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Abstract. This study aims to examine intuitionistic fuzzy congruences and intuitionistic fuzzy submodules on an $R$-module (near-ring module). The relationship between intuitionistic fuzzy congruences and intuitionistic fuzzy submodules of an $R$-module is also obtained. Furthermore, the intuitionistic fuzzy quotient $R$-module of an $R$-module over an intuitionistic fuzzy submodule is defined. The correspondence between intuitionistic fuzzy congruences on an $R$-module and intuitionistic fuzzy congruences on the intuitionistic fuzzy quotient $R$-module of an $R$-module over an intuitionistic fuzzy submodule of an $R$-module is also obtained.

2020 AMS Classification: 08A72, 08A30, 03F55, 16Y30, 16D10.

Keywords: Congruence; $R$-module, Intuitionistic fuzzy submodule, Quotient module, Intuitionistic fuzzy congruence.

Corresponding Author: P. K. Sharma (pksharma@davjalandhar.com )

## 1. Introduction

The concept of a fuzzy set was introduced by Zadeh [1] in 1965. Since then, there has been a tremendous interest in the subject due to its diverse applications ranging from engineering and computer science to social behaviour studies. The concept of fuzzy relations on a set was defined by Zadeh [1, 2]. Fuzzy relations on group has been studied by Bhattacharya and Mukharjee [3], and those in rings and groups by Malik and Modeson in [4]. The detailed applications of fuzzy relations are given by Baets and Kerre in [5]. The construction of a fuzzy congruence relation generated by a fuzzy relation on a vector space was given by Khosravi et al. in [6]. Dutta and Biswas [7] applied the concept of fuzzy congruence in the near-ring module. As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [8] in 1983. After that time, several researchers [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] applied the notion of intuitionistic fuzzy sets to relations, algebra, topology, and topological structures. In particular, Bustince and Burillo [20], and Deschrijver and Kerre [21] applied the concept of intuitionistic fuzzy sets to relations. Also, Hur et al. [14] investigated several properties of intuitionistic fuzzy congruences. Moreover, Hur and his colleagues [22] introduce the notion of intuitionistic fuzzy congruences on a lattice and a semigroup, and investigate some of their properties. Basnet in [11, 12] studied many properties of intuitionistic fuzzy relations with respect to level cut sets. The notion of intuitionistic fuzzy congruence on groups was introduced by Emam [23], and that of universal algebra was studied by Cuvalcioglu and Tarsuslu (Yilmaz) in [24, 25]. Rasuli in [26, 27] studied intuitionistic fuzzy congruence on groups and rings under the $t$-norm, respectively.

Since the correspondence theorem provides a bridge between algebraic structures, and their quotients, allowing us to study properties, factorizations, and relationships in a more manageable way. Its applications extend beyond pure algebra and have implications in various mathematical areas. The main objective of this paper is to establish a connection between intuitionistic fuzzy congruences and intuitionistic fuzzy submodules on an $R$-module (where $R$ is a near-ring) and quotient $R$-module over an intuitionistic fuzzy submodule of an $R$-module. This is achieved by establishing a one-to-one correspondence between the set of intuitionistic fuzzy submodules and the set of intuitionistic fuzzy congruences of an $R$-module. Lastly, we study the intuitionistic fuzzy congruence of a quotient $R$-module over an intuitionistic fuzzy submodule of an $R$-module and obtain a correspondence theorem. .

## 2. Preliminaries

We recall some definitions and results that are used in this paper. For details, see the references quoted therein.

**Definition 2.1** ([28, 29])**.** A *near-ring* $R$ is a system with two binary operations, addition and multiplication, such that:
(i) $(R, +)$ is a group,
(ii) $(R, \cdot)$ is a semigroup,
(iii) $x(y + z) = xy + xz$ for all $x,\ y, z \in R$.

**Definition 2.2** ([28])**.** An *$R$-module* (i.e. *near-ring module*) $M$ is a system consisting of an additive group $M$, a near-ring $R$, and a mapping $(m, r) \rightarrowtail mr$ of $M \times R$ into $M$ such that
(i) $m(x + y) = mx + my$ for all $m \in M$ and for all $x,\ y \in R$,
(ii) $m(xy) = (mx)y$ for all $m \in M$ and for all $x,\ y \in R$.

**Definition 2.3** ([28])**.** An *$R$-homomorphism* $f$ of an $R$-module $M$ into an $R$-module $M^{'}$ is a mapping from $M$ to $M^{'}$ such that for all $m; m_1,\ m_2 \in M$ and for all $r \in R$,
(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$,
(ii) $f(m)r = f(mr)$.

A non-empty subset $N$ of an $R$-module $M$ that forms with the restrictions of the operations on $M$ (addition and scalar multiplication) to $N$ itself an $R$-module is called a *submodule* of $M$. The *kernel* of an $R$-module homomorphism $f : M \to M^{'}$, denoted by $kerf$ is defined as:

$$kerf = \{x \in M : f(x) = 0^{'}\}.$$

It may be noted that $kerf$ is a submodule of $M$.

The submodules of an $R$-module $M$ are defined to be the kernels of $R$-homomorphisms.

**Proposition 2.4** ([28])**.** *An additive normal subgroup $N$ of an $R$-module $M$ is a submodule if and only if $(m + b)r - mr \in N$ for each $m \in M,\ b \in N$ and $r \in R$.*

**Definition 2.5** ([14])**.** A relation $\rho$ on an $R$-module $M$ is called a *congruence* on $M$, if it is an equivalence relation on $M$ such that $(a, b) \in \rho$ and $(c, d) \in \rho$ imply that $(a + c, b + d) \in \rho$ and $(ar, br) \in \rho$ for all $a,\ b,\ c,\ d$ in $M$ and for all $r$ in $R$.

**Definition 2.6** ([8, 9, 10])**.** An *intuitionistic fuzzy set* (IFS) $A$ in $X$ can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to $A$ respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

The *intutionistic fuzzy whole* [resp. *empty*] *set* in $X$, denoted by $\overline{1}$ [resp. $\overline{0}$], is an IFS in $X$ defined as follows: for each $x \in X$,

$$\overline{1}(x) = (1, 0) \text{ [resp. } \overline{0}(x) = (0, 1)].$$

The set of all IFSs of $X$ will be written as $IFS(X)$.

**Remark 2.7** ([10, 12])**.** (1) When $\mu_A(x) + \nu_A(x) = 1\ \forall x \in X$, $A$ is a fuzzy set.

(2) If $p, q \in [0, 1]$ such that $p + q \le 1$, then $A \in IFS(X)$ defined by $\mu_A(x) = p$ and $\nu_A(x) = q$ for all $x \in X$, is called a *constant IFS* of $X$.

If $A,\ B \in IFS(X)$, then $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)\ \forall x \in X$. For any subset $Y$ of $X$, the intuitionistic fuzzy characteristic function (IFCF) $\chi_Y$ is an intuitionistic fuzzy set of $X$, defined as $\chi_Y(x) = (1, 0)\ \forall x \in Y$ and $\chi_Y(x) = (0, 1)\ \forall x \in X \backslash Y$. Let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \le 1$. Then the crisp set

$$A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \ge \alpha \text{ and } \nu_A(x) \le \beta\}$$

is called the $(\alpha, \beta)$-*level subset* of $A$ (See[12]).

**Definition 2.8** ([13])**.** A nonempty IFS of an additive group $G$ is called an *intuitionistic fuzzy normal subgroup* of $G$, if for all $x,\ y$ in $G$,

(i) $\mu_A(x + y) \ge \min\{\mu_A(x), \mu_A(y)\}$, $\nu_A(x + y) \le \max\{\nu_A(x), \nu_A(y)\}$,

(ii) $\mu_A(-x) = \mu_A(x)$, $\nu_A(-x) = \nu_A(x)$,

(iii) $\mu_A(y + x - y) = \mu_A(x)$, $\nu_A(y + x - y) = \nu_A(x)$.

**Definition 2.9** ([13])**.** Let $A$ be an intuitionistic fuzzy normal subgroup of an additive group $G$ and $x \in G$. Then the IFS $x + A$ in $G$ defined by

$$\mu_{(x+A)}(y) = \mu_A(y - x) \text{ and } \nu_{(x+A)}(y) = \nu_A(y - x) \text{ for all } y \text{ in } G$$

is called the *intuitionistic fuzzy coset of $A$ with respect to $x$.*

## 3. Intuitionistic fuzzy submodule (IFSM)

**Definition 3.1.** Let $A$ be a non-empty IFS of an $R$-module $M$. Then $A$ is said to be an *intuitionistic fuzzy submodule* (IFSM) of $M$, if for all $x,\ y \in M,\ r \in R$,

(i) $A$ is an IF normal subgroup of $M$,

(ii) $\mu_A((x+y)r - xr) \geq \mu_A(y)$ and $\nu_A((x+y)r - xr) \leq \nu_A(y)$.

**Example 3.2.** Consider the additive group $(M = \{0,1,2,3\}, +_4)$ and the near-ring $R = (M_{2\times 2}, +, .)$ of $2\times 2$ matrices over the real numbers under the usual operation of addition and multiplication of matrices. Let $. : M \times R \to M$ be the mapping defined as $mX = m + m + m + ...... + m(|X|$ times), where $|X|$ denotes the determinant of matrix $X \in R$. Then $M$ is an $R$-module.
Define the IFS $A$ on $M$ as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0,2 \\ 0.6, & \text{if } x = 1,3 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0,2 \\ 0.3, & \text{if } x = 1,3. \end{cases}$$

Then it is a routine matter to check that the conditions (i) and (ii) of Definition 3.1 hold. Thus $A$ is an IFSM of an $R$-module $M$.

**Proposition 3.3.** *Let $K$ be a non-empty subset of an $R$-module $M$. Then the IF characteristic function $\chi_K$ is an IFSM of $M$ if and only if $K$ is a submodule of $M$.*

The proposition can be directly verified.

**Proposition 3.4.** *Let $A$ be an IFSM of an $R$-module $M$. Then the $(\alpha,\beta)$-level set $A_{(\alpha,\beta)} = \{x \in M : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is a submodule of $M$*

*Proof.* The proof is omitted. □

**Definition 3.5.** Let $A$ be an IFSM of an $R$-module $M$. Then the submodule $A_{(\alpha,\beta)}$ is called the $(\alpha,\beta)$-*level submodule* of $M$.

**Proposition 3.6.** *For a non-empty IFS $A$ of an $R$-module $M$, the following assertion are equivalent:*

(1) *$A$ is an IFSM of $M$,*

(2) *the $(\alpha,\beta)$-level set $A_{(\alpha,\beta)}$ are submodules of $M$*

*Proof.* Since the proof is a simple matter of verification, we omit it. □

**Proposition 3.7.** *Let $A$ is an IFNSG of an additive group $G$. Then $x + A = y + A$ if and only if $\mu_A(y-x) = \mu_A(0), \nu_A(y-x) = \nu_A(0)$ for all $x, y \in G$.*

*Proof.* The proof is straightforward. □

**Theorem 3.8.** *Let $A$ be an IFSM of an $R$-module $M$. Then the set $M/A$ of all IF cosets of $A$ is an $R$-module w.r.t. the operation defined by*
*$(x+A)+(y+A) = (x+y)+A$ and $(x+A)r = (xr+A)$ for all $x, y \in M, r \in R$.*

*If $f : M \to M/A$ is a surjective mapping defined by $f(x) = x + A$ for all $x \in M$, then $f$ is an $R$-homomorphism with $Kerf = \{x \in M : \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$.*

*Proof.* We shall first show that the given operations are well-defined.
Let $x, y, u, v \in M$ be such that $x+A = u+A$ and $y+A = v+A$. Then $\mu_A(x-u) = \mu_A(0), \nu_A(x-u) = \nu_A(0)$ and $\mu_A(y-v) = \mu_A(0), \nu_A(y-v) = \nu_A(0)$.
On the other hand, we have

$$\begin{aligned}\mu_A(x+y-v-u) &= \mu_A(-u+x+y-v)\\ &\geq \min\{\mu_A(-u+x), \mu_A(y-v)\}\\ &= \mu_A(0)\\ &\geq \mu_A(x+y-v-u).\end{aligned}$$

Thus $\mu_A(x+y-v-u) = \mu_A(0)$. Similarly, we can show that $\nu_A(x+y-v-u) = \nu_A(0)$. So $x+y+A = u+v+A$, i.e., $(x+A)+(y+A) = (u+A)+(v+A)$. Hence the first operation is well-defined.

Let $x, y$ be two elements of $M$ such that $x+A = y+A$. Let $r \in R$. Then $\mu_A(x-y) = \mu_A(0),\ \nu_A(x-y) = \nu_A(0)$. Thus we get

$$\mu_A(xr-yr) = \mu_A((y-y+x)r-yr) \geq \mu_A(y-x) = \mu_A(x-y) = \mu_A(0).$$

Again $\mu_A(0) = \mu_A((0+xr-yr)0_R - 00_R) \geq \mu_A(xr-yr)$. So $\mu_A(xr-yr) = \mu_A(0)$. Similarly, we can show that $\nu_A(xr-yr) = \nu_A(0)$. Hence $xr+A = yr+A$. Therefore the second operation is well-defined.

Now let $f$ be the mapping from $M$ to $M/A$ defined by $f(x) = x+A$ for all $x$ in $M$. Then we have

$$f(x+y) = x+y+A = (x+A)+(y+A) = f(x)+f(y).$$

Also, we get

$$f(xr) = xr+A = (x+A)r = f(x)r \text{ for } x, y \in M \text{ and } r \in R.$$

Obviously $f$ is surjective. Thus $f$ is an $R$-epimorphism.
Lastly, $x \in Kerf \Leftrightarrow f(x) = 0+A \Leftrightarrow x+A = 0+A \Leftrightarrow \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)$.
So $Kerf = \{x \in M : \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$. □

**Definition 3.9.** The $R$-module $M/A$ is called the *quotient $R$-module* of $M$ over its intuitionistic fuzzy submodule $A$.

## 4. Intuitionistic fuzzy congruence

**Definition 4.1** ([11, 14])**.** Let $M$ be an $R$-module. A nonempty intuitionistic fuzzy relation (IFR) $\rho$ on $M$ [i.e., a mapping $\rho : M \times M \to [0,1] \times [0,1]$ ] is called an *intuitionistic fuzzy equivalence relation* (IFER), if for all $x,\ y \in M$,
(i) $\mu_\rho(x,x) = 1, \nu_\rho(x,x) = 0$ (IF reflective),
(ii) $\mu_\rho(x,y) = \mu_\rho(y,x)$ and $\nu_\rho(x,x) = \nu_\rho(y,x)$ (IF symmetric),
(iii) $\mu_\rho(x,y) \geq \sup_{z\in M}\{\mu_\rho(x,z), \mu_\rho(z,y)\}$ and $\nu_\rho(x,y) \leq \inf_{z\in M}\{\nu_\rho(x,z), \nu_\rho(z,y)\}$ (IF transitive).

Let $\rho$ be an IFER on a set $M$ and $a \in M$ be any element. Then the IFS $\rho_a$ on $M$ defined by $\mu_{\rho_a}(x) = \mu_\rho(a,x)$ and $\nu_{\rho_a}(x) = \nu_\rho(a,x)\ \forall x \in M$ is called the *intuitionistic fuzzy equivalence class* of $\rho$ containing $a$. The set $\{\rho_a : a \in M\}$ is called the *intuitionistic fuzzy quotient set* of $M$ by $\rho$ and is denoted by $M/\rho$.

**Definition 4.2** ([14]). An IFER $\rho$ on an $R$-module $M$ is called an *intuitionistic fuzzy congruence* (IFC), if for all $a,\ b,\ c,\ d \in M$ and all $r \in R$.

(i) $\mu_\rho(a+c, b+d) \geq \min\{\mu_\rho(a,b), \mu_\rho(c,d)\}$, $\nu_\rho(a+c, b+d) \leq \max\{\nu_\rho(a,b), \nu_\rho(c,d)\}$,

(ii) $\mu_\rho(ar, br) \geq \mu_\rho(a,b)$, $\nu_\rho(ar, br) \leq \nu_\rho(a,b)$.

**Example 4.3.** Consider $M = \mathbf{Z_4}$, $R = (\{0,1\}, +_2, \cdot_2)$. Then $M$ is an $R$-module. Now we define the IFS $\rho$ on $M \times M$ by

$$\mu_\rho((x,y)) = \begin{cases} 1, & \text{if } x = y \\ 0.5, & \text{if } (x,y) \in \{(2,3),(3,2)\} \\ 0.3, & \text{otherwise,} \end{cases}$$

$$\nu_\rho((x,y)) = \begin{cases} 0, & \text{if } x = y \\ 0.4, & \text{if } (x,y) \in \{(2,3),(3,2)\} \\ 0.6, & \text{otherwise.} \end{cases}$$

Then $\rho$ is an IFER on $M$, but it is not an IFC, for $\mu_\rho(2+2, 3+2) \ngeq \min\{\mu_\rho(2,3), \mu_\rho(2,2)\}$.

**Example 4.4.** Consider the $R$-module $M$ as in Example 3.2, we define the IFS $\rho$ on $M \times M$ as follows

$$\mu_\rho((x,y)) = \begin{cases} 1, & \text{if } x = y \\ 0.6, & \text{if } x \neq y, \end{cases} \qquad \nu_\rho((x,y)) = \begin{cases} 0, & \text{if } x = y \\ 0.3, & \text{if } x \neq y. \end{cases}$$

Then it is a routine matter to verify that the conditions of Definition 4.2 hold. Thus $\rho$ is an IFC on $M$.

**Theorem 4.5.** *Let $\lambda$ be a relation on an $R$-module $M$ and $\chi_\lambda$ be its intuitionistic fuzzy characteristic function. Then $\lambda$ is a congruence relation on $M$ if and only if $\chi_\lambda$ is an IFC on $M$.*

*Proof.* The proof is omitted. □

**Definition 4.6** ([11, 14]). Let $\rho$ be an IFR on an $R$-module $M$. For each $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, the set

$$\rho_{(\alpha,\beta)} = \{(a,b) \in M \times M : \mu_\rho(a,b) \geq \alpha, \nu_\rho(a,b) \leq \beta\}$$

is called a $(\alpha,\beta)$-*level set* of $\rho$.

It is shown in [11] that $\rho$ is an IFER if and only if $\rho_{(\alpha,\beta)}$ is an equivalence relation on $M$ for all $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$."

**Theorem 4.7.** *Let $\rho$ be an IFER on an $R$-module $M$. Then $\rho$ is an IFC on $M$ if and only if $\rho_{(\alpha,\beta)}$ is a congruence on $M$ for each $(\alpha,\beta) \in img(\rho)$ (Image of $\rho$).*

**Proposition 4.8.** *Let $\rho$ be an IFC on an $R$-module $M$ and $A_\rho$ be an IFS of $M$ defined by*

$$\mu_{A_\rho}(a) = \mu_\rho(a,0) \text{ and } \nu_{A_\rho}(a) = \nu_\rho(a,0) \text{ for all } a \in M.$$

*Then $A_\rho$ is an IFSM of $M$.*

*Proof.* Since $\mu_{A_\rho}(0) = \mu_\rho(0,0) = 1$ and $\nu_{A_\rho}(0) = \nu_\rho(0,0) = 0$.
Also, $\mu_{A_\rho}(a+b) = \mu_\rho(a+b, 0) \geq \min\{\mu_\rho(a,0), \mu_\rho(b,0)\} = \min\{\mu_{A_\rho}(a), \mu_{A_\rho}(b)\}$

Similarly, $\nu_{A_\rho}(a+b) = \nu_\rho(a+b,0) \leq \max\{\nu_\rho(a,0), \nu_\rho(b,0)\} = \max\{\nu_{A_\rho}(a), \nu_{A_\rho}(b)\}$. Then we have

$$\begin{aligned} \mu_{A_\rho}(-a) &= \mu_\rho(-a,0) = \mu_\rho(-a+0, -a+a) \\ &\geq \min\{\mu_\rho(-a,-a), \mu_\rho(0,a)\} \\ &= \mu_\rho(0,a) = \mu_\rho(a,0) \\ &= \mu_{A_\rho}(a). \end{aligned}$$

Similarly, we can show $\mu_{A_\rho}(a) \geq \mu_{A_\rho}(-a)$. Thus $\mu_{A_\rho}(-a) = \mu_{A_\rho}(a)$. In the same way, we can show that $\nu_{A_\rho}(-a) = \nu_{A_\rho}(a)$. Also, we have

$$\mu_{A_\rho}(a+b-a) = \mu_\rho(a+b-a,0) = \mu_\rho(a+b-a, a+0-a) \geq \mu_\rho(b,0) = \mu_{A_\rho}(b).$$

Similarly, we can show that $\nu_{A_\rho}(a+b-a) \leq \nu_{A_\rho}(b)$. So $A_\rho$ is an IFNSG of an $R$-module $M$.
Now for $a,\ b \in M$ and $r \in R$, we have

$$\begin{aligned} \mu_{A_\rho}((a+b)r - ar) &= \mu_\rho((a+b)r - ar, 0) = \mu_\rho((a+b)r - ar, ar - ar) \\ &\geq \min\{\mu_\rho((a+b)r, ar), \mu_\rho(-ar,-ar)\} = \mu_\rho((a+b)r, ar) \\ &\geq \mu_\rho((a+b), a) \\ &\geq \mu_\rho(b,0) \\ &= \mu_{A_\rho}(b). \end{aligned}$$

Similarly, we can show that $\nu_{A_\rho}((a+b)r - ar) \leq \nu_{A_\rho}(b)$.
Hence $A_\rho$ is an IFSM of $M$. □

**Example 4.9.** Consider the $R$-module $M$ as in Example 3.2, we define the IFS $\rho$ on $M \times M$ as follows

$$\mu_\rho((x,y)) = \begin{cases} 1, & \text{if } x = y \\ 0.6, & \text{if } x \neq y \end{cases}; \quad \nu_\rho((x,y)) = \begin{cases} 0, & \text{if } x = y \\ 0.3, & \text{if } x \neq y. \end{cases}$$

Then $\rho$ is an IFC on $M$ (See Example 4.4). Let $A_\rho$ be an IFS of $M$, defined by

$\mu_{A_\rho}(a) = \mu_\rho(a,0)$ and $\nu_{A_\rho}(a) = \nu_\rho(a,0)$ for all $a \in M$. Then it is a easy to check that

$$\mu_{A_\rho}(a) = \begin{cases} 1, & \text{if } a = 0 \\ 0.6, & \text{if } a \neq 0, \end{cases} \quad \nu_{A_\rho}(a) = \begin{cases} 0, & \text{if } a = 0 \\ 0.3, & \text{if } a \neq 0. \end{cases}$$

Thus $A_\rho$ is an IFSM of $M$.

**Remark 4.10.** From Examples 3.2 and 4.9, one can easily check that $(\rho)_{A_\rho} \neq A$.

**Proposition 4.11.** *Let $A$ be an IFSM of an $R$-module $M$. Let $\rho_A$ be an IFR on $M$ defined by*

$$\mu_{\rho_A}(x,y) = \mu_A(x-y) \text{ and } \nu_{\rho_A}(x,y) = \nu_A(x-y) \text{ for all } x,\ y \in M.$$

*Then $\rho_A$ is an IFC on $M$.*

*Proof.* Let $x \in M$. Then we have

$$\mu_{\rho_A}(x,x) = \mu_A(x-x) = \mu_A(0) = 1,\ \nu_{\rho_A}(x,x) = \nu_A(x-x) = \nu_A(0) = 0.$$

Thus $\rho_A$ is IF reflexive. It is clear that $\rho_A$ is IF symmetric. Now let $x,\ y \in M$. Then we get

$$\begin{aligned}\mu_{\rho_A}(x,y) &= \mu_A(x-y) = \mu_A(x-z+z-y)\\ &\geq \min\{\mu_A(x-z), \mu_A(z-y)\}\\ &= \min\{\mu_{\rho_A}(x,z), \mu_{\rho_A}(z,y)\}\ \forall z \in M.\end{aligned}$$

Thus $\mu_{\rho_A}(x,y) \geq \sup_{z\in M} \min\{\mu_{\rho_A}(x,z), \mu_{\rho_A}(z,y)\}$. Similarly, we can show that $\nu_{\rho_A}(x,y) \leq \inf_{z\in M} \max\{\nu_{\rho_A}(x,z), \nu_{\rho_A}(z,y)\}$. So $\rho_A$ is an IF equivalence relation on $M$. On the other hand, we have

$$\begin{aligned}\mu_{\rho_A}(x+u, y+v) &= \mu_A(x+u-v-y) = \mu_A(-y+x+u-v)\\ &\geq \min\{\mu_A(-y+x), \mu_A(u-v)\} = \min\{\mu_A(x-y), \mu_A(u-v)\}\\ &= \min\{\mu_{\rho_A}(x,y), \mu_{\rho_A}(u,v)\}.\end{aligned}$$

Similarly, we can show that $\nu_{\rho_A}(x+u, y+v) \leq \max\{\nu_{\rho_A}(x,y), \nu_{\rho_A}(u,v)\}$. Again

$$\begin{aligned}\mu_{\rho_A}(xr, yr) &= \mu_A(xr-yr) = \mu_A((y-y+x)r-yr)\\ &\geq \mu_A(-y+x) = \mu_A(x-y)\\ &= \mu_{\rho_A}(x,y).\end{aligned}$$

Similarly, we can show that $\nu_{\rho_A}(xr-yr) \leq \nu_{\rho_A}(x,y)$. Hence $\rho_A$ is an IFC on $M$. □

**Remark 4.12.** Note $\rho_A$ is called the *IFC induced by* $A$ and $A_\rho$ is called the *IFSM induced by* $\rho$.

**Example 4.13.** Consider $R$, $M$ and IFSM $A$ as in Example 3.2. Define the IFS $\rho_A$ on $M \times M$ as

$$\mu_{\rho_A}(x,y) = \mu_A(x-y) \text{ and } \nu_{\rho_A}(x,y) = \nu_A(x-y) \text{ for all } x,y \in M.$$

Then we have

$$\mu_{\rho_A}((x,y)) = \begin{cases}1, & \text{if } x-y \in \{0,2\}\\ 0.6, & \text{if } x-y \in \{1,3\},\end{cases} \quad \nu_{\rho_A}((x,y)) = \begin{cases}0, & \text{if } x-y \in \{0,2\}\\ 0.3, & \text{if } x-y \in \{1,3\}.\end{cases}$$

It is easy to check that $\rho_A$ is an IFC on $M$ induced by $A$.

**Remark 4.14.** It can be easily seen that $(A_\rho)_A \neq \rho$.

**Theorem 4.15.** *Let $M$ be an $R$-module. Then there exists an inclusion preserving bijection from the set $\mathcal{IFM}(\mathcal{M})$ of all IFSMs of $M$ to the set of $\mathcal{IFC}(\mathcal{M})$ of all IFCs.*

*Proof.* We define mappings $\psi : \mathcal{IFM}(\mathcal{M}) \to \mathcal{IFC}(\mathcal{M})$ and $\phi : \mathcal{IFC}(\mathcal{M}) \to \mathcal{IFM}(\mathcal{M})$, respectively by $\psi(A) = \rho_A$ and $\phi(\rho) = A_\rho$ for $A \in \mathcal{IFM}(\mathcal{M})$ and $\rho \in \mathcal{IFC}(\mathcal{M})$. It is obvious that

$$(\phi \circ \psi)(A) = \phi(\psi(A)) = \phi(\rho_A) = A_{(\rho_A)}.$$

Let $a \in M$. Then we get

$$\mu_{A_{(\rho_A)}}(a) = \mu_{\rho_A}(a,0) = \mu_A(a-0) = \mu_A(a),\ \nu_{A_{(\rho_A)}}(a) = \nu_{\rho_A}(a,0) = \nu_A(a-0) = \nu_A(a).$$

Thus $A_{(\rho_A)} = A$. So $(\phi \circ \psi)(A) = A = \mathrm{Id}(\mathcal{IFC}(\mathcal{M}))(A)$. Hence $\psi$ is injective.

Let $A_1,\ A_2 \in \mathcal{IFM}(\mathcal{M})$ be such that $A_1 \subseteq A_2$. Then for all $(x, y) \in M \times M$,

$$\mu_{\rho_{A_2}}(x, y) = \mu_{A_2}(x - y) \geq \mu_{A_1}(x - y) = \mu_{\rho_{A_1}}(x, y)$$

and

$$\nu_{\rho_{A_2}}(x, y) = \nu_{A_2}(x - y) \leq \nu_{A_1}(x - y) = \nu_{\rho_{A_1}}(x, y).$$

Thus $\rho_{A_1} \subseteq \rho_{A_2}$. So $\psi(A_1) \subseteq \psi(A_2)$. Hence $\psi$ is inclusion preserving.

Now let $\rho \in \mathcal{IFC}(\mathcal{M})$. Then $A_\rho$ is an IFSM of $M$. Note that $\rho_{A_\rho}$ is an IFC of $M$. It is clear that

$$(\psi \circ \phi)(\rho) = \psi(\phi(\rho)) = \psi(A_\rho) = \rho_{A_\rho}.$$

Let $(x, y) \in M \times M$. Then we have

$$\mu_{\rho_{A_\rho}}(x, y) = \mu_{A_\rho}(x - y) = \mu_\rho(x - y, 0) = \mu_\rho(x, y)$$

and

$$\nu_{\rho_{A_\rho}}(x, y) = \nu_{A_\rho}(x - y) = \nu_\rho(x - y, 0) = \nu_\rho(x, y).$$

Thus $\rho_{A_\rho} = \rho$. So $(\psi \circ \phi)(\rho) = \psi(\phi(\rho)) = \rho = Id(\mathcal{IFC}(\mathcal{M}))(\rho)$ for all $\rho \in \mathcal{IFC}(\mathcal{M})$. Hence $\phi \circ \psi = \mathrm{Id}(\mathcal{IFC}(\mathcal{M}))$ which implies that $\psi$ is surjective. Therefore $\psi$ is an inclusion preserving bijection from $\mathcal{IFM}(\mathcal{M})$ to $\mathcal{IFC}(\mathcal{M})$. □

**Proposition 4.16.** *Let $\rho$ be an IFC on an $R$-module $M$ and $A_\rho$ be an IFSM induced by $\rho$. Let $(\alpha, \beta) \in Img(\rho)$. Then*

$$A_{\rho_{(\alpha,\beta)}} = \{x \in M : x \equiv 0(\rho_{(\alpha,\beta)})\}$$

*is a submodule induced by the congruence $\rho_{(\alpha,\beta)}$.*

*Proof.* Let $a \in M$. Then $a \in A_{\rho_{(\alpha,\beta)}} \Leftrightarrow \mu_{A_\rho}(a) \geq \alpha, \nu_{A_\rho}(a) \leq \beta$
$\Leftrightarrow \mu_\rho(a, 0) \geq \alpha, \nu_\rho(a, 0) \leq \beta \Leftrightarrow (a, 0) \in \rho_{(\alpha,\beta)} \Leftrightarrow a \in \{x \in M : x \equiv 0(\rho_{(\alpha,\beta)})\}$.
Thus $A_{\rho_{(\alpha,\beta)}}$ is a submodule induced by the congruence $\rho_{(\alpha,\beta)}$. □

**Proposition 4.17.** *Let $A$ be an IFSM of an $R$-module $M$ and $\rho_A$ be the IFC induced by $A$. Let $(\alpha, \beta) \in Img(A)$. Then $\rho_{A_{(\alpha,\beta)}}$ is the congruence on $M$ induced by $A_{(\alpha,\beta)}$.*

*Proof.* Let $\lambda$ be the congruence on $M$ induced by $A_{(\alpha,\beta)}$. Then

$$(x, y) \in \lambda \Leftrightarrow x - y \in A_{(\alpha,\beta)}.$$

Let $(x, y) \in \rho_{A_{(\alpha,\beta)}}$. Then $\mu_{\rho_A}(x, y) \geq \alpha,\ \nu_{\rho_A}(x, y) \leq \beta$. Thus we have

$$\mu_A(x - y) \geq \alpha,\ \nu_A(x - y) \leq \beta\ \Rightarrow\ x - y \in A_{(\alpha,\beta)}\ \Rightarrow\ x - y \in \lambda.$$

So $\rho_{A_{(\alpha,\beta)}} \subseteq \lambda$. By reversing the above argument, we get $\lambda \subseteq \rho_{A_{(\alpha,\beta)}}$. Hence $\rho_{A_{(\alpha,\beta)}} = \lambda$. □

**Definition 4.18.** Let $M$ be an $R$-module and $\rho$ be an IFC on $M$. An IFC $\lambda$ on $M$ is said to be *$\rho$-invariant*, if for all $(x, y), (u, v) \in M \times M$,

$$\mu_\rho(x, y) = \mu_\rho(u, v) \text{ and } \nu_\rho(x, y) = \nu_\rho(u, v) \Rightarrow \mu_\lambda(x, y) = \mu_\lambda(u, v) \text{ and } \nu_\lambda(x, y) = \nu_\lambda(u, v).$$

**Lemma 4.19.** *Let $M$ be an $R$-module and $A$ be an IFSM of $M$. Let $\rho_A$ be the IFC on $M$ induced by $A$. Then the IFR $\rho_A/\rho_A$ on $M/A$ defined by*

$$(\rho_A/\rho_A)(x + A, y + A) = \rho_A(x, y)$$

*is an IFC on $M/A$.*

*Proof.* Assume that $x+A=u+A$ and $y+A=v+A$. Then we have

$$\mu_A(x-u)=\mu_A(0)=1,\ \nu_A(x-u)=\nu_A(0)=0$$

and

$$\mu_A(y-v)=\mu_A(0)=1,\ \nu_A(y-v)=\nu_A(0)=0.$$

Thus we get

$$\begin{aligned}\mu_{\rho_A}(x,y) &\geq \min\{\mu_{\rho_A}(x,u),\mu_{\rho_A}(u,y)\}\\ &=\mu_{\rho_A}(u,y)\\ &\geq \min\{\mu_{\rho_A}(u,v),\mu_{\rho_A}(v,y)\}\\ &=\mu_{\rho_A}(u,v).\end{aligned}$$

Similarly, we can show $\mu_{\rho_A}(u,v)\geq \mu_{\rho_A}(x,y)$. So $\mu_{\rho_A}(x,y)=\mu_{\rho_A}(u,v)$. Similarly, we get $\nu_{\rho_A}(x,y)=\nu_{\rho_A}(u,v)$. Hence $\rho_A/\rho_A$ is meaningful.

Now the rest of the proof is a routine matter of verification. □

**Theorem 4.20.** *Let $M$ be an $R$-module and $A$ be an IFSM of $M$. Let $\rho_A$ be the IFC on $M$ induced by $A$. Then there exists a one-to-one correspondence between the set $\mathcal{IFC}_{\rho_{\mathcal{A}}}(\mathcal{M})$ of $\rho_A$-invariant IFCs of $M$ and the set $\mathcal{IFC}_{\rho_{\mathcal{A}}/\rho_{\mathcal{A}}}(\mathcal{M}/\mathcal{A})$ of $\rho_A/\rho_A$-invariant IFCs on $M/A$.*

*Proof.* Let $\lambda$ be an $\rho_A$-invariant IFC on $M$. Define

$$(\lambda/\rho_A)(x+A,y+A)=\lambda(x,y) \text{ for all } x,y\in M.$$

Let $x+A=u+A$ and $y+A=v+A$. These imply that $\rho_A(x,y)=\rho_A(u,v)$. Since $\lambda$ is $\rho_A$-invariant, we have $\lambda(x,y)=\lambda(u,v)$. Then the definition of $\lambda/\rho_A$ is meaningful. It is easy to show that $\lambda/\rho_A$ is $\rho_A/\rho_A$-invariant IFC on $M/A$. We define a map $\theta:\mathcal{IFC}_{\rho_{\mathcal{A}}}(\mathcal{M})\to\mathcal{IFC}_{\rho_{\mathcal{A}}/\rho_{\mathcal{A}}}(\mathcal{M}/\mathcal{A})$ by $\theta(\lambda)=\lambda/\rho_A$. Let $\lambda_1,\ \lambda_2\in\mathcal{IFC}_{\rho_{\mathcal{A}}}(\mathcal{M})$ such that $\lambda_1(x,y)\neq\lambda_2(x,y)$. Then we have

$$(\lambda_1/\rho_A)(x+A,y+A)=\lambda_1(x,y)\neq\lambda_2(x,y)=(\lambda_2/\rho_A)(x+A,y+A).$$

Thus $\theta$ is injective.

Let $\lambda^{'}$ be a $\rho_A/\rho_A$-invariant IFC on $M/A$. We define an IFR $\lambda$ on $M$ as follow:

$$\lambda(x,y)=\lambda^{'}(x+A,y+A).$$

Then $\mu_\lambda(x,x)=\mu_{\lambda'}(x+A,x+A)=1$ and $\nu_\lambda(x,x)=\nu_{\lambda'}(x+A,x+A)=0$. Similarly, we can show that $\lambda(x,y)=\lambda(y,x)$. Also, we get

$$\mu_\lambda(x,y)\geq \sup_{z\in M}\min\{\mu_\lambda(x,z),\mu_\lambda(z,y)\}$$

and

$$\nu_\lambda(x,y)\leq \inf_{z\in M}\max\{\nu_\lambda(x,z),\nu_\lambda(z,y)\}.$$

Thus $\lambda$ is an IFR on $M$. On the other hand, we get

$$\begin{aligned}\mu_\lambda(x+a,y+b) &= \mu_{\lambda'}(x+a+A,y+b+A)\\ &= \mu_{\lambda'}(x+A+a+A,y+A+b+A)\\ &\geq \min\{\mu_{\lambda'}(x+A,y+A),\mu_{\lambda'}(a+A,b+A)\}\\ &= \min\{\mu_\lambda(x,y),\mu_\lambda(a,b)\}.\end{aligned}$$

Similarly, we can show that $\nu_\lambda(x+a, y+b) \leq \max\{\nu_\lambda(x,y), \nu_\lambda(a,b)\}$. Also, we have

$$\begin{aligned}\mu_\lambda(xr, yr) &= \mu_{\lambda'}(xr+A, yr+A)\\ &= \mu_{\lambda'}((x+A)r, (y+A)r)\\ &\geq \mu_{\lambda'}(x+A, y+A).\end{aligned}$$

Similarly, we can show that $\nu_\lambda(xr, yr) \leq \nu_\lambda(x,y)$. So $\lambda$ is an IFC on $M$. Since $\rho_A(x,y) = \rho_A(u,v)$, $\rho_A/\rho_A(x+A, y+A) = \rho_A/\rho_A(u+A, v+A)$. This implies that $\lambda^{'}(x+A, y+A) = \lambda^{'}(u+A, v+A)$. Hence $\lambda(x,y) = \lambda(u,v)$. Therefore $\lambda$ is $\rho_A$-invariant.

Now let $(x+A, y+A) \in M/A \times M/A$. Then $(\lambda/\rho)(x+A, y+A) = \lambda(x,y) = \lambda^{'}(x+A, y+A)$. Thus $\lambda^{'} = \lambda/\rho_A = \theta(\lambda)$. So $\theta$ is surjective. Hence $\theta$ is bijective. □

## Conclusion

In this paper, we develop and study the notions of intuitionistic fuzzy submodules, intuitionistic fuzzy congruences of an $R$-module (where $R$ is a near-ring) and quotient $R$-modules over an intuitionistic fuzzy submodule of an $R$-module. We exhibit a one-to-one correspondence between the set of intuitionistic fuzzy submodules and the set of intuitionistic fuzzy congruences of an $R$-module. Finally, we established an intuitionistic fuzzy congruence of the quotient $R$-module over an intuitionistic fuzzy submodule of the $R$-module and obtained a correspondence theorem.

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P.K.Sharma (pksharma@davjalandhar.com)
Post-Graduate Department of mathemathics, D.A.V.College, Jalandhar, Punjab, India