_A(y,x))$. Let $D,\ \alpha \in L^X$. Since

$$\begin{aligned} C^r_{d^r_A}(\alpha\oplus D)(y) &= \bigvee_{x\in X}((\alpha\oplus D(x))\ominus d^r_A(y,x)) \\ &\leq \bigvee_{x\in X}(\alpha\oplus(D(x)\ominus d^r_A(y,x)) \leq \alpha\oplus C^r_{d^r_A}(D)(y), \end{aligned}$$

$C^r_{d^r_A}$ is a right closure operator with

$$\begin{aligned} C^r_{d^r_A}(D\oslash\alpha)(y) &= \bigvee_{x\in X}((D(x)\oslash\alpha)\ominus d^r_A(y,x)) \\ &= \bigvee_{x\in X}(D(x)\ominus d^r_A(y,x))\oslash\alpha = C^r_{d^r_A}(D)(y)\oslash\alpha, \end{aligned}$$

$$\begin{aligned} C^r_{d^r_A}(\textstyle\bigvee_{i\in I} D_i)(y) &= \bigvee_{x\in X}(\bigvee_{i\in I} D_i(x)\ominus d^r_A(y,x)) \\ &= \bigvee_{i\in I}(\bigvee_{x\in X}(D_i(x)\ominus d^r_A(y,x))) = \bigvee_{i\in I} C^r_{d^r_A}(D_i)(y). \end{aligned}$$

For $B = ((\frac{4}{5},1),(\frac{2}{3},3),(\frac{3}{4},1))$,

$$\begin{aligned} C^r_{G^r_A}(B) &= \bigwedge_{\alpha\in L}\{\alpha\oplus A\in G^r_A \mid B\leq \alpha\oplus A\} = \bigwedge_{C\in G^r_A}(d^r_{L^X}(B,C)\oplus C) \\ &= (\tfrac{3}{4},-3)_X\oplus A = ((\tfrac{9}{10},-\tfrac{14}{5}),(1,0),(\tfrac{3}{4},1)), \\ C^r_{d^r_A}(B) &= \bigvee_{x\in X}(B(x)\ominus d^r_A(-,x)) = ((\tfrac{4}{5},1),(\tfrac{4}{5},1),(\tfrac{2}{3},\tfrac{25}{6})), \\ C^r_{G^r_A}(B) &\neq C^r_{d^r_A}(B). \end{aligned}$$

Since $H^l_{G^r_A} = \{n_1(D) \mid D\in G^r_A\} = \{n_1(A)\ominus\alpha \mid \alpha\in L\}$ and $(n_1(A)(x)\ominus\alpha)\ominus(n_1(A)(y)\ominus\alpha)\leq n_1(A)(x)\ominus n_1(A)(y)$, where $n_1(A) = (\frac{5}{6},-\frac{2}{3}),(\frac{2}{3},\frac{5}{3}),(1,-4))$,

$$\begin{aligned} d^r_{C^r_{G^r_A}}(x,y) &= n_1C^r_{G^r_A}(n_2(\perp_y))(x) = n_1(\bigwedge_{D\in G^r_A}(d^r_{L^X}(n_2(\perp_y),D)\oplus D(x)) \\ &= \bigvee_{D\in G^r_A}(n_1(D)(x)\ominus d^r_{L^X}(n_2(\perp_y),D)) \\ &= \bigvee_{D\in G^r_A}(n_1(D)(x)\ominus d^l_{L^X}(n_1(D),\perp_y)) \\ &= \bigvee_{D\in G^r_A}(n_1(D)(x)\ominus n_1(D)(y)) \\ &= e^l_{H^l_{G^r_A}}(y,x) = n_1(A)(x)\ominus n_1(A)(y) \\ &= d^l_{n_1(A)}(y,x) = n_1(n_1(d^r_A(x,y))). \end{aligned}$$

For each $C\in L^X$, since $C^r_{G^r_A}(C)\in G^r_A$,

$$\begin{aligned} C^r_{d^r_A}(C^r_{G^r_A}(C))(x) &= \bigvee_{y\in X}(C^r_{G^r_A}(C)(y)\ominus d^r_A(x,y)) \\ &= \bigvee_{y\in X}(C^r_{G^r_A}(C)(y)\ominus \bigvee_{D\in G^r_A}(D(y)\oslash D(x))) \\ &\leq \bigvee_{y\in X}(C^r_{G^r_A}(C)(y)\ominus(C^r_{G^r_A}(C)(y)\oslash C^r_{G^r_A}(C)(x))) \\ &\leq C^r_{G^r_A}(C)(x), \\ C^r_{d^r_A}(C^r_{G^r_A}(C))(x) &= \bigvee_{y\in X}(C^r_{G^r_A}(C)(y)\ominus d^r_A(x,y)) \\ &\geq C^r_{G^r_A}(C)(x)\ominus d^r_A(x,x) = C^r_{G^r_A}(C)(x). \end{aligned}$$

Then $C^r_{G^r_A}(C) = C^r_{d^r_A}C^r_{G^r_A}(C)\geq C^r_{d^r_A}(C)$ for each $C\in L^X$. Since $n_1(n_1(d^r_A(x,y))) = d^l_{n_1(A)}(y,x)$ for any $x,\ y\in X$,

$$\begin{aligned} I^l_{C^r_{d^r_A}}(D)(y) &= n_1(C^r_{d^r_A}(n_2(D)))(y) = n_1(\bigvee_{x\in X}(n_2(D)(x)\ominus d^r_A(y,x))) \\ &= n_1(\bigvee_{x\in X}(n_1(d^r_A(y,x))\oslash D(x)) = \bigwedge_{x\in X}(n_1(n_1(d^r_A(y,x)))\oplus D(x)) \\ &= \bigwedge_{x\in X}(d^l_{n_1(A)}(x,y)\oplus D(x)). \end{aligned}$$

(4) $G^l_A = \{A \oplus \alpha \mid \alpha \in L\}$ is a left closure system. Since $A(y) \ominus A(x) \geq (A(y) \oplus \alpha) \ominus (A(x) \oplus \alpha)$,

$$e^l_{G^l_A}(x,y) \quad = \bigvee_{B \in G^l_A} (B(y) \ominus B(x)) = A(y) \ominus A(x) = d^l_A(x,y).$$

Define $C^l_{d^l_A} : L^X \to L^X$ as $C^l_{d^l_A}(D)(y) = \bigvee_{x \in X}(D(x) \oslash d^l_A(y,x))$. By a similar way in (3), $C^l_{d^l_A}$ is a left closure operator.

Since for $D,\ \alpha \in L^X$

$$\begin{aligned} C^l_{d^l_A}(D \oplus \alpha)(y) &= \bigvee_{x \in X}((D \oplus \alpha) \oslash d^l_A(y,x)) \\ &\leq \bigvee_{x \in X}((D(x) \oslash d^l_A(y,x)) \oplus \alpha) \leq C^l_{d^l_A}(D)(y) \oplus \alpha, \end{aligned}$$

$C^l_{d^l_A}$ is a left closure operator with

$$\begin{aligned} C^l_{d^l_A}(D \ominus \alpha)(y) &= \bigvee_{x \in X}((D(x) \ominus \alpha) \oslash d^l_A(y,x)) \\ &= \bigvee_{x \in X}(D(x) \oslash d^l_A(y,x)) \ominus \alpha = C^l_{d^l_A}(D)(y) \ominus \alpha, \end{aligned}$$

$$\begin{aligned} C^l_{d^l_A}(\bigvee_{i \in I} D_i)(y) &= \bigvee_{x \in X}(\bigvee_{i \in I} D_i(x) \oslash d^l_A(y,x)) \\ &= \bigvee_{i \in I}(\bigvee_{x \in X}(D_i(x) \oslash d^l_A(y,x))) = \bigvee_{i \in I} C^l_{d^l_A}(D_i)(y). \end{aligned}$$

For $B = ((\frac{4}{5},1),(\frac{2}{3},3),(\frac{3}{4},1))$,

$$\begin{aligned} C^l_{G^l_A}(B) &= \bigwedge_{\alpha \in L}\{A \oplus \alpha \in G^l_A \mid B \leq A \oplus \alpha\} \\ &= \bigwedge_{C \in G^l_A}(C \oplus d^l_{L^X}(B,C)) \\ &= A \oplus (\tfrac{3}{4},-5)_X = ((\tfrac{9}{10},-\tfrac{7}{2}),(1,0),(\tfrac{3}{4},1)), \\ C^l_{d^l_A}(B) &= \bigvee_{x \in X}(B(x) \oslash d^l_A(-,y)) \\ &= ((\tfrac{4}{5},1),(\tfrac{4}{5},1),(\tfrac{2}{3},5)). \end{aligned}$$

Since $H^r_{G^l_A} = \{n_2(D) \mid D \in G^l_A\} = \{n_2(A) \oslash \alpha \mid \alpha \in L\}$ and $(n_2(A)(x) \oslash \alpha) \oslash (n_2(A)(y) \oslash \alpha) \leq n_2(A)(x) \oslash n_2(A)(y)$, where $n_2(A) = ((\frac{5}{6},-\frac{5}{3}),(\frac{2}{3},\frac{8}{3}),(1,-8))$ for any $x,\ y \in X$,

$$\begin{aligned} d^l_{C^l_{G^l_A}}(x,y) &= n_2 C^l_{G^l}(n_1(\perp_y))(x) = n_2(\bigwedge_{D \in G^l_A}(d^l_{L^X}(n_1(\perp_y),D) \oplus D(x))) \\ &= \bigvee_{D \in G^l_A}(n_2(D)(x) \oslash d^l_{L^X}(n_1(\perp_y),D)) \\ &= \bigvee_{D \in G^l_A}(n_2(D)(x) \oslash d^r_{L^X}(n_2(D),\perp_y)) \\ &= \bigvee_{D \in G^l_A}(n_2(D)(x) \oslash n_2(D)(y)) \\ &= e^r_{H^r_{G^l_A}}(y,x) = n_2(A)(x) \oslash n_2(A)(y) \\ &= d^r_{n_2(A)}(y,x) = n_2(n_2(d^l_A(x,y))). \end{aligned}$$

By a similar way in (3), $C^l_{G^l_A}(D) = C^l_{d^l_A}(C^l_{G^l_A}(D)) \geq C^l_{d^l_A}(D)$ for each $D \in L^X$. Since $n_2(n_2(d^l_A(x,y))) = d^r_{n_2(A)}(y,x)$ for any $x,\ y \in X$,

$$\begin{aligned} I^r_{C^l_{d^l_A}}(D)(y) &= n_2(C^l_{d^l_A}(n_1(D)))(y) = n_2(\bigvee_{x \in X}(n_1(D)(x) \oslash d^l_A(y,x))) \\ &= n_2(\bigvee_{x \in X}(n_2(d^l_A(y,x)) \ominus D(x)) = \bigwedge_{x \in X}(D(x) \oplus n_2(n_2(d^l_A(y,x)))) \\ &= \bigwedge_{x \in X}(D(x) \oplus d^r_{n_2(A)}(x,y)). \end{aligned}$$

(5) By Theorem 3.13 (7) and (8),

$$K^r_{d^r_A} = \{ \bigvee_{x\in X} (B(x) \ominus d^r_A(-,x) \mid B \in L^X\} = \{ \bigwedge_{x\in X} (B(x) \oplus d^r_A(x,-) \mid B \in L^X\}$$

is the smallest right extensional system such that $H^r_A \subset K^r_{d^r_A}$ and $G^r_A \subset K^r_{d^r_A}$ in (1) and (2). Since $A(y) \oslash A(x) \geq (A(y) \oslash \alpha) \oslash (A(x) \oslash \alpha)$ and $A(y) \oslash A(x) \geq (\alpha \oplus A(y)) \oslash (\alpha \oplus A(x))$ from Lemma 2.2 (9) and (10),

$$\begin{aligned} e^r_{K^r_{d^r_A}}(x,y) &= \bigvee_{D\in K^r_{d^r_A}} (D(y) \oslash D(x)) = e^r_{H^r_A}(x,y) = e^r_{G^r_A}(x,y) \\ &= d^r_A(x,y) = A(y) \oslash A(x). \end{aligned}$$

For $B = ((\frac{4}{5},1),(\frac{2}{3},3),(\frac{3}{4},1))$,

$$\begin{aligned} I^r_{K^r_{d^r_A}}(B) &= \bigvee\{C \in K^r_{d^r_A} \mid C \leq B\} = \bigwedge_{x\in X}(B(x) \oplus d^r_A(x,-)) = I^r_{d^r_A}(B) \\ &= ((\tfrac{2}{3},3),(\tfrac{2}{3},3),(\tfrac{2}{3},3)), \\ C^r_{K^r_{d^r_A}}(B) &= \bigvee_{y\in X}(B(y) \ominus d^r_A(-,y)) = C^r_{d^r_A}(B) = ((\tfrac{4}{5},1),(\tfrac{4}{5},1),(\tfrac{3}{4},1)). \end{aligned}$$

(6) By Corollary 3.14 (7) and (8),

$$K^l_{d^l_A} = \{ \bigvee_{x\in X} (B(x) \oslash d^l_A(-,x) \mid B \in L^X\} = \{ \bigwedge_{x\in X} (d^l_A(x,-) \oplus B(x) \mid B \in L^X\}$$

is the smallest left extensional system such that $H^l_A \subset K^l_{d^l_A}$ and $G^l_A \subset K^l_{d^l_A}$ in (1) and (2). Since $A(y) \ominus A(x) \geq (A(y) \ominus \alpha) \ominus (A(x) \ominus \alpha)$ and $A(y) \ominus A(x) \geq (A(y) \oplus \alpha) \ominus (A(x) \oplus \alpha)$ from Lemma 2.2 (9) and (11),

$$\begin{aligned} e^l_{K^l_{d^l_A}}(x,y) &= \bigvee_{D\in K^l_{d^l_A}} (D(y) \ominus D(x)) = e^l_{H^l_A}(x,y) \\ &= e^l_{G^l_A}(x,y) = d^l_A(x,y) = A(y) \ominus A(x). \end{aligned}$$

For $B = ((\frac{4}{5},1),(\frac{2}{3},3),(\frac{3}{4},1))$,

$$\begin{aligned} I^l_{K^l_{d^l_A}}(B) &= \bigwedge_{x\in X}(d^l_A(x,-) \oplus B(x)) = I^l_{d^l_A}(B) = ((\tfrac{2}{3},3),(\tfrac{2}{3},3),(\tfrac{2}{3},3)), \\ C^l_{K^l_{d^l_A}}(B) &= \bigvee_{y\in X}(B(y) \ominus d^r_A(-,y)) = C^l_{d^l_A}(B) = ((\tfrac{4}{5},1),(\tfrac{4}{5},1),(\tfrac{3}{4},1)). \end{aligned}$$

4. Conclusion

We have studied the relations among the two types (right, left) of interior (closure) operators, interior (closure, extensional) systems and distance functions based on a generalized co-residuated lattice as a noncommutative structure. Moreover, we have discussed their properties.

In the future, we plan to investigate fuzzy rough sets, fuzzy automata, information systems and decision rules by using the concepts of two types in a generalized co-residuated lattice. In particular, algebraic structures (interior (closure) operators, interior (closure, extensional) systems, distance functions, fuzzy rough sets) can be used to classify big data into small groups.

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