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Abstract. The main goal of this paper is to define and study two novel types of regular and normal space called πge-regular and πge-normal space. The ideas of πge-regularity and πge-normality are introduced by πge-closed sets defined by Özkoç and Ayhan, which are general variations of the separation axioms known from general topology. Moreover, we examine not only the relations of these spaces with other types of spaces but also some of their basic features.

2020 AMS Classification: 54C08, 54C10.

Keywords: ge-closed set, πge-closed set, ge-regular space, πge-regular space, ge-normal space, πge-normal space.

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1. Introduction

In recent years, many topologists have worked on different strong and weak forms of regular and normal space. In 2010, Caldas defined the notion of πgp-normality [1] in topological spaces. In 2014, Ravi et al. worked on $\pi g\gamma$-normality [2]. In 2015, Kumar et al. introduced the concept of $\pi g\beta$-normality [3]. In 2019, Sharma and Kumar introduced and studied πgb-normality [4]. In 2024, Korkmaz introduced the termed of πgs-normality [5]. On the other hand, Tahiliani defined the notion of $\pi g\beta$-regularity [6] in topological spaces.

This study aims to introduce and investigate the notions of πge-regular and πge-normal space which are associated with the concepts of e-regular [7] and e-normal [8] space, respectively. In addition, we examine characterizations of πge-regular and πge-normal space and some basic properties of its.

2. Preliminaries

In this section, we will recall some definitions and results used in this paper. Throughout this paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (resp. open) sets of X is denoted by $C(X)$ (resp. $O(X)$).

A subset A of a space (X, τ) is said to be *regular open* [9], if $A = int(cl(A))$. The family of all regular open sets of X will be denoted by $RO(X)$. The finite union of regular open sets is said to be *π-open* [10]. The family of all π-open sets of X will be denoted by $\pi O(X)$.

A subset A is said to be *δ-open* [11], if for each $x \in A$ there exists a regular open set B such that $x \in B \subseteq A$. A point x of X is called a *δ-cluster point* of A, if $A \cap int(cl(U)) \neq \varnothing$ for each open set U containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $cl_\delta(A)$. The set $\{x|(U \in RO(X))(x \in U \subseteq A)\}$ is called the *δ-interior* of A and is denoted by $int_\delta(A)$.

A subset A of a space X is said to be *e-open* [12], if $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$. The complement of e-open set is said to be *e-closed* [12]. The intersection of all e-closed sets containing A is called the *e-closure* of A and is denoted by e-$cl(A)$. The union of all e-open sets contained in A is called the *e-interior* of A and is denoted by e-$int(A)$. A subset A is said to be *e-regular* [7], if it is e-open and e-closed. The collection of all e-open (resp. e-closed, e-regular) subsets of X is denoted by $eO(X)$ (resp. $eC(X)$, $eR(X)$). The collection of all e-open (resp. e-closed, e-regular) subsets of X containing a point x of X is denoted by $eO(X, x)$ (resp. $eC(X, x)$, $eR(X, x)$).

A subset A of a topological space X is said to be *generalized closed* [13] (briefly, g-closed), if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X. A subset A of X is said to be *ge-closed* [14] (resp. *πge-closed* [15]), if e-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open (resp. π-open). The complement of ge-closed (resp. πge-closed) set is said to be *ge-open* (resp. *πge-open*). The collection of all ge-closed (resp. ge-open, πge-closed, πge-open) subsets of X is denoted by $geC(X)$ (resp. $geO(X)$, $\pi geC(X)$, $\pi geO(X)$).

Also, it is noted in [7, 14, 15] that

$$e\text{-regular} \Rightarrow e\text{-closed} \Rightarrow ge\text{-closed} \Rightarrow \pi ge\text{-closed}$$

Lemma 2.1 ([7]). *Let A be a subset of a topological space X. Then*
(1) *$A \in eO(X)$ if and only if e-$cl(A) \in eR(X)$,*
(2) *$A \in eC(X)$ if and only if e-$int(A) \in eR(X)$.*

Definition 2.2. A topological space X is said to be:
(i) *e-regular* [7], if for each closed set F of X and each point $x \in X \setminus F$, there exist disjoint e-open sets U, $V \subseteq X$ such that $x \in U$ and $F \subseteq V$,
(ii) *e-normal* [8], if for each pair of nonempty disjoint closed sets can be separated by disjoint e-open sets,
(iii) *strongly e-normal* [16], if for each pair of disjoint e-closed subsets A and B of X, there exist disjoint e-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Definition 2.3. A function $f : X \to Y$ is said to be:

(i) *contra e-continuous* [8], if $f^{-1}[F]$ is e-open in X for each closed set F of Y,
(ii) *πge-open* [15], if $f[U]$ is πge-closed in Y for each πge-closed set U of X.

3. πge-regular Spaces

Definition 3.1. A space X is said to be *generalized e-regular* (briefly, ge-regular), if for each ge-closed set F of X and each point $x \in X \setminus F$, there exist disjoint e-open sets $U,\ V \subseteq X$ such that $x \in U$ and $F \subseteq V$.

Definition 3.2. A space X is said to be *π-generalized e-regular* (briefly, πge-regular), if for each πge-closed set F of X and each point $x \in X \setminus F$, there exist disjoint e-open sets $U,\ V \subseteq X$ such that $x \in U$ and $F \subseteq V$.

Remark 3.3. From Definitions 2.2(i), 3.1 and 3.2, we have the following diagram. The converse of these implications is not true in general as shown by the following examples.

$$\pi ge\text{-regular} \Rightarrow ge\text{-regular} \Rightarrow e\text{-regular}$$

Example 3.4. Let $X = \{a, b, c, d\}$ and let the topology on X be $\tau = \{\varnothing, X, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. It can be easily seen that $eC(X) = geC(X) = 2^X \setminus \{\{c, d\}, \{b, c, d\}, \{a, c, d\}\}$ and $\pi geC(X) = 2^X \setminus \{\{b, c, d\}, \{a, c, d\}\}$. Then X is ge-regular but not πge-regular since $\{c, d\}$ is πge-closed but for b not in $\{c, d\}$, there does not exist any pair of disjoint e-open sets containing them.

Question. Is there any e-regular space which is not ge-regular?

Definition 3.5. A space X is said to be *πge-$T_{1/2}$* [15], if every πge-closed set is e-closed.

Example 3.6. Let $X = \{a, b, c, d\}$ and let the topology on X be $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. It can be easily seen that $eC(X) = \pi geC(X) = 2^X \setminus \{\{a, b, c\}\}$. Then X is πge-$T_{1/2}$ since every πge-closed set is e-closed.

Lemma 3.7. *A space X is πge-$T_{1/2}$ if and only if $eC(X) = geC(X) = \pi geC(X)$.*

Proof. It is obvious from Theorem 4.7(2), [15]. □

Theorem 3.8. *A space X is πge-regular if and only if X is ge-regular and πge-$T_{1/2}$.*

Proof. Necessity. It is clear since every πge-regular space is ge-regular.
Sufficiency. It is obvious from Lemma 3.7. □

Theorem 3.9. *For a topological space X, the following statements are equivalent:*
(1) *X is πge-regular,*
(2) *every πge-open set is a union of e-regular sets,*
(3) *every πge-closed set is an intersection of e-regular sets.*

Proof. $(1) \Rightarrow (2)$: Let $A \in \pi geO(X, x)$.

$$\left.\begin{array}{r} A \in \pi geO(X,x) \Rightarrow (X \setminus A \in \pi geC(X))(x \notin X \setminus A) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists U \in eO(X,x))(\exists V \in eO(X))(X \setminus A \subseteq V)(U \cap V = \varnothing) \\ W := e\text{-}cl(U) \end{array}\right\} \overset{\text{Lemma 2.1(1)}}{\Rightarrow}$$

$\Rightarrow (W \in eR(X,x))(W \cap (X \setminus A) \subseteq W \cap V = \varnothing)$
$\Rightarrow (W \in eR(X,x))(W \subseteq A)$.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let $F \in \pi geC(X)$ and $x \notin F$.

$$\left.\begin{array}{r}(F \in \pi geC(X))(x \notin F) \overset{\text{Hypothesis}}{\Rightarrow} (\exists V \in eR(X))(F \subseteq V)(x \notin V)\\ U := X \setminus V\end{array}\right\} \Rightarrow$$

$\Rightarrow (U \in eO(X,x))(V \in eO(X))(F \subseteq V)(U \cap V = \varnothing)$. □

Theorem 3.10. *For a topological space X, the following statements are equivalent:*
(1) *X is πge-regular,*
(2) *for each point x in X and each πge-open set U of X containing x, there exists $V \in eO(X,x)$ such that e-$cl(V) \subseteq U$;*

Proof. (1) $\Rightarrow$ (2) : Let $U \in \pi geO(X,x)$.

$$\left.\begin{array}{r}U \in \pi geO(X,x) \Rightarrow (X \setminus U \in \pi geC(X))(x \notin X \setminus U)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in eO(X,x))(\exists W \in eO(X))(X \setminus U \subseteq W)(V \cap W = \varnothing)$
$\Rightarrow (V \in eO(X,x))(W \in eO(X))(e\text{-}cl(V) \cap W = \varnothing)$
$\Rightarrow (V \in eO(X,x))(e\text{-}cl(V) \subseteq U)$.

(2) $\Rightarrow$ (1) : Let $F \in \pi geC(X)$ and $x \notin F$.

$$\left.\begin{array}{r}(F \in \pi geC(X))(x \notin F) \Rightarrow X \setminus F \in \pi geO(X,x))\\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists V \in eO(X,x))(e\text{-}cl(V) \subseteq X \setminus F)\\ W := X \setminus e\text{-}cl(V)\end{array}\right\} \Rightarrow$$

$\Rightarrow (V \in eO(X,x))(W \in eO(X))(F \subseteq W)(V \cap W = \varnothing)$. □

4. πge-normal Spaces

Definition 4.1. A space X is said to be *generalized e-normal* (briefly,ge-normal) , if for each pair of nonempty disjoint ge-closed sets A and B of X, there exist disjoint e-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Definition 4.2. A space X is said to be *π-generalized e-normal* (briefly, πge-normal), if for each pair of nonempty disjoint πge-closed sets A and B of X, there exist disjoint e-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Remark 4.3. From Definitions 2.2, 4.1 and 4.2, we have the following diagram. The converse of these implications is not true in general as shown by the following examples.

πge-normal $\Rightarrow$ ge-normal $\Rightarrow$ strongly e-normal $\Rightarrow$ e-normal

Example 4.4. Let $X = \{a,b,c,d\}$ and let the topology on X be $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}\}$. It can be easily seen that $eC(X) = geC(X) =$

$2^X \setminus \{\{a,b\},\{b,c\},\{a,b,c\}\}$ and $\pi geC(X) = 2^X \setminus \{\{a,b\},\{a,b,c\}\}$. Then X is *ge*-normal but not πge-normal since $\{a,d\}$ and $\{b,c\}$ are disjoint πge-closed sets in X, but there does not exist any pair of disjoint e-open sets containing them.

Example 4.5. Let $X = \{a,b,c,d\}$ and let the topology on X be $\tau = \{\varnothing, X, \{a\}, \{b,c\}, \{a,b,c\}\}$. It can be easily seen that $eC(X) = 2^X \setminus \{\{d\},\{b,d\},\{c,d\}\}$ and $geC(X) = \pi geC(X) = 2^X$. Then X is strongly e-normal but not ge-normal since $\{d\}$ and $\{a,b,c\}$ are disjoint ge-closed sets in X, but there does not exist any pair of disjoint e-open sets containing them.

Example 4.6. Consider the Example 4.4. Then X is e-normal but not strongly e-normal since $\{b\}$ and $\{d\}$ are disjoint e-closed sets in X, but there does not exist any pair of disjoint e-open sets containing them.

Remark 4.7. A πge-normal space need not be πge-regular, as the following example shows:

Example 4.8. Let $X = \{a,b,c,d\}$ and let the topology on X be $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$. It can be easily seen that $eC(X) = geC(X) = \pi geC(X) = 2^X \setminus \{\{a,b\},\{a,b,c\},\{a,b,d\}\}$. Then X is πge-normal but not πge-regular since $\{a,b\}$ is πge-closed but for c not in $\{a,b\}$, there does not exist any pair of disjoint e-open sets containing them.

Theorem 4.9. *For a topological space X, the following statements are equivalent:*
(1) *X is πge-normal,*
(2) *for every πge-closed set A and every πge-open set B containing A, there is an e-regular set V such that $A \subseteq V \subseteq B$.*

Proof. (1) $\Rightarrow$ (2) : Let A be πge-closed set and B be πge-open set with $A \subseteq B$.

$$\left.\begin{array}{r}(A \in \pi geC(X))(B \in \pi geO(X) \Rightarrow X \setminus B \in \pi geC(X))(A \subseteq B)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists W_1, W_2 \in eO(X))(A \subseteq W_1)(X \setminus B \subseteq W_2)(A \cap (X \setminus B) \subseteq W_1 \cap W_2 = \varnothing)\\ V := e\text{-}cl(W_1)\end{array}\right\} \Rightarrow$$

$$\overset{\text{Lemma 2.1(1)}}{\Rightarrow} (V \in eR(X))(A \subseteq W_1 \subseteq V \subseteq B).$$

(2) $\Rightarrow$ (1) : *Obvious.* □

Definition 4.10. A topological space X is said to be *weakly πge-normal*, if disjoint πge-closed set can be separated by disjoint closed sets.

Example 4.11. Let $X = \{a,b,c\}$ and let the topology on X be discrete topological space. It can be easily seen that $C(X) = \pi geC(X) = 2^X$. Then X is weakly πge-normal.

Theorem 4.12. *Let $f : X \to Y$ be a contra e-continuous and πge-open injection. If Y is weakly πge-normal, then X is πge-normal.*

Proof. Let A and B be disjoint πge-closed sets in X.

$$\left.\begin{array}{r}(A, B \in \pi geC(X))(A \cap B = \varnothing)\\ f \text{ is } \pi ge\text{-open injection}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (f[A], f[B] \in \pi geC(Y))(f[A \cap B] = f[A] \cap f[B] = \varnothing) \\ Y \text{ is weakly } \pi ge\text{-normal}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists U, V \in C(Y))(f[A] \subseteq U)(f[B] \subseteq V)(U \cap V = \varnothing) \\ f \text{ is contra } e\text{-continuous}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f^{-1}[U], f^{-1}[V] \in eO(X))(A \subseteq f^{-1}[U])(B \subseteq f^{-1}[V])(f^{-1}[U] \cap f^{-1}[V] = \varnothing). \qquad \square$$

Definition 4.13. A topological space X is said to be $(e, \pi ge)$-R_0, if $e\text{-}cl(\{x\}) \subseteq U$, whenever U is πge-open and $x \in U$.

Example 4.14. Let $X = \{a, b, c\}$ and let the topology on X be $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. It can be easily seen that $eC(X) = 2^X \setminus \{\{a, b\}\}$ and $\pi geO(X) = 2^X \setminus \{\{c\}\}$. Then X is $(e, \pi ge)$-R_0.

Theorem 4.15. *Every πge-normal and $(e, \pi ge)$-R_0 space is πge-regular.*

Proof. Let $F \in \pi geC(X)$ and $x \in X \setminus F$.

$$\left.\begin{array}{r}(F \in \pi geC(X))(x \in X \setminus F) \Rightarrow X \setminus F \in \pi geO(X, x) \\ X \text{ is } (e, \pi ge)\text{-}R_0\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow X \setminus F \supseteq e\text{-}cl(\{x\}) \in \pi geC(X) \\ X \text{ is } \pi ge\text{-normal}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U, V \in eO(X))(F \subseteq U)(e\text{-}cl(\{x\}) \subseteq V)(U \cap V = \varnothing)$$
$$\Rightarrow (\exists U \in eO(X, x))(\exists V \in eO(X))(F \subseteq V)(U \cap V = \varnothing). \qquad \square$$

5. Conclusion

The study of different types of generalized regular and normal spaces has been one of the main areas of research in general topology during the last several decades. This paper has been written to continue this line of research.

In this study, we have introduced and studied two new space concepts via πge-closed sets. Also, we have examined the relationships between some regular and normal spaces defined in the literature and newly defined spaces (cf. Remarks 3.3 and 4.3). Moreover, we shed light on their place in topological spaces through examples.

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