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Abstract. In this paper, we show the existences of the residuated and Galois connections for various transformations on adjoint triples. Moreover, we investigate residuated connections and Galois connections on adjoint triples. Using the properties of residuated connections and Galois connections, we solve fuzzy relation equations and define various fuzzy concepts. we give their examples.

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1. Introduction

Ward et al. [1] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool as algebraic structures for many valued logics (See [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). However, this structure is very restrictive. As a weak condition, Abdel-Hamid [14] introduced the notion of adjoint triples. By using this concepts, Medina et al. [15, 16, 17] introduced the notion of formal concepts with $R \in L^{X\times Y}$ on an adjoint triple $(\&, \nearrow, \searrow)$ with respect to partially ordered sets $(L_i, \leq_i)_{i\in\{1,2,3\}}$.

Sanchez [18] introduced the theory of fuzzy relation equations with various types of composition: max-min, min-max, min-α. Fuzzy relation equations with new types of composition (See [19] for continuous t-norm and [6, 7, 20] for residuated lattice) is developed. Perfilieva [13, 21, 22, 23, 24] introduced the theory of fuzzy transform and inverse fuzzy transform in complete residuated lattices which is similar to the ones of well-known transform such as the Fourier, Laplace, Hilbert and wavelet transforms. It is used in signal and image processing, data analysis and neural network approaches (See [13, 24, 25]).

In this paper, we introduced residuated and Galois connections for various transformations on adjoint triples. Using the properties of residuated and Galois connections, we solve fuzzy relation equations with various operations. Moreover, we show that the families of fuzzy closure operators and fuzzy interior operators are complete lattices. We define various fuzzy concepts in Definition 3.10 using the residuated and Galois connections. The fuzzy concepts are complete lattices and we give their examples.

2. Preliminaries

Definition 2.1 ([14, 15, 16, 17])**.** Let $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$ be complete lattices. We say that the mappings $\& : L_1 \times L_2 \to L_3$, $\searrow: L_2 \times L_3 \to L_1$ and $\nearrow: L_1 \times L_3 \to L_2$ is called an *adjoint triple*, if it satisfies the following conditions:

$$x \leq_1 y \searrow z \text{ iff } x\&y \leq_3 z \text{ iff } y \leq_2 x \nearrow z \text{for each } x \in L_1,\ y \in L_2,\ z \in L_3.$$

Example 2.2 ([15, 16, 17])**.** Let $[0,1]_m$ be a regular partition of $[0,1]$ in m pieces with $[0,1]_m = \{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\}$. Let $\odot : [0,1] \times [0,1] \to [0,1]$ be a left continuous t-norm and $x \to y = \bigvee\{z \in [0,1] \mid x \odot z \leq y\}$. We define the operator $\&$: $[0,1]_m \times [0,1]_n \to [0,1]_k$ as $x\&y = \frac{[k(x\odot y)]}{k}$, where $[x] = \bigwedge\{n \in Z \mid x \leq n\}$ is the ceiling function. For this operator, the corresponding implication operators $\searrow: [0,1]_n \times [0,1]_k \to [0,1]_m$ and $\nearrow: [0,1]_m \times [0,1]_k \to [0,1]_n$ defined as

$$y \searrow z = \frac{\langle m(y \to z)\rangle}{m},\ \ x \nearrow z = \frac{\langle n(x \to z)\rangle}{n},$$

where $\langle x\rangle = \bigvee\{n \in Z \mid n \leq x\}$ is the floor function.

Let $x \leq y \searrow z = \frac{\langle m(y\to z)\rangle}{m}$. Since $x - 1 \leq \langle x\rangle \leq x$, $x \leq \frac{\langle m(y\to z)\rangle}{m} \leq \frac{m(y\to z)}{m} = y \searrow z$. Then $x \odot y \leq z$. Since $x \leq [x] < x + 1$,

$$x \odot y = \frac{[k(x \odot y)]}{k} < \frac{k(x \odot y) + 1}{k} \leq z + \frac{1}{k}.$$

Since $x \odot y \in [0,1]_k$ and $z = \frac{p}{k}$ for $p \in Z$, $x \odot y = \frac{[k(x\odot y)]}{k} \leq z$.

Let $x \odot y = \frac{[k(x\odot y)]}{k} \leq z$. Since $k(x \odot y) \leq [k(x \odot y)]$, $x \odot y \leq z$ iff $y \leq x \to z$. Then

$$x \nearrow z = \frac{\langle n(x \to z)\rangle}{n} > \frac{\langle n(x \to z)\rangle}{n} \geq y.$$

Other cases are similarly proved.

Example 2.3 ([15, 16, 17])**.** Let $\& : [0,1] \times [0,1] \to [0,1]$ be defined by $x\&y = x^2y$. We can obtain $\nearrow, \searrow: [0,1] \times [0,1] \to [0,1]$ as $x \nearrow z = \bigvee\{y \in [0,1] \mid x^2y \leq z\}$, $y \searrow z = \bigvee\{x \in [0,1] \mid x^2y \leq z\}$. Then $(\&, \searrow, \nearrow)$ is an adjoint triple with

$$x \nearrow z = \begin{cases} 1, & \text{if } x = 0 \\ \frac{z}{x^2} \wedge 1, & \text{otherwise,} \end{cases} \qquad y \searrow z = \begin{cases} 1, & \text{if } y = 0 \\ \sqrt{\frac{z}{y}} \wedge 1, & \text{otherwise.} \end{cases}$$

Lemma 2.4 ([6, 15, 16, 17])**.** *Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. For any x, y, z, x_i, y_i, $w \in L$, we have the following properties.*

(1) *If $x_1 \leq x_2$, then $x_1\&y \leq x_2\&y$.*
(2) *If $y_1 \leq y_2$, then $x\&y_1 \leq x\&y_2$.*

(3) $\searrow, \nearrow$ *are order-preserving on the second argument and order-reversing on the first argument.*
(4) $y \leq_2 x \nearrow (x\&y),\ x \leq_1 y \searrow (x\&y)$.
(5) $x\&(x \nearrow z) \leq_3 z,\ (y \searrow z)\&y \leq_3 z$.
(6) $y \leq_2 (y \searrow z) \nearrow z,\ x \leq_1 (x \nearrow z) \searrow z$.
(7) $(\bigvee_i x_i)\&y = \bigvee_i (x\&y_i)$ *and* $x\&(\bigvee_i y_i) = \bigvee_i (x\&y_i)$.
(8) $x \nearrow (\bigwedge_i z_i) = \bigwedge_i (x \searrow z_i)$ *and* $(\bigvee_i x_i) \nearrow z = \bigwedge_i (x_i \nearrow z)$.
(9) $y \searrow (\bigwedge_i z_i) = \bigwedge_i (y \searrow z_i)$ *and* $(\bigvee_i y_i) \searrow z = \bigwedge_i (y_i \searrow z)$.

Definition 2.5 ([2])**.** Let X be a set and $(L, \leq)$ be a complete lattice.
(i) An operator $C : L^X \to L^X$ is called a *fuzzy closure operator* on X, if it satisfies the following conditions: for all $f,\ g \in L^X$,
(C1) $f \leq C(f)$ and $C(C(f)) = C(f)$,
(C2) $f \leq g$ implies $C(f) \leq C(g)$.
(ii) An operator $I : L^X \to L^X$ is called a *fuzzy interior operator* on X, if it satisfies the conditions: for all $f,\ g \in L^X$,
(I1) $I(f) \leq f$ and $I(f) = I(I(f))$,
(I2) $f \leq g$implies $I(f) \leq I(g)$.

3. Various fuzzy transformations and fuzzy concept lattices on adjoint triples

Definition 3.1 ([2, 26])**.** Let X and Y be two sets and $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be complete lattices. Let $H,\ K_1 : L_1^X \to L_2^Y$ and $J, K_2 : L_2^Y \to L_1^X$.
(i) (L_1^X, H, J, L_2^Y) is called a *residuated connection*, if $H(f) \leq_2 g$ iff $f \leq_1 J(g)$ for all $f \in L_1^X,\ g \in L_2^Y$.
(2) (L_1^X, K_1, K_2, L_1^Y) is called a *Galois connection*, if $g \leq_2 K_1(f)$ iff $f \leq_1 K_2(g)$ for all $f \in L_1^X,\ g \in L_2^Y$.

Definition 3.2. Let $(L_1, \leq, \vee, \wedge)$ and $(L_2, \leq, \vee, \wedge)$ be two complete lattices.
(i) L_1 and L_2 are *isomorphic*, if there exists a bijective function $h : L_1 \to L_2$ such that $h(\bigvee_{i\in\Gamma} x_i) = \bigvee_{i\in\Gamma} h(x_i)$ and $h(\bigwedge_{i\in\Gamma} x_i) = \bigwedge_{i\in\Gamma} h(x_i)$ for each $i \in \Gamma,\ x_i \in L_1$.
(ii) L_1 and L_2 are *anti-isomorphic*, if there exists a bijective function $h : L_1 \to L_2$ such that $h(\bigvee_{i\in I} x_i) = \bigwedge_{i\in I} h(x_i)$ and $h(\bigwedge_{i\in I} x_i) = \bigvee_{i\in I} h(x_i)$ for all $x_i,\ i \in I$.

Definition 3.3. Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$.
(i) For $f \in L_1^X$ and $h \in L_3^Y$, define $\Delta_f : (L_2^Y)^X \to L_3^Y$ and $\Theta_h : (L_2^Y)^X \to L_1^X$ as follows: for each $y \in Y,\ x \in X$,

$$\Delta_f(\psi)(y) = \bigvee_{x\in X} (f(x)\&\psi(x)(y)), \quad \Theta_h(\psi)(x) = \bigwedge_{y\in X} (\psi(x)(y) \searrow h(y)).$$

(ii) For $f \in L_2^X$ and $h \in L_3^Y$, define $\Delta^f : (L_1^Y)^X \to L_3^Y$ and $\Theta^h : (L_1^Y)^X \to L_2^X$ as follows: for each $y \in Y,\ x \in X$,

$$\Delta^f(\psi)(y) = \bigvee_{x\in X} (\psi(x)(y)\&f(x)), \quad \Theta^h(\psi)(x) = \bigwedge_{y\in X} (\psi(x)(y) \nearrow h(y)).$$

(3) For $f \in L_1^X$ and $g \in L_2^Y$, define $\Lambda : (L_3^Y)^X \to L_2^Y$ and $\Psi : (L_3^Y)^X \to L_1^X$ as follows: for each $y \in Y,\ x \in X$,

$$\Lambda^f(\psi)(y) = \bigwedge_{x\in X} (f(x) \nearrow \psi(x)(y)), \quad \Psi_g(\psi)(x) = \bigwedge_{y\in X} (g(y) \searrow \psi(x)(y)).$$

Theorem 3.4. *Let* $(\&, \searrow, \nearrow)$ *be an adjoint triple with respect to* $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. *Then the following properties hold.*

(1) $\Delta_f : (L_2^Y)^X \to L_3^Y$ *is a join preserving map. Then there exists a unique meet preserving map* $\mathcal{J}_{\Delta_f} : L_3^Y \to (L_2^Y)^X$ *such that* $\mathcal{J}_{\Delta_f}(h)(x)(y) = f(x) \nearrow h(y)$. *Moreover,* $((L_2^Y)^X, \Delta_f, \mathcal{J}_{\Delta_f}, L_3^Y)$ *is a residuated connection.*

(2) $\Delta^f : (L_1^Y)^X \to L_3^Y$ *is a join preserving map. Then there exists a unique meet preserving map* $\mathcal{J}_{\Delta^f} : L_3^Y \to (L_1^Y)^X$ *such that* $\mathcal{J}_{\Delta^f}(g)(x)(y) = f(x) \searrow g(y)$. *Moreover,* $((L_1^Y)^X, \Delta^f, \mathcal{J}_{\Delta^f}, L_3^Y)$ *is a residuated connection.*

(3) $\Theta_h : (L_2^Y)^X \to L_1^X$ *is a join-meet preserving map. Then there exists a unique join-meet preserving map* $\mathcal{K}_{\Theta_h} : L_1^X \to (L_2^Y)^X$ *such that* $\mathcal{K}_{\Theta_h}(f)(x)(y) = f(x) \nearrow h(y) = K^h(f)(x)$. *Moreover,* $((L_2^Y)^X, \Theta_h, \mathcal{K}_{\Theta_h}, L_1^X)$ *is a Galois connection.*

(4) $\Theta^h : (L_1^Y)^X \to L_2^X$ *is a join-meet preserving map. Then there exists a unique join-meet preserving map* $\mathcal{K}_{\Theta^h} : L_2^X \to (L_1^Y)^X$ *such that* $\mathcal{K}_{\Theta^h}(f)(x)(y) = f(x) \searrow h(y)$. *Moreover,* $((L_1^Y)^X, \Theta^h, \mathcal{K}_{\Theta^h}, L_2^X)$ *is a Galois connection.*

(5) $\Lambda^f : (L_3^Y)^X \to L_2^Y$ *is a meet preserving map. Then there exists a unique join preserving map* $\mathcal{H}_{\Lambda^f} : L_2^Y \to (L_3^Y)^X$ *such that* $\mathcal{H}_{\Lambda^f}(g)(x)(y) = f(x)\&g(y)$. *Moreover,* $(L_2^Y, \mathcal{H}_{\Lambda^f}, \Lambda^f, (L_3^Y)^X)$ *is a residuated connection.*

(6) $\Psi_g : (L_3^Y)^X \to L_1^X$ *is a meet preserving map. Then there exists a unique join preserving map* $\mathcal{H}_{\Psi_g} : L_1^Y \to (L_3^Y)^X$ *such that* $\mathcal{H}_{\Psi_g}(x)(y) = f(x)\&g(y)$. *Moreover,* $(L_1^Y, \mathcal{H}_{\Psi_g}, \Psi_g, (L_3^Y)^X)$ *is a residuated connection.*

Proof. (1) By Lemma 2.4 (7), $\Delta_f(\bigvee_{i\in\Gamma} \psi_i) = \bigvee_{i\in\Gamma} \Delta_f(\psi_i)$. Define $\mathcal{J}_{\Delta_f} : L_3^Y \to (L_2^Y)^X$ as

$$\mathcal{J}_{\Delta_f}(h)(x) = \bigvee\{\psi(x) \in L_2^Y \mid \Delta_f(\psi)(y) \leq_3 h(y)\}.$$

Since $\bigvee_{x\in X}(f(x)\&\psi(x)(y)) \leq_3 h(y)$, we have $\psi(x)(y) \leq_2 f(x) \nearrow h(y)$. Then $\mathcal{J}_{\Delta_f}(h)(x)(y) \leq_2 f(x) \nearrow h(y)$.

Since

$$\Delta_f(f \nearrow h)(y) = \bigvee_{x\in X}(f(x)\&(f(x) \to h(y))) \leq_3 h(y),$$

we get $\mathcal{J}_{\Delta_f}(h)(x)(y) \geq_2 f(x) \nearrow h(y)$. Thus $\mathcal{J}_{\Delta_f}(h)(x)(y) = f(x) \nearrow h(y)$.

Since

$$\begin{aligned} \mathcal{J}_{\Delta_f}(\textstyle\bigwedge_{i\in\Gamma} h_i)(x)(y) &= f(x) \nearrow \textstyle\bigwedge_{i\in\Gamma} h_i(y) \\ &= \textstyle\bigwedge_{i\in\Gamma}(f(x) \nearrow h_i(y)) = \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(h_i)(x)(y), \end{aligned}$$

we have $\mathcal{J}_{\Delta_f}(\bigwedge_{i\in\Gamma} h_i) = \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(h_i)$. Moreover, for $\psi \in (L_2^Y)^X, h \in L_3^Y$,

$$\begin{aligned} \Delta_f(\psi)(y) \leq_3 h(y) \quad &\text{iff} \ \ \textstyle\bigvee_{x\in X}(f(x)\&\psi(x)(y)) \leq_3 h(y) \\ &\text{iff} \ \ \psi(x)(y) \leq_2 (f(x) \nearrow h(y)) = \mathcal{J}_{\Delta_f}(h)(x)(y). \end{aligned}$$

(3) The map Θ_h is a join-meet preserving map because

$$\begin{aligned} \Theta_h(\textstyle\bigvee_{i\in\Gamma} \psi_i)(x) &= \textstyle\bigwedge_{y\in Y}(\bigvee_{i\in\Gamma} \psi_i(x)(y) \searrow h(y)) \\ &= \textstyle\bigwedge_{y\in X}(\bigwedge_{i\in\Gamma}(\psi_i(x)(y) \searrow h(y))) \ \ \text{(by Lemma 2.4 (8))} \\ &= \textstyle\bigwedge_{i\in\Gamma}(\bigwedge_{y\in X}(\psi_i(x)(y) \searrow h(y))) = \bigwedge_{i\in\Gamma}(\Theta_h(\psi_i)). \end{aligned}$$

Define $\mathcal{K}_{\Theta_h} : L_1^X \to (L_2^Y)^X$ as $\mathcal{K}_{\Theta_h}(f)(x) = \bigvee\{\psi(x) \in L_2^Y \mid f(x) \leq_1 \Theta_h(\psi)(x)\}$. Since $f(x) \leq_1 \bigwedge_{y\in X}(\psi(x)(y) \searrow h(y))$, we have $\psi(x)(y) \leq_2 f(x) \nearrow h(y)$. Then $\mathcal{K}_{\Theta_h}(f)(x)(y) \leq_2 f(x) \nearrow h(y)$.

Since $\Theta_h(f \nearrow h)(x) = \bigwedge_{y\in Y}((f \nearrow h)(x)(y) \searrow h(y)) \geq_1 (f(x) \nearrow h(y)) \searrow h(y) \geq_1 f(x)$, $\mathcal{K}_{\Theta_h}(f)(x)(y) \geq_2 f(x) \nearrow h(y)$. Thus $\mathcal{K}_{\Theta_h}(f)(x)(y) = f(x) \nearrow h(y)$. So we have

$$\mathcal{K}_{\Theta_h}(\bigvee_{i\in\Gamma} f_i)(x)(y) = (\bigvee_{i\in\Gamma} f_i(x)) \searrow h(y) = \bigwedge_{i\in\Gamma}(f_i(x) \searrow h(y)) = \bigwedge_{i\in\Gamma} \mathcal{K}_{\Theta_h}(f_i)(x)(y).$$

Moreover, for $\psi \in (L_2^Y)^X, h \in L_3^Y$,

$$\begin{array}{ll} f(x) \leq_1 \Theta_h(\psi)(x) & \text{iff } f(x) \leq_1 \bigwedge_{y\in Y}(\psi(x)(y) \searrow h(y))) \\ & \text{iff } \psi(x)(y) \leq_2 (f(x) \nearrow h(y)) = \mathcal{K}_{\Theta_h}(f)(x)(y). \end{array}$$

(5) By Lemma 2.4 (8), $\Lambda^f(\bigwedge_{i\in\Gamma} \psi_i) = \bigwedge_{i\in\Gamma} \Lambda^f(\psi_i)$. Define $\mathcal{H}_{\Lambda^f} : L_2^Y \to (L_3^Y)^X$ as

$$\mathcal{H}_{\Lambda^f}(g)(x) = \bigwedge\{\psi(x) \in L_3^Y \mid g(y) \leq_2 \Lambda^f(\psi)(y)\}.$$

Since $g(y) \leq_2 \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y))$, we have $f(x)\&g(y) \leq_3 \psi(x)(y)$. Then $\mathcal{H}_{\Lambda^f}(g)(x)(y) \geq_3 f(x)\&g(y)$. Since $\Lambda^f(f\&g)(y) = \bigwedge_{x\in X}(f(x) \nearrow f(x)\&g(y)) \geq_2 g(y)$, $\mathcal{H}_{\Lambda^f}(g)(x)(y) \leq_3 f(x)\&g(y)$. Thus we get

$$\mathcal{H}_{\Lambda^f}(\bigvee_{i\in\Gamma} g_i)(x)(y) = f(x)\&(\bigvee_{i\in\Gamma} g_i(y)) = \bigvee_{i\in\Gamma}(f(x)\&g_i(y)) = \bigvee_{i\in\Gamma} \mathcal{H}_{\Lambda^f}(g_i)(y).$$

Moreover,

$$\begin{array}{ll} g(y) \leq_2 \Lambda^f(\psi)(y) & \text{iff } g(y) \leq_2 \bigwedge_{x\in Y}(f(x) \nearrow \psi(x)(y)) \\ & \text{iff } f(x)\&g(y) \leq_3 \psi(x)(y) \\ & \text{iff } \mathcal{H}_{\Lambda^f}(g)(x)(y) \leq_3 \psi(x)(y). \end{array}$$

So $(L_2^Y, \mathcal{H}_{\Lambda^f}, \Lambda^f, (L_3^Y)^X)$ is a residuated connection.

The proofs of (2), (4) and (6) are similarly proved as (1),(3) and(5), respectively. □

Theorem 3.5. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Define operators as follows:*

$$\begin{array}{lll} H_f(g)(x)(y) & = f(x)\&g(y) & \text{for all } f \in L_1^X, \\ H^g(f)(x)(y) & = f(x)\&g(y) & \text{for all } g \in L_2^X, \\ G^f(g)(x)(y) & = f(x) \nearrow g(y) & \text{for all } f \in L_1^X, \\ G_f(g)(x)(y) & = f(x) \searrow g(y) & \text{for all } f \in L_2^X, \\ K^f(g)(x)(y) & = g(y) \nearrow f(x) & \text{for all } f \in L_3^X, \\ K_f(g)(x)(y) & = g(y) \searrow f(x) & \text{for all } f \in L_3^X. \end{array}$$

(1) $H_f : L_2^Y \to (L_3^Y)^X$ is a join preserving map. Then there exists a unique meet preserving map $\mathcal{J}_{H_f} : (L_3^Y)^X \to L_2^Y$ such that

$$\mathcal{J}_{H_f}(\psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)) = \Lambda^f(\psi)(y).$$

Moreover, $(L_2^Y, H_f, \mathcal{J}_{H_f}, (L_3^Y)^X)$ is a residuated connection.

(2) $H^g : L_1^X \to (L_3^Y)^X$ is a join preserving map. Then there exists a unique meet preserving map $\mathcal{J}_{H^g} : (L_3^Y)^X \to L_1^X$ such that

$$\mathcal{J}_{H^g}(\psi)(y) = \bigwedge_{x\in X}(g(y) \searrow \psi(x)(y)) = \Lambda^g(\psi)(y).$$

Moreover, $(L_1^Y, H^g, \mathcal{J}_{H^g}, (L_3^Y)^X)$ *is a residuated connection.*

(3) $G^f : L_3^Y \to (L_2^Y)^X$ *is a meet preserving map. Then there exists a unique join preserving map* $\mathcal{H}_{G^f} : (L_2^Y)^X \to L_3^Y$ *such that*

$$\mathcal{H}_{G^f}(\psi)(y) = \bigvee_{x\in X} (f(x)\&\psi(x)(y)) = \Delta^f(\psi)(y).$$

Moreover, $((L_2^Y)^X, \mathcal{H}_{G^f}, G^f, L_3^Y)$ *is a residuated connection.*

(4) $G_f : L_3^Y \to (L_1^Y)^X$ *is a meet preserving map. Then there exists a unique join preserving map* $\mathcal{H}_{G_f} : (L_1^Y)^X \to L_3^Y$ *such that*

$$\mathcal{H}_{G^f}(\psi)(y) = \bigvee_{x\in X} (\psi(x)(y)\&f(x)).$$

Moreover, $((L_1^Y)^X, \mathcal{H}_{G_f}, G_f, L_3^Y)$ *is a residuated connection.*

(5) $K^f : L_1^Y \to (L_2^Y)^X$ *is a join-meet preserving map. Then there exists a unique meet preserving map* $\mathcal{K}_{K^f} : (L_2^Y)^X \to L_1^Y$ *such that*

$$\mathcal{K}_{K^f}(\psi)(y) = \bigwedge_{x\in X} (\psi(x)(y) \searrow f(x)).$$

Moreover, $(L_1^Y, K^f, \mathcal{K}_{K^f}, (L_2^Y)^X)$ *is a Galois connection.*

(6) $K_f : L_2^Y \to (L_1^Y)^X$ *is a join-meet preserving map. Then there exists a unique meet preserving map* $\mathcal{K}_{K_f} : (L_1^Y)^X \to L_2^Y$ *such that*

$$\mathcal{K}_{K_f}(\psi)(y) = \bigwedge_{x\in X} (\psi(x)(y) \nearrow f(x)).$$

Moreover, $(L_2^Y, K_f, \mathcal{K}_{K_f}, (L_1^Y)^X)$ *is a Galois connection.*

Proof. (1) Since $H_f(\bigvee_{i\in I} g_i)(x)(y) = f(x)\&(\bigvee_{i\in I} g_i)(y) = \bigvee_{i\in I}(f(x)\&g_i(y)) = \bigvee_{i\in I} H_f(g_i)(x)(y)$, we define $\mathcal{J}_{H_f} : (L_3^Y)^X \to L_2^Y$ as

$$\mathcal{J}_{H_f}(\psi) = \bigvee\{g \in L_2^Y \mid H_f(g) \leq_3 \psi\}.$$

Since $H_f(g)(x)(y) = f(x)\&g(y) \leq_3 \psi(x)(y)$, $g(y) \leq_2 f(x) \nearrow \psi(x)(y)$. Then

$$\mathcal{J}_{H_f}(\psi)(y) \leq \bigwedge_{x\in X} (f(x) \nearrow \psi(x)(y)) = \Lambda^f(\psi)(y).$$

Since $H_f(\bigwedge_{x\in X}(f(x) \nearrow \psi(x))(y) = f(x)\& \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)) \leq \psi(x)(y)$,

$$\mathcal{J}_{H_f}(\psi)(y) \geq \bigwedge_{x\in X} (f(x) \nearrow \psi(x)(y)) = \Lambda^f(\psi)(y).$$

Thus $\mathcal{J}_{H_f}(\psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)) = \Lambda^f(\psi)(y)$. By Theorem 3.4 (5), $\mathcal{J}_{H_f} = \Lambda^f$ is a meet preserving map. Moreover, it holds that $(L_2^Y, H_f, \mathcal{J}_{H_f}, (L_3^Y)^X)$ is a residuated connection.

(3) Since $G^f(\bigwedge_{i\in I} g_i)(x)(y) = f(x) \nearrow (\bigwedge_{i\in I} g_i)(y) = \bigwedge_{i\in I}(f(x) \nearrow g_i(y)) = \bigwedge_{i\in I} G^f(g_i)(x)(y)$, we define $\mathcal{H}_{G^f} : (L_2^Y)^X \to L_3^Y$ as

$$\mathcal{H}_{G^f}(\psi) = \bigwedge\{g \in L_3^Y \mid \psi \leq_2 G^f(g)\}.$$

Since $\psi(x)(y) \leq_2 f(x) \nearrow g(y)$, $f(x)\&\psi(x)(y) \leq_3 g(y)$. Then

$$\mathcal{H}_{G^f}(y) \geq \bigvee_{x\in X} (f(x)\&\psi(x)(y)) = \Delta_f(\psi)(y).$$

Since $G^f(\bigvee_{x\in X}(f(x)\&\psi(x)))(y) = f(x) \nearrow \bigvee_{x\in X}(f(x)\&\psi(x)(y)) \geq \psi(x)(y)$,

$$\mathcal{H}_{G^f}(y) \leq \bigvee_{x\in X} (f(x)\&\psi(x)(y)) = \Delta_f(\psi)(y).$$

Thus $\mathcal{H}_{G^f}(y) = \bigvee_{x\in X}(f(x)\&\psi(x)(y)) = \Delta_f(\psi)(y)$. By Theorem 3.4 (1), $\mathcal{H}_{G^f} = \Delta_f$ is a join preserving map. Moreover, it holds that $((L_2^Y)^X, \mathcal{H}_{G^f}, G^f, L_3^Y)$ is a residuated connection.

(5) By Lemma 2.4 (8),

$$\begin{aligned} K^f(\textstyle\bigvee_{i\in I} g_i)(x)(y) &= (\textstyle\bigvee_{i\in I} g_i(y)) \nearrow f(x) \\ &= \textstyle\bigwedge_{i\in I}(g_i(y) \nearrow f(x)) = \bigwedge_{i\in I} K^f(g_i)(x)(y). \end{aligned}$$

Define $\mathcal{K}_{K^f} : (L_1^Y)^X \to L_2^Y$ as $\mathcal{K}_{K^f}(\psi) = \bigvee\{g \in L_2^Y \mid \psi \leq_2 K^f(g)\}$. Since $\psi(x)(y) \leq_2 K^f(g)(x)(y) = g(y) \nearrow f(x)$, $g(y) \leq_1 \psi(x)(y) \searrow f(x)$. Then we have $\mathcal{K}_{K^f}(g)(y) \leq \bigwedge_{x\in X}(\psi(x)(y) \searrow f(x))$. Moreover, $\mathcal{K}_{K^f}(g)(y) \geq \bigwedge_{x\in X}(\psi(x)(y) \searrow f(x))$ from

$$\begin{aligned} K^f(\textstyle\bigwedge_{x\in X}(\psi(x) \searrow f(x)))(y) &= \textstyle\bigwedge_{x\in X}(\psi(x)(y) \searrow f(x)) \nearrow f(x) \\ &= \textstyle\bigvee_{x\in X}((\psi(x)(y) \searrow f(x)) \nearrow f(x)) \geq_2 \psi(x)(y). \end{aligned}$$

Thus $\mathcal{K}_{K^f}$ is a join-meet preserving map. So $(L_2^Y, K^f, \mathcal{K}_{K^f}, (L_1^Y)^X)$ is a Galois connection.

The proofs of (2), (4) and (6) are similarly proved as (1),(3) and (5), respectively. □

Theorem 3.6. *Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Then the following properties hold.*

(1) *$((L_2^Y)^X, \Delta_f, \mathcal{J}_{\Delta_f}, L_3^Y)$ is a residuated connection. Moreover $\bigvee_{i\in\Gamma} \Delta_f(\psi_i) = \Delta_f(\bigvee_{i\in\Gamma} \psi_i)$ and $\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(h_i) = \mathcal{J}_{\Delta_f}(\bigwedge_{i\in\Gamma} h_i)$ for all $\psi_i \in (L_2^Y)^X$ and $h_i \in L_3^Y$.*

(2) *$\Delta_f(\mathcal{J}_{\Delta_f}(h)) \leq_3 h$ iff $\psi \leq_2 \mathcal{J}_{\Delta_f}(\Delta_f(\psi))$ for all $\psi \in (L_2^Y)^X, h \in L_3^Y$,*

(3) *If $\psi_1 \leq_2 \psi_2$ and $h_1 \leq_3 h_2$, then $\Delta_f(\psi_1) \leq_3 \Delta_f(\psi_2)$ for all $\psi_1, \psi_2 \in (L_2^Y)^X$ and $\mathcal{J}_{\Delta_f}(h_1) \leq_2 \mathcal{J}_{\Delta_f}(h_2)$ for all $h_1, h_2 \in L_3^Y$,*

(4) *$\Delta_f(\psi) = \Delta_f(\mathcal{J}_{\Delta_f}(\Delta_f(\psi)))$ for all $f \in (L_2^Y)^X$. If $\psi = \psi_0$ is a solution of $\Delta_f(\psi) = h$, then $\mathcal{J}_{\Delta_f}(h)$ is a solution of $\Delta_f(\psi) = h$ such that $\psi_0 \leq_2 \mathcal{J}_{\Delta_f}(h)$.*

(5) *$\mathcal{J}_{\Delta_f}(\Delta_f(\mathcal{J}_{\Delta_f}(h))) = \mathcal{J}_{\Delta_f}(h)$ for all $h \in L_3^Y$. If $h = h_1$ is a solution of $\mathcal{J}_{\Delta_f}(h) = \psi$, then $\Delta_f(\psi)$ is a solution of $\mathcal{J}_{\Delta_f}(h) = \psi$ such that $\Delta_f(\psi) \leq_3 h_1$.*

(6) *$\Delta_f \circ \mathcal{J}_{\Delta_f} : L_3^Y \to L_3^Y$ is a fuzzy interior operator.*

(7) *$\mathcal{J}_{\Delta_f} \circ \Delta_f : (L_2^Y)^X \to (L_2^Y)^X$ is a fuzzy closure operator.*

(8) *Define $\sqcap_{i\in\Gamma} g_i = \Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i)), \bigvee_{i\in\Gamma} g_i$ for all $\{g_i\}_{i\in\Gamma} \subseteq I(L_3^Y) = \{g \in L_3^Y \mid g = \Delta_f \circ \mathcal{J}_{\Delta_f}(g)\}$. Then $(I(L_3^Y), \sqcap, \bigvee)$ is a complete lattice.*

(9) *Define $\bigwedge_{i\in\Gamma} \psi_i, \sqcup_{i\in\Gamma}\psi_i = \mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\psi_i))$ for all $\{\psi_i\}_{i\in\Gamma} \subseteq C((L_2^Y)^X) = \{\psi \in (L_2^Y)^X \mid \psi = \mathcal{J}_{\Delta_f} \circ \Delta_f(\psi)\}$. Then $(C((L_2^Y)^X), \bigwedge, \sqcup)$ is a complete lattice.*

(10) *$I(L_3^Y)$ and $C((L_2^Y)^X)$ are isomorphic.*

(11) $\Delta_f(\mathcal{J}_{\Delta_f}(g)) = \bigvee\{h \in I(L_3^Y) \mid h \leq_3 g\}$ *for all* $g \in L_3^Y$ *and* $\mathcal{J}_{\Delta_f}(\Delta_f(\psi)) = \bigwedge\{\rho \in C((L_2^Y)^X) \mid \psi \leq_3 \rho\}$ *for all* $\psi \in (L_2^Y)^X$.

Proof. (1) It follows from Theorem 3.4 (1).

(2) For $\psi \in (L_2^Y)^X$, $\Delta_f(\psi)(y) \leq_3 \Delta_f(\psi)(y)$ iff $\psi(x)(y) \leq_2 \mathcal{J}_{\Delta_f}(\Delta_f(\psi))(x)(y)$.

For $h \in L_3^Y$, $\mathcal{J}_{\Delta_f}(h)(x)(y) \leq_2 \mathcal{J}_{\Delta_f}(h)(x)(y)$ iff $\Delta_f(\mathcal{J}_{\Delta_f}(h))(y) \leq_3 h(y)$.

(3) Since $\psi_1(x) \leq_2 \psi_2(x) \leq_2 \mathcal{J}_{\Delta_f}(\Delta_f(\psi_2))(x)$, $\Delta_f(\psi_1)(y) \leq \Delta_f(\psi_2)(y)$. Moreover, since $\Delta_f(\mathcal{J}_{\Delta_f}(h_1))(y) \leq_3 h_1(y) \leq_3 h_2(y)$, $\mathcal{J}_{\Delta_f}(h_1)(x) \leq_2 \mathcal{J}_{\Delta_f}(h_2)(x)$.

(4) By (2), $\Delta_f(\psi) = \Delta_f(\mathcal{J}_{\Delta_f}(\Delta_f(\psi)))$ for all $\psi \in (L_2^Y)^X$. If $\Delta_f(\psi_0) = h$, then $\Delta_f(\mathcal{J}_{\Delta_f}(\Delta_f(\psi_0))) = \Delta_f(\mathcal{J}_{\Delta_f}(h)) = \Delta_f(\psi_0) = h$. Moreover, $\psi_0 \leq_2 \mathcal{J}_{\Delta_f}(\Delta_f(\psi_0)) = \mathcal{J}_{\Delta_f}(h)$.

(5) It is similarly proved as (4).

(6) For each $h, h_1, h_2 \in L_3^Y$, $\Delta_f \circ \mathcal{J}_{\Delta_f}(h) \leq_3 h$ and $(\Delta_f \circ \mathcal{J}_{\Delta_f})(\Delta_f \circ \mathcal{J}_{\Delta_f})(h) = \Delta_f \circ \mathcal{J}_{\Delta_f}(h)$. If $h_1 \leq_3 h_2$, then

$$(\Delta_f \circ \mathcal{J}_{\Delta_f})(h_1) \leq_3 (\Delta_f \circ \mathcal{J}_{\Delta_f})(h_2)$$

(7) It is similarly proved as (6).

(8) By (5), since $\Delta_f(\mathcal{J}_{\Delta_f}(\Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i)))) = \Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i))$, $\sqcap_{i\in\Gamma} g_i = \Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i)) \in I(L_3^Y)$.

Let $g \leq_3 g_i$ for all g, $g_i \in I(L_3^Y)$. Then $\mathcal{J}_{\Delta_f}(g) \leq_2 \mathcal{J}_{\Delta_f}(g_i)$ for all i. Thus $\mathcal{J}_{\Delta_f}(g) \leq_2 \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i)$. So $g = \Delta_f(\mathcal{J}_{\Delta_f}(g)) \leq_3 \Delta_f(\bigwedge \mathcal{J}_{\Delta_f}(g_i)) \leq_3 \Delta_f(\mathcal{J}_{\Delta_f}(g_i)) = g_i$. Hence $\sqcap_{i\in\Gamma} g_i = \Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i))$ is an infimum of for all g_i.

(9) By (6), since $\mathcal{J}_{\Delta_f}(\Delta_f(\mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\psi_i)))) = \mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\psi_i))$, $\sqcup_{i\in\Gamma} \psi_i = \mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\psi_i)) \in C((L_2^Y)^X)$.

Let $\psi_i \leq_2 \psi$ for each ψ_i, $i \in \Gamma$, $\psi \in C(L_2^X)$. Then $\Delta_f(\psi_i) \leq_3 \Delta_f(\psi)$ for each $i \in \Gamma$. Thus $\bigvee_{i\in\Gamma} \Delta_f(\psi_i) \leq_3 \Delta_f(\psi)$ implies $\mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\psi_i)) \leq_2 \mathcal{J}_{\Delta_f}(\Delta_f(\psi)) = \psi$. Moreover, $\mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\psi_i)) \geq_2 \mathcal{J}_{\Delta_f}(\Delta_f(\psi_i)) = \psi_i$. So $\sqcup_{i\in\Gamma} \psi_i = \mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\psi_i))$ is a supremum of for all ψ_i, $i \in \Gamma$.

(10) Define $\mathcal{J}_{\Delta_f} : I(L_3^Y) \to C((L_2^Y)^X)$ as $\mathcal{J}_{\Delta_f}(g)(x) = \bigwedge_{y\in X}(\psi_y(x) \nearrow g(y))$. If $g_1 = \Delta_f(\mathcal{J}_{\Delta_f}(g_1)) = g_2 = \Delta_f(\mathcal{J}_{\Delta_f}(g_2)) \in I(L_3^Y)$, then

$$\mathcal{J}_{\Delta_f}(g_1) = \mathcal{J}_{\Delta_f}(\Delta_f(\mathcal{J}_{\Delta_f}(g_1)) = \mathcal{J}_{\Delta_f}(\Delta_f(\mathcal{J}_{\Delta_f}(g_2)) = \mathcal{J}_{\Delta_f}(g_1).$$

Thus $\mathcal{J}_{\Delta_f}$ is well defined. If $\mathcal{J}_{\Delta_f}(g_1) = \mathcal{J}_{\Delta_f}(g_2)$, then $g_1 = \Delta_f(\mathcal{J}_{\Delta_f}(g_1)) = \Delta_f(\mathcal{J}_{\Delta_f}(g_2)) = g_2$. Thus $\mathcal{J}_{\Delta_f}$ is injective. For $\psi \in C((L_2^Y)^X)$, $\psi = \mathcal{J}_{\Delta_f}(\Delta_f(\psi))$. So $\mathcal{J}_{\Delta_f}$ is surjective. Hence $\mathcal{J}_{\Delta_f}$ is bijective.

On the other hand, since $\mathcal{J}_{\Delta_f} \circ \Delta_f$ is a fuzzy closure operator,

$$\begin{array}{ll} \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i) & \leq_2 \mathcal{J}_{\Delta_f}(\Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i))), \\ \mathcal{J}_{\Delta_f}(\Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i))) & \leq_2 \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(\Delta_f(\mathcal{J}_{\Delta_f}(g_i))) = \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i). \end{array}$$

Then $\mathcal{J}_{\Delta_f}(\Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i))) = \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i) \in I(L_3^Y)$. Thus

$$\begin{array}{ll} \mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} g_i) & = \mathcal{J}_{\Delta_f}(\bigvee_{i\in\Gamma} \Delta_f(\mathcal{J}_{\Delta_f}(g_i))) = \sqcup_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i), \\ \mathcal{J}_{\Delta_f}(\sqcap_{i\in\Gamma} g_i) & = \mathcal{J}_{\Delta_f}(\Delta_f(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i)))) = \bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta_f}(g_i). \end{array}$$

So $I(L_3^Y)$ and $C(L_2^X)$ are isomorphic.

(11) Put $p = \bigvee\{h \in I(L_3^Y) \mid h \leq_3 g\}$. By (10), $p \in I(L_3^Y)$. Since $p \leq_3 g$, $p = \Delta_f(\mathcal{J}_{\Delta_f}(p)) \leq_3 \Delta_f(\mathcal{J}_{\Delta_f}(g))$.

Since $\Delta_f(\mathcal{J}_{\Delta_f}(\Delta_f(\mathcal{J}_{\Delta_f}(g)))) = \Delta_f(\mathcal{J}_{\Delta_f}(g))$, $\Delta_f(\mathcal{J}_{\Delta_f}(g)) \in I(L_3^Y)$. Then by (7), $\Delta_f(\mathcal{J}_{\Delta_f}(g)) \leq_3 g$, $\Delta_f(\mathcal{J}_{\Delta_f}(g)) \leq_3 p$. Thus $\Delta_f(\mathcal{J}_{\Delta_f}(g)) = \bigvee\{h \in I(L_3^Y) \mid h \leq_3 g\}$.

Other case is similarly proved. □

Remark 3.7. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Then the residuated connections in Theorems 3.4 and 3.5 hold the similar properties in Theorem 3.6.

Theorem 3.8. *Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Then the following properties hold.*

(1) *$((L_1^Y)^X, \Theta^h, \mathcal{K}_{\Theta^h}, L_2^X)$ is a Galois connection and $\bigwedge_{i\in\Gamma} \Theta^h(\psi_i) = \Theta^h(\bigvee_{i\in\Gamma} \psi_i)$ and $\bigwedge_{i\in\Gamma} \mathcal{K}_{\Theta^h}(f_i) = \mathcal{K}_{\Theta^h}(\bigvee_{i\in\Gamma} f_i)$ for all $\psi_i \in (L_1^Y)^X$ and $f_i \in L_2^X$.*

(2) *$f \leq_2 \Theta^h(\mathcal{K}_{\Theta^h}(f))$ iff $\psi \leq_1 \mathcal{K}_{\Theta^h}(\Theta^h(\psi))$ for all $\psi \in (L_1^Y)^X$ and $f \in L_2^X$.*

(3) *If $\psi_1 \leq_1 \psi_2$ for all $\psi_1, \psi_2 \in (L_1^Y)^X$, then $\Theta^h(\psi_2)(y) \leq_2 \Theta^h(\psi_1)(y)$.*

(4) *If $f_1 \leq_2 f_2$ for all $f_1, f_2 \in L_2^X$, then $\mathcal{K}_{\Theta^h}(f_2) \leq_1 \mathcal{K}_{\Theta^h}(f_1)$.*

(5) *$\Theta^h(\psi) = \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi)))$ for all $\psi \in (L_1^Y)^X$. If $\psi = \psi_1$ is a solution of $\Theta^h(\psi) = f_1$, then $\psi = \mathcal{K}_{\Theta^h}(f_1)$ is a solution of $\Theta^h(\psi) = f_1$ such that $\psi_1 \leq_1 \mathcal{K}_{\Theta^h}(f_1)$.*

(6) *$\mathcal{K}_{\Theta^h}(\Theta^h(\mathcal{K}_{\Theta^h}(g))) = \mathcal{K}_{\Theta^h}(g)$ for all $g \in L_2^Y$. If $f = f_1$ is a solution of $\mathcal{K}_{\Theta^h}(g) = \psi_1$, then $f = \Theta^h(\psi_1)$ is a solution of $\mathcal{K}_{\Theta^h}(f) = \psi_1$ such that $f_1 \leq_2 \Theta^h(\psi_1)$.*

(7) *$\mathcal{K}_{\Theta^h} \circ \Theta^h : (L_1^Y)^X \to (L_1^Y)^X$ and $\Theta^h \circ \mathcal{K}_{\Theta^h} : L_2^X \to L_2^X$ are fuzzy closure operators.*

(8) *Define $\sqcup^k_{i\in\Gamma}\psi_i = \mathcal{K}_{\Theta^h}(\bigwedge_{i\in\Gamma} \Theta^h(\psi_i))$ for all $\{\psi_i\}_{i\in\Gamma} \subseteq K((L_1^Y)^X) = \{\psi \in (L_1^Y)^X \mid \mathcal{K}_{\Theta^h}(\Theta^h(\psi)) = \psi\}$. Then $(K((L_1^Y)^X), \bigwedge, \sqcup^k)$ is a complete lattice.*

(9) *Define $\sqcup^h_{i\in\Gamma} g_i = \Theta^h(\bigwedge_{i\in\Gamma} \mathcal{K}_{\Theta^h}(g_i))$ for all $\{g_i\}_{i\in\Gamma} \subseteq H(L_2^Y) = \{g \in L_2^Y \mid \Theta^h(\mathcal{K}_{\Theta^h}(g)) = g\}$. Then $(H(L_2^Y), \bigwedge, \sqcup^h)$ is a complete lattice.*

(10) *$K((L_1^Y)^X)$ and $H(L_2^Y)$ are anti-isomorphic.*

(11) *$\mathcal{K}_{\Theta^h}(\Theta^h(f)) = \bigwedge\{p \in K((L_1^Y)^X) \mid f \leq_1 p\}$ for all $f \in (L_1^Y)^X$ and $\Theta^h(\mathcal{K}_{\Theta^h}(g)) = \bigwedge\{q \in H(L_2^Y) \mid g \leq_2 q\}$ for all $g \in L_2^Y$.*

Proof. (1) It follows from Theorem 3.4 (4).

(2) It follows from $\mathcal{K}_{\Theta^h}(f) \leq_1 \mathcal{K}_{\Theta^h}(f)$ iff $f \leq_2 \Theta^h(\mathcal{K}_{\Theta^h}(f))$ and $\Theta^h(\psi) \leq_2 \Theta^h(\psi)$ iff $\psi \leq_1 \mathcal{K}_{\Theta^h}(\Theta^h(\psi))$.

(3) Since $\psi_1 \leq_1 \psi_2 \leq_1 \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2))$, $\Theta^h(\psi_2) \leq_2 \Theta^h(\psi_1)$.

(4) Since $f_1 \leq_2 f_2 \leq_2 \Theta^h(\mathcal{K}_{\Theta^h}(f_2))$, $\mathcal{K}_{\Theta^h}(f_2) \leq_1 \mathcal{K}_{\Theta^h}(f_1)$.

(5) By (2), $\Theta^h(\psi) = \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi)))$ for all $\psi \in (L_1^Y)^X$. If $\Theta^h(\psi_1) = f$, then $\Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_1))) = \Theta^h(\mathcal{K}_{\Theta^h}(f)) = \Theta^h(\psi_1) = f$. Moreover $\psi_1 \leq \mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) = \mathcal{K}_{\Theta^h}(f)$.

(6) It is similarly proved as (5).

(7) For any ψ, ψ_1, $\psi_2 \in L_2^X$, $\psi \leq_2 \Theta^h \circ \mathcal{K}_{\Theta^h}(\psi)$ and $(\Theta^h \circ \mathcal{K}_{\Theta^h}) \circ (\Theta^h \circ \mathcal{K}_{\Theta^h}(\psi)) = \Theta^h \circ \mathcal{K}_{\Theta^h}(\psi)$. If $\psi_1 \leq_2 \psi_2$, then $\mathcal{K}_{\Theta^h}(\psi_2) \leq_1 \mathcal{K}_{\Theta^h}(\psi_1)$. Moreover $(\Theta^h \circ \mathcal{K}_{\Theta^h})(\psi_1) \leq_2 (\Theta^h \circ \mathcal{K}_{\Theta^h})(\psi_2)$. Thus $\Theta^h \circ \mathcal{K}_{\Theta^h} : L_2^Y \to L_2^Y$ is a fuzzy closure operator. Similarly, $\mathcal{K}_{\Theta^h} \circ \Theta^h$ is a fuzzy closure operator.

(8) By (6), since $\mathcal{K}_{\Theta^h}(\Theta^h(\mathcal{K}_{\Theta^h}(\bigwedge_{i\in\Gamma} \Theta^h(\psi_i)) = \mathcal{K}_{\Theta^h}(\bigwedge_{i\in\Gamma} \Theta^h(\psi_i))$, $\sqcup^k_{i\in\Gamma}\psi_i = \mathcal{K}_{\Theta^h}(\bigwedge_{i\in\Gamma} \Theta^h(\psi_i)) \in K(L_1^X)$.

Let $\psi_i \leq \psi$ for all $\psi,\ \psi_i \in K((L_1^Y)^X)$. Then $\Theta^h(\psi) \leq_2 \Theta^h(\psi_i)$ for all i. Thus $\Theta^h(\psi) \leq_2 \bigwedge_i \Theta^h(\psi_i)$. So $\psi = \mathcal{K}_{\Theta^h}(\Theta^h(\psi)) \geq_1 \mathcal{K}_{\Theta^h}(\bigwedge \Theta^h(\psi_i)) \geq_1 \mathcal{K}_{\Theta^h}(\Theta^h(\psi_i)) = \psi_i$. Hence $\sqcup^k_{i\in\Gamma}\psi_i = \mathcal{K}_{\Theta^h}(\bigwedge_{i\in\Gamma} \Theta^h(\psi_i))$ is a supremum of for all $\psi_i \in K((L_1^Y)^X)$.

(9) It is similarly proved as in (8).

(10) Define $\Theta^h : K((L_1^Y)^X) \to H(L_2^Y)$ as $\Theta^h(\psi)(y) = \bigwedge_{y\in X}(\psi(x)(y) \nearrow h(y))$.

If $\psi_1 = \mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) = \psi_2 = \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)) \in K(L_1^X)$, then by (5),

$$\Theta^h(\psi_1) = \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_1))) = \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_2))) = \Theta^h(\psi_1).$$

Thus Θ^h is well defined. If $\Theta^h(\psi_1) = \Theta^h(\psi_2)$, then $\psi_1 = \mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) = \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)) = \psi_2$. Thus Θ^h is injective. For $g \in H(L_2^Y)$, $g = \Theta^h(\mathcal{K}_{\Theta^h}(g))$. So Θ^h is surjective. Hence Θ^h is bijective.

Let $\psi_i \in K((L_1^Y)^X)$ for each $i \in I$. Since $\Theta^h \circ \mathcal{K}_{\Theta^h}$ is an increasing function,

$$\Theta^h(\mathcal{K}_{\Theta^h}(\bigwedge_{i\in I} \Theta^h(\psi_i))) \quad \leq_2 \bigwedge_{i\in I} \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_i))) = \bigwedge_{i\in I} \Theta^h(\psi_i).$$

Since $\Theta^h \circ \mathcal{K}_{\Theta^h}$ is a fuzzy closure operator, $\bigwedge_{i\in I} \Theta^h(\psi_i) \leq_2 \Theta^h(\mathcal{K}_{\Theta^h}(\bigwedge_{i\in I} \Theta^h(\psi_i)))$. Then $\bigwedge_{i\in I} \Theta^h(\psi_i) \in H(L_2^Y)$. Thus

$$\Theta^h(\sqcup^k_{i\in I}\psi_i) = \Theta^h(\mathcal{K}_{\Theta^h}(\bigwedge_{i\in I} \Theta^h(\psi_i))) = \bigwedge_{i\in I} \Theta^h(\psi_i),$$
$$\sqcup^r_{i\in I}\Theta^h(\psi_i) = \Theta^h(\bigwedge_{i\in I} \mathcal{K}_{\Theta^h}(\Theta^h(\psi_i))) = \Theta^h(\bigwedge_{i\in I} \psi_i).$$

So $K((L_1^Y)^X)$ and $H(L_2^Y)$ are anti-isomorphic.

(11) Put $r = \bigwedge\{p \in K(L_1^X) \mid \psi \leq_1 p\}$. Then by (8), $r \in K((L_1^Y)^X)$. Since $\psi \leq r$, $\psi = \mathcal{K}_{\Theta^h}(\Theta^h(\psi)) \leq \mathcal{K}_{\Theta^h}(\Theta^h(r))$. Since $\mathcal{K}_{\Theta^h}(\Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi)))) = \mathcal{K}_{\Theta^h}(\Theta^h(\psi))$, $\mathcal{K}_{\Theta^h}(\Theta^h(\psi)) \in K(L_1^X)$ and $\psi \leq_1 \mathcal{K}_{\Theta^h}(\Theta^h(\psi))$, $r \leq_1 \mathcal{K}_{\Theta^h}(\Theta^h(\psi))$. Thus we have $\mathcal{K}_{\Theta^h}(\Theta^h(\psi)) = \bigwedge\{p \in K(L_1^X) \mid \psi \leq_1 p\}$.

Other case is similarly proved. □

Remark 3.9. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Then the Galois connections in Theorems 3.4 and 3.5 hold the similar properties in Theorem 3.8.

Definition 3.10. Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$.

(i) Let $((L_2^Y)^X, \Delta_f, \mathcal{J}_{\Delta_f}, L_3^Y)$ be a residuated connection. A family $\mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$ is called an $((L_2^Y)^X, L_3^Y)$-*fuzzy concept lattice*, where

$$\mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f}) \quad = \{(\psi, g) \in (L_2^Y)^X \times L_3^Y \mid \Delta_f(\psi) = g, \mathcal{J}_{\Delta_f}(g) = \psi\}$$

and $(\psi_1, g_1) \leq (\psi_2, g_2)$ iff $\psi_1 \leq_2 \psi_2$(or $g_1 \leq_3 g_2$). Moreover, the pair (ψ, g) is called an $((L_2^Y)^X, L_3^Y)$-*fuzzy concept*.

(ii) Let $(L_1^Y, \mathcal{H}_{\Psi_g}, \Psi_g, (L_3^Y)^X)$ be a residuated connection. A family $\mathcal{R}(\mathcal{H}_{\Psi_g}, \Psi_g)$ is called an $(L_1^Y, (L_3^Y)^X)$-*fuzzy concept lattice*, where

$$\mathcal{R}(\mathcal{H}_{\Psi_g}, \Psi_g) \quad = \{(f, g) \in L_1^Y \times (L_3^Y)^X \mid \mathcal{H}_{\Psi_g}(f) = \psi, \Psi_g(\psi) = f\}$$

and $(f_1, \psi_1) \leq (f_2, \psi_2)$ iff $f_1 \leq_1 f_2$(or $\psi_1 \leq_3 \psi_2$). Moreover, the pair (f, ψ) is called an $(L_1^Y, (L_3^Y)^X)$-*fuzzy concept*.

(iii) Let $((L_1^Y)^X, \Theta^h, \mathcal{K}_{\Theta^h}, L_2^X)$ be a Galois connection. A family $\mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$ is called an $((L_1^Y)^X, L_2^X)$- *fuzzy concept lattice* where

$$\mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h}) = \{(\psi, g) \in (L_1^Y)^X \times L_2^Y \mid \Theta^h(\psi) = g, \mathcal{K}_{\Theta^h}(g) = \psi\}$$

and $(\psi_1, g_1) \le (\psi_2, g_2)$ iff $\psi_1 \le_1 \psi_2$(or $g_2 \le_2 g_1$). Moreover, (ψ, g) is called an $((L_1^Y)^X, L_2^X)$*-fuzzy concept.*

Theorem 3.11. *Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \le_1)$, $(L_2, \le_2)$, $(L_3, \le_3)$. Let $((L_2^Y)^X, \Delta_f, \mathcal{J}_{\Delta_f}, L_3^Y)$ be a residuated connection. Then the following properties hold.*

(1) *For all $\psi \in (L_2^Y)^X$ and $g \in L_3^Y$,*

$$\begin{array}{rl}\mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f}) & = \{(\psi, \Delta_f(\psi)) \in (L_2^Y)^X \times L_3^Y \mid \mathcal{J}_{\Delta_f}(\Delta_f(\psi)) = \psi\} \\ & = \{(\mathcal{J}_{\Delta_f}(g), g) \in (L_2^Y)^X \times L_3^Y \mid \Delta_f(\mathcal{J}_{\Delta_f}(g)) = g\}.\end{array}$$

(2) *For all (ψ_1, g_1), $(\psi_2, g_2) \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$, $\psi_1 \le_2 \psi_2$ iff $g_1 \le_3 g_2$.*

(3) *For all (ψ_1, g_1), (ψ_2, g_2), $(\psi_i, g_i)_{i\in I} \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$, define as*

$$(\psi_1, g_1) \le (\psi_2, g_2) \text{ iff } \psi_1 \le_2 \psi_2 (\text{or } g_1 \le_3 g_2),$$

$$\begin{array}{rl}\bigvee_{i\in I}(\psi_i, g_i) & = (\bigsqcup_{i\in I} \psi_i, \bigvee_{i\in I} g_i) \\ & = (\mathcal{J}_{\Delta_f}(\Delta_f(\bigvee_{i\in I} \psi_i)), \bigvee_{i\in I} g_i) \\ & = (\mathcal{J}_{\Delta_f}(\bigvee_{i\in I} g_i), \bigvee_{i\in I} g_i), \\ \bigwedge_{i\in I}(\psi_i, g_i) & = (\bigwedge_{i\in I} \psi_i, \sqcap g_i) = (\bigwedge_{i\in I} \psi_i, \Delta_f(\mathcal{J}_{\Delta_f}(\bigwedge_{i\in I} g_i))) \\ & = (\bigwedge_{i\in I} \psi_i, \Delta_f(\bigwedge_{i\in I} \psi_i)).\end{array}$$

Then $\mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$ forms a complete lattice.

(4) *Define $\gamma : C((L_2^Y)^X) \to \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$ as $\gamma(\psi) = (\psi, \Delta_f(\psi))$. Then γ is an isomorphism and $\mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f}) = \{\gamma(\psi) \mid \psi \in C((L_2^Y)^X)\}$.*

Proof. (1) Put $\mathcal{B} = \{(\psi, \Delta_f(\psi)) \in (L_2^Y)^X \times L_3^Y \mid \mathcal{J}_{\Delta_f}(\Delta_f(\psi)) = \psi\}$. Let $(\psi, g) \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$. Then $\mathcal{J}_{\Delta_f}(\Delta_f(\psi)) = \mathcal{J}_{\Delta_f}(g) = \psi$, $g = \Delta_f(\psi)$. Thus $(\psi, \Delta_f(\psi)) \in \mathcal{B}$. Since $\mathcal{J}_{\Delta_f}(\Delta_f(\psi)) = \psi$, $(\psi, \Delta_f(\psi)) \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$.

Put $\mathcal{C} = \{(\mathcal{J}_{\Delta_f}(g), g) \in (L_2^Y)^X \times L_3^Y \mid \Delta_f(\mathcal{J}_{\Delta_f}(g)) = g\}$. Let $(\psi, g) \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$. Then $\Delta_f(\mathcal{J}_{\Delta_f}(g)) = \Delta_f(\psi) = g, f = \mathcal{J}_{\Delta_f}(g)$. Thus $(\mathcal{J}_{\Delta_f}(g), g) \in \mathcal{C}$. Since $\Delta_f(\mathcal{J}_{\Delta_f}(g)) = g$, $(\mathcal{J}_{\Delta_f}(g), g) \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$.

(2) Let $\psi_1 \le_2 \psi_2$. By Theorem 3.6 (3), $g_1 = \Delta_f(\psi_1) \le_3 \Delta_f(\psi_2) = g_2$. Let $g_1 \le_3 g_2$. Then $\psi_1 = \mathcal{J}_{\Delta_f}(g_1) \le_2 \mathcal{J}_{\Delta_f}(g_2) = \psi_2$.

(3) For all $\{(\psi_i, g_i)\}_{i\in I} \subseteq \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$, by Theorem 3.6 (8), we have

$$\begin{array}{rl}\sqcup_{i\in I}\psi_i & = \mathcal{J}_{\Delta_f}(\bigvee_{i\in I} \Delta_f(\psi_i))) = \mathcal{J}_{\Delta_f}(\Delta_f(\bigvee_{i\in I} \psi_i)), \\ \Delta_f(\mathcal{J}_{\Delta_f}(\Delta_f(\bigvee_{i\in I} \psi_i))) & = \Delta_f(\bigvee_{i\in I} \psi_i) = \bigvee_{i\in I} \Delta_f(\psi_i) = \bigvee_{i\in I} g_i, \\ \mathcal{J}_{\Delta_f}(\Delta_f(\bigvee_{i\in I} \psi_i)) & = \mathcal{J}_{\Delta_f}(\bigvee_{i\in I} F(\psi_i)) = \mathcal{J}_{\Delta_f}(\bigvee_{i\in I} g_i).\end{array}$$

Then $\bigvee_{i\in I}(\psi_i, g_i) \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$. Since $g_i \le_3 \bigvee_{i\in I} g_i$ and $\Delta_f(\bigvee_{i\in I} \psi_i) = \bigvee_{i\in I} \Delta_f(\psi_i)$ $= \bigvee_{i\in I} g_i$, from Theorem 3.6 (1), $\psi_i = \mathcal{J}_{\Delta_f}(g_i) \le_2 \mathcal{J}_{\Delta_f}(\bigvee_{i\in I} g_i) = \mathcal{J}_{\Delta_f}(\Delta_f(\bigvee_{i\in I} \psi_i))$. Thus $(\psi_i, g_i) \le \bigvee_{i\in I}(\psi_i, g_i)$.

If $(\psi_i, g_i) \le (\psi, g)$ for all $i \in I$, then $\bigvee_{i\in I} g_i \le_3 g$ and $\mathcal{J}_{\Delta_f}(\bigvee_{i\in I} g_i) \le_2 \mathcal{J}_{\Delta_f}(g) = \psi$. Thus $\bigvee_{i\in I}(\psi_i, g_i) = (\mathcal{J}_{\Delta_f}(\bigvee_{i\in I} g_i), \bigvee_{i\in I} g_i) \le (\psi, g)$. So $\bigvee_{i\in I}(\psi_i, g_i)$ is a supremum of (ψ_i, g_i) for each $i \in I$.

For all $\{(\psi_i, g_i)\}_{i\in I} \subseteq \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$, we get

$$\begin{array}{rl}\sqcap_{i\in I} g_i & = \Delta_f(\bigwedge_{i\in I} \mathcal{J}_{\Delta_f}(g_i)) = \Delta_f(\mathcal{J}_{\Delta_f}(\bigwedge_{i\in I} g_i)), \\ \mathcal{J}_{\Delta_f}(\Delta_f(\mathcal{J}_{\Delta_f}(\bigwedge_{i\in I} g_i))) & = \mathcal{J}_{\Delta_f}(\bigwedge_{i\in I} g_i) = \bigwedge_{i\in I} \mathcal{J}_{\Delta_f}(g_i) = \bigwedge_{i\in I} \psi_i, \\ \Delta_f(\mathcal{J}_{\Delta_f}(\bigwedge_{i\in I} \psi_i)) & = \Delta_f(\bigwedge_{i\in I} \mathcal{J}_{\Delta_f}(g_i)) = \Delta_f(\bigwedge_{i\in I} \psi_i).\end{array}$$

Then $\bigwedge_{i\in I}(\psi_i, g_i) \in \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f})$. Moreover, since $\bigwedge_{i\in I}\psi_i \leq \psi_i$, we have

$$\Delta_f(\mathcal{J}_{\Delta_f}(\bigwedge_{i\in I} g_i)) \leq \bigwedge_{i\in I}\Delta_f(\mathcal{J}_{\Delta_f}(g_i)) = \bigwedge_{i\in I} g_i \leq g_i.$$

Thus $\bigwedge_{i\in I}(\psi_i, g_i) \leq (\psi_i, g_i)$ for all $i \in I$.

If $(\psi, g) \leq (\psi_i, g_i)$ for all $i \in I$, then $f \leq_2 \bigwedge_{i\in I}\psi_i$ and $g = \Delta_f(\psi) \leq_3 \Delta_f(\bigwedge_{i\in I}\psi_i)$. Thus $(\psi, g) \leq \bigwedge_{i\in I}(\psi_i, g_i) = (\bigwedge_{i\in I}\psi_i, \Delta_f(\bigwedge_{i\in I}\psi_i))$. So $\bigwedge_{i\in I}(\psi_i, g_i)$ is an infimum of (ψ_i, g_i) for each $i \in I$.

(4) We easily prove that γ is bijective. For all $\{\psi_i \mid i \in I\} \subseteq C((L_2^Y)^X)$,

$$\begin{aligned}
\gamma(\bigwedge_{i\in I}\psi_i) &= (\bigwedge_{i\in I}\psi_i, \Delta_f(\bigwedge_{i\in I}\psi_i)) = \bigwedge_{i\in I}(\psi_i, \Delta_f(\psi_i)) = \bigwedge_{i\in I}\gamma(\psi_i),\\
\gamma(\sqcup_{i\in I}\psi_i) &= (\sqcup_{i\in I}\psi_i, \Delta_f(\sqcup_{i\in I}\psi_i)) = (\sqcup_{i\in I}\psi_i, \Delta_f(\mathcal{J}_{\Delta_f}(\Delta_f(\bigvee_{i\in I}\psi_i))))\\
&= (\sqcup_{i\in I}\psi_i, \Delta_f(\bigvee_{i\in I}\psi_i)) = (\sqcup_{i\in I}\psi_i, \bigvee_{i\in I}\Delta_f(\psi_i))\\
&= \bigvee_{i\in I}(\psi_i, \Delta_f(\psi_i)) = \bigvee_{i\in I}\gamma(\psi_i).
\end{aligned}$$

□

The following corollary be similarly obtained from Theorem 3.11.

Corollary 3.12. *Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Let $(L_1^Y, \mathcal{H}_{\Psi_g}, \Psi_g, (L_3^Y)^X)$ be a residuated connection. Then the following properties hold.*

(1) *For all $f \in L_1^Y$ and $\psi \in (L_3^Y)^X$,*

$$\begin{aligned}
\mathcal{R}(\mathcal{H}_{\Psi_g}, \Psi_g) &= \{(f, \mathcal{H}_{\Psi_g}(f)) \in L_1^Y \times (L_3^Y)^X \mid \Psi_g(\mathcal{H}_{\Psi_g}(f)) = f\}\\
&= \{(\Psi_g(\psi), \psi) \in L_1^Y \times (L_3^Y)^X \mid \mathcal{H}_{\Psi_g}(\Psi_g(\psi)) = \psi\}.
\end{aligned}$$

(2) *For all (f_1, ψ_1), $(f_2, \psi_2) \in \mathcal{R}(\mathcal{H}_{\Psi_g}, \Psi_g)$, $f_1 \leq f_2$ iff $\psi_1 \leq \psi_2$.*

(3) *For all (f_1, ψ_1), (f_2, ψ_2), $(f_i, \psi_i)_{i\in I} \in \mathcal{R}(\mathcal{H}_{\Psi_g}, \Psi_g)$, define as*

$$(f_1, \psi_1) \leq (f_2, \psi_2) \text{ iff } f_1 \leq f_2 (\text{or } \psi_1 \leq \psi_2)$$

and

$$\begin{aligned}
\bigvee_{i\in I}(f_i, \psi_i) &= (\bigvee_{i\in I} f_i, \sqcup_{i\in I}\psi_i) = (\bigvee_{i\in I} f_i, \mathcal{H}_{\Psi_g}(\Psi_g(\bigvee_{i\in I}\psi_i)))\\
&= (\bigvee_{i\in I} f_i, \mathcal{H}_{\Psi_g}(\bigvee_{i\in I} f_i)),\\
\bigwedge_{i\in I}(f_i, \psi_i) &= (\sqcap_{i\in I} f_i, \bigwedge_{i\in I}\psi_i) = (\Psi_g(\mathcal{H}_{\Psi_g}(\bigwedge_{i\in I} f_i)), \bigwedge_{i\in I}\psi_i)\\
&= (\Psi_g(\bigwedge_{i\in I}\psi_i), \bigwedge_{i\in I}\psi_i).
\end{aligned}$$

Then $\mathcal{R}(\mathcal{H}_{\Psi_g}, \Psi_g)$ forms a complete lattice.

Remark 3.13. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Then the residuated connections in Theorems 3.4 and 3.5 hold the similar properties in Theorem 3.11 or Corollary 3.12.

Theorem 3.14. *Let X, Y be sets and L_i be complete lattices. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Let $((L_1^Y)^X, \Theta^h, \mathcal{K}_{\Theta^h}, L_2^Y)$ be a Galois connection. Then the following properties hold.*

(1) *For all $\psi \in (L_1^Y)^X$ and $B \in L_2^Y$, a family $\mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$ is an $((L_1^Y)^X, L_2^Y)$-fuzzy concept lattice such that*

$$\begin{aligned}
\mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h}) &= \{(\psi, \Theta^h(\psi)) \in (L_1^Y)^X \times L_2^Y \mid \mathcal{K}_{\Theta^h}(\Theta^h(\psi)) = \psi\}\\
&= \{(\mathcal{K}_{\Theta^h}(g), g) \in (L_1^Y)^X \times L_2^Y \mid \Theta^h(\mathcal{K}_{\Theta^h}(g)) = g\}.
\end{aligned}$$

(2) *For all (ψ_1, g_1), $(\psi_2, g_2) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$, $\psi_1 \leq_1 \psi_2$ iff $g_2 \leq_2 g_1$.*

(3) *For all* $(\psi_1, g_1),\ (\psi_2,\ g_2), (\psi_i, g_i)_{i\in I} \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$, *define as*

$$(\psi_1, g_1) \leq (\psi_2, g_2) \text{ iff } \psi_1 \leq_1 \psi_2 (\text{or } g_1 \leq_2 g_2)$$

and

$$\begin{aligned}\bigwedge_{i\in I}(\psi_i, g_i) &= (\bigwedge_{i\in I}\psi_i, \Theta^h(\mathcal{K}_{\Theta^h}(\bigvee_{i\in I} g_i))) = (\bigwedge_{i\in I}\psi_i, \Theta^h(\bigwedge_{i\in I}\psi_i)) \\ &= (\bigwedge_{i\in I}\psi_i, \sqcup^h_{i\in I} g_i)), \\ \bigvee_{i\in I}(\psi_i, g_i) &= (\mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i)), \bigwedge_{i\in I} g_i) = (\mathcal{K}_{\Theta^h}(\bigwedge_{i\in I} g_i), \bigwedge_{i\in I} g_i) \\ &= (\sqcup_{i\in I}\psi_i, \bigwedge_{i\in I} g_i).\end{aligned}$$

Then $\mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$ *forms a complete lattice.*

(4) *Define* $\alpha : K((L_1^Y)^X) \to \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$ *as* $\alpha(\psi) = (\psi, \Theta^h(\psi))$. *Then* α *is an isomorphism and* $\mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h}) = \{\alpha(\psi) \mid \psi \in K((L_1^Y)^X)\}$.

Proof. (1) Put $\mathcal{B} = \{(\psi, \Theta^h(\psi)) \in (L_1^Y)^X \times L_2^Y \mid \mathcal{K}_{\Theta^h}(\Theta^h(\psi)) = \psi\}$. Let $(\psi, g) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$. Then $\mathcal{K}_{\Theta^h}(\Theta^h(\psi)) = \mathcal{K}_{\Theta^h}(g) = \psi, g = \Theta^h(\psi)$. Thus $(\psi, \Theta^h(\psi)) \in \mathcal{B}$. Since $\mathcal{K}_{\Theta^h}(\Theta^h(\psi)) = \psi$, $(\psi, \Theta^h(\psi)) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$.

Put $\mathcal{C} = \{(\mathcal{K}_{\Theta^h}(g), g) \in (L_1^Y)^X \times L_2^Y \mid \Theta^h(\mathcal{K}_{\Theta^h}(g)) = g\}$. Let $(\psi, g) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$. Then $\Theta^h(\mathcal{K}_{\Theta^h}(g)) = \Theta^h(\psi) = g, \psi = \mathcal{K}_{\Theta^h}(g)$. Thus $(\mathcal{K}_{\Theta^h}(g), g) \in \mathcal{C}$. Since $\Theta^h(\mathcal{K}_{\Theta^h}(g)) = g$, $(\mathcal{K}_{\Theta^h}(g), g) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$.

(2) Let $\psi_1 \leq_1 \psi_2$. By Theorem 3.8 (3), $g_2 = \Theta^h(\psi_2) \leq_2 \Theta^h(\psi_1) = g_1$. Let $g_2 \leq_2 g_1$. By Theorem 3.8 (4), $\psi_1 = \mathcal{K}_{\Theta^h}(g_1) \leq_1 \mathcal{K}_{\Theta^h}(g_2) = \psi_2$.

(3) For all $\{(\psi_i, g_i)\}_{i\in I} \subseteq \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$,

$$\begin{aligned}\Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i))) &= \Theta^h(\bigvee_{i\in I}\psi_i) = \bigwedge_{i\in I}\mathcal{K}_{\Theta^h}(g_i) = \bigwedge_{i\in I} g_i, \\ \mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i)) &= \mathcal{K}_{\Theta^h}(\bigwedge_{i\in I}\Theta^h(\psi_i)) = \sqcup_{i\in I}\psi_i = \mathcal{K}_{\Theta^h}(\bigwedge_{i\in I} g_i).\end{aligned}$$

Then $\bigwedge_{i\in I}(\psi_i, g_i) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$. Moreover, $\psi_i \geq_1 \bigwedge_{i\in I}\psi_i$ and $g_i = \Theta^h(\psi_i) \leq_2 \Theta^h(\bigwedge_{i\in I}\psi_i) = \Theta^h(\mathcal{K}_{\Theta^h}(\bigvee_{i\in I} g_i))$. Thus $(\psi_i, g_i) \geq \bigwedge_{i\in I}(\psi_i, g_i)$.

If $(\psi_i, g_i) \geq (\psi, g)$ for each $i \in I$, then $\bigwedge_{i\in I}\psi_i \geq_1 \psi$ and $\Theta^h(\bigwedge_{i\in I}\psi_i) \leq_2 \Theta^h(\psi) = g$. Thus $\bigwedge_{i\in I}(\psi_i, g_i) = (\bigwedge_{i\in I}\psi_i, \Theta^h(\bigwedge_{i\in I}\psi_i)) \geq (\psi, g)$. So $\bigwedge_{i\in I}(\psi_i, g_i)$ is an infimum of $\{(\psi_i, g_i)\}_{i\in I}$.

For all $\{(\psi_i, g_i)\}_{i\in I} \subseteq \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$,

$$\begin{aligned}\Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i))) &= \Theta^h(\bigvee_{i\in I}\psi_i) = \bigwedge_{i\in I}\Theta^h(\psi_i) = \bigwedge_{i\in I} g_i, \\ \mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i)) &= \mathcal{K}_{\Theta^h}(\bigwedge_{i\in I}\Theta^h(\psi_i)) = \sqcup_{i\in I}\psi_i = \mathcal{K}_{\Theta^h}(\bigwedge_{i\in I} g_i).\end{aligned}$$

Then $\bigvee_{i\in I}(\psi_i, g_i) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$. Moreover, since $\bigwedge_{i\in I} g_i \leq_2 g_i, \mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i)) \geq \bigvee_{i\in I}\psi_i \geq \psi_i$. Thus, $\bigvee_{i\in I}(\psi_i, g_i) \geq (\psi_i, g_i)$ for all$i \in I$.

If $(\psi, g) \geq (\psi_i, g_i)$ for all $i \in I$, then $g \leq_2 \bigwedge_{i\in I} g_i$ and $\psi = \mathcal{K}_{\Theta^h}(g) \geq_1 \mathcal{K}_{\Theta^h}(\bigwedge_{i\in I} g_i) = \mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i))$. Thus

$$(\psi, g) \geq \bigwedge_{i\in I}(\psi_i, g_i) = (\mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i)), \bigwedge_{i\in I} g_i).$$

So $\bigvee_{i\in I}(\psi_i, g_i)$ is a supremum of $\{(\psi_i, g_i)\}_{i\in I}$.

(4) We easily prove that α is bijective. For all $\{\psi_i \mid i \in I\} \subseteq K((L_1^Y)^X)$,

$$\begin{aligned}\alpha(\bigwedge_{i\in I}\psi_i) &= (\bigwedge_{i\in I}\psi_i, \Theta^h(\bigwedge_{i\in I}\psi_i)) = \bigwedge_{i\in I}(\psi_i, \Theta^h(\psi_i)) = \bigwedge_{i\in I}\alpha(\psi_i), \\ \alpha(\sqcup_{i\in I}\psi_i) &= (\sqcup_{i\in I}\psi_i, \Theta^h(\sqcup_{i\in I}\psi_i)) = (\sqcup_{i\in I}\psi_i, \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\bigvee_{i\in I}\psi_i)))) \\ &= (\sqcup_{i\in I}\psi_i, \Theta^h(\bigvee_{i\in I}\psi_i)) = (\sqcup_{i\in I}\psi_i, \bigwedge_{i\in I}\Theta^h(\psi_i)) \\ &= \bigvee_{i\in I}(\psi_i, \Theta^h(\psi_i)) = \bigvee_{i\in I}\alpha(\psi_i).\end{aligned}$$

□

Remark 3.15. Let $(\&, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Then the Galois connections in Theorems 3.4 and 3.5 hold the similar properties in Theorem 3.15.

Example 3.16. Let $X = \{x, y, z\}$ be a set of cars and $Y = \{a, b\}$ be a set of attributes. Let $([0,1], \odot, \rightarrow, 0, 1)$ be a continuous t-norm (See [2, 14, 15, 16, 23]) as

$$x \odot y = \max\{0, x + y - 1\}, \ x \rightarrow y = \min\{1 - x + y, 1\}.$$

Let $[0,1]_m$ be a regular partition of $[0,1]$ in m pieces with $[0,1]_m = \{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\}$.

Let $\& : [0,1]_3 \times [0,1]_4 \rightarrow [0,1]_2$, $\nearrow: [0,1]_3 \times [0,1]_2 \rightarrow [0,1]_4$, $\searrow: [0,1]_4 \times [0,1]_2 \rightarrow [0,1]_3$ defined as

$$x \& y = \frac{[2(x \odot y)]}{2}, \ x \nearrow y = \frac{\langle 4(x \rightarrow y) \rangle}{4}$$

$$x \searrow y = \frac{\langle 3(x \rightarrow y) \rangle}{3},$$

where $[x] = \bigwedge\{n \in Z \mid x \leq n\}, \langle x \rangle = \bigvee\{n \in Z \mid n \leq x\}$,

$\&$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
$\frac{1}{3}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{2}{3}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1

$\nearrow$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{3}$	$\frac{1}{2}$	1	1
$\frac{2}{3}$	$\frac{1}{4}$	$\frac{3}{4}$	1
1	0	$\frac{1}{2}$	1

$\searrow$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{4}$	$\frac{2}{3}$	1	1
$\frac{1}{2}$	$\frac{1}{3}$	1	1
$\frac{3}{4}$	0	$\frac{2}{3}$	1
1	0	$\frac{1}{3}$	1

(1) For $f_1 = (\frac{2}{3}, \frac{1}{3}, 0) \in [0,1]_3^X$ and $\psi(x) : Y \rightarrow [0,1]_4$ for $x \in \{x, y, z\}$ with $\psi(x)(a) = \psi(a, x)$ as

$$\psi = \begin{pmatrix} \frac{1}{4} & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}.$$

By Theorem 3.3, $\Delta_{f_1}(\psi) = \bigvee_{x \in X}(f_1(x) \& \psi(x)(-)) = (\frac{1}{2}, \frac{1}{2})$. Moreover,

$$\mathcal{J}_{\Delta_{f_1}}(\Delta_{f_1}(\psi))(x)(a) = f_1(x) \nearrow \Delta_{f_1}(\psi)(a)$$

with

$$\mathcal{J}_{\Delta_{f_1}}(\Delta_{f_1}(\psi)) = \begin{pmatrix} \frac{3}{4} & 1 & 1 \\ \frac{3}{4} & 1 & 1 \end{pmatrix}$$

Then $\psi \leq \mathcal{J}_{\Delta_{f_1}}(\Delta_{f_1}(\psi))$ and $\mathcal{J}_{\Delta_{f_1}}(\Delta_{f_1}(\psi))$ is the greatest solution of $\Delta_{f_1}(\psi) = (\frac{1}{2}, \frac{1}{2})$. Moreover, $(\mathcal{J}_{\Delta_{f_1}}(\Delta_{f_1}(\psi)), \Delta_{f_1}(\psi)) \in \mathcal{R}(\Delta_{f_1}, \mathcal{J}_{\Delta_{f_1}})$, however $(\psi, \Delta_{f_1}(\psi)) \notin \mathcal{R}(\Delta_{f_1}, \mathcal{J}_{\Delta_{f_1}})$.

Let $h_2 = (\frac{1}{2}, 0)$ be a solution of $\mathcal{J}_{\Delta_f}(h_2) = \phi$ with

$$\phi = \begin{pmatrix} \frac{3}{4} & 1 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}.$$

Since

$$\begin{array}{ll} \Delta_{f_1}(\phi) & = \bigvee_{x\in X}(f_1(x)\&\phi(x)(-)) = (0,0), \\ \mathcal{J}_{\Delta_{f_1}}(\Delta_{f_1}(\mathcal{J}_{\Delta_{f_1}}(h_2))) & = \mathcal{J}_{\Delta_{f_1}}(h_2) = \phi, \end{array}$$

$\Delta_{f_1}(\phi)$ is a solution of $\mathcal{J}_{\Delta_{f_1}}(h) = \phi$ such that $\Delta_{f_1}(\phi) \leq_3 h_2$.

$(\mathcal{J}_{\Delta_{f_1}}(h_2), \Delta_{f_1}(\mathcal{J}_{\Delta_{f_1}}(h_2)) = (\phi, (0,0)) \in \mathcal{R}(\Delta_{f_1}, \mathcal{J}_{\Delta_{f_1}})$ but $(\phi, h_2) \notin \mathcal{R}(\Delta_{f_1}, \mathcal{J}_{\Delta_{f_1}})$ because $\Delta_{f_1}(\phi) = (0,0) \neq h_2$.

For $h \in [0,1]_2^Y$ with $h(a) = 0, h(b) = \frac{1}{2}$. By Definition 3.6 (2), we get $\Theta_h(\psi) = \bigwedge_{a\in X}(\psi(a)(-) \searrow h(a)) = (\frac{2}{3}, 0, 0)$. By Theorem 3.7 (4), we get $\mathcal{K}_{\Theta_h}(\Theta_h(\psi))(x)(a) = \Theta_h(\psi)(x) \nearrow h(a)$ such that

$$\mathcal{K}_{\Theta_h}(\Theta_h(\psi)) = \begin{pmatrix} \frac{3}{4} & 1 & 1 \\ \frac{1}{4} & 1 & 1 \end{pmatrix}.$$

Since $\Theta_h(\mathcal{K}_{\Theta_h}(\Theta_h(\psi))) = \Theta_h(\psi) = (\frac{2}{3}, 0, 0)$, $\mathcal{K}_{\Theta_h}(\Theta_h(\psi))$ and ψ are solutions of $\Theta_h(\psi) = (\frac{2}{3}, 0, 0)$ with $\psi \leq_1 \mathcal{K}_{\Theta_h}(\Theta_h(\psi))$. Moreover, $(\mathcal{K}_{\Theta_h}(\Theta_h(\psi)), \Theta_h(\psi)) \in \mathcal{G}(\Theta_h, \mathcal{K}_{\Theta_h})$ but $(\psi, \Theta_h(\psi)) \notin \mathcal{G}(\Theta_h, \mathcal{K}_{\Theta_h})$.

$\mathcal{K}_{\Theta_h}(f_1)(x)(a) = f_1(x) \nearrow h(a)$ such that

$$\mathcal{K}_{\Theta_h}(f_1) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since $\mathcal{K}_{\Theta_h}(\Theta_h(\mathcal{K}_{\Theta_h}(f_1))) = \mathcal{K}_{\Theta_h}(f_1)$, we have $\Theta_h(\mathcal{K}_{\Theta_h}(f_1)) = f_1 = (\frac{2}{3}, \frac{1}{3}, 0)$ and $(\mathcal{K}_{\Theta_h}(f_1), f_1) \in \mathcal{G}(\Theta_h, \mathcal{K}_{\Theta_h})$.

(2) Let

$$\psi_1 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 1 \\ \frac{1}{2} & 1 & \frac{1}{4} \end{pmatrix}, \psi_2 = \begin{pmatrix} 0 & \frac{1}{4} & 1 \\ \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix},$$
$$\psi_3 = \begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}, \psi_4 = \begin{pmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}.$$

For $f \in [0,1]_3^X$ with $f(x) = \frac{1}{3}, f(y) = \frac{2}{3}, f(z) = 1$. By Definition 3.6 (1), $\Delta_f(\psi_1) = \bigvee_{x\in X}(f(x)\&\psi(-)(x)) = (1,1)$, $\Delta_f(\psi_2) = (1, \frac{1}{2})$, $\Delta_f(\psi_3) = (\frac{1}{2}, 1)$ and $\Delta_f(\psi_4) = (\frac{1}{2}, 0)$. By Theorem 3.7 (1), we obtain $\mathcal{J}_{\Delta_f}(\Delta_f(\psi))(x)(y) = f(x) \nearrow \Delta_f(\psi)(y)$ such that

$$\begin{array}{l} \mathcal{R}(\Delta_f, \mathcal{J}_{\Delta_f}) = \\ \{(\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}, (0,0)), (\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & \frac{3}{4} & \frac{1}{2} \end{pmatrix}, (0, \frac{1}{2})), (\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & 1 & 1 \end{pmatrix}, (0,1)), \\ (\begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}, (\frac{1}{2}, 0)), (\begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ 1 & \frac{3}{4} & \frac{1}{2} \end{pmatrix}, (\frac{1}{2}, \frac{1}{2})), (\begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}, (\frac{1}{2}, 1)), \\ (\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}, (1,0)), (\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{3}{4} & \frac{1}{2} \end{pmatrix}, (1, \frac{1}{2})), (\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, (1,1))\} \end{array}$$

$$\mathcal{J}_{\Delta_f}(\Delta_f(\psi_1)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathcal{J}_{\Delta_f}(\Delta_f(\psi_2)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{3}{4} & \frac{1}{2} \end{pmatrix},$$
$$\mathcal{J}_{\Delta_f}(\Delta_f(\psi_3)) = \begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}, \mathcal{J}_{\Delta_f}(\Delta_f(\psi_4)) = \begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}.$$

Then $((L_2^Y)^X, \Delta_f, \mathcal{J}_{\Delta_f}, L_3^Y)$ is a residuated connection.

For $h \in [0,1]_2^Y$ with $h(a) = 0, h(b) = \frac{1}{2}$. By Definition 3.6 (i), $\Theta_h(\psi_1) = \bigwedge_{a\in Y}(\psi_1(a)(-) \searrow h(a)) = (0, \frac{1}{3}, 0)$, $\Theta_h(\psi_2) = (\frac{2}{3}, \frac{2}{3}, 0)$, $\Theta_h(\psi_3) = (0, 0, \frac{1}{3})$ and

$\Theta_h(\psi_4)=(0,0,\frac{2}{3})$. Since $\mathcal{K}_{\Theta_h}(\Theta_h(\psi_1))(x)(a)=\Theta_h(\psi_1)(x)\nearrow h(a)$, by Theorem 3.7 (1), we obtain

$$\mathcal{K}_{\Theta_h}(\Theta_h(\psi_1))=\begin{pmatrix}1&\frac{1}{2}&1\\1&1&1\end{pmatrix},\mathcal{K}_{\Theta_h}(\Theta_h(\psi_2))=\begin{pmatrix}\frac{1}{4}&\frac{1}{4}&1\\\frac{3}{4}&\frac{3}{4}&1\end{pmatrix},$$
$$\mathcal{K}_{\Theta_h}(\Theta_h(\psi_3))=\begin{pmatrix}1&1&\frac{1}{2}\\1&1&1\end{pmatrix},\mathcal{K}_{\Theta_h}(\Theta_h(\psi_4))=\begin{pmatrix}1&1&\frac{1}{4}\\1&1&\frac{3}{4}\end{pmatrix}.$$

Then we get

$$\begin{aligned}&(\mathcal{K}_{\Theta_h}(\Theta_h(\psi_1))\wedge\mathcal{K}_{\Theta_h}(\Theta_h(\psi_3)),\Theta_h(\psi_1)\sqcup\Theta_h(\psi_3))\\&=(\begin{pmatrix}1&\frac{1}{2}&\frac{1}{2}\\1&1&1\end{pmatrix},(0,\frac{1}{3},\frac{1}{3}))\in\mathcal{G}(\Theta_h,\mathcal{K}_{\Theta_h}),\end{aligned}$$

$$\begin{aligned}&(\mathcal{K}_{\Theta_h}(\Theta_h(\psi_2))\wedge\mathcal{K}_{\Theta_h}(\Theta_h(\psi_3)),\Theta_h(\psi_2)\sqcup\Theta_h(\psi_3))\\&=(\begin{pmatrix}\frac{1}{4}&\frac{1}{4}&\frac{1}{2}\\\frac{3}{4}&\frac{3}{4}&1\end{pmatrix},(\frac{2}{3},\frac{2}{3},\frac{1}{3}))\in\mathcal{G}(\Theta_h,\mathcal{K}_{\Theta_h}),\end{aligned}$$

$$\begin{aligned}&(\mathcal{K}_{\Theta_h}(\Theta_h(\psi_1))\sqcup\mathcal{K}_{\Theta_h}(\Theta_h(\psi_4)),\Theta_h(\psi_1)\wedge\Theta_h(\psi_4))\\&=(\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix},(0,0,0))\in\mathcal{G}(\Theta_h,\mathcal{K}_{\Theta_h}).\end{aligned}$$

Thus $((L_2^Y)^X,\Theta_h,\mathcal{K}_{\Theta_h},L_1^Y)$ is a Galois connection.

(3) Define $\psi(x):Y\to[0,1]_3$ for $x\in\{x,y,z\}$ with $\psi(x)(a)=\psi(a,x)$ as

$$\psi=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&1\\\frac{1}{3}&\frac{2}{3}&0\end{pmatrix}$$

For $f\in[0,1]_4^X$ with $f(x)=\frac{1}{4},f(y)=\frac{3}{4},f(z)=\frac{1}{2}$. By Definition 3.3 (ii), $\Delta^f(\psi)=\bigvee_{x\in X}(\psi(x)(-)\&f(x))=(\frac{1}{2},\frac{1}{2})$. By Theorem 3.4 (2), we obtain $\mathcal{J}_{\Delta^f}(g)(x)(y)=f(x)\searrow g(y)$ such that $((L_1^Y)^X,\Delta^f,\mathcal{J}_{\Delta^f},L_3^Y)$ is a residuated connection.

$\mathcal{J}_{\Delta^f}(\Delta^f(\psi))(x)(y)=f(x)\searrow\Delta^f(\psi)(y)$ as

$$\mathcal{J}_{\Delta^f}(\Delta^f(\psi))=\begin{pmatrix}1&\frac{2}{3}&1\\1&\frac{2}{3}&1\end{pmatrix}$$

$(\mathcal{J}_{\Delta^f}(\Delta^f(\psi)),\Delta^f(\psi))\in\mathcal{R}(\Delta^f,\mathcal{J}_{\Delta^f})$ but we have $(\psi,\Delta^f(\psi))\notin\mathcal{R}(\Delta^f,\mathcal{J}_{\Delta^f})$ because $\mathcal{J}_{\Delta^f}(\Delta^f(\psi))\neq\psi$.

For $h\in[0,1]_2^Y$ with $h(a)=\frac{1}{2},h(b)=0$. By Definition 3.3 (ii), we have $\Theta^h(\psi)=\bigwedge_{a\in X}(\psi(-)(a)\nearrow h(a))=(\frac{1}{2},\frac{1}{4},\frac{1}{2})$. By Theorem 3.4 (4), we get $\mathcal{K}_{\Theta^h}(\Theta^h(\psi))(x)(a)=\Theta^h(\psi)(x)\searrow h(a)$ such that

$$\mathcal{K}_{\Theta^h}(\Theta^h(\psi))=\begin{pmatrix}1&1&1\\\frac{1}{3}&\frac{2}{3}&\frac{1}{3}\end{pmatrix}.$$

Since $\Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi)))=\Theta^h(\psi)=(\frac{1}{2},\frac{1}{4},\frac{1}{2})$, $\mathcal{K}_{\Theta^h}(\Theta^h(\psi))$ and ψ are solutions of $\Theta^h(\psi)=(\frac{1}{2},\frac{1}{4},\frac{1}{2})$ with $\psi\leq_1\mathcal{K}_{\Theta^h}(\Theta^h(\psi))$. Moreover, $(\mathcal{K}_{\Theta^h}(\Theta^h(\psi)),\Theta^h(\psi))\in\mathcal{G}(\Theta^h,\mathcal{K}_{\Theta^h})$ but $(\psi,\Theta^h(\psi))\notin\mathcal{G}(\Theta^h,\mathcal{K}_{\Theta^h})$.

$\mathcal{K}_{\Theta^h}(f)(x)(a)=f(x)\searrow h(a)$ such that

$$\mathcal{K}_{\Theta^h}(\Theta^h(\psi))=\begin{pmatrix}1&\frac{2}{3}&1\\\frac{2}{3}&0&\frac{1}{3}\end{pmatrix}.$$

Since $\mathcal{K}_{\Theta^h}(\Theta^h(\mathcal{K}_{\Theta^h}(f))) = \mathcal{K}_{\Theta^h}(f)$, we have $\Theta^h(\mathcal{K}_{\Theta^h}(f)) = (\frac{1}{2}, \frac{3}{4}, \frac{1}{2}) > f = (\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$. Then $(\mathcal{K}_{\Theta^h}(f), f) \in \mathcal{G}(\Theta^h, \mathcal{K}_{\Theta^h})$.

(4) By a similar method in Theorems 3.3 (2) and 3.4 (4), for $g \in [0,1]^Y_4$ with $g(a) = \frac{1}{4}, g(b) = \frac{3}{4}$, define $\Delta^g(\psi_1) = \bigvee_{a\in Y}(\psi(a)(-)\&g(a))$ and $(\mathcal{J}_{\Delta^g}(\Delta^g(\psi))(x)(a)) = (g(a) \searrow \Delta^g(\psi)(x))$. Then $((L_1^Y)^X, \Delta^g, \mathcal{J}_{\Delta^g}, L_3^X)$ is a residuated connection. Let $\psi_i(x)(a) = \psi_i(a,x)$ as

$$\psi_1 = \begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & 1 \end{pmatrix}, \psi_2 = \begin{pmatrix} 0 & \frac{1}{3} & 1 \\ \frac{2}{3} & 0 & 0 \end{pmatrix}, \psi_3 = \begin{pmatrix} 1 & 1 & \frac{2}{3} \\ 1 & \frac{2}{3} & 0 \end{pmatrix}.$$

Then $\Delta^g(\psi_1) = \bigvee_{a\in Y}(\psi(a)(-)\&g(a)) = (0, \frac{1}{2}, 1)$, $\Delta^g(\psi_2) = (\frac{1}{2}, 0, \frac{1}{2})$ and $\Delta^g(\psi_3) = (1, \frac{1}{2}, 0)$. Moreover, we obtain $(\mathcal{J}_{\Delta^g}(\Delta^g(\psi))(x)(a)) = (g(a) \searrow \Delta^g(\psi)(x))$ such that

$$\mathcal{J}_{\Delta^g}(\Delta^g(\psi_1)) = \begin{pmatrix} \frac{2}{3} & 1 & 1 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}, \mathcal{J}_{\Delta^g}(\Delta^g(\psi_2)) = \begin{pmatrix} 1 & \frac{2}{3} & 1 \\ \frac{2}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

and $(\mathcal{J}_{\Delta^g}(\Delta^g(\psi_3)) = \psi_3$. Thus $(\mathcal{J}_{\Delta^g}(\Delta^g(\psi_i)), \Delta^g(\psi_i)) \in \mathcal{R}(\Delta^g, \mathcal{J}_{\Delta^g})$.

For all $\{\phi_i\}_{i\in\Gamma} \subseteq C((L_1^Y)^X) = \{\phi \in (L_1^Y)^X \mid \phi = \mathcal{J}_{\Delta^g} \circ \Delta^g(\phi)\}$,

$$\bigwedge_{i\in\Gamma} \phi_i, \sqcup_{i\in\Gamma}\phi_i = \mathcal{J}_{\Delta^g}(\bigvee_{i\in\Gamma} \Delta^g(\phi_i))$$

and

$$\begin{aligned} \mathcal{J}_{\Delta^g}(\Delta^g(\psi_1)) \sqcup \mathcal{J}_{\Delta^g}(\Delta^g(\psi_2)) &= \mathcal{J}_{\Delta^g}(\Delta^g(\mathcal{J}_{\Delta^g}(\Delta^g(\psi_1))) \vee \Delta^g(\mathcal{J}_{\Delta^g}(\Delta^g(\psi_2)))) \\ &= \mathcal{J}_{\Delta^g}(\Delta^g(\psi_1) \vee \Delta^g(\psi_2)), \\ \mathcal{J}_{\Delta^g}(\Delta^g(\psi_1)) \sqcup \mathcal{J}_{\Delta^g}(\Delta^g(\psi_3)) &= \mathcal{J}_{\Delta^g}(\Delta^g(\psi_1) \vee \Delta^g(\psi_3)), \\ \mathcal{J}_{\Delta^g}(\Delta^g(\psi_2)) \sqcup \mathcal{J}_{\Delta^g}(\Delta^g(\psi_3)) &= \mathcal{J}_{\Delta^g}(\Delta^g(\psi_2) \vee \Delta^g(\psi_3)). \end{aligned}$$

So $(C((L_1^Y)^X), \bigwedge, \sqcup)$ is a complete lattice.

For all $\{f_i\}_{i\in\Gamma} \subseteq I(L_3^X) = \{f \in L_3^X \mid f = \Delta^g \circ \mathcal{J}_{\Delta^g}(f)\}$, define

$$\sqcap_{i\in\Gamma} f_i = \Delta^g(\bigwedge_{i\in\Gamma} \mathcal{J}_{\Delta^g}(f_i)), \bigvee_{i\in\Gamma} f_i.$$

Then

$$\begin{aligned} \Delta^g(\psi_1) \sqcap \Delta^g(\psi_2) &= \Delta^g(\mathcal{J}_{\Delta^g}(\Delta^g(\psi_1)) \wedge \mathcal{J}_{\Delta^g}(\Delta^g(\psi_2))) = \Delta^g(\psi_1) \wedge \Delta^g(\psi_2), \\ \Delta^g(\psi_1) \sqcap \Delta^g(\psi_3) &= \Delta^g(\psi_1) \wedge \Delta^g(\psi_3), \\ \Delta^g(\psi_2) \sqcap \Delta^g(\psi_3) &= \Delta^g(\psi_2) \wedge \Delta^g(\psi_3). \end{aligned}$$

Thus $(I(L_3^Y), \sqcap, \bigvee)$ is a complete lattice.

For $h \in [0,1]^Y_2$ with $h(a) = 0, h(b) = \frac{1}{2}$. By Definition 3.3 (ii), $\Theta^h(\psi_1) = \bigwedge_{a\in Y}(\psi(a)(-) \nearrow h(a)) = (\frac{1}{2}, 0, \frac{1}{4})$, $\Theta^h(\psi_2) = (\frac{3}{4}, \frac{1}{2}, 0)$ and $\Theta^h(\psi_3) = (0, 0, \frac{1}{4})$. By Theorem 3.4 (4), we obtain $\mathcal{K}_{\Theta^h}(f)(x)(a) = f(x) \searrow h(a)$ such that $((L_1^Y)^X, \Theta^h, \mathcal{K}_{\Theta^h}, L_2^Y)$ is a Galois connection. Since $(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_i))(x)(a)) = (\Theta^h(\psi_i)(x) \searrow h(a))$,

$$\begin{aligned} \mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) &= \begin{pmatrix} \frac{1}{3} & 1 & \frac{2}{3} \\ 1 & 1 & 1 \end{pmatrix}, \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)) = \begin{pmatrix} 0 & \frac{1}{3} & 1 \\ \frac{2}{3} & 1 & 1 \end{pmatrix}, \\ \mathcal{K}_{\Theta^h}(\Theta^h(\psi_3)) &= \begin{pmatrix} 1 & 1 & \frac{2}{3} \\ 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Then $(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_i)), \Theta^h(\psi_i)) \in \mathcal{G}(\mathcal{K}_{\Theta^h}, \Theta^h)$.

Since

$$\begin{aligned} \mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) \sqcup \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)) &= \mathcal{K}_{\Theta^h}(\Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_1))) \wedge \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)))) \\ &= \mathcal{K}_{\Theta^h}(\Theta^h(\psi_1) \wedge \Theta^h(\psi_2)), \end{aligned}$$

$$(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) \sqcup \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)), \Theta^h(\psi_1) \wedge \Theta^h(\psi_2))$$
$$= (\begin{pmatrix} \frac{1}{3} & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, (\frac{1}{2}, 0, 0)) \in \mathcal{G}(\mathcal{K}_{\Theta^h}, \Theta^h),$$

$$(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)) \sqcup \mathcal{K}_{\Theta^h}(\Theta^h(\psi_3)), \Theta^h(\psi_1) \wedge \Theta^h(\psi_2))$$
$$= (\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, (0, 0, 0)) \in \mathcal{G}(\mathcal{K}_{\Theta^h}, \Theta^h).$$

Since $\Theta^h(\psi_1) \sqcup \Theta^h(\psi_2) = \Theta^h(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) \wedge \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2))) = (\frac{3}{4}, \frac{1}{2}, \frac{1}{4})$,

$$(\mathcal{K}_{\Theta^h}(\Theta^h(\psi_1)) \wedge \mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)), \Theta^h(\psi_1) \sqcup \Theta^h(\psi_2))$$
$$= (\mathcal{K}_{\Theta^h}(\Theta^h(\psi_2)) \wedge \mathcal{K}_{\Theta^h}(\Theta^h(\psi_3)), \Theta^h(\psi_2) \sqcup \Theta^h(\psi_3))$$
$$= (\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 1 & 1 \end{pmatrix}, (\frac{3}{4}, \frac{1}{2}, \frac{1}{4})) \in \mathcal{G}(\mathcal{K}_{\Theta^h}, \Theta^h).$$

(5) Define $\psi(c) : X \to [0,1]_2$ for $c \in \{a, b\}$ with $\psi(x)(a) = \psi(a, x)$ as

$$\psi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \end{pmatrix}.$$

For $f \in [0,1]_3^X$ with $f(x) = \frac{1}{3}, f(y) = 1, f(z) = \frac{2}{3}$. By Definition 3.3 (iii), $\Lambda^f(\psi) = \bigwedge_{x \in X}(f(x) \nearrow \psi(x)(-)) = (\frac{1}{2}, \frac{1}{4})$. By Theorem 3.4 (5), $\mathcal{H}_{\Lambda_f}(g)(x)(a) = f(x) \& g(a)$ such that $(L_2^Y, \mathcal{H}_{\Lambda^f}, \Lambda^f, (L_3^Y)^X)$ is a residuated connection. Then $\mathcal{H}_{\Lambda_f}(\Lambda^f(\psi))(x)(a) = f(x) \& \Lambda^f(\psi)(a)$ as

$$\mathcal{H}_{\Lambda_f}(\Lambda^f(\psi)) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Since $\Lambda^f(\mathcal{H}_{\Lambda_f}(\Lambda^f(\psi))) = \Lambda^f(\psi) = (\frac{1}{2}, \frac{1}{4})$, $\mathcal{H}_{\Lambda_f}(\Lambda^f(\psi))$ and ψ are solution of $\Lambda^f(\psi) = (\frac{1}{2}, \frac{1}{4})$ and $\mathcal{H}_{\Lambda_f}(\Lambda^f(\psi)) < \psi$. Moreover, $(\Lambda^f(\psi), \mathcal{H}_{\Lambda_f}(\Lambda^f(\psi))) \in \mathcal{R}(\mathcal{H}_{\Lambda^f}, \Lambda^f)$ but $(\Lambda^f(\psi), \psi) \in \mathcal{R}(\mathcal{H}_{\Lambda^f}, \Lambda^f)$ because $\mathcal{H}_{\Lambda_f}(\Lambda^f(\psi)) \neq \psi$.

For $h \in [0,1]_4^Y$ with $h(a) = \frac{1}{4}$, $h(b) = \frac{3}{4}$. By Definition 3.3 (iii), $\Psi_h(\psi) = \bigwedge_{a \in Y}(h(a) \searrow \psi(a)(-)) = (\frac{2}{3}, 1, 0)$. By Theorem 3.4 (6), $\mathcal{H}_{\Psi_h}(f)(x)(a) = f(x) \& h(a)$ such that $(L_1^X, \mathcal{H}_{\Psi_h}, \Psi_h, (L_3^Y)^X)$ is a residuated connection. Then $\mathcal{H}_{\Psi_h}(\Psi_h(\psi))$ as

$$\mathcal{H}_{\Psi_h}(\Psi_h(\psi)) = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Since $\Psi_h(\mathcal{H}_{\Psi_h}(\Psi_h(\psi))) = \Psi_h(\psi) = (\frac{2}{3}, 1, 0)$, $\mathcal{H}_{\Psi_h}(\Psi_h(\psi))$ and ψ are solutions of $\Psi_h(\psi) = (\frac{2}{3}, 1, 0)$ and $\mathcal{H}_{\Psi_h}(\Psi_h(\psi)) < \psi$. Moreover, $(\Psi_h(\psi), \mathcal{H}_{\Psi_h}(\Psi_h(\psi))) \in \mathcal{R}(\mathcal{H}_{\Psi_h}, \Psi_h)$ but $(\Psi_h(\psi), \psi) \notin \mathcal{R}(\mathcal{H}_{\Psi_h}, \Psi_h)$ because $\mathcal{H}_{\Psi_h}(\Psi_h(\psi)) \neq \psi$.

For $f_2 = (1, \frac{2}{3}, \frac{1}{3}) \in L_1^X$, $\mathcal{H}_{\Psi_h}(f_2)(x)(a) = f_2(x) \& h(a)$ such that

$$\mathcal{H}_{\Psi_h}(f_2) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and $\Psi_h(\mathcal{H}_{\Psi_h}(f_2)) = (1, \frac{2}{3}, \frac{2}{3})$. Then $(\Psi_h(\mathcal{H}_{\Psi_h}(f_2)), \mathcal{H}_{\Psi_h}(f_2)) \in \mathcal{R}(\mathcal{H}_{\Psi_h}, \Psi_h)$. Moreover,

$$(\Psi_h(\psi), \mathcal{H}_{\Psi_h}(\Psi_h(\psi))) \vee (\Psi_h(\mathcal{H}_{\Psi_h}(f_2)), \mathcal{H}_{\Psi_h}(f_2))$$
$$= (\Psi_h(\mathcal{H}_{\Psi_h}(\Psi_h(\psi)) \vee \mathcal{H}_{\Psi_h}(f_2)), \mathcal{H}_{\Psi_h}(\Psi_h(\psi)) \vee \mathcal{H}_{\Psi_h}(f_2))$$
$$= ((1, 1, \frac{2}{3}), \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & \frac{1}{2} \end{pmatrix}) \in \mathcal{R}(\mathcal{H}_{\Psi_h}, \Psi_h),$$

$$(\Psi_h(\psi), \mathcal{H}_{\Psi_h}(\Psi_h(\psi))) \wedge (\Psi_h(\mathcal{H}_{\Psi_h}(f_2)), \mathcal{H}_{\Psi_h}(f_2)) = (\Psi_h(\psi) \wedge \Psi_h(\mathcal{H}_{\Psi_h}(f_2)),$$
$$\mathcal{H}_{\Psi_h}(\Psi_h(\psi) \wedge \Psi_h(\mathcal{H}_{\Psi_h}(f_2)))) = ((\tfrac{2}{3}, \tfrac{2}{3}, 0), \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}) \in \mathcal{R}(\mathcal{H}_{\Psi_h}, \Psi_h).$$

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