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Abstract. The main purpose of this study is to define the concept of rough limit set of a triple sequence space of Bernstein-Stancu polynomials of Borel summability of fuzzy numbers. We obtain the relation between the set of rough limit and the extreme limit points of a triple sequence space of Bernstein-Stancu polynomials of Borel summability method of fuzzy numbers. Finally, we investigate some properties of the rough limit set of Bernstein-Stancu polynomials under which Borel summable sequence of fuzzy numbers are convergent. Also, we give the results for Borel summability method of series of fuzzy numbers.

2020 AMS Classification: 03E72, 08A72

Keywords: Triple sequences, Rough convergence, Closed and convex, Cluster points and rough limit points, Sequences of fuzzy numbers, Bernstein-Stancu polynomials, Borel summability method.

Corresponding Author: A. Esi (ayhan.esi@ozal.edu.tr)

1. Introduction

A triple sequence (S_{mnk}) of complex numbers is called Borel summable to S if the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^{m+n+k}}{(m+n+k)!}S_{mnk}$ converges for all $x \in \mathbb{R}$ and

$$e^{-x}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^{m+n+k}}{(m+n+k)!}S_{mnk} \to S \in \mathbb{R} \text{ for } x \to \infty.$$

In this paper, we define Borel summability method for triple sequences and series of fuzzy numbers.

Definition 1.1. Let (u_{mnk}) be a triple sequence of fuzzy numbers. Then the expression $\sum\sum\sum u_{mnk}$ is called a series of fuzzy numbers. Throughout the paper S_{rst} will be denoted by $S_{rst} = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} u_{mnk}$ for all $r, s, t \in \mathbb{N}$. If the sequence (S_{rst}) converges to a fuzzy number u, then we say that the series $\sum\sum\sum u_{mnk}$ of fuzzy numbers converges to u and write $\sum\sum\sum u_{mnk} = u$ which implies that $\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} u_{mnk}^{-}(\lambda) \to u^{-}(\lambda)$ and $\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} u_{mnk}^{+}(\lambda) \to u^{+}(\lambda)$ as $r, s, t \to \infty$, uniformly in $\lambda \in [0, 1]$.

Conversely, if the fuzzy numbers $u_{mnk} = \left\{\left(u_{mnk}^{-}(\lambda), u_{mnk}^{+}(\lambda)\right) : \lambda \in [0, 1]\right\}$, $\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} u_{mnk}^{-}(\lambda) \to u^{-}(\lambda)$ and $\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} u_{mnk}^{+}(\lambda) \to u^{+}(\lambda)$ converge uniformly in λ, then $u = \{(u^{-}(\lambda), u^{+}(\lambda)) : \lambda \in [0, 1]\}$ defines a fuzzy number such that $u = \sum\sum\sum u_{mnk}$.

Otherwise we say that the series of fuzzy numbers diverges. Additionally, if the triple sequence (S_{rst}) is bounded then we say that the series $\sum\sum\sum u_{mnk}$ of fuzzy numbers is bounded. We denote the set of all bounded series of fuzzy numbers by $bs(F)$.

Definition 1.2. A triple sequence (u_{mnk}) of fuzzy numbers is called Borel summable to $\zeta \in E^{`}$ if the series

$$f(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^{m+n+k}}{(m+n+k)!} u_{mnk}$$

converges for $x \in (0, \infty)$ and $\lim_{x\to\infty} e^{-x} f(x) = \zeta$.

The idea of rough convergence was first introduced by [1, 2, 3] in finite dimensional normed spaces. He showed that the set $\mathrm{LIM}^{r} x$ is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathrm{LIM}^{r} x$ on the roughness of degree r.

Aytar [4] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, it was studied in [5] that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dündar and Çakan [6] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence. The notion of I-convergence of a triple sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence. Tripathy et al. [7] introduced the notion of rough I- statistical convergence in probabilistic n-normed spaces. Kişi, and Choudhury [8, 9] introduced and investigated the concept of statistical convergence for triple sequences and rough I- deferred statistical convergence of sequences in Gradual Normed Linear Spaces. Kişi and Dündar [10] introduced and studied the notion of rough I_2-lacunary statistical convergence of double sequences in normed linear spaces. Mohiuddine et al. [11] introduced concept of weighted statistical convergence and strong weighted summability for sequences of fuzzy numbers. Hazarika et al. [12] presented a Korovkin-type approximation theorem for Bernstein polynomials of rough statistical convergence of triple sequences. Indumathi et al. [13] defined Borel rough summable of triple sequences and discuss some fundamental results related to Borel rough summable of triple Bernstein-Stancu operators based on (p,q)-integers.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and

$$K_{ijl}=\{(m,n,k)\in K: m\leq i, n\leq j, k\leq l\}.$$

Then the natural density of K is given by

$$\delta(K)=\lim_{i,j,\ell\to\infty}\frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

First applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. Later, based on (p,q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

Recently, Khalid [14] introduced an insightful application in computer-aided geometric design, utilizing Bernstein basis for constructing (p,q)-Bezier curves and surfaces based on (p,q)-Bezier curves and surfaces based on q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p,q)-approximation and its application. In this paper we study statistical approximation properties of Bernstein-Stancu operators based on (p,q)-integers.

Now we recall some basic definitions about (p,q)-integers. For any $u,v,w\in\mathbb{N}$, the (p,q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q}:=0 \text{ and } [uvw]_{p,q}=\frac{p^{uvw}-q^{uvw}}{p-q} \text{ if } u,v,w\geq 1,$$

where $0<q<p\leq 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}!:=1 \text{ and } [uvw]!_{p,q}=[1]_{p,q}[2]_{p,q}\cdots[uvw]_{p,q} \text{ if } u,v,w\geq 1 \text{ and } u,v,w\in\mathbb{N}.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q}=\frac{[u]!_{p,q}}{[m]!_{p,q}\,[u-m]!_{p,q}}\;\frac{[v]!_{p,q}}{[n]!_{p,q}\,[v-n]!_{p,q}}\;\frac{[w]!_{p,q}}{[k]!_{p,q}\,[w-k]!_{p,q}}$$

for all $u,v,w,m,n,k\in\mathbb{N}$ with $u\geq m$, $v\geq n$, $w\geq k$.

The formula for (p,q)-binomial expansion is as follows:

$$\begin{aligned}&(ax+by)^{uvw}_{p,q}\\ &=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}}q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}\\ &\quad\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q}a^{(u-m)+(v-n)+(w-k)}b^{m+n+k}x^{(u-m)+(v-n)+(w-k)}y^{m+n+k},\end{aligned}$$

$$\begin{aligned}&(x+y)^{uvw}_{p,q}\\ &=(x+y)(px+qy)\left(p^2x+q^2y\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x+q^{(u-1)+(v-1)+(w-1)}y\right),\end{aligned}$$

$$(1-x)_{p,q}^{uvw}$$
$$=(1-x)\,(p-qx)\,\left(p^2-q^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}-q^{(u-1)+(v-1)+(w-1)}x\right),$$

and

$$(x)_{p,q}^{mnk}=x\,(px)\,\left(p^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right)=p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator of order rst is given by

$$B_{rst}(f,x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}\,(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

The (p,q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}\,(f,x)$$
$$=\frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}x^{m+n+k}$$

(1.1)

$$\prod_{u_1=0}^{(r-m-1)}\left(p^{u_1}-q^{u_1}x\right)\prod_{u_2=0}^{(s-n-1)}\left(p^{u_2}-q^{u_2}x\right)\prod_{u_3=0}^{(t-k-1)}\left(p^{u_3}-q^{u_3}x\right)$$
$$f\left(\frac{[m]_{p,q}\,[n]_{p,q}\,[k]_{p,q}}{p^{(m-r)+(n-s)+(k-t)}\,[r]_{p,q}\,[s]_{p,q}\,[t]_{p,q}+\mu}\right),x\in[0,1]$$

Also, we have

$$(1-x)_{p,q}^{rst}$$
$$=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k}\,p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{6}}q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}$$
$$\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}.$$

(p,q)-Bernstein-Stancu operators are defined as follows:

$$S_{rst,p,q}\,(f,x)$$
$$=\frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}x^{m+n+k}$$

(1.2)

$$\prod_{u_1=0}^{(r-m-1)}\left(p^{u_1}-q^{u_1}x\right)\prod_{u_2=0}^{(s-n-1)}\left(p^{u_2}-q^{u_2}x\right)\prod_{u_3=0}^{(t-k-1)}\left(p^{u_3}-q^{u_3}x\right)$$
$$f\left(\frac{p^{(r-m)+(s-n)+(t-k)}\,[m]_{p,q}\,[n]_{p,q}\,[k]_{p,q}+\eta}{[r]_{p,q}\,[s]_{p,q}\,[t]_{p,q}+\mu}\right),x\in[0,1]$$

Note that for $\eta = \mu = 0$, (p, q)-Bernstein-Stancu operators given by (1.2) reduces into (p, q)-Bernstein-Stancu operators. Also for $p = 1$, (p, q)-Bernstein-Stancu operators given by (1.1) turn out to be q-Bernstein-Stancu operators.

Throughout the paper, $\mathbb{R}$ denotes the real with metric (X, d). Consider a triple sequence of Bernstein stancu polynomials $(B_{mnk}(f, x))$ such that $(B_{mnk}(f, x)) \in \mathbb{R}$, $m, n, k \in \mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is called statistically convergent to $0 \in \mathbb{R}$, written as $st - \lim\ x = 0$, provided that the set

$$K_\epsilon := \left\{(m, n, k) \in \mathbb{N}^3 : |S_{rst,p,q}(f, x) - (f, x)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein-Stancu polynomials. i.e., $\delta(K_\epsilon) = 0$. That is,

$$\lim_{r,s,t\to\infty} \frac{1}{pqj} |\{m \leq p, n \leq q, k \leq j : |S_{rst,p,q}(f, x) - (f, x)| \geq \epsilon\}| = 0.$$

In this case, we write $\delta - \lim S_{rst,p,q}(f, x) = (f, x)$ or $S_{rst,p,q}(f, x) \to^{S_S} (f, x)$.

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by [15, 16, 17, 18, 19, 20, 21, 22], [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] and many others.

A triple sequence $x = (x_{mnk})$ is called triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3.

The Borel summability of fuzzy real numbers is denoted by $(\zeta, X)(\mathbb{R})$, and d denotes the supremum metric on $(\zeta, X)(\mathbb{R}^3)$. Now let r be nonnegative real number. A Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ of fuzzy numbers is r-convergent to a fuzzy number (ζ, X) and we write

$$S_{rst,p,q}(\zeta, X) \to^r (\zeta, X) \text{ as } m, n, k \to \infty,$$

provided that for every $\epsilon > 0$ there is an integer $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d(S_{rst,p,q}(\zeta, X), (\zeta, X)) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

The set $\mathrm{LIM}^r S_{rst,p,q}(\zeta, X) := \big\{(\zeta, X) \in (\zeta, X)(\mathbb{R}^3) : S_{rst,p,q}(\zeta, X) \to^r (\zeta, X),$ as $m, n, k \to \infty\big\}$ is called the r-limit set of the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$.

A Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of fuzzy numbers which is divergent can be convergent with a certain roughness degree. For instance, let us define

$$S_{rst,p,q}(\zeta, X) = \begin{cases} \eta(X), & \text{if } m, n, k \text{ are odd integers} \\ \mu(X), & \text{otherwise,} \end{cases}$$

where

$$\eta(X) = \begin{cases} X, & \text{if } X \in [0,1] \\ -X+2, & \text{if } X \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu(X) = \begin{cases} X-3, & \text{if } X \in [3,4] \\ -X+5, & \text{if } X \in [4,5] \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\text{LIM}^r S_{rst,p,q}(\zeta, X) = \begin{cases} \phi, & \text{if } r < \frac{3}{2} \\ [\mu - r_1, \eta + r_1], & \text{otherwise,} \end{cases}$$

where r_1 is nonnegative real number with

$$[\mu - r_1, \eta + r_1] := \left\{ S_{rst,p,q}(\zeta, X) \in (\zeta, X)\left(\mathbb{R}^3\right) : \mu - r_1 \leq S_{rst,p,q}(\zeta, X) \leq \eta + r_1 \right\}.$$

The ideal of rough convergence of a Borel summability of triple sequence space of Bernstein-Stancu polynomials can be interpreted as follows:

Let $(S_{rst,p,q}(\zeta, Y))$ be a convergent triple sequence space of Bernstein-Stancu polynomials of fuzzy numbers. Assume that $(S_{rst,p,q}(\zeta, Y))$ cannot be determined exactly for every $(m,n,k) \in \mathbb{N}^3$. That is, $(S_{rst,p,q}(\zeta, Y))$ cannot be calculated so we can use approximate value of $(S_{rst,p,q}(\zeta, Y))$ for simplicity of calculation. We only know that $(S_{rst,p,q}(\zeta, Y)) \in [\mu_{mnk}, \lambda_{mnk}]$, where $d(\mu_{mnk}, \lambda_{mnk}) \leq r$ for every $(m,n,k) \in \mathbb{N}^3$. The Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ satisfying $(S_{rst,p,q}(\zeta, X)) \in [\mu_{mnk}, \lambda_{mnk}]$, for all $m,n,k \in \mathbb{N}$. Then the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ may not be convergent, but the inequality

$$\begin{aligned} d(S_{rst,p,q}(\zeta, X), (\zeta, X)) &\leq d(S_{rst,p,q}(\zeta, X), S_{rst,p,q}(\zeta, Y)) + d(S_{rst,p,q}(\zeta, Y), (\zeta, Y)) \\ &\leq r + d(S_{rst,p,q}(\zeta, Y), (\zeta, Y)) \end{aligned}$$

implies that the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ is r-convergent.

In this paper, we first define the concept of rough convergence of a Borel summability of triple sequence space of Bernstein-Stancu polynomials of fuzzy numbers. Also obtain the relation between the set of rough limit and the extreme limit points of a Borel summability of triple sequence space of Bernstein-Stancu polynomials of fuzzy numbers. We show that the rough limit set of a Borel summability of triple sequence space of Bernstein-Stancu polynomials is closed, bounded and convex.

2. Definitions and Preliminaries

A fuzzy number X is a fuzzy subset of the real $\mathbb{R}^3$, which is normal fuzzy convex, upper semi-continuous, and the X^0 is bounded where $X^0; = \text{cl}\left\{x \in \mathbb{R}^3 : X(x) > 0\right\}$ and cl is the closure operator. These properties imply that for each $\alpha \in (0,1]$, the *α-level set* X^α defined by

$$X^\alpha = \left\{x \in \mathbb{R}^3 : X(x) \geq \alpha\right\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$$

is a non empty compact convex subset of $\mathbb{R}^3$.

The *supremum metric* d on the set $L\left(\mathbb{R}^3\right)$ is defined by

$$d\left(X,Y\right)=\sup_{\alpha\in[0,1]}\max\left(\left|\underline{X}^{\alpha}-\underline{Y}^{\alpha}\right|,\left|\overline{X}^{\alpha}-\overline{Y}^{\alpha}\right|\right).$$

Now, given $X,\ Y\in L\left(\mathbb{R}^3\right)$, we define $X\leq Y$ if $\underline{X}^{\alpha}\leq\underline{Y}^{\alpha}$ and $\overline{X}^{\alpha}\leq\overline{Y}^{\alpha}$ for each $\alpha\in[0,1]$.

We write $X\leq Y$ if $X\leq Y$ and there exists an $\alpha_0\in[0,1]$ such that $\underline{X}^{\alpha_0}\leq\underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0}\leq\overline{Y}^{\alpha_0}$.

A subset E of $L\left(\mathbb{R}^3\right)$ is called *bounded above*, if there exists a fuzzy number μ, called an *upper bound* of E, such that $X\leq\mu$ for every $X\in E$. μ is called the *least upper bound* of E, if μ is an upper bound and $\mu\leq\mu^{'}$ for all upper bounds $\mu^{'}$.

A lower bound and the greatest lower bound are defined similarly. E is called *bounded*, if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E\subset L\left(\mathbb{R}^3\right)$ is bounded then its supremum and infimum exist.

The *limit infimum* and *limit supremum* of a triple sequence spaces (X_{mnk}) is defined by

$$\lim_{m,n,k\to\infty}\inf\ X_{mnk}:=\inf\ A_X,$$

$$\lim_{m,n,k\to\infty}\sup\ X_{mnk}:=\inf\ B_X,$$

where

$$A_X:=\left\{\mu\in L\left(\mathbb{R}^3\right):\text{the set }\left\{(m,n,k)\in\mathbb{N}^3:X_{mnk}<\mu\right\}\text{ is infinite}\right\}$$

$$B_X:=\left\{\mu\in L\left(\mathbb{R}^3\right):\text{the set }\left\{(m,n,k)\in\mathbb{N}^3:X_{mnk}>\mu\right\}\text{ is infinite}\right\}.$$

Now, given two fuzzy numbers $X,Y\in L\left(\mathbb{R}^3\right)$, we define their sum as $Z=X+Y$, where $\underline{Z}^{\alpha}:=\underline{X}^{\alpha}+\underline{Y}^{\alpha}$ and $\overline{Z}^{\alpha}:=\overline{X}^{\alpha}+\overline{Y}^{\alpha}$ for all $\alpha\in[0,1]$.

To any real number $a\in\mathbb{R}^3$, we can assign a fuzzy number $a_1\in L\left(\mathbb{R}^3\right)$, which is defined by

$$a_1\left(x\right)=\begin{cases}1, & \text{if } x=a\\ 0, & \text{otherwise.}\end{cases}$$

An order interval in $L\left(\mathbb{R}^3\right)$ is defined by $[X,Y]:=\left\{Z\in L\left(\mathbb{R}^3\right):X\leq Z\leq Y\right\}$, where $X,\ Y\in L\left(\mathbb{R}^3\right)$.

A set E of fuzzy numbers is called *convex*, if $\lambda\mu_1+(1-\lambda)\mu_2\in E$ for all $\lambda\in[0,1]$ and $\mu_1,\ \mu_2\in E$.

3. Main Results

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A Borel summability of rough triple sequence of Bernstein-Stancu polynomials of $\left(S_{rst,p,q}\left(\zeta,X\right)\right)$ of real numbers. If $\left(\zeta,X\right)\in\mathrm{LIM}^{r}S_{rst,p,q}\left(\zeta,X\right)$, then* $\mathrm{diam}\left(\limsup\ S_{rst,p,q}\left(\zeta,X\right),\left(\zeta,X\right)\right)\leq r$ *and* $\mathrm{diam}\left(\liminf\ S_{rst,p,q}\left(\zeta,X\right),\left(\zeta,X\right)\right)\leq r$.

Proof. We assume that diam $(\limsup\ S_{rst,p,q}(\zeta, X), (\zeta, X)) > r$. We define

$$\tilde{\epsilon} := \frac{(\limsup\ S_{rst,p,q}(\zeta, X), (\zeta, X)) - r}{2}.$$

By definition of limit supremum, we have that given $m'_{\tilde{\epsilon}},\ n'_{\tilde{\epsilon}},\ k'_{\tilde{\epsilon}} \in \mathbb{N}$, there exist some integers $m, n, k \in \mathbb{N}$ with $m \geq m'_{\tilde{\epsilon}}$, $n \geq n'_{\tilde{\epsilon}}$, $k \geq k'_{\tilde{\epsilon}}$ such that

$$\text{diam}\,(\limsup\ S_{rst,p,q}(\zeta, X), (\zeta, X)) \leq \tilde{\epsilon}.$$

Also, since $S_{rst,p,q}(\zeta, X) \to^r (\zeta, X)$ as $m,\ n,\ k \to \infty$, there are some integers $m''_{\tilde{\epsilon}},\ n''_{\tilde{\epsilon}},\ k''_{\tilde{\epsilon}}$ so that $d(S_{rst,p,q}(\zeta, X), (\zeta, X)) < r + \tilde{\epsilon}$, whenever $m \geq m''_{\tilde{\epsilon}}$, $n \geq n''_{\tilde{\epsilon}}$, $k \geq k''_{\tilde{\epsilon}}$. Let

$$m_{\tilde{\epsilon}} = \max\left\{m'_{\tilde{\epsilon}}, m''_{\tilde{\epsilon}}\right\}, n_{\tilde{\epsilon}} = \max\left\{n'_{\tilde{\epsilon}}, n''_{\tilde{\epsilon}}\right\}, k_{\tilde{\epsilon}} = \max\left\{k'_{\tilde{\epsilon}}, k''_{\tilde{\epsilon}}\right\}.$$

Then there exist integers $m,\ n,\ k \in \mathbb{N}$ such that $m \geq m_{\tilde{\epsilon}}$, $n \geq n_{\tilde{\epsilon}}$, $k \geq k_{\tilde{\epsilon}}$ and

$$\begin{aligned}
\text{diam}(\limsup S_{rst,p,q}(\zeta,X),(\zeta,X)) &\leq (\zeta,X)\ \text{diam}(\limsup S_{rst,p,q}(\zeta,X), S_{rst,p,q}(\zeta,X)) \\
&\quad + \text{diam}\,(S_{rst,p,q}, (\zeta, X)) \\
&< \tilde{\epsilon} + r + \tilde{\epsilon} \\
&< r + 2\tilde{\epsilon} \\
&= r + \text{diam}\,(\limsup\ S_{rst,p,q}(\zeta, X), (\zeta, X)) - r \\
&= \text{diam}\,(\limsup\ S_{rst,p,q}(\zeta, X), (\zeta, X)).
\end{aligned}$$

Thus the contradiction proves the theorem. Similarly, diam $(\liminf\ S_{rst,p,q}(\zeta, X), (\zeta, X)) \leq r$ can be proved using definition of limit infimum. □

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0, 1]$. A Borel summability of rough triple sequence of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ of real numbers. If* $\text{LIM}^r S_{rst,p,q}(\zeta, X) \neq \phi$, *then we have*

$$\text{LIM}^r S_{rst,p,q}(\zeta, X) \subseteq [(\limsup S_{rst,p,q}(\zeta, X)) - r_1, (\liminf S_{rst,p,q}(\zeta, X)) + r_1].$$

Proof. To prove that $(\zeta,X) \in [(\limsup S_{rst,p,q}(\zeta,X)) - r_1, (\liminf S_{rst,p,q}(\zeta,X)) + r_1]$ for an arbitrary $(\zeta, X) \in \text{LIM}^r S_{rst,p,q}(\zeta, X)$, i.e.,

$$(\limsup S_{rst,p,q}(\zeta, X)) - r_1 \leq (\zeta, X) \leq (\liminf S_{rst,p,q}(\zeta, X)) + r_1.$$

Let us assume that $(\limsup S_{rst,p,q}(\zeta, X)) - r_1 \leq (\zeta, X)$ does not hold. Then there exists an $\alpha_0 \in [0, 1]$ such that

$$\left(\underline{\limsup S_{rst,p,q}(\zeta, X)}^{\alpha_0}\right) - r_1 > \underline{(\zeta, X)}^{\alpha_0} \text{ or } \left(\overline{\limsup S_{mnk}(\zeta, X)}^{\alpha_0}\right) - r_1 > \leq \overline{(\zeta, X)}^{\alpha_0}$$

holds i.e.,

$$\left(\underline{\limsup S_{rst,p,q}(\zeta,X)}^{\alpha_0}\right) - \underline{(\zeta,X)}^{\alpha_0} > r_1 \text{ or } \left(\overline{\limsup S_{rst,p,q}(\zeta,X)}^{\alpha_0}\right) - \leq \overline{(\zeta, X)}^{\alpha_0} > r_1.$$

On the other hand, by theorem 3.1 we have

$$\left|\left(\underline{\limsup S_{rst,p,q}(\zeta, X)}^{\alpha_0}\right) - \underline{(\zeta, X)}^{\alpha_0}\right| \leq r_1$$

and

$$\left|\left(\overline{\limsup S_{rst,p,q}(\zeta, X)}^{\alpha_0}\right) - \leq \overline{(\zeta, X)}^{\alpha_0}\right| \leq r_1.$$

Thus we obtain a contradiction. So we get $(\limsup S_{rst,p,q}(\zeta, X)) - r_1 \leq (\zeta, X)$. By using the similar arguments and get it for second part. □

Note 3.3. The converse inclusion in this theorem holds for f be a continuous function defined on the closed interval $[0, 1]$. A Borel summability of rough triple sequence of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ of real numbers, but it may not hold for Borel summability of rough triple sequences of Bernstein-Stancu polynomials of fuzzy numbers as in the following example:

Example 3.4. Define

$$S_{rst,p,q}(\zeta, X) = \begin{cases} \frac{-1}{2(mnk)}X + 1, & \text{if } X \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and

$$(\zeta, X) = \begin{cases} 1, & \text{if } X \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $\left|\overline{(\zeta, X)}^1 - \overline{S_{rst,p,q}(\zeta, X)}^1\right| = |1 - 0| = 1$, i.e., $d(S_{rst,p,q}(\zeta, X), (\zeta, X)) \geq 1$ for all $(m, n, k) \in \mathbb{N}^3$. Although the Borel summability of rough triple sequence spaces of Bernstein-Stancu poynomials of $(S_{rst,p,q}(\zeta, X))$ is not convergent to (ζ, X), $\limsup S_{rst,p,q}(\zeta, X)$ and $\liminf S_{rst,p,q}(\zeta, X)$ of this Borel summability of rough triple sequence space of Bernstein-Stancu polynomials are equal to (ζ, X). Thus we get

$$L \in \left[\limsup S_{rst,p,q}(\zeta, X) - \left(\frac{1}{2}\right)_1, \liminf S_{rst,p,q}(\zeta, X) + \left(\frac{1}{2}\right)_1\right],$$

but $(\zeta, X) \notin \mathrm{LIM}^{\frac{1}{2}} S_{rst,p,q}(\zeta, X)$.

Theorem 3.5. *Let f be a continuous function defined on the closed interval $[0, 1]$. A Borel summability of rough triple sequence of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ of real numbers converges to the fuzzy number (f, X), then*

$$\mathrm{LIM}^r S_{rst,p,q}(\zeta, X) = \bar{S}_r((\zeta, X)) := \left\{\mu \in (\zeta, X)\left(\mathbb{R}^3\right) : d(\mu, (\zeta, X)) \leq r\right\}$$

Proof. Let $\epsilon > 0$. Since the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ is convergent to (ζ, X), there are integers $m_\epsilon,\ n_\epsilon,\ k_\epsilon$ so that

$$d(S_{rst,p,q}(\zeta, X), (\zeta, X)) < \epsilon, \text{ whenever } m \geq m_\epsilon,\ n \geq n_\epsilon,\ k \geq k_\epsilon.$$

Let $Y \in \bar{S}_r((\zeta, X))$. Then we have

$$d(S_{rst,p,q}(\zeta, X), Y) \leq d(S_{rst,p,q}(\zeta, X), (\zeta, X)) + d((\zeta, X), Y) < \epsilon + r$$

for all $m \geq m_\epsilon$, $n \geq n_\epsilon$, $k \geq k_\epsilon$. Thus we have $Y \in \mathrm{LIM}^r S_{rst,p,q}(\zeta, X)$.

Now let $Y \in \mathrm{LIM}^r S_{rst,p,q}(\zeta, X)$. Then there are some integers $m_\epsilon^{'}, n_\epsilon^{'}, k_\epsilon^{'}$ so that

$$d(S_{rst,p,q}(\zeta, X), Y) < r + \epsilon,$$

whenever $m \geq m_\epsilon^{'}$, $n \geq n_\epsilon^{'}$, $k \geq k_\epsilon^{'}$. Let

$$m_\epsilon^{''} = \max\left\{m_\epsilon, m_\epsilon^{'}\right\}, n_\epsilon^{''} = \max\left\{n_\epsilon, n_\epsilon^{'}\right\}, k_\epsilon^{''} = \max\left\{k_\epsilon, k_\epsilon^{'}\right\}.$$

Then we obtain

$$d(Y, \zeta(X)) \leq d(Y, S_{rst,p,q}(\zeta, X)) + d(S_{rst,p,q}(\zeta, X), (\zeta, X)) < r + \epsilon + \epsilon = r + 2\epsilon.$$

Since ϵ is arbitrary, we have $d(Y, (\zeta, X)) \leq r$. Thus we get $Y \in \bar{S}_r((\zeta, X))$. So, if the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X)) \to^r (\zeta, X)$, then $\mathrm{LIM}^r S_{rst,p,q}(\zeta, X) = \bar{S}_r((\zeta, X))$. □

Theorem 3.6. *Let f be a continuous function defined on the closed interval $[0,1]$. The diameter of $\mathrm{LIM}\, S_{rst,p,q}(\zeta, X)$ of triple sequence of Bernstein-Stancu polynomials $S_{rst,p,q}(\zeta, X)$ is not greater than $3\boldsymbol{r}$.*

Proof. We have to prove that

$$\sup\{d(W, Z) : W, Y, Z \in \mathrm{LIM}^r S_{rst,p,q}(\zeta, X)\} \leq 3r.$$

Assume on the contrary that

$$\sup\{d(W, Z) : W,\ Y,\ Z \in \mathrm{LIM}^r S_{rst,p,q}(\zeta, X)\} > 3r.$$

By this assumption, there exists, $W, Y, Z \in \mathrm{LIM}^r S_{rst,p,q}(\zeta, X)$ satisfying $\lambda := d(W, Z) > 3r$. For an arbitrary $\epsilon \in \left(0, \frac{\lambda}{3} - r\right)$, we have

$$\exists \left(m_\epsilon^{'}, n_\epsilon^{'}, k_\epsilon^{'}\right) \in \mathbb{N}^3 : \forall m \geq m_\epsilon^{'}, n \geq n_\epsilon^{'}, k \geq k_\epsilon^{'} \Longrightarrow d(S_{rst,p,q}(\zeta, X), W) \leq r + \epsilon,$$

$$\exists \left(m_\epsilon^{''}, n_\epsilon^{''}, k_\epsilon^{''}\right) \in \mathbb{N}^3 : \forall m \geq m_\epsilon^{''}, n \geq n_\epsilon^{''}, k \geq k_\epsilon^{''} \Longrightarrow d(S_{rst,p,q}(\zeta, X), Y) \leq r + \epsilon,$$

$$\exists \left(m_\epsilon^{'''}, n_\epsilon^{'''}, k_\epsilon^{'''}\right) \in \mathbb{N}^3 : \forall m \geq m_\epsilon^{'''}, n \geq n_\epsilon^{'''}, k \geq k_\epsilon^{'''} \Longrightarrow d(S_{rst,p,q}(\zeta, X), Z) \leq r + \epsilon.$$

Define

$$m_\epsilon = \max\left\{m_\epsilon^{'}, m_\epsilon^{''}, m_\epsilon^{'''}\right\}, n_\epsilon = \max\left\{n_\epsilon^{'}, n_\epsilon^{''}, n_\epsilon^{'''}\right\}, k_\epsilon := \max\left\{k_\epsilon^{'}, k_\epsilon^{''}, k_\epsilon^{'''}\right\}.$$

Then we get

$$\begin{aligned} d(W, Z) &\leq d(S_{rst,p,q}(\zeta, X), W) + d(S_{rst,p,q}(\zeta, X), Y) + d(S_{rst,p,q}(\zeta, X), Z) \\ &< (r + \epsilon) + (r + \epsilon) + (r + \epsilon) \\ &< 3(r + \epsilon) \\ &< 3r + 3\left(\frac{\lambda}{3} - r\right) < 3r + \lambda - 3r \\ &= \lambda \end{aligned}$$

for all $m \geq m_\epsilon$, $n \geq n_\epsilon$, $k \geq k_\epsilon$, which contradicts to the fact that $\lambda = d(W, Z)$. □

Theorem 3.7. *Let f be a continuous function defined on the closed interval $[0,1]$. A Borel summability of rough triple sequence of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ of real numbers is analytic if and only if there exists an $r \geq 0$ such that $\mathrm{LIM}^r S_{rst,p,q}(\zeta, X) \neq \phi$.*

Proof. **Necessity:** Let the set of all triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$ be analytic and the set by

$$s := \sup\left\{d\left(S_{rst,p,q}(\zeta, X)^{1/m+n+k}, 0\right) : (m, n, k) \in \mathbb{N}^3\right\} < \infty.$$

Then we have $0 \in \mathrm{LIM}^s S_{rst,p,q}(\zeta, X)$, i.e., $\mathrm{LIM}^r S_{rst,p,q}(\zeta, X) \neq \phi$, where $r = s$.

Sufficiency: If $\text{LIM}^r\, S_{rst,p,q}(\zeta, X) \neq \phi$ for some $r \geq 0$, then there exists $(\zeta, X) \in \text{LIM}^r\, S_{rst,p,q}(\zeta, X)$. By definition, for every $\epsilon > 0$ there are some integers $m_\epsilon,\ n_\epsilon,\ k_\epsilon$ so that

$$d\left(S_{rst,p,q}(\zeta, X), (\zeta, X)\right) < r + \epsilon$$

whenever $m \geq m_\epsilon$, $n \geq n_\epsilon$, $k \geq k_\epsilon$. Define

$$t = t(\epsilon) := \max\left\{d\left((\zeta, X), 0\right), d\left(S_{111,p,q}(\zeta, X), 0\right), \cdots, d\left(S_{r_\epsilon s_\epsilon t_\epsilon,p,q}(\zeta, X), 0\right), r + \epsilon\right\}.$$

Then we have

$$S_{rst,p,q} \in \left\{\mu \in (\zeta, X)\left(\mathbb{R}^3\right) : d(\mu, 0) \leq t + r + \epsilon\right\}$$

for every $(m, n, k) \in \mathbb{N}^3$, which proves the boundedness of the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$. □

Theorem 3.8. *Let f be a continuous function defined on the closed interval $[0, 1]$. A Borel summability of rough triple sequence of Bernstein-Stancu polynomials of $(S_{u_m v_n w_k,p,q}(\zeta, X))$ of real numbers is a sub sequence of a Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, X))$, then $\text{LIM}^r\, S_{rst,p,q}(\zeta, X) \subset \text{LIM}^r\, S_{u_m v_n w_k,p,q}(\zeta, X)$.*

Proof. The proof of this theorem is clear from the fact that every subsequence of a convergent sequence is also convergent. □

Theorem 3.9. *Let f be a continuous function defined on the closed interval $[0, 1]$. The set $\text{LIM}^r\, S_{rst,p,q}(\zeta, X)$ of triple sequence of Bernstein-Stancu polynomials $S_{rst,p,q}(\zeta, X)$ is closed.*

Proof. Let $(Y_{mnk}) \subset \text{LIM}^r\, S_{rst,p,q}(\zeta, Y)$ and $S_{rst,p,q}(\zeta, Y) \to (\zeta, Y)$ as $m, n, k \to \infty$. Let $\epsilon > 0$. Since the Borel summability of rough triple sequence space of Bernstein-Stancu polynomials of $(S_{rst,p,q}(\zeta, Y)) \to^r (\zeta, Y)$, there are some integers $i_\epsilon,\ j_\epsilon,\ \ell_\epsilon$ so that

$$d\left(S_{rst,p,q}(\zeta, Y), (\zeta, Y)\right) < \frac{\epsilon}{2},$$

whenever $m \geq i_\epsilon$, $n \geq j_\epsilon$, $k \geq \ell_\epsilon$.
Since $S_{i_\epsilon j_\epsilon \ell_\epsilon,p,q}(\zeta, Y) \in \text{LIM}^r\, S_{rst,p,q}(\zeta, X)$, there is an integer $(m_\epsilon n_\epsilon k_\epsilon)$ so that

$$d\left(S_{rst,p,q}(\zeta, X), S_{i_\epsilon j_\epsilon \ell_\epsilon,p,q}(\zeta, Y)\right) < r + \frac{\epsilon}{2},$$

whenever $m \geq m_\epsilon$, $n \geq n_\epsilon$, $k \geq k_\epsilon$.
Then we have

$$d\left(S_{rst,p,q}(\zeta, X), (\zeta, X)\right) \leq d\left(S_{rst,p,q}(\zeta, X), S_{i_\epsilon j_\epsilon \ell_\epsilon,p,q}(\zeta, Y)\right) < r + \frac{\epsilon}{2} + \frac{\epsilon}{2} = r + \epsilon$$

for every $m \geq m_\epsilon$, $n \geq n_\epsilon$, $k \geq k_\epsilon$.
Thus $L \in \text{LIM}^r\, S_{rst,p,q}(\zeta, X)$. So $\text{LIM}^r\, S_{rst,p,q}(\zeta, X)$ is closed. □

4. Conclusions and Future Work

In this paper we studied statistical approximation properties of Bernstein-Stancu operators and introduced Borel summability of triple sequence space of Bernstein-Stancu polynomials of rough convergence of fuzzy numbers. For the reference sections, consider the following introduction described the main results are motivating the research.

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Ayhan Esi (aesi23@hotmail.com)
Department of Basic Engineering Sciences, Malatya Turgut Ozal University, 44040, Malatya, Turkey

Nagarajan Subramanian (nsmaths@gmail.com)
Department of Mathematics, SASTRA University, Thanjavur-613 401, India

M. Kemal Ozdemir (kozdemir73@gmail.com)
Department of Mathematics, Inonu University, 44200 Malatya, Turkey