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New approaches for solving fuzzy data envelopment analysis

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Abstract. One of the most appropriate approaches for determining the efficiency of decision-making units is Data Envelopment Analysis (DEA). When data is described uncertainty and vaguely, the necessity of using fuzzy theory appears. In this study, new approaches are proposed to solve the fuzzy DEA model. The fuzzy DEA model is first transformed into an interval DEA by the nearest interval approximation of a fuzzy number. The proposed approach is based on the marked distance for comparison and ranking of efficiency. Also, we study fuzzy DEA models, which use possibility and necessity to measure fuzzy events. The necessity measure estimates the amount of necessity for each fuzzy set. In this research, because the possibility measure and necessity measure are each other dual, comparing the results of their efficiency, it is more reliable to determine efficient units. We give numerical examples to examine the proposed approaches and compare the models.

2020 AMS Classification: 03E72, 08A72

Keywords: Fuzzy DEA, Interval DEA, Marked Distance, Possibility Measure, Necessity Measure.

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1. Introduction

One of the most appropriate approaches for determining the efficiency of decision-making units is Data Envelopment Analysis (DEA). In recent years, DEA has been used to assess the performance of institutions and other common activities in evaluating the organizations and industries such as the banking industry, post offices, hospitals, educational organizations, power plants, refineries, and so on [1]. In the standard DEA models, input and output data are examined only on the condition of certainty. Cooper et al. [2] and Zhu [3] employed Data Envelopment Analysis technique for uncertain data. Fuzzy DEA solutions are divided into four main classes: tolerance approach, α-cut approach, ranking approach, and possibility approach. The tolerance approach can be found in Sengupta [4]. Kao and Liu are among the pioneers of α-cut approach. The efficiency results are defined as intervals [5]. Saati et al. [6] employed α-cut approach as an interval programming approach to transforming the form of the fuzzy CCR model definitely. Then they solved it by modifying an appropriate variable. The application of DEA has proven to be invaluable in assessing the efficiency of decision-making units across various industries and organizations. By adopting DEA, institutions such as banks, post offices, hospitals, and educational organizations can gain valuable insights into their performance and identify areas for improvement. Moreover, the advancement of DEA techniques has led to the emergence of fuzzy DEA solutions, allowing for the consideration of uncertain data. This opens up new possibilities for accurately evaluating efficiency in situations where there is a degree of ambiguity or imprecision. By incorporating fuzzy DEA approaches such as the tolerance, α-cut, ranking, and possibility approaches, decision-makers can obtain a more comprehensive understanding of their organization's performance. These approaches offer flexible and robust methodologies to handle uncertain data, enabling a more accurate efficiency assessment. The main goal of Mishra et al.'s article in 2023 introduce a new sequence of linear positive operator, i.e., α-Schurer Durrmeyer operator and their approximation behavior based on function $\eta(z)$, where η is infinitely differentiable on $[0, 1]$, $\eta(z) = 0$, $\eta(1) = 1$ and $\eta^{'}(z) > 0$ for all $z \in [0, 1]$. Further, they calculate central moments and basic estimates for the sequence of the operators. Moreover, they discuss the rate of convergence and order of approximation in terms of modulus of continuity, smoothness, Korovkin theorem, and Peeter's K-functional [7]. Mishra et al. [8] prove the existence and uniqueness of common fixed point for Ciri'c-Riech-Rus contraction mapping in the setting of quasi-partial b-metric space. The ambiguity of a fuzzy number is a characteristic that plays a central role in this article. In most fuzzy DEA solutions with α-cut approach, the problem is transformed into an interval linear programming problem. Despotis and Smirlis [9] evaluated interval or boundary data in DEA, presenting a three-class classification of efficiency. Wang et al. [10] developed a new couple of interval DEA models to deal with imprecise data, such as interval data, sequential preference-based data, and fuzzy data. Their interval DEA models are easier and more comprehensible than Cooper's imprecise DEA model [2]. Furthermore, their interval DEA models use constant and uniform generation boundaries as a benchmark to measure the efficiency of all DMUs, which makes it more reasonable and reliable than Despotis and Smirlis's interval DEA models. In addition, their approach with sequential preference-based data seems more reasonable than Zhu's Method [11]. The notion of fuzzy set theory has not been directed over medical diagnosis. There are some added applications, such as image processing, pattern identification, and many medical devices. Research paper by Sharma et al. [12] introduced a new mediative fuzzy ranking technique as the fuzzy extension in decision making. The proposed mediative fuzzy logic-based technique is more relevant and applicable to incomplete and doubtful situations or some contradictions present in the expert knowledge. The value of the contradictory degree for mediative fuzzy sets used in the extension principle is defined. The proposed mediative fuzzy ranking method is easily implemented in the medical field, and the proposed mediative fuzzy extension-based measured technique is useful to medical experts and doctors in many decision-making situations. Entani et al. [13] considered the efficiency of DEA both optimistically and pessimistically. In their DEA models, an interval is made by using optimistic or pessimistic efficiencies. However, their model has a significant defect, which is that it does not consider some input and output data because only the data of one input or one output of DMU is evaluated, and the rest of the data is not used [13]. Mishra et al. [14] define the weighted mean summability method of double sequences in intuitionistic fuzzy normed spaces (IFNS) and obtain necessary and sufficient tauberian conditions under which convergence of double sequences in IFNS follows from their weighted mean summability. this study also reveals Tauberian results for some known summation methods in special cases. Understanding and addressing the ambiguity inherent in fuzzy numbers is essential in the realm of fuzzy DEA. Researchers have made significant strides in advancing fuzzy DEA solutions by utilizing the α-cut approach, which transforms the problem into an interval linear programming problem. The work of Despotis and Smirlis [9] provides a valuable framework for categorizing efficiency based on interval or boundary data, offering insights into the assessment of DMUs. Wang et al. [10] have further enhanced the field by developing more simpler and more comprehensible interval DEA models that accommodate imprecise data. Their models introduce constant and uniform generation boundaries as benchmarks for measuring efficiency, resulting in a more reasonable and reliable approach compared to previous models. Mishra et al.'s research emphasizes the basic notions regarding the Neutrosophic Fuzzy Sets (NFSs) with operations and their applicability in the medical diagnostic process. They developed a neutrosophic fuzzy set-based Monte Carlo simulation technique for the decision-making in medical diagnostic processing fuzzy environment. In this work, they managed the waiting time and idle time of the doctor during the treatment process of the patients. The various parameters are stated as linguistic variable in the form of NFSs. The developed neutrosophic Monte Carlo simulation technique (NMCST) is extended in the planning strategy of a doctor to treat the patient in a neutrosophic fuzzy environment [15].

Incorporating sequential preference-based data, Wang et al.'s approach demonstrate a higher level of reasonability compared to alternative methods proposed by Zhu [11]. By considering both optimistic and pessimistic efficiencies, Entani et al. [13] shed light on different perspectives of efficiency evaluation. However, their model's limitation in evaluating only a subset of input or output data should be addressed in future research. The delving into the intricacies of fuzzy DEA and exploring these advancements can contribute to understanding and proficiency in the field. By embracing and further developing these methodologies, can make valuable contributions to the field of fuzzy DEA, enabling more accurate and comprehensive assessments of efficiency in real-world decision-making units. Zhou and Xu [16] found out the researching on the theoretical development and practical applications of the FDEA is valuable, because it has been successfully applied in many factual fields; the results have been shown it is suitable for the different real fields for efficiency analysis and alternative improvement. Also, FDEA can effectively address some real decision-making issues based on the relative efficiency principle and provide quantitative improvement suggestions. Sharma et al. [17] stated that in the real world, not all the parameters, i.e., cost, demand, and supply related to the Transportation Problem (TP) need to be known precisely. One of the recent ways to tackle the impreciseness is the Fermatean fuzzy set (FFS), an extension of the Pythagorean fuzzy set (PFS). First, they established a score function for grading FFS in this research paper. The main aim of their research article is to solve the TP in a Fermatean fuzzy environment. To optimize the TP using Fermatean fuzzy parameters, they presented an algorithm for three types of Fermatean fuzzy transportation problems (ty-1 FFTP, ty-2 FFTP, and ty-3 FFTP).

The primary purpose of this study is to develop a new model of fuzzy DEA models, which can define efficiency in an interval manner. Suggested approach can, such as the alpha cut approach proposed by Saati et al. [6], the fuzzy arithmetic approach proposed by Wang et al. [10], and many others, transform fuzzy DEA into a linear programming problem. In other words, in this study, with a new approach, fuzzy DEA can be converted into a linear programming problem. In order to compare and rank the efficiency of DMUs, we present a symptomatic interval-based approach. In 2011, Tawana et al. [18] provided a taxonomy and review of the fuzzy DEA methods. they present a classification scheme with four primary categories, namely, the tolerance approach, the α-level based approach, the fuzzy ranking approach and the possibility approach. they discuss each classification scheme and group the fuzzy DEA papers published in the literature over the past 20 years. To the best of their knowledge, this paper appears to be the only review and complete source of references on fuzzy DEA.

Another class (for solving) of fuzzy DEA solutions is to use the theory of possibility. The possibility theory was first introduced by Zadeh [19] as an extension of the theory of fuzzy sets. Dubois and Prade [20] further contributed to its development. The possibility theory is an uncertainty theory devoted to managing the incomplete information and is an alternative to probability theory. Most researchers use the concept of α-cut to solve the possibility fuzzy programming problems, which can be solved by converting it to interval programming and comparing the intervals. Zadeh [19] introduced the possibility theory in modeling the conditions which are confronted with uncertainty. In fact, this theory defines a measure called 'possibility' for a fuzzy space. The utilization of the "measure" concept in a space is excellent, especially in possibility space which is much more extended than fuzzy space. By incorporating the possibility theory into fuzzy DEA solutions, researchers have opened up new avenues for addressing complex problems with incomplete information. This theory, which builds upon the foundation of fuzzy sets, offers a fresh perspective on managing uncertainty. By embracing the concept of "possibility", researchers can navigate through the challenges posed by uncertain conditions and gain valuable insights. The utilization of this theory expands the possibilities within the field, unlocking the greater potential for finding effective solutions. So, whether you are an aspiring researcher or a practitioner in the field, exploring the theory of possibility can enrich your understanding and empower you to tackle complex problems in innovative ways. Embrace the power of possibility and let it guide you toward novel solutions and valuable discoveries.

The rest of the current study is as follows: In the second section, we review the concept of preliminary fuzzy set and some mathematical operations on them. We present the fuzzy DEA model and the most common conventional methods for solving fuzzy DEA models in the third section. In Section 4, we propose an approach to solve the fuzzy DEA model and rank them, and also compare the proposed approach with the conventional methods with the results in numerical examples. In section 5, the extension of fuzzy DEA models is expressed in terms of the possibility measure and necessity measure. Finally, we use a numerical example to express our idea further. In these methods, we simplify the work and manage fuzzy numbers. The structure of the study you are reading reflects a systematic approach to exploring and addressing complex problems in the realm of fuzzy DEA models. By reviewing the fundamental concepts of fuzzy set theory and mathematical operations, the study establishes a strong foundation for understanding subsequent sections. As you progress to the third section, you will discover common methods used to solve fuzzy DEA models, allowing you to gain insights into established approaches. However, the study continues beyond there. Section 4 introduces a novel approach that promises to simplify the problem-solving process and provide meaningful comparisons with conventional methods through numerical examples. The study also explores the possibilities of extending fuzzy DEA models by incorporating possibility and necessity measures, showcasing the breadth of potential applications. By delving into the numerical example, you will witness the practical implications of these methods firsthand. Embrace this journey of discovery, as it equips you with valuable knowledge and innovative techniques for managing fuzzy numbers and solving complex problems in the field of fuzzy DEA.

2. Preliminaries

Firstly, we review some of the basic concepts of fuzzy sets. Suppose that X is a reference set with real numbers whose members are represented as x. Each fuzzy subset A from X is defined by membership function $\mu_A : X \to [0,1]$ that relates each member x from X to the membership degree $\mu_A(x)$; when $\mu_A(x) = 0$, it can be certainly said that this member does not belong to A.

Definition 2.1. (i) A fuzzy set $\widetilde{a}$ on $\mathbb{R}$ is called a *fuzzy number*, if it satisfies the following conditions:

(a) $\widetilde{a}$ is normal, i.e., $\sup_{x\in\mathbb{R}} \mu_{\widetilde{a}}(\mathbf{x}) = 1$,
(b) $\widetilde{a}$ is normal, i.e., for all $\lambda \in [0,1]$,

$$\mu_{\widetilde{a}}\{\lambda x_1 + (1-\lambda)x_2\} \geq \min\{\mu_{\widetilde{a}}(x_1), \mu_{\widetilde{a}}(x_2)\}.$$

(ii) A fuzzy number $\widetilde{a}$ is called a *fuzzy number L-R*, denoted by $(a^L, a^R, \gamma^L, \gamma^R)_{LR}$, if its membership function is defined as follows: for each $\mathbf{x} \in \mathbb{R}$,

$$\mu_{\widetilde{a}}(\mathbf{x}) = \begin{cases} (L(\frac{a^L - x}{\gamma^L})) & (x \leq a^L) \\ 1 & (a^L \leq x \leq a^R) \\ (R(\frac{x - a^R}{\gamma^R})) & (x \geq a^R) \end{cases}$$

where $a^L \leq a^R$, $\gamma^L \geq 0$ and $\gamma^R \geq 0$.

Definition 2.2. The *general trapezoidal fuzzy number* $(GTrFN)$ $\widetilde{A}$ can be as $\widetilde{A} = (a_1, a_2, a_3, a_4; h_1, h_2)$ in which $a_1 \leq a_2 \leq a_3 \leq a_4$ and $h_1 \leq h_2$ or vice versa $(h_1, h_2 \in [0, 1])$. Then the membership function is defined as follows: for each $\mathbf{x} \in \mathbb{R}$,

$$\mu_{\widetilde{A}}(\mathbf{x}) = \begin{cases} h_1(\frac{x-a_1}{a_2-a_1}) & a_1 \leq x \leq a_2 \\ (h_2 - h_1)\frac{x-a_2}{a_3-a_2} & a_2 \leq x \leq a_3 \\ h_2(\frac{a_4-x}{a_4-a_3}) & a_3 \leq x \leq a_4 \\ 0 & x \leq a_1 x \geq a_4 \end{cases}$$

In the above definition, if two heights $h_1 = h_2$, the general trapezoidal fuzzy number will convert to a flat trapezoidal fuzzy number. Moreover, if $h_1 = h_2 < 1$, we will have the flat trapezoidal fuzzy sub-number.

Definition 2.3. The *general triangular fuzzy number* $(GTrFN)$ $\widetilde{A}$, which is a special kind of the general trapezoidal fuzzy number of $(a_2 = a_3)$, and on the interval $[a_1, a_3]$, is $\widetilde{A} = (a_1, a_2, a_3; h)$ in which $a_1 \leq a_2 \leq a_3$ and $(h \in [0, 1]$. The membership function of a general triangular fuzzy number is as follows: for each $\mathbf{x} \in \mathbb{R}$,

$$\mu_{\widetilde{A}}(\mathbf{x}) = \begin{cases} h(\frac{x-a_1}{a_2-a_1}) & a_1 \leq x \leq a_2 \\ h(\frac{a_3-x}{a_3-a_2}) & a_2 \leq x \leq a_3 \\ 0 & x \leq a_1,\ x \geq a_3 \end{cases}$$

Remark 2.4. (1) If $h = 1$, we call a general triangular fuzzy number as a triangular fuzzy number.

(2) If $h < 1$, we call a general triangular fuzzy number as a triangular fuzzy sub-number.

Definition 2.5. The *core* of a fuzzy number $\widetilde{A}$, denoted by Core $(\widetilde{A})$, is defined as follows:

$$\text{Core}(\widetilde{A}) = \{\mathbf{x} \in \mathbb{R} | \mu_{\widetilde{A}}(\mathbf{x}) = 1\}.$$

Definition 2.6 (See [21, 22])**.** The *η-cut* of a fuzzy number $\widetilde{A}$, denoted by $\widetilde{A}_\eta$, is defined as follows:

$$\widetilde{A}_\eta = \{\mathbf{x} \in \mathbb{R} | \mu_{\widetilde{A}}(\mathbf{x}) \geq \eta\} = [\check{A}(\eta), \hat{A}(\eta)],$$

where $\check{A}(\eta) = inf\{\mathbf{x} \in \mathbb{R} | \mu_{\widetilde{A}}(\mathbf{x}) \geq \eta\}$ and $\hat{A}(\eta) = sup\{\mathbf{x} \in \mathbb{R} | \mu_{\widetilde{A}}(\mathbf{x}) \geq \eta\}$. Subsequently, the fuzzy $\widetilde{a}$ is convex, if all of the η-cut sets are convex.

Suppose that $\widetilde{A}$ is a triangular fuzzy number and $[\check{A}(\eta), \hat{A}(\eta)]$ is its $\eta - cut$. The approximation of the nearest interval for $\widetilde{A}$ is a closed interval $I(\widetilde{A}) = [\check{I}, \hat{I}]$, in which $\check{I} = \int_0^1 \check{A}(\eta) d\eta$ and $\hat{I} = \int_0^1 \hat{A}(\eta) d\eta$ [23].

Kushwaha et al.'s article in 2022 determined the degree of approximation of functions belonging to the Lipschitz class and weighted class by using $(N, p)(C, \theta, \beta)$ means of Fourier series and conjugate series of Fourier series which in particular becomes $(E, q)(C, \alpha, \beta)$ [24].

Theorem 2.7. *The approximation of the nearest interval for triangular fuzzy number* $\widetilde{A} = (a_1, a_2, a_3)$ *is* $[\frac{a_1+a_2}{2}, \frac{a_3+a_2}{2}]$.

Proof. The membership function of the triangular fuzzy number $\widetilde{A}$ is as follows: for each $\mathbf{x} \in \mathbb{R}$,

$$\mu_{\widetilde{A}}(\mathbf{x}) = \begin{cases} h(\frac{x-a_1}{a_2-a_1}) & a_1 \leq x \leq a_2 \\ h(\frac{a_3-x}{a_3-a_2}) & a_2 \leq x \leq a_3 \\ 0 & x \leq a_1,\ x \geq a_3 \end{cases}$$

Then η-cut of a triangular fuzzy number is as follows:

$$\widetilde{A}_\eta = [\check{A}(\eta), \hat{A}(\eta)] = [a_1 + \eta(a_2 - a_1), a_3 - \eta(a_3 - a_2)].$$

Thus we complete our proof by considering the concept of the nearest interval approximation.

$$\begin{aligned} I(\widetilde{A}) = [\check{I}, \hat{I}] &= [\int_0^1 \check{A}(\eta)d\eta, \int_0^1 \hat{A}(\eta)d\eta] \\ &= [\int_0^1 (a_1 + \eta(a_2 - a_1))d\eta, \int_0^1 (a_3 - \eta(a_3 - a_2))d\eta] \\ &= [\frac{a_1 + a_2}{2}, \frac{a_3 + a_2}{2}]. \end{aligned}$$

□

The fuzzy calculation is applied on fuzzy numbers. Fuzzy numbers must be used for fuzzy sets. On the other hand, since we use triangular fuzzy numbers in this article, we will present some mathematical operators on general triangular fuzzy numbers.

2.1. **Probability theory.** Probability theory is a fundamental branch of mathematics that deals with the study of uncertainty and randomness. It provides a framework for quantifying and analyzing the likelihood of events occurring and understanding the patterns and behavior of random phenomena. By examining the principles and concepts of probability theory, we can gain valuable insights into various fields, such as statistics, risk assessment, decision theory, and machine learning.

At its core, probability theory enables us to make informed predictions and decisions in situations where outcomes are uncertain. By assigning probabilities to different events, we can assess the likelihood of their occurrence and estimate the potential outcomes. This information empowers us to make rational choices, assess risks, and optimize strategies in various domains.

Probability theory offers a range of powerful tools and techniques for analyzing and modeling uncertain events. It encompasses concepts such as random variables, probability distributions, conditional probabilities, and statistical inference. These tools allow us to quantify uncertainty, measure the variability of outcomes, and make probabilistic predictions based on available information.

Moreover, probability theory provides a rigorous framework for evaluating the validity and reliability of scientific experiments and observations. It allows us to assess the likelihood of observed data occurring purely by chance or due to some underlying cause. This aspect is crucial in fields such as physics, biology, and social sciences, where empirical evidence needs to be interpreted in a statistically sound manner.

By studying probability theory, you can develop essential skills for critical thinking, problem-solving, and decision-making. It equips you with the ability to assess risks, evaluate uncertain situations, and make sound judgments based on available information. Understanding probability theory empowers you to navigate complex scenarios with confidence, optimize strategies, and make informed choices.

Whether you are interested in fields like finance, data science, engineering, or any domain where uncertainty is inherent, probability theory will be an invaluable tool. It enables you to analyze and interpret data, identify patterns, and make predictions with a quantitative and evidence-based approach.

Embracing probability theory opens up a world of possibilities, allowing you to make sense of the uncertain nature of our lives and harness its insights for better decision-making. So, dive into the fascinating realm of probability theory, and unlock the potential to unravel the mysteries of uncertainty while gaining a competitive edge in your chosen field.

2.2. **Fuzzy necessity and possibility theory.** Probability theory and Fuzzy Necessity and Possibility Theory are two distinct mathematical frameworks that deal with uncertainty and reasoning. While they approach uncertainty from different perspectives, they share common goals and can be complementary in certain applications.

Probability theory is widely used to model and quantify uncertainty in various fields, such as statistics, engineering, and finance. It provides a rigorous framework for dealing with random events and enables the calculation of probabilities and expected values. Probability theory assumes that uncertainty can be represented by assigning probabilities to different outcomes, where the sum of all probabilities equals 1.

On the other hand, Fuzzy Necessity and Possibility Theory, often referred to as fuzzy logic, focuses on representing and reasoning with imprecise and vague information. It acknowledges that in many real-world scenarios, uncertainties are not easily quantifiable or can be better described using linguistic terms rather than precise probabilities. Fuzzy logic allows for the representation of uncertainty through linguistic variables and membership functions, which assign degrees of membership to different categories or sets.

The relationship between these two theories lies in their common aim of addressing uncertainty. While probability theory deals with the uncertainty by assigning precise probabilities to events, fuzzy logic tackles uncertainty by allowing for degrees of membership or truth values. This distinction makes fuzzy logic particularly useful when dealing with situations that involve subjective judgments or when precise data is unavailable.

In some cases, these two theories can be used together to enhance the modeling and reasoning capabilities. For example, in decision-making problems, fuzzy logic can be used to represent imprecise criteria and preferences, while probability theory can be employed to calculate the likelihood of different outcomes based on available data. This combination allows for a more comprehensive analysis of uncertainty and a better understanding of the problem at hand.

Probability theory provides a solid foundation for statistical inference, risk assessment, and decision analysis. Its mathematical rigor and wide acceptance make it a valuable tool in various fields. Fuzzy logic, on the other hand, offers a flexible framework for handling uncertainty in situations where precise data is limited or when human reasoning and linguistic terms play a significant role. Its applications range from control systems and pattern recognition to expert systems and artificial intelligence.

By understanding and applying both probability theory and fuzzy logic, individuals can broaden their perspectives on uncertainty and gain powerful tools for decision-making and problem-solving in complex and ambiguous situations. The ability to navigate and reason under uncertainty is increasingly crucial in today's world, where data is often incomplete, noisy, or subject to interpretation. Embracing these theories can lead to more robust and nuanced analyses and, ultimately, better-informed decisions.

Fuzzy Necessity Theory, also known as Fuzzy Essentiality Theory, was introduced by Zadeh in 1978. The theory aimed to quantify the degree of necessity or requirement for an element to belong to a fuzzy set. By focusing on the essential characteristics of a set, this theory provided a means to reason about the crucial criteria for membership and to make decisions based on necessity [19].

In 1978, Zadeh also introduced Possibility Theory as an extension of the Fuzzy Set Theory. Possibility Theory aims to address the challenges of managing incomplete or uncertain information in a more flexible and intuitive way. It provided a measure of possibility to quantify the degree to which an element can be considered possible or plausible within a fuzzy set. Possibility theory offered a robust framework for reasoning, decision-making, and handling imprecision.

Both Fuzzy Necessity and Possibility Theory have undergone further development and refinement over the years. Researchers and scholars have expanded upon the foundational concepts, explored their applications in various domains, and developed computational methods and algorithms to support their practical use.

Today, Fuzzy Necessity and Possibility Theory continue to be actively researched and applied in fields such as artificial intelligence, decision analysis, optimization, pattern recognition, and expert systems. They provide valuable tools for handling uncertainty, vagueness, and imprecision, enabling more robust and flexible modeling and analysis of complex systems.

Possibility theory is a mathematical theory in the fuzzy environment that is actually an alternative to probability theory. The content of this theory is that in the analysis of environmental conditions and events, we are not only looking for possible events but in uncertain conditions, we are looking for all the possibility of the events that are introduced by the degree (measure) of these events' possibility. While in probability theory, only one number (probability) is used to describe the amount of the chance that the event will occur as a result of an experiment. In the possibility theory, another measure is defined as the "necessity measure", which is coupled with the possibility measure and estimates the necessity measure of each fuzzy set.

Suppose that $\widetilde{a}$ and $\widetilde{b}$ are two fuzzy numbers with membership functions $\mu_{\widetilde{a}}(\mathbf{x})$ and $\mu_{\widetilde{b}}(\mathbf{x})$, respectively. Then the *possibility*, denoted by $\mathrm{Pos}(\widetilde{a} \circledast \widetilde{b})$ and the *necessity* of $\widetilde{a}$, denoted by $\text{Nes}(\widetilde{a} \divideontimes \widetilde{b})$ and $\widetilde{b}$, are defined, respectively as follows:

$$\text{Pos}(\widetilde{a} \divideontimes \widetilde{b}) = sup\{min(\mu_{\widetilde{a}}(\mathbf{x}), \mu_{\widetilde{b}}(\mathbf{x}))|\mathbf{x}, \mathbf{y} \in \mathbb{R}, \mathbf{x} \divideontimes \mathbf{y}\},$$

$$\text{Nes}(\widetilde{a} \divideontimes \widetilde{b}) = inf\{max(1 - \mu_{\widetilde{a}}(\mathbf{x}), \mu_{\widetilde{b}}(\mathbf{x}))|\mathbf{x}, \mathbf{y} \in \mathbb{R}, \mathbf{x} \divideontimes \mathbf{y}\},$$

where $\divideontimes$ is one of $>, <, =, \geq, \leq$ [20].

The dual relationship between the possibility and the necessity is as follows:

$$\text{Nes}(\widetilde{a} \divideontimes \widetilde{b}) = 1 - \text{Pos}\overline{(\widetilde{a} \divideontimes \widetilde{b})}.$$

The necessity measure is established in the following conditions:

$$\min\{\text{Nes}(\widetilde{a} \divideontimes \widetilde{b}), \text{Nes}\overline{(\widetilde{a} \divideontimes \widetilde{b})}\} = 0.$$

Also, the relationships between the measures of the possibility and the necessity are as follows:

$$\begin{gathered} \text{Pos}(\widetilde{a} \divideontimes \widetilde{b}) \geq \text{Nes}(\widetilde{a} \divideontimes \widetilde{b}), \\ \text{Nes}(\widetilde{a} \divideontimes \widetilde{b}) > 0 \Longrightarrow \text{Pos}(\widetilde{a} \divideontimes \widetilde{b}) = 1, \\ \text{Pos}(\widetilde{a} \divideontimes \widetilde{b}) < 1 \Longrightarrow \text{Nes}(\widetilde{a} \divideontimes \widetilde{b}) = 0. \end{gathered}$$

If the right side, $\widetilde{b}$, has a crisp value, the measure of the possibility and the necessity of a fuzzy event is represented as follows:

$$\text{Pos}(\widetilde{a} \leq b) = sup_{\mathbf{x} \in \mathbb{R}}\{\mu_{\widetilde{a}}(\mathbf{x})|, \mathbf{x} \leq \mathbf{b}\},\ \text{Nes}(\widetilde{\mathbf{a}} \leq \mathbf{b}) = \mathbf{1} - \mathbf{sup}_{\mathbf{x} \in \mathbb{R}}\{\mu_{\widetilde{\mathbf{a}}}(\mathbf{x})|, \mathbf{x} > \mathbf{b}\},$$

$$\text{Pos}(\widetilde{a} \geq b) = sup_{\mathbf{x} \in \mathbb{R}}\{\mu_{\widetilde{a}}(\mathbf{x})|, \mathbf{x} \geq \mathbf{b}\},\ \text{Nes}(\widetilde{\mathbf{a}} \geq \mathbf{b}) = \mathbf{1} - \mathbf{sup}_{\mathbf{x} \in \mathbb{R}}\{\mu_{\widetilde{\mathbf{a}}}(\mathbf{x})|, \mathbf{x} < \mathbf{b}\},$$

$$\text{Pos}(\widetilde{a} = b) = \mu_{\widetilde{a}}(\mathbf{b}),\ \text{Nes}(\widetilde{a} = b) = 1 - \mu_{\widetilde{a}}(\mathbf{b}).$$

Suppose that $\widetilde{a}_1, \cdots, \widetilde{a}_n$ are fuzzy variables and $\varphi_i : \mathbb{R}^n \to \mathbb{R}(i = 1, \cdots, m)$ is a real function of the value. The possibility and necessity of fuzzy event $\varphi_i : \mathbb{R}^n \to \mathbb{R}(i = 1, \cdots, m)$ is as follows:

$$\begin{aligned} &Pos_{i=1,\cdots,m}(\varphi_i(\widetilde{a}_1, \cdots, \widetilde{a}_n) \leq 0 \\ &= \sup_{x_1,\cdots,x_n}\{\min\{\mu_{\widetilde{a}_j}(\mathbf{x_j})\}|\varphi_\mathbf{i}(\mathbf{x_1}, \cdots, \mathbf{x_n}) \leq \mathbf{0}, \mathbf{i} = \mathbf{1}, \cdots, \mathbf{m}\}, \end{aligned}$$

$$\begin{aligned} &Nes_{i=1,\cdots,m}(\varphi_i(\widetilde{a}_1, \cdots, \widetilde{a}_n) \leq 0) \\ &= 1 - \sup_{x_1,\cdots,x_n}\{\min\{\mu_{\widetilde{a}_j}(\mathbf{x_j})\}|\varphi_\mathbf{i}(\mathbf{x_1}, \cdots, \mathbf{x_n}) \leq \mathbf{0}, \mathbf{i} = \mathbf{1}, \cdots, \mathbf{m}\}. \end{aligned}$$

Proposition 2.8 (See [20, 25])**.** *Suppose that* $\widetilde{a} = (a^L, a^R, \gamma^L, \gamma^R)$ *and* $\widetilde{b} = (b^L, b^R, \theta^L, \theta^R)$ *are two ITrFNs. Then*

$$\text{Pos}(\widetilde{a} \leq \widetilde{b}) = \begin{cases} 0 & a^L - b^R > \gamma^L + \theta^R \\ \frac{b^R - a^L + \gamma^L + \theta^R}{\gamma^L + \theta^R} & 0 < a^L - b^R \leq \gamma^L + \theta^R \\ 1 & a^L - b^R \leq 0 \end{cases}$$

Lemma 2.9. *Suppose that* $\widetilde{a} = (a^L, a^R, \gamma^L, \gamma^R)$ *and* $\widetilde{b} = (b^L, b^R, \theta^L, \theta^R)$ *are two ITrFNs,* $p \in [0, 1]$*. Then* $\text{Pos}(\widetilde{a} \leq \widetilde{b})$ *if only if* $b^R - a^L \geq (p - 1)(\gamma^L + \theta^R)$.

Proof. If $p = 1$, due to $\text{Pos}(\widetilde{a} \leq \widetilde{b}) \geq 1$, it can be said that $a^L - b^R \leq 0$, given the value of p, we have $b^R - a^L \geq (p-1)(\gamma^L + \theta^R)$.

If $0 < p < 1$, then $\frac{b^R - a^L + \gamma^L + \theta^R}{\gamma^L + \theta^R} \geq p$, we have $b^R - a^L \geq (p-1)(\gamma^L + \theta^R)$.

Conversely, we will have the reversible process of the above equation. □

Lemma 2.10. *Suppose that* $\widetilde{a}_1, \cdots, \widetilde{a}_2$ *are fuzzy numbers and* $a_j^L - L^{-1}(\eta)\gamma_j^L$ *and* $a_j^R - R^{-1}(\eta)\gamma_j^R$ *denote the lower and upper bounds of* η*-cuts* $\widetilde{a}_j$*,* $j = 1, \cdots, n$*, respectively. Then for each possibility degree,* η_1*,* η_2 *and* η_3 *with* $0 \leqslant \eta_1, \eta_2, \eta_3 \leqslant 1$*, we have*

(1) $\text{Pos}(\sum_{j=1}^n \widetilde{a}_j \leq b) \geq \eta_1$ *if and only if* $\sum_{j=1}^n (a_j^L - L^{-1}(\eta)\gamma_j^L) \leq b$,

(2) $\text{Pos}(\sum_{j=1}^n \widetilde{a}_j \geq b) \geq \eta_2$ *if and only if* $\sum_{j=1}^n (a_j^R - R^{-1}(\eta)\gamma_j^R) \geq b$,

(3) $\text{Pos}(\sum_{j=1}^n \widetilde{a}_j = b) \geq \eta_3$ *if and only if* $\sum_{j=1}^n (a_j^L - L^{-1}(\eta)\gamma_j^L) \leq b \leq \sum_{j=1}^n (a_j^R - R^{-1}(\eta)\gamma_j^R)$.

Proof. (1) Suppose $\text{Pos}(\sum_{j=1}^n \widetilde{a}_j \leq b) \geq \eta_1$. Then we have

$$\sup\{\min\{\mu_{\widetilde{a}_1}(\mathbf{x}_1), \cdots, \mu_{\widetilde{a}_n}(\mathbf{x}_n) | \mathbf{x}_1 + \cdots + \mathbf{x}_n \leq b\}\} \geq \eta_1.$$

Assuming that $(\mathbf{x}_1^*, \cdots, \mathbf{x}_n^*) = arg \sup\{\min\{\mu_{\widetilde{a}_1(\mathbf{x}_1)}), \cdots, \mu_{\widetilde{a}_n}(\mathbf{x}_n) | \mathbf{x}_1 + \cdots + \mathbf{x}_n \leq b\}$. It results that $\min\{\mu_{\widetilde{a}_1}(\mathbf{x}_1^*), \cdots, \mu_{\widetilde{a}_n}(\mathbf{x}_n^*)\} \geq \eta_1$ and $\mathbf{x_1^*} + \cdots + \mathbf{x_n^*} \leq \mathbf{b}$. Since $\min\{\mu_{\widetilde{a}_1}(\mathbf{x}_1^*), \cdots, \mu_{\widetilde{a}_n}(\mathbf{x}_n^*)\} \geq \eta_1$, we get

$$\mu_{\widetilde{a}_1}(\mathbf{x}_1^*) \geq \eta_1, \cdots, \mu_{\widetilde{a}_n}(\mathbf{x}_n^*) \geq \eta_1.$$

Then we have $\mathbf{x}_1^* \in [a_1^L - L^{-1}(\eta_1)\gamma_1^L, a_1^R - R^{-1}(\eta_1)\gamma_1^R], \cdots, \mathbf{x}_n^* \in [a_n^L - L^{-1}(\eta_1)\gamma_n^L, a_n^R - R^{-1}(\eta_1)\gamma_n^R]$. Thus $\mathbf{x}_1^* + \cdots + \mathbf{x}_n^* \leq b$ implies that $\sum_{j=1}^n (a_j^L - L^{-1}(\eta_1)\gamma_j^L) \leq b$. On the contrary, if $\sum_{j=1}^n (a_j^L - L^{-1}(\eta_1)\gamma_j^L) \leq b$, then there exists $\eta_1^{'}$ with $\eta_1 \leq \eta_1^{'} \leq 1$. Thus $\sum_{j=1}^n (a_j^L - L^{-1}(\eta_1^{'})\gamma_j^L) \leq b$. This is also true that

$$\mu_{\widetilde{a}_1}(a_1^L - L^{-1}(\eta_1^{'})\gamma_1^L \geq \eta_1, \cdots, \mu_{\widetilde{a}_n}(a_1^L - L^{-1}(\eta_1^{'})\gamma_1^L \geq \eta_1.$$

This is equivalent to

$$\min\{\mu_{\widetilde{a}_1}(a_1^L - L^{-1}(\eta_1^{'})\gamma_1^L), \cdots, \mu_{\widetilde{a}_n}(a_1^L - L^{-1}(\eta_1^{'})\gamma_1^L) | \sum_{j=1}^n (a_j^L - L^{-1}(\eta_1^{'})\gamma_j^L) \leq b\} \geq \eta_1.$$

As a result, we have

$$\min\{\mu_{\widetilde{a}_1}(a_1^L - L^{-1}(\eta_1^{'})\gamma_1^L), \cdots, \mu_{\widetilde{a}_n}(a_1^L - L^{-1}(\eta_1^{'})\gamma_1^L) | \sum_{j=1}^n (a_j^L - L^{-1}(\eta_1^{'})\gamma_j^L) \leq b\} \geq \eta_1.$$

The proofs of (2) and (3) are similar to (1). □

Lemma 2.11. *Suppose that* $\widetilde{a}_1, \cdots, \widetilde{a}_2$ *are fuzzy numbers and* $a_j^L - L^{-1}(\eta)\gamma_j^L$ *and* $a_j^R - R^{-1}(\eta)\gamma_j^R$ *denote the lower and upper bounds of* η*-cuts* $\widetilde{a}_j$*,* $j = 1, \cdots, n$*, respectively. Then for each necessity degree,* η_1*,* η_2 *and* η_3 *with* $0 \leqslant \eta_1, \eta_2, \eta_3 \leqslant 1$*, we have*

(1) $\text{Nes}(\sum_{j=1}^n \widetilde{a}_j \leq b) \geq \eta_1$ *if and only if* $\sum_{j=1}^n (a_j^R - R^{-1}(\eta)\gamma_j^R) \leq b$,

(2) $\text{Nes}(\sum_{j=1}^n \widetilde{a}_j \geq b) \geq \eta_2$ *if and only if* $\sum_{j=1}^n (a_j^L - L^{-1}(\eta)\gamma_j^L) \geq b$,

(3) $\text{Nes}(\sum_{j=1}^n \widetilde{a}_j = b) \geq \eta_3$ *if and only if* $\sum_{j=1}^n (a_j^R - R^{-1}(\eta)\gamma_j^R) \leq b$, $\sum_{j=1}^n (a_j^L - L^{-1}(\eta)\gamma_j^L) \geq b$.

Proof. Similar to 2.10. □

Most researchers use triangular fuzzy numbers with the following membership function:

$$\mu_{\widetilde{a}}(\mathbf{x}) = \begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\ 1 & x = a_2 \\ \frac{a_3-x}{a_3-a_2} & a_2 \leq x \leq a_3 \end{cases}$$

If $\widetilde{a}_j,\ j = 1, \cdots, n$ are triangular fuzzy numbers, according to the above lemma, the following equations are obtained for the possibility and the necessity measures:

(**Possibility**)

$$\mathrm{Pos}(\textstyle\sum_{j=1}^{n} \widetilde{a}_j \leq \widetilde{b}) \geq \eta_1 \quad \text{if and only if} \quad \sum_{j=1}^{n} a_{1j} + \eta_1 \sum_{j=1}^{n} (a_{2j} - a_{1j}) \leq b,$$

$$\mathrm{Pos}(\textstyle\sum_{j=1}^{n} \widetilde{a}_j \geq \widetilde{b}) \geq \eta_2 \quad \text{if and only if} \quad \sum_{j=1}^{n} a_{3j} - \eta_2 \sum_{j=1}^{n} (a_{3j} - a_{2j}) \leq b,$$

$$\mathrm{Pos}(\textstyle\sum_{j=1}^{n} \widetilde{a}_j = \widetilde{b}) \geq \eta_3 \quad \text{if and only if} \quad \left\{ \begin{smallmatrix} \sum_{j=1}^{n} a_{1j} + \eta_3 \sum_{j=1}^{n} (a_{2j} - a_{1j}) \leq b \\ \sum_{j=1}^{n} a_{3j} - \eta_3 (\sum_{j=1}^{n} (a_{3j} - a_{2j}) \geq b \end{smallmatrix} \right. .$$

(**Neccessity**)

$$\mathrm{Nes}(\textstyle\sum_{j=1}^{n} \widetilde{a}_j \leq \widetilde{b}) \geq \eta_1 \quad \text{if and only if} \quad \sum_{j=1}^{n} a_{2j} + \eta_1 (\sum_{j=1}^{n} (a_{3j} - a_{2j}) \leq b,$$

$$\mathrm{Nes}(\textstyle\sum_{j=1}^{n} \widetilde{a}_j \geq \widetilde{b}) \geq \eta_2 \quad \text{if and only if} \quad \sum_{j=1}^{n} a_{2j} - \eta_2 (\sum_{j=1}^{n} (a_{2j} - a_{1j}) \leq b,$$

$$\mathrm{Nes}(\textstyle\sum_{j=1}^{n} \widetilde{a}_j = \widetilde{b}) \geq \eta_3 \quad \text{if and only if} \quad \left\{ \begin{smallmatrix} \sum_{j=1}^{n} a_{2j} + \eta_3 (\sum_{j=1}^{n} (a_{3j} - a_{2j}) \leq b, \\ \sum_{j=1}^{n} a_{2j} - \eta_3 (\sum_{j=1}^{n} (a_{2j} - a_{1j}) \geq b \end{smallmatrix} \right. .$$

The laws of Fuzzy Necessity and Possibility Theory presented here describe mathematical operations and relationships between fuzzy numbers. Fuzzy Necessity and Possibility Theory is a branch of mathematics that extends the classical set theory to handle uncertainty and imprecision in data. The laws outlined in the content define measures of possibility (Pos) and necessity (Nes) for fuzzy numbers and fuzzy events. These measures provide a way to quantify the degree to which an event or a comparison between fuzzy numbers is possible or necessary. One important aspect of these laws is the duality relationship between possibility and necessity. The necessity measure can be expressed in terms of the possibility measure and vice versa. This duality allows for a complementary understanding of uncertainty and provides a comprehensive framework for reasoning about fuzzy events. The laws also establish conditions for the necessity measure, ensuring that it satisfies certain properties. These properties help ensure consistency and coherence in the theory. Additionally, the content presents specific formulas for calculating the possibility and necessity measures in different scenarios, such as when comparing fuzzy numbers or evaluating fuzzy functions.

By studying and applying Fuzzy Necessity and Possibility Theory, researchers and practitioners gain a powerful toolset for modeling and analyzing uncertain and imprecise data. The theory finds applications in various fields, including decision-making, pattern recognition, data analysis, and control systems, among others. Understanding and utilizing Fuzzy Necessity and Possibility Theory can enhance decision-making processes, improve risk assessment, and provide more robust solutions in scenarios where uncertainty is prevalent. Its flexibility and ability to handle ambiguity make it a valuable addition to the toolbox of anyone working with complex and uncertain information.

3. Fuzzy DEA Model

Data Envelopment Analysis measures the relative efficiency of the DMUs set. Suppose that there are n decision-making units (DMUs) for evaluation. Each DMU uses m inputs to generate s outputs. Especially, DMU_j consumes $\widetilde{x}_{ij}(i = 1, \cdots ., m)$ fuzzy input values and produces $\widetilde{y}_{rj}$ fuzzy output values.

The Fuzzy CCR model for evaluation is as follows:
(**Primal FCCR DEA**)

$$
\begin{array}{lll}
\min & \theta & \\
s.t. & \sum_{j=1}^{n} \lambda_j \widetilde{x}_{ij} \leq \theta \widetilde{x}_{io} & i = 1, 2, \cdots, m, \\
& \sum_{j=1}^{n} \lambda_j \widetilde{y}_{rj} \geq \widetilde{y}_{ro} & r = 1, 2, \cdots, s, \\
& \lambda_j \geq 0 & j = 1, 2, \cdots, n, \\
& \theta \quad \text{free}. &
\end{array} \tag{3.1}
$$

(**Dual FCCR DEA**)

$$
\begin{array}{lll}
\max & \sum_{r=1}^{s} u_r \widetilde{y}_{ro} & \\
s.t. & \sum_{i=1}^{m} v_i \widetilde{x}_{io} = 1 & \\
& \sum_{r=1}^{s} u_r \widetilde{y}_{rj} - \sum_{i=1}^{m} v_i \widetilde{x}_{ij} \leq 0 & j = 1, 2, \cdots, n, \\
& v_i \geq 0 & i = 1, 2, \cdots, m, \\
& u_r \geq 0 & r = 1, 2, \cdots, s.
\end{array} \tag{3.2}
$$

In the following, we present a method to solve the model 3.2. We consider the fuzzy inputs and outputs of $DMUs$ as triangular fuzzy numbers.

Suppose $\widetilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3)$ and $\widetilde{y}_{ij} = (y_{rj}^1, y_{rj}^2, y_{rj}^3)$, which are $x_{ij}^1 \geq 0$ and $y_{rj}^1 \geq 0$ for $i = 1, \cdots, m$ and $r = 1, \cdots, s$. Hence, the model 3.2 can be rewritten as follows:

$$
\begin{array}{lll}
\max & \widetilde{z}_o = \sum_{r=1}^{s} u_r (y_{ro}^1, y_{ro}^2, y_{ro}^3) & \\
s.t. & \sum_{i=1}^{m} v_i (x_{io}^1, x_{io}^2, x_{io}^3) = (1^1, 1^2, 1^3) & \\
& \sum_{r=1}^{s} u_r (y_{rj}^1, y_{rj}^2, y_{rj}^3) - \sum_{i=1}^{m} v_i (x_{ij}^1, x_{ij}^2, x_{ij}^3) \leq 0 & j = 1, 2, \cdots, n, \\
& v_i \geq 0 & i = 1, 2, \cdots, m, \\
& u_r \geq 0 & r = 1, 2, \cdots, s,
\end{array} \tag{3.3}
$$

where $1^1 \leq 1^2$ and $1^3 \geq 1^2$ are real numbers. There are several methods to solve the above model. The η-cut Method has been the most focused among the other methods, in which the model is transformed into an interval linear programming problem. There are many methods to compare the intervals. In this section, we propose our suggested approach and compare it with the two methods mentioned above.

3.1. **Despotis and Smirlis's method.** Suppose that all data of inputs and outputs $x_{ij}, y_{rj}(i = 1, \cdots, m; r = 1, \cdots, n.)$ are an interval data $[\check{x}_{ij}, \hat{x}_{ij}]$ and $[\check{y}_{rj}, \hat{y}_{rj}]$ respectively, which $\check{x}_{ij} \geq 0$ and $\check{y}_{rj} \geq 0$. In order to deal with such uncertainty, Despotis and Smirlis [9] developed LP models to generate upper and lower efficiency boundaries for each DMU.

$$
\begin{aligned}
\max \quad & \hat{H}_o = \sum_{r=1}^{s} u_r \hat{y}_{ro} \\
s.t. \quad & \sum_{i=1}^{m} v_i \check{x}_{io} = 1 \\
& \sum_{r=1}^{s} u_r \hat{y}_{ro} - \sum_{i=1}^{m} v_i \check{x}_{io} \leq 0 \\
& \sum_{r=1}^{s} u_r \check{y}_{rj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \leq 0, \quad && j = 1, 2, \cdots, n, j \neq 0, \\
& v_i \geq 0 && i = 1, 2, \cdots, m, \\
& u_r \geq 0 && r = 1, 2, \cdots, s.
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
\max \quad & \check{H}_o = \sum_{r=1}^{s} u_r \check{y}_{ro} \\
s.t. \quad & \sum_{i=1}^{m} v_i \hat{x}_{io} = 1 \\
& \sum_{r=1}^{s} u_r \check{y}_{ro} - \sum_{i=1}^{m} v_i \hat{x}_{io} \leq 0 \\
& \sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{i=1}^{m} v_i \check{x}_{ij} \leq 0 \quad && j = 1, 2, \cdots, n, j \neq 0, \\
& v_i \geq 0 && i = 1, 2, \cdots, m, \\
& u_r \geq 0 && r = 1, 2, \cdots, s.
\end{aligned} \tag{3.5}
$$

[9]$\hat{H}_o$ and $\check{H}_o$ are the best relative efficiency for DMU_O under the most favorable and the most unfavorable conditions, respectively. Considering the upper and lower boundaries in DEA models, we may find out that the set of constraints used to measure the DMU_s efficiency varies from one DMU to another one, and even the set of constraints used to measure the upper and lower efficiency boundaries of some DMU_s are different from the other DMU_s. The main drawback of using different sets of constraints to measure the efficiency of DMU_s is the lack of comparison between efficiencies because different generation boundaries are accepted in the efficiency measurement process. The Despotis and Smirlis method presented in the article aims to address the issue of uncertainty in DEA by considering interval data for inputs and outputs. They propose two linear programming (LP) models to generate upper and lower efficiency boundaries for each Decision Making Unit (DMU).

The first LP model, represented by equation 3.4, maximizes the upper-efficiency boundary, denoted as $\hat{H}_o$, which is the sum of weighted outputs under the upper limits. The model includes constraints to ensure that the weighted sum of inputs equals one and that the weighted sum of outputs minus the weighted sum of inputs is less than or equal to zero for each DMU. Additionally, constraints are included to handle the uncertainty in the upper limits of the outputs.

The second LP model, represented by equation 3.5, maximizes the lower efficiency boundary, denoted as $\check{H}_o$, which is the sum of weighted outputs under the lower limits. This model is similar to the first model but deals with the lower limits of the outputs.

One of the disadvantages of this method is the variation in the set of constraints used to measure the efficiency of different DMUs. Each DMU may have a different set of constraints, and even the set of constraints used to measure the upper and lower efficiency boundaries for a specific DMU may differ from those used for other DMUs. This inconsistency in the constraints makes it difficult to compare the efficiencies of different DMUs since different generation boundaries are accepted in the efficiency measurement process.

3.2. **Wang et al.'s method.** Wang et al. [10] developed another interval DEA model to prevent the use of different generation boundaries to measure the efficiency of different DMUs.

$$\text{(3.6)} \qquad \begin{array}{lll} \max & \hat{\theta}_o = \sum\limits_{r=1}^{s} u_r \hat{y}_{ro} & \\ s.t. & \sum\limits_{i=1}^{m} v_i \check{x}_{io} = 1 & \\ & \sum\limits_{r=1}^{s} u_r \hat{y}_{rj} - \sum\limits_{i=1}^{m} v_i \check{x}_{ij} \leq 0 & j = 1, 2, \cdots, n, \\ & v_i \geq 0 & i = 1, 2, \cdots, m, \\ & u_r \geq 0 & r = 1, 2, \cdots, s. \end{array}$$

$$\text{(3.7)} \qquad \begin{array}{lll} \max & \check{\theta}_o = \sum\limits_{r=1}^{s} u_r \check{y}_{ro} & \\ s.t. & \sum\limits_{i=1}^{m} v_i \hat{x}_{io} = 1 & \\ & \sum\limits_{r=1}^{s} u_r \hat{y}_{rj} - \sum\limits_{i=1}^{m} v_i \check{x}_{ij} \leq 0 & j = 1, 2, \cdots, n, \\ & v_i \geq 0 & i = 1, 2, \cdots, m, \\ & u_r \geq 0 & r = 1, 2, \cdots, s. \end{array}$$

When all DMU_s are at their best generation state, $\hat{\theta}_o$ obtains the best relative possible efficiency for DMU_O, whereas $\check{\theta}_o$ obtains the best relative possible efficiency for the lower boundary. They form the relative efficiency intervals $[\check{\theta}_o, \hat{\theta}_o]$. Compared to the Despotis and Smirlis' interval DEA models, Wang's model uses a constant generation boundary as a benchmark to measure the efficiency of all DMUs, which makes their model to be more reasonable and reliable. Also, their approach does not seem to be more reasonable with sequential preference-based data.

The Wang et al.'s method, presented in the article, aims to address the issue of using different generation boundaries to measure the efficiency of different DMU_s in interval data envelopment analysis (DEA). They propose two interval DEA models that use a constant generation boundary as a benchmark for measuring efficiency.

The first model, represented by equation 3.6, maximizes the upper-efficiency boundary, denoted as $\hat{\theta}_o$, which is the sum of weighted outputs under the upper limits. The model includes constraints to ensure that the weighted sum of inputs equals one and that the weighted sum of outputs minus the weighted sum of inputs is less than or equal to zero for each DMU. The uncertainty in the upper limits of the outputs is also considered.

The second model, represented by equation 3.7, maximizes the lower efficiency boundary, denoted as $\check{\theta}_o$, which is the sum of weighted outputs under the lower limits. Similar to the first model, this model uses a constant generation boundary and includes constraints to handle the uncertainty in the lower limits of the outputs.

One advantage of Wang et al.'s method is that it avoids the use of different generation boundaries for measuring efficiency across DMUs. Instead, a constant benchmark is employed, which makes the model more reasonable and reliable. However, the article mentions that their approach may not be as suitable for sequential preference-based data, indicating a potential limitation or drawback of the Method in certain scenarios. Further analysis and evaluation would be required to fully understand the implications and limitations of this approach.

4. Proposed approach

One of the most suitable methods for providing imprecise information is to use fuzzy set theory. In most methods of solving decision-making models with fuzzy data, the data transformation method to definite is used. The Method of the closest definite approximation of fuzzy data with the least deletion of information is one of the new and appropriate methods.

If we use a defuzzification operator that replaces the fuzzy set with a definite number, we generally lose a lot of important information. Therefore, a definite set approximation of a fuzzy set is proposed. In this approach, we replace a given fuzzy set with a deterministic set that is somehow close to the previous set. The most popular approximation operator is the so-called closest normal set operator. This concept is used in many fields, such as fuzzy pattern recognition, image processing, fuzzification, etc.

In this research, to check the efficiency of decision-making units with fuzzy data, we use a type of interval approximation which is a continuous and simple approximation operator called nearest interval approximation. In other words, interval approximation is used to obtain the nearest interval approximation, although different methods are used to find the interval approximation of fuzzy sets.

In fact, the approximation of the nearest interval can preserve the characteristics and importance of fuzzy numbers from a theoretical and practical point of view. It is also computationally efficient and additive, but the proposed approach with the approximation of the closest interval can be of great help to some extent in practical and theoretical progress. Where it is recommended to simplify the data (expressed by fuzzy numbers), but the ambiguity should be preserved (See [26]).

The nearest interval approximation operator follows the interval approximation rule and the distance issue. Therefore, this operator results in a distance that is the best distance according to a certain measurement of distances between fuzzy numbers. Therefore, nearest interval approximation can preserve the characteristics and importance of fuzzy numbers from a theoretical and practical point of view. Also, this operator is simple and natural, and continuous (See[27]).

The idea behind models is to propose a new approach for measuring the efficiency of DMU_s in a fuzzy environment. The approach involves transforming fuzzy inputs and outputs into interval numbers using the nearest interval approximation method. In the proposed approach, the fuzzy inputs $\widetilde{x}_{ij}$ and fuzzy outputs $\widetilde{y}_{rj}$ are represented as intervals $[\check{x}_{ij}, \hat{x}_{ij}]$ and $[\check{y}_{rj}, \hat{y}_{rj}]$, respectively. These intervals are obtained by taking the averages of the lower and upper bounds of the fuzzy numbers. Therefore, given the approximation of the nearest interval, fuzzy inputs $\widetilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij})$ and fuzzy outputs $\widetilde{y}_{rj} = (y^1_{rj}, y^2_{rj}, y^3_{rj})$ are as follows:

$$[\frac{x^1_{ij}+x^2_{ij}}{2}, \frac{x^3_{ij}+x^2_{ij}}{2}] = [\check{x}_{ij}, \hat{x}_{ij}], (i = 1, \cdots, m; j = 1, \cdots, n;),$$

$$[\frac{y^1_{rj}+y^2_{rj}2}{,}\frac{y^3_{rj}+y^2_{rj}2}{]} = [\check{y}_{rj}, \hat{y}_{rj}], (r = 1, \cdots, s; j = 1, \cdots, n;).$$

Here, in order to prevent the use of different generation boundaries in measuring the efficiency of different DMU_s, we present new models.

Suppose $\widetilde{z}_j = \frac{\sum_{r=1}^{s} u_r \widetilde{y}_{rj}}{\sum_{i=1}^{m} v_i \widetilde{x}_{ij}} (j = 1, \cdots, n)$ is the efficiency of $DMU_j (j = 1, \cdots, n)$. The efficiency of a $DMU_j (j = 1, \cdots, n)$, denoted as $\widetilde{Z}_j$, is calculated as the ratio of the aggregated fuzzy outputs to the aggregated fuzzy inputs. Using interval calculations, the efficiency z_j of $DMU_j (j = 1, \cdots, n)$ is expressed as $[\check{Z}_j, \hat{Z}_j]$, where $\check{Z}_j$ represents the lower bound, and $\hat{Z}_j$ represents the upper bound of the efficiency. Then given the approximation of the nearest interval and interval calculations, we have the following:

$$Z_j = \frac{\sum_{r=1}^{s} u_r [\check{y}_{rj}, \hat{y}_{rj}]}{\sum_{i=1}^{m} v_i [\check{x}_{ij}, \hat{x}_{ij}]} = \frac{[\sum_{r=1}^{s} u_r \check{y}_{rj}, \sum_{r=1}^{s} u_r \hat{y}_{rj}]}{[\sum_{i=1}^{m} v_i \check{x}_{ij}, \sum_{i=1}^{m} v_i \hat{x}_{ij}]} = [\frac{\sum_{r=1}^{s} u_r \check{y}_{rj}}{\sum_{i=1}^{m} v_i \hat{x}_{ij}}, \frac{\sum_{r=1}^{s} u_r \hat{y}_{rj}}{\sum_{i=1}^{m} v_i \check{x}_{ij}}].$$

Thus Z_j is also an interval number $[\check{Z}_j, \hat{Z}_j] (j = 1, ..., n)$. Also we have

$$0 < Z_j = [\check{Z}_j, \hat{Z}_j] = [\frac{\sum_{r=1}^{s} u_r \check{y}_{rj}}{\sum_{i=1}^{m} v_i \hat{x}_{ij}}, \frac{\sum_{r=1}^{s} u_r \hat{y}_{rj}}{\sum_{i=1}^{m} v_i \check{x}_{ij}}] \leq 1, j = 1, \cdots, n,$$

To measure the upper and lower boundaries of DMU_O (a specific DMU), frictional programming models are used. These models aim to maximize the upper boundary $\hat{Z}_o$ and minimize the lower boundary $\check{Z}_o$ of $DMU'_O s$ relative efficiency, considering the best and worst values of the objective function in interval linear programming.

$$(4.1) \qquad \begin{array}{lll} \max & \hat{Z}_o = \frac{\sum_{r=1}^{s} u_r \hat{y}_{ro}}{\sum_{i=1}^{m} v_i \check{x}_{io}} & \\ s.t. & \frac{\sum_{r=1}^{s} u_r \hat{y}_{rj}}{\sum_{i=1}^{m} v_i \check{x}_{ij}} \leq 1 & j = 1, \cdots, n, \\ & v_i \geq 0 & i = 1, 2, \cdots, m, \\ & u_r \geq 0 & r = 1, 2, \cdots, s. \end{array}$$

$$
\text{(4.2)} \qquad
\begin{array}{lll}
\max & \check{Z}_o = \frac{\sum_{r=1}^{s} u_r \check{y}_{ro}}{\sum_{i=1}^{m} v_i \hat{x}_{io}} & \\
s.t. & \frac{\sum_{r=1}^{s} u_r \check{y}_{rj}}{\sum_{i=1}^{m} v_i \hat{x}_{ij}} \le 1 & j = 1, \cdots, n, \\
 & v_i \ge 0 & i = 1, 2, \cdots, m, \\
 & u_r \ge 0 & r = 1, 2, \cdots, s.
\end{array}
$$

The above models are based on the best and worst value of the objective function in interval linear programming as well as the CCR model in data coverage analysis. The frictional programming models can be transformed into the linear programming model. These constraints ensure that the efficiency of DMU_o, represented by $\hat{Z}_o$ or $\check{Z}_o$, does not exceed one and that the aggregated fuzzy outputs and inputs are balanced.

$$
\text{(4.3)} \qquad
\begin{array}{lll}
\max & \hat{Z}_o = \sum\limits_{r=1}^{s} u_r \hat{y}_{ro} & \\
s.t. & \sum\limits_{i=1}^{m} v_i \hat{x}_{io} = 1 & \\
 & \sum\limits_{r=1}^{s} u_r \hat{y}_{rj} - \sum\limits_{i=1}^{m} v_i \check{x}_{ij} \le 0 & j = 1, 2, \cdots, n, \\
 & v_i \ge 0 & i = 1, 2, \cdots, m, \\
 & u_r \ge 0 & r = 1, 2, \cdots, s.
\end{array}
$$

$$
\text{(4.4)} \qquad
\begin{array}{lll}
\max & \check{Z}_o = \sum\limits_{r=1}^{s} u_r \check{y}_{ro} & \\
s.t. & \sum\limits_{i=1}^{m} v_i \hat{x}_{io} = 1 & \\
 & \sum\limits_{r=1}^{s} u_r \check{y}_{rj} - \sum\limits_{i=1}^{m} v_i \hat{x}_{ij} \le 0 & j = 1, 2, \cdots, n, \\
 & v_i \ge 0 & i = 1, 2, \cdots, m, \\
 & u_r \ge 0 & r = 1, 2, \cdots, s,
\end{array}
$$

where $\hat{Z}_o$ denotes the best relative efficiency of DMU_o when all DMU_s are at their best generation activity, while $\check{Z}_o$ presents the worst relative efficiency of DMU_o when all DMUs are at their worst generation activity. Therefore, the meanings of $\hat{H}_o$, $\check{H}_o$, $\hat{\theta}_o$, and $\check{\theta}_o$ differ from the meanings of $\hat{Z}_o$ and $\check{H}_o$.
In the following, the efficiency of DMU can be expressed as an interval number.

Definition 4.1. DMU_o is called the *DEA interval strong efficiency*, if the efficiency of upper and lower boundaries equals 1 in the interval of efficiency $[\check{Z}_o^*, \hat{Z}_o^*]$, i.e., $\check{Z}_o^* = \hat{Z}_o^* = 1$.

Definition 4.2. DMU_o is called the *DEA interval weak efficiency*, if the efficiency of the upper boundary equals 1, i.e., $\hat{Z}_o^* = 1$ and DMU_o is called *DEA interval inefficiency*, if $\hat{Z}_o^* < 1$.

Sometimes in evaluating DMU_s, the efficiency of the lower boundary of some DMU_s is higher than the efficiency of their upper boundary. In such circumstances, we need a secondary goal to establish the efficiency interval. This secondary goal is achieved by adding the additional constraint $\sum_{r=1}^{s} u_r \check{y}_{ro} \leq \sum_{r=1}^{s} u_r^* \hat{y}_{ro}$ in order to establish the efficiency interval.

Theorem 4.3. *For each DMU in which the efficiency of the lower boundary is higher than the efficiency of the upper boundary, the efficiency model of the lower boundary* (4.4) *needs an additional constraint of* $\sum_{r=1}^{s} u_r \check{y}_{ro} \leq \sum_{r=1}^{s} u_r^* \hat{y}_{ro}$. *In the optimal solution, this constraint is established by the equivalent.*

Proof. By adding the constraint of $\sum_{r=1}^{s} u_r \check{y}_{ro} \leq \sum_{r=1}^{s} u_r^* \hat{y}_{ro}$ to the model (4.4), it is written as follows:

$$
\begin{array}{lll}
\max & \check{Z}_o = \sum_{r=1}^{s} u_r \check{y}_{ro} & \\
s.t. & \sum_{r=1}^{s} u_r \check{y}_{ro} \leq \sum_{r=1}^{s} u_r^* \hat{y}_{ro} & \\
 & \sum_{i=1}^{m} v_i \hat{x}_{io} = 1 & \\
 & \sum_{r=1}^{s} u_r \check{y}_{rj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \leq 0 & j = 1, 2, \cdots, n, \\
 & v_i \geq 0 & i = 1, 2, \cdots, m, \\
 & u_r \geq 0 & r = 1, 2, \cdots, s.
\end{array}
\tag{4.5}
$$

Now given to the objective function $\check{Z}_o = \sum_{r=1}^{s} u_r \check{y}_{ro}$ and $\sum_{r=1}^{s} u_r \check{y}_{ro} \leq \sum_{r=1}^{s} u_r^* \hat{y}_{ro}$, it is obvious that the optimal solution of the lower boundary efficiency equals $\sum_{r=1}^{s} u_r^* \hat{y}_{ro}$. □

Theorem 4.3 states that in DEA model, if the efficiency of the lower boundary is higher than the efficiency of the upper boundary, an additional constraint is required in the efficiency model of the lower boundary. This additional constraint is given by the inequality $\sum_{r=1}^{s} u_r \check{y}_{ro} \leq \sum_{r=1}^{s} u_r^* \hat{y}_{ro}$ Theorem 4.3 provides valuable insights into the field of data envelopment analysis, a widely used method for measuring the relative efficiency of decision-making units (DMU_s). By understanding and applying this theorem, researchers, and practitioners can improve the accuracy and reliability of their efficiency assessments.

The significance of this theorem lies in its ability to address situations where the efficiency of the lower boundary exceeds that of the upper boundary. In such cases, it is necessary to introduce an additional constraint to the efficiency model of the lower boundary, ensuring that the optimal solution aligns with the reference output quantities. This constraint enhances the validity of efficiency evaluations and helps capture the true efficiency levels of the DMU_s under analysis.

By being aware of Theorem 4.3 and its implications, researchers and practitioners can avoid potential errors or inaccuracies in their efficiency evaluations. This theorem serves as a reminder of the importance of incorporating appropriate constraints into DEA models, especially when dealing with scenarios where the lower boundary efficiency surpasses the upper boundary efficiency. By adhering to the principles outlined in this theorem, analysts can enhance the robustness and reliability of their efficiency assessments, leading to more accurate decision-making processes and resource allocation strategies.

Overall, Theorem 4.3 underscores the critical role of constraints in DEA models and highlights the need for their careful consideration to ensure valid and meaningful efficiency evaluations. By embracing this theorem, researchers and practitioners can enhance the effectiveness of their analyses and contribute to improved decision-making in various domains, such as operations research, management science, and economics.

Theorem 4.4. *If $\check{Z}_o^*$ and $\hat{Z}_o^*$ are the optimal values of the objective function of the models (4.3) and (4.4), $\check{H}_o^*$ and $\hat{H}_o^*$ are the optimal values of the objective function of the models (3.4) and (3.5), $\check{\theta}_o^*$ and $\hat{\theta}_o^*$ are the optimal values of objective function of the models (3.6) and (3.7), then $\check{\theta}_o^* \leq \check{H}_o^* \leq \check{Z}_o^*$ and $\hat{\theta}_o^* = \hat{Z}_o^* \leq \hat{H}_o^*$.*

Proof. Let us v_i^* and u_r^* $(i = 1, \cdots, m; r = 1, \cdots, s)$ be the optimal solutions of the model (4.3). Then we have $\sum_{r=1}^{s} u_r^* \hat{y}_{rj} - \sum_{i=1}^{m} v_i^* \check{x}_{ij} \leq 0, j = 1, 2, \cdots, n$. Specially, for DMU_o, we have

$$\sum_{r=1}^{s} u_r^* \check{y}_{ro} - \sum_{i=1}^{m} v_i^* \hat{x}_{io} \leq \sum_{r=1}^{s} u_r^* \hat{y}_{ro} - \sum_{i=1}^{m} v_i^* \check{x}_{io} \leq 0, j = 1, 2, \cdots, n.$$

Thus v_i^* and u_r^* $(i = 1, \cdots, m; r = 1, \cdots, s)$ are possible solutions for the models (3.5) and (4.4), respectively. So $\check{\theta}_o^* = \sum_{r=1}^{s} u_r^* \check{y}_{rj} = \check{H}_o \leq \check{H}_o^*$. On the other hand, given that model (3.5) has the additional constraint $\sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{i=1}^{m} v_i \check{x}_{ij} \leq 0$. Hence compared to model (4.4), $\check{\theta}_o^* \leq \check{H}_o^* \leq \check{Z}_o^*$. The proof $\hat{\theta}_o^* = \hat{Z}_o^* \leq \hat{H}_o^*$ is confirmed with respect to (3.5). □

Theorem 4.4 provides relationships between the optimal values of the objective functions in different models, showing that the values of H, θ, and Z (efficiency measure in the proposed approach) are related in certain orderings.

Theorem 4.4 has important implications for understanding the relationships between different models in DEA and the optimal values of their objective functions. By studying and applying this theorem, researchers, and practitioners can gain a deeper understanding of the efficiency evaluations and comparisons performed using these models.

The significance of this theorem lies in its ability to establish bounds on the optimal values of the objective functions in various DEA models. It provides insights into the relationships between these models and their respective efficiency measures, such as θ and H. By establishing inequalities between these measures, and the theorem provides a framework for understanding the relative efficiency levels and performance of decision-making units (DMU_s).

Understanding Theorem 4.4 enables researchers and practitioners to interpret and compare the efficiency measures obtained from different DEA models. It highlights the fact that θ, H, and Z, as represented by the objective functions in models (3.6), (3.7), (3.4), (3.5), (4.3) and (4.4), are interrelated and subject to certain inequalities. These relationships help ensure consistency and coherence in efficiency evaluations across different models and provide a deeper understanding of the efficiency assessments conducted in DEA.

By being aware of Theorem 4.4 and its implications, researchers and practitioners can make informed decisions when selecting and applying DEA models for efficiency analysis. They can use this theorem as a guiding principle to ensure that their chosen model aligns with the desired objectives and provides reliable and meaningful efficiency evaluations. Furthermore, the theorem can assist in interpreting and comparing the results obtained from different models, facilitating better decision-making processes and resource allocation strategies.

Overall, Theorem 4.4 serves as a valuable tool in the field of DEA, shedding light on the relationships between different models and their corresponding efficiency measures. By understanding and utilizing this theorem, researchers, and practitioners can enhance the quality and reliability of their efficiency evaluations, leading to more informed decision-making and improved resource management in various domains, including operations research, management science, and economics.

So far, the advantages of the proposed approach can be listed as follows: Information Preservation: Unlike traditional defuzzification methods that result in a significant loss of important information, the proposed approach aims to approximate fuzzy sets while preserving their characteristics and importance. This ensures that the ambiguity inherent in the data is maintained, providing a more accurate representation of the underlying uncertainty.

Computational Efficiency: The nearest interval approximation operator employed in this research offers a computationally efficient and simple method for obtaining interval approximations. It allows for the transformation of fuzzy inputs and outputs into interval numbers, facilitating further analysis and decision-making processes.

Theoretical Relevance: The proposed approach builds upon well-established concepts and operators in fuzzy set theory. Utilizing the closest normal set operator and interval approximation rule aligns with existing theoretical foundations and extends their applicability to various domains such as fuzzy pattern recognition, image processing, and fuzzification.

Practical Applicability: The proposed approach not only has theoretical merits but also holds practical value. It offers a means to simplify data expressed by fuzzy numbers while preserving the underlying ambiguity. This can be highly beneficial in real-world scenarios where decision-making units need to handle imprecise or uncertain information effectively.

Measurement of Efficiency: The proposed approach introduces frictional programming models that measure the upper and lower boundaries of the efficiency of decision-making units (DMU_s). These models provide valuable insights into the relative efficiency of DMU_s under different scenarios, enabling performance evaluation and benchmarking.

Secondary Goal Establishment: In cases where the efficiency of the lower boundary of certain DMUs exceeds that of their upper boundary, the proposed approach introduces a secondary goal to establish an efficiency interval. This ensures a more comprehensive evaluation of DMU_s, considering both upper and lower boundary efficiencies.

Theoretical Comparisons: Theorems within the proposed approach establish relationships between different objective functions and efficiency measures. This allows for meaningful comparisons between different models and provides a deeper understanding of the efficiency bounds and the associated interpretations. In summary, the proposed approach combining fuzzy set theory, interval approximation, and frictional programming models offers several advantages, including effective handling of imprecise information, information preservation, computational efficiency, theoretical relevance, practical applicability, and comprehensive efficiency evaluation. These advantages serve as strong motivations for researchers and practitioners interested in advancing decision-making models with fuzzy data.

4.1. **Proposed approach for ranking.** Efficiency ranking is a crucial aspect of DEA, as it allows decision-makers to identify and prioritize DMU_s based on their performance and effectiveness. By studying and applying the ranking approaches presented in the text, researchers and practitioners can gain valuable insights into how to compare and rank the efficiency of different DMU_s, even in the presence of fuzzy and imprecise data.

In most past methods, like other standard DEA ranking methods, fuzzy and imprecise data are not considered because the efficiency evaluation is expressed as interval efficiency in the proposed approach for solving fuzzy DEA. Therefore, a simple and efficient ranking approach is required to compare and rank the efficiency of different DMU_s. Nowadays, there are a few approaches to ranking interval efficiency. In the following, we present two ranking approaches: the ranking approach of weak efficiency units and the ranking approach of strong efficiency units.

The motivation for developing a simple and efficient ranking approach lies in the limitations of existing methods. The text acknowledges that current standard DEA ranking methods do not consider fuzzy and imprecise data, which can be prevalent in real-world scenarios. Therefore, the proposed approach for solving fuzzy DEA fills this gap by expressing efficiency evaluation as interval efficiency. This approach allows for a more comprehensive and realistic assessment of DMU efficiency by accounting for uncertainty and imprecision in the data.

By introducing two ranking approaches, namely the ranking approach of weak efficiency units and the ranking approach of strong efficiency units, the text provides alternative methods for comparing and ranking DMU_s based on their interval efficiency. These approaches offer flexibility and adaptability to different contexts and preferences, allowing decision-makers to choose the most appropriate method for their specific needs.

Understanding and applying these ranking approaches can have several benefits. Firstly, it enables researchers and practitioners to account for fuzzy and imprecise data in their efficiency evaluations, leading to more accurate and reliable rankings. This is particularly important in domains where data uncertainty is prevalent, such as healthcare, finance, and environmental analysis.
Secondly, a simple and efficient ranking approach saves time and computational resources, making it practical for large-scale efficiency evaluations. Decision-makers

can quickly compare and rank DMU_s without being burdened by complex methodologies or excessive computational requirements. This streamlines the decision-making process and facilitates effective resource allocation.

Lastly, having multiple ranking approaches provides flexibility and robustness in assessing DMU efficiency. Different approaches may emphasize different aspects or criteria, allowing decision-makers to gain a comprehensive understanding of DMU performance from multiple perspectives. This enhances the decision-making process by considering various factors and considerations relevant to the specific context.

In conclusion, the need for a simple and efficient ranking approach in DEA is evident, especially when dealing with fuzzy and imprecise data. The presented ranking approaches offer valuable tools for comparing and ranking DMU efficiency, taking into account interval efficiency and accommodating different preferences and contexts. By utilizing these approaches, researchers and practitioners can make more informed decisions, allocate resources effectively, and drive continuous improvement in various fields of application.

Definition 4.5. Suppose $m, 0 \in \mathbb{R}$. We define the marked distance $d^*(m,0) = m$. If $m > 0$, then m is the right side of 0, and the interval is $d^*(m,0) = m$. If $m < 0$, then m is the left side of 0, and the interval is $d^*(m,0) = -m$. Thus $d^*(m,0) = m$ is the marked distance of m from 0.

Now we use the concept of marked distance to rank the weak interval efficiency. Since $0 < \check{Z}_o^*$, the efficiency interval is positive, and the total interval is placed on the right side of 0. Hence, the marked distance of definite interval $[\check{Z}_o^*, \hat{Z}_o^*]$ from 0 can be defined as follows:

$$d^*([\check{Z}_o^*, \hat{Z}_o^*], 0) = \frac{1}{2}[d^*(\check{Z}_o^*, 0) + d^*(\hat{Z}_o^*, 0)] = \frac{1}{2}[\check{Z}_o^* + \hat{Z}_o^*].$$

Definition 4.6. *Ranking* for weak interval efficiency DMUs are defined as follows:

$$[\check{Z}_i^*, \hat{Z}_i^*] < [\check{Z}_j^*, \hat{Z}_j^*] \ if \ and \ only \ if \ d^*([\check{Z}_i^*, \hat{Z}_i^*], 0) < d^*([\check{Z}_j^*, \hat{Z}_j^*], 0),$$

where $j = 1, \cdots, n, \ i \neq j$.

In evaluating DMU_s, some DMUs may be strong interval efficiency. Hence, there is a need to rank efficient DMU_s. One Method to rank them is first to solve the super-efficiency interval models, which is:

$$(4.6) \quad \begin{array}{lll} \max & S\hat{Z}_o = \sum_{r=1}^{s} u_r \hat{y}_{ro} & \\ s.t. & \sum_{i=1}^{m} v_i \hat{x}_{io} = 1 & \\ & \sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{i=1}^{m} v_i \check{x}_{ij} \leq 0 & j = 1, 2, \cdots, n, j \neq o \\ & v_i \geq 0 & i = 1, 2, \cdots, m, \\ & u_r \geq 0 & r = 1, 2, \cdots, s. \end{array}$$

$$
\begin{array}{llll}
 & \max & S\check{Z}_o = \sum_{r=1}^{s} u_r \check{y}_{ro} & \\
 & s.t. & \sum_{i=1}^{m} v_i \hat{x}_{io} = 1 & \\
(4.7) & & \sum_{r=1}^{s} u_r \check{y}_{rj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \leq 0 & j = 1, 2, \cdots, n, j \neq o \\
 & & v_i \geq 0 & i = 1, 2, \cdots, m, \\
 & & u_r \geq 0 & r = 1, 2, \cdots, s.
\end{array}
$$

For DMU_o, the lower boundary efficiency of the above super-efficiency model is $S\check{Z}_o^* \geq 1$. Now the definition (definition number of ranking for weak efficiency) can be used to rank the strong interval efficiency DMU_s.

Definition 4.7. *Ranking* for interval strong efficiency DMUs are defined as follows:

$$[S\check{Z}_i^*, S\hat{Z}_i^*] < [S\check{Z}_j^*, S\hat{Z}_j^*] \text{ if and only if } d^*([S\check{Z}_i^*, S\hat{Z}_i^*], 0) < d^*([S\check{Z}_j^*, S\hat{Z}_j^*], 0),$$

where $j = 1, 2, \cdots, n, \ i \neq j$.

Understanding marked distance and interval efficiency allows decision-makers to identify strong and weak performers, facilitating the identification of areas for improvement and potential strategies for enhancing efficiency. By implementing the ranking methods outlined in the text, decision-makers can prioritize resources, investments, or interventions to achieve greater efficiency and effectiveness in their operations.

Additionally, acquiring knowledge about these concepts can broaden one's understanding of optimization techniques and mathematical models used in various fields, such as economics, operations research, and management science. This knowledge can be applied to a wide range of real-world scenarios, including resource allocation, performance evaluation, and benchmarking, leading to improved organizational outcomes.

In summary, by delving into the concepts presented in the text and embracing the motivation to understand and apply them, the reader can enhance their decision-making abilities, identify areas for improvement, and contribute to achieving greater efficiency and effectiveness in their endeavors.

4.2. **Numerical examples.** In this section, three numerical examples are presented to show the fuzzy DEA models by different approaches. In all three numerical examples, the interval efficiency and ranking are determined. In addition, the examples are solved by the models of Despotis and Smirlis and Wang et al., and then, their results are compared with the proposed approach.

Example 4.8. Numerical example used is taken from Guo and Tanaka [28]. Information on the numerical example containing five DMUs with two fuzzy inputs and two fuzzy outputs is presented in Table 4.8. The fuzzy inputs and outputs are considered symmetric triangular fuzzy numbers, which are a special type of triangular fuzzy numbers.

Table 4.8 shows the results of interval efficiency based on Despotis and Smirlis's model. The results show that $DMU_s\{2, 3, 5\}$ are weak interval efficiency while DMU4 is strong interval efficiency. The results of interval efficiency for Wang's

Table 1. Tanaka and Gue data (2001)

DMU	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
Input1	$(3.5, 4.0, 4.5)$	$(2.9, 2.9, 2.9)$	$(4.4, 4.9, 5.4)$	$(3.4, 4.1, 4.8)$	$(5.9, 6.5, 7.1)$
Input2	$(1.9, 2.1, 2.3)$	$(1.4, 1.5, 1.6)$	$(2.2, 2.6, 3.0)$	$(2.2, 2.3, 2.4)$	$(3.6, 4.1, 4.6)$
Output1	$(2.4, 2.6, 2.8)$	$(2.2, 2.2, 2.2)$	$(2.7, 3.2, 3.7)$	$(2.5, 2.9, 3.3)$	$(4.4, 5.1, 5.8)$
Output2	$(3.8, 4.1, 4.4)$	$(3.3, 3.5, 3.7)$	$(4.3, 5.1, 5.9)$	$(5.5, 5.7, 5.9)$	$(6.5, 7.4, 8.3)$

model are presented in Table 4.8. $DMU_s\{2, 4, 5\}$ are weak interval efficiency. In this model, unlike Despotis and Smirlis's model, DMU4 is transformed from strong interval efficiency into weak one. The results of our proposed approach are presented in Table 4.8. In this example, with the help of the nearest interval approximation, each decision unit that has fuzzy information is attributed to an interval, which causes continuity, uniformity, and linearity. These results show that $DMU_s\{2, 4\}$ are strong interval efficiency, while in Despotis and Smirlis's model, DMU5 is weak interval efficiency. DMU3, like Wang's model, is interval inefficiency. Also, the results of these three views show that $\check{\theta}^*_o \leq \check{H}^*_o \leq \check{Z}^*_o$ and $\hat{\theta}^*_o = \hat{Z}^*_o \leq \check{H}^*_o$. For the results of our proposed approach, efficiency DMU_s are ranked, and the ranking results are presented in Table 4.8. Table 4.8 presents a comparison and ranking of the interval efficiencies of the five DMU_s. The ranking is based on the marked distance (d^*) from the proposed approach. DMU 2 obtains the highest ranking, followed by DMU 4, while DMU 5 ranks third.

In summary, the text provides a detailed analysis of three numerical examples utilizing fuzzy DEA models to determine interval efficiency and ranking for DMUs. It demonstrates the application of different models and compares their results. The proposed approach introduces the concept of nearest interval approximation to enhance the evaluation process. The analysis presented in the text allows readers to gain insights into the performance of DMUs, make informed decisions, and allocate resources effectively.

Table 2. Interval efficiency Results by Despotis and Smirlis's model

DMU	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
$[\check{H}^*_o, \hat{H}^*_o]$	$[0.7579, 0.9633]$	$[0.9904, 1]$	$[0.7165, 1]$	$[1, 1]$	$[0.8450, 1]$

Table 3. Interval efficiency Results by Wang's model

DMU	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
$[\check{\theta}^*_o, \hat{\theta}^*_o]$	$[0.7579, 0.9093]$	$[0.9567, 1]$	$[0.7165, 0.9571]$	$[0.9244, 1]$	$[0.7973, 1]$

Table 4. Interval efficiency Results by ourselves a model

DMU	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
$[\check{Z}^*_o, \hat{Z}^*_o]$	$[0.8129, 0.9093]$	$[1, 1]$	$[0.7722, 0.9577]$	$[1, 1]$	$[0.9208, 1]$

Table 5. compare and rank the interval efficiencies of the five DMU_s

DMU	$[S\check{Z}_o^*, S\hat{Z}_o^*]$	d^*	$Rank$
$DMU2$	$[1.0624, 1.0624]$	1.0624	2
$DMU4$	$[1.0864, 1.2257]$	1.1560	1
$DMU5$	$[0.9208, 1]$	0.9604	3

Understanding the interval efficiency of DMU_s is crucial for decision-makers as it helps identify both weak and strong performers. The examples presented in the text provide tables (Table 4.8, Table 4.8, and Table 4.8) that show the interval efficiency results obtained by different models for each DMU. By comparing these results, the reader can grasp the differences and similarities among the models and evaluate their effectiveness in capturing the efficiency of the DMU_s.

Furthermore, the proposed approach in the examples introduces the concept of nearest interval approximation, which allows decision units with fuzzy information to be attributed to an interval. This approach aims to provide continuity, uniformity, and linearity in the assessment process.

Example 4.9. The example presented demonstrates the practical application of interval efficiency analysis in evaluating the performance of manufacturing industries. By using interval inputs and output, decision-makers can gain valuable insights into the efficiency of different units within the industry. This numerical example was used by Wang et al. [10]. The information for the numerical example contains 7 DMU_s with two interval inputs and one interval output, which is presented in Table 4.9. The data is from seven manufacturing industries that participated in different cities to evaluate the performance with regard to capital and labor as inputs and gross production value as output.

Table 6. Wang et al.'s data (2005)

	$Input$		$Output$
DMU	$Capital$	$Labor$	Gross output value
1	$[564403, 621755]$	$[674111, 743281]$	$[806549, 866063]$
2	$[614371, 669665]$	$[685943, 742345]$	$[917507, 985424]$
3	$[762203, 798427]$	$[762207, 805677]$	$[1117142, 1195562]$
4	$[862016, 937044]$	$[779894, 846496]$	$[1206179, 1261031]$
5	$[1016898, 1082662]$	$[799714, 877137]$	$[1381315, 1462543]$
6	$[1164350, 1267970]$	$[807172, 889416]$	$[1497679, 1652787]$
7	$[1731916, 1816008]$	$[818090, 895746]$	$[1702249, 1812655]$

Wang et al. (2005) solved the models with weights higher than $\epsilon = 10^{-10}$. There are drawbacks to some of the results presented in Table 4.8 of Wang's article. Therefore, all of the models are presented with weights higher than $\epsilon = 10^{-10}$ in this numerical example, and the results are as follows:

For Despotis and Smirlis's model, the results are presented in the second column of Table 4.9. The results show that all DMU_s (DMU1-DMU7) are weak interval efficiency. The third column of Table 4.9 shows the interval efficiency results based on Wang's model. In this column, $DMU_s\{2,3,6,7\}$ are weak interval efficiency, and the other DMU_s are interval inefficiency. The results of solving the models are presented in the fourth column of Table 4.9, in which $DMU_s\{3,6,7\}$ are strong interval efficiency while DMU2 is weak interval efficiency. In this numerical example, we use the secondary goal for $DMU_s\{4,5\}$. In other words, the lower boundary efficiency is higher than the upper boundary efficiency in these two $DMU_s\{4,5\}$, and with respect to theorem 4.4, the lower boundary efficiency is assumed to be equal to the upper boundary efficiency. Also, the results of these three views show $\check{\theta}^*_o \leq \check{H}^*_o \leq \check{Z}^*_o$ and $\hat{\theta}^*_o = \hat{Z}^*_o \leq \hat{H}^*_o$. For the results of our proposed approach, efficient DMU_s are ranked, and the results of the ranking are presented in Table 4.9.

Table 7. Interval efficiency Results by Wang et al.'s data (2005)

DMU	$[\check{H}^*_o, \hat{H}^*_o]$	$[\check{\theta}^*_o, \hat{\theta}^*_o]$	$[\check{Z}^*_o, \hat{Z}^*_o]$
$DMU1$	$[0.8088, 1.0000]$	$[0.8088, 0.9567]$	$[0.9271, 0.9567]$
$DMU2$	$[0.8735, 1.0000]$	$[0.8555, 1.0000]$	$[0.9792, 1.0000]$
$DMU3$	$[0.8986, 1.0000]$	$[0.8986, 1.0000]$	$[0.1.0000, 1.0000]$
$DMU4$	$[0.8460, 1.0000]$	$[0.8460, 0.9610]$	$[0.9610, 0.9610]$
$DMU5$	$[0.8642, 1.0000]$	$[0.8642, 0.9819]$	$[0.9819, 0.9819]$
$DMU6$	$[0.8866, 1.0000]$	$[0.8866, 1.0000]$	$[1.0000, 1.0000]$
$DMU7$	$[0.9280, 1.0000]$	$[0.8664, 1.0000]$	$[1.0000, 1.0000]$

Table 8. Compare and rank the interval efficiencies of the seven DMUs

DMU	$[S\check{Z}^*_o, S\hat{Z}^*_o]$	d^*	$Rank$
$DMU2$	$[0.9792, 1.0000]$	0.9896	4
$DMU3$	$[1.0099, 1.0099]$	1.0092	3
$DMU6$	$[1.0185, 1.0737]$	1.0461	2
$DMU7$	$[1.0820, 1.0820]$	1.0820	1

Table 4.9 presents the ranking of the interval efficiencies for the seven DMU_s based on the proposed approach. The ranking is determined using the marked distance (d^*) from the proposed approach. DMU 7 achieves the highest ranking, followed by DMU 6, DMU 3, and DMU 2.

In summary, Example 4.9 showcases the evaluation of manufacturing industries using interval inputs and output. Different models are applied to assess the interval efficiency of the DMU_s, and the results are compared. The proposed approach provides a ranking of the efficient DMU_s based on their interval efficiencies, allowing decision-makers to identify the best-performing units.

By presenting the numerical example and its results, you can consider interval efficiency analysis as a valuable tool for evaluating and benchmarking the performance of manufacturing industries. The example highlights the importance of using interval inputs and output in capturing the inherent uncertainty and variability in real-world scenarios. The ranking of efficient DMU_s provides a practical and actionable way to assess performance and identify areas for improvement, ultimately contributing to informed decision-making and enhancing overall industry efficiency.

Example 4.10. This numerical example is taken from Saati et al. (2002). Data for this numerical example containing ten DMU_s with two fuzzy inputs and two fuzzy outputs are expressed in Table 4.9. In this example, unlike Example 4.9, the asymmetric triangular fuzzy numbers are considered. The results of interval efficiency for 10 DMU_s are presented in Table 4.10. The interval efficiency of Wang's model, Despotis and Smirlis' model, and our model are in columns 2, 3, and 4, respectively. The results of our approach show that $DMU_s\{1, 2, 10\}$ are strong interval efficiency, while in Despotis and Smirlis's model, DMU2 is strong interval efficiency, and $DMU_s\{1, 10\}$ are weak interval efficiency. In addition, in Wang's model, $DMU_s\{1, 2, 10\}$ are weak interval efficiency. In this example, on the other hand, $\check{\theta}^*_o \leq \check{H}^*_o \leq \check{Z}^*_o$ and $\hat{\theta}^*_o = \hat{Z}^*_o \leq \hat{H}^*_o$ are established for DMUs. Table 4.10 shows the ranking results, for Example 4.10. In summary, Example 4.10 showcases the application of fuzzy inputs and fuzzy outputs in assessing the efficiency of DMU_s. The use of asymmetric triangular fuzzy numbers adds a level of uncertainty representation in the analysis. The interval efficiency results obtained from different models highlight the variations in classification. The ranking of DMU_s based on the proposed approach provides decision-makers with insights into the relative performance of the units. Overall, this example demonstrates the potential of interval efficiency analysis in fuzzy decision-making contexts and its relevance in real-world scenarios.

Table 9. Saati et al.'s data (2002)

Data	*Input*1	*Input*2	*Output*1	*Output*2
$DMU1$	$(6, 7, 8)$	$(29, 30, 32)$	$(35.5, 38, 41)$	$(409, 411, 416)$
$DMU2$	$(5.5, 6, 6.5)$	$(33, 35, 36.5)$	$(39, 40, 43)$	$(478, 480, 484)$
$DMU3$	$(7.5, 9, 10.5)$	$(43, 45, 48)$	$(32, 35, 38)$	$(297, 299, 301)$
$DMU4$	$(7, 8, 10)$	$(37.5, 39, 42)$	$(28, 31, 31)$	$(347, 352, 360)$
$DMU5$	$(9, 11, 12)$	$(43, 44, 45)$	$(33, 35, 38)$	$(406, 411, 415)$
$DMU6$	$(10, 10, 10)$	$(53, 55, 57.5)$	$(36, 38, 40)$	$(282, 286, 289)$
$DMU7$	$(10, 12, 14)$	$(107, 110, 113)$	$(34.5, 36, 38)$	$(396, 400, 405)$
$DMU8$	$(9, 13, 16)$	$(95, 100, 101)$	$(37, 41, 46)$	$(387, 393, 402)$
$DMU9$	$(12, 14, 15)$	$(120, 125, 131)$	$(24, 27, 28)$	$(400, 404, 406)$
$DMU10$	$(5, 8, 10)$	$(35, 38, 39)$	$(48, 50, 51)$	$(470, 470, 470)$

Table 10. Interval efficiency Results by Saati et al.'s data (2002)

Data	$[\check{H}^*_o, \hat{H}^*_o]$	$[\check{\theta}^*_o, \hat{\theta}^*_o]$	$[\check{Z}^*_o, \hat{Z}^*_o]$
$DMU1$	$[0.9485, 1.0000]$	$[0.9369, 1.0000]$	$[1.0000, 1.0000]$
$DMU2$	$[1.0000, 1.0000]$	$[0.9451, 1.0000]$	$[1.0000, 1.0000]$
$DMU3$	$[0.5207, 0.7270]$	$[0.5207, 0.5996]$	$[0.5983, 0.5996]$
$DMU4$	$[0.6087, 0.7067]$	$[0.6087, 0.6573]$	$[0.6463, 0.6573]$
$DMU5$	$[0.6475, 0.7164]$	$[0.6475, 0.6716]$	$[0.6716, 0.6716]$
$DMU6$	$[0.4762, 0.6367]$	$[0.4762, 0.5220]$	$[0.5220, 0.5220]$
$DMU7$	$[0.3720, 0.5322]$	$[0.3720, 0.4555]$	$[0.4290, 0.4555]$
$DMU8$	$[0.3541, 0.6555]$	$[0.3541, 0.5332]$	$[0.4256, 0.5332]$
$DMU9$	$[0.3307, 0.4065]$	$[0.3307, 0.3716]$	$[0.3617, 0.3716]$
$DMU10$	$[0.9505, 1.0000]$	$[0.9274, 1.0000]$	$[1.0000, 1.0000]$

Table 11. compare and rank the interval efficiencies of the ten DMUs

DMU	$[S\check{Z}^*_o, S\hat{Z}^*_o]$	d^*	*Rank*
$DMU1$	$[1.0188, 1.0328]$	1.0258	3
$DMU2$	$[1.1593, 1.1593]$	1.1593	1
$DMU10$	$[1.0826, 1.1146]$	1.0986	2

5. DEA model with possibility and necessity measures

Simultaneously with the formation of fuzzy logic, mathematical theories are invented and developed to understand and identify aspects of uncertainty in the decision environment and its possible outcomes. In possibility programming models, each fuzzy parameter is considered as a fuzzy variable, and each fuzzy constraint is considered as a fuzzy event. In the possibility theory, the possibility degree of each fuzzy event is calculated. The possibility theory is closely related to the theory of random sets and confidence intervals in a way that provides the simplest and most widely used structure and framework for statistical reasoning with probability distributions. In addition, it is considered a tool to propagate uncertainty in problems with limited subjective information or statistics. In fact, possibility theory should be considered as the regulator between fuzzy and possible sets. The theory of possibility has received the attention of a large number of researchers and has been used in various and wide fields of science with uncertain and ambiguous data. For this reason, it has many relative advantages compared to other events in the face of uncertain and ambiguous data.

Despite the advantages that we have listed in using the theory of possibility in fuzzy logic, the use of the theory of possibility in fuzzy logic and because of one of the main challenges in this research, we tried to calculate the efficiency of decision-making units with fuzzy data, then NesPCCR models and NesDCCR is used for calculation. On the other hand, since the theory of possibility makes the problems more realistic, as a result, the combination of DEA models with fuzzy problems based on the theory of possibility will improve the performance and efficiency of other models.

In this article, considering that the theory of possibility makes problems more realistic, the combination of fuzzy DEA models based on the theory of possibility improves the performance and efficiency of fuzzy DEA models. In this section, we present a new possibility approach to solving fuzzy DEA models. This approach offers a solution to confront the uncertainty of fuzzy goals and constraints through possibility and necessity measures. The necessity (Nes) and possibility (Pos) measures are dual to each other. Using the concept of possibility and necessity of the fuzzy event, the fuzzy CCR model converts to the possibility and necessity CCR model. First, we describe the possibility and necessity CCR models for the primal CCR model:

(**PosCCR DEA**)

$$
\begin{aligned}
&\min \quad \theta \\
&s.t. \quad Pos(\sum_{j=1}^{n} \lambda_j \widetilde{x}_{ij} \leq \theta \widetilde{x}_{io}) \geq \beta_i \quad && i = 1, 2, \cdots, m, \\
&\qquad\; Pos(\sum_{j=1}^{n} \lambda_j \widetilde{y}_{rj} \geq \widetilde{y}_{ro}) \geq \alpha_r && r = 1, 2, \cdots, s, \\
&\qquad\; \lambda_j \geq 0 && j = 1, 2, \cdots, n, \\
&\qquad\; \theta \quad \text{free}.
\end{aligned} \tag{5.1}
$$

(**NesCCR DEA**)

$$
\begin{aligned}
&\min \quad \theta \\
&s.t. \quad Nes(\sum_{j=1}^{n} \lambda_j \widetilde{x}_{ij} - \theta \widetilde{x}_{io} \leq 0) \geq \beta_i \quad && i = 1, 2, \cdots, m, \\
&\qquad\; Pos(\sum_{j=1}^{n} \lambda_j \widetilde{y}_{rj} - \widetilde{y}_{ro} \geq 0) \geq \alpha_r && r = 1, 2, \cdots, s, \\
&\qquad\; \lambda_j \geq 0 && j = 1, 2, \cdots, n, \\
&\qquad\; \theta \quad \text{free}.
\end{aligned} \tag{5.2}
$$

In the above models, β_i and $\alpha_i (i = 1, \cdots, m; r = 1, \cdots, s)$ denote the minimum possibility level for the first and second constraints of the model 5.1, respectively. β_i and α_i present the minimum necessity level for the first and second constraints of the model 5.2, respectively.

The possibility and necessity CCR model for dual CCR is as follows:

(**PosCCR DEA**)

$$
\begin{aligned}
&\max \quad \overline{\phi} \\
&s.t. \quad Pos(\sum_{r=1}^{s} u_r \widetilde{y}_{ro} \geq \overline{\phi}) \geq \beta_o \\
&\qquad\; Pos(\sum_{r=1}^{m} v_i \widetilde{x}_{io} = 1) \geq \alpha_o && i = 1, 2, \cdots, m, \\
&\qquad\; Pos(\sum_{r=1}^{s} u_r \widetilde{y}_{rj} - \sum_{r=1}^{m} v_i \widetilde{x}_{ij} \leq 0) \geq \alpha_j \quad && j = 1, 2, \cdots, n, \\
&\qquad\; u_r \geq 0 && r = 1, 2, \cdots, s, \\
&\qquad\; v_i \geq 0 && i = 1, 2, \cdots, m,
\end{aligned} \tag{5.3}
$$

(**NesCCR DEA**)

$$
\begin{aligned}
(5.4)\quad & \max \quad \overline{\phi} \\
& s.t. \quad Nes(\sum_{r=1}^{s} u_r \widetilde{y}_{ro} \geq \overline{\phi}) \geq \beta_o \\
& \qquad Nes(\sum_{r=1}^{m} v_i \widetilde{x}_{io} = 1) \geq \alpha_o \qquad i = 1, 2, \cdots, m, \\
& \qquad Nes(\sum_{r=1}^{s} u_r \widetilde{y}_{rj} - \sum_{r=1}^{m} v_i \widetilde{x}_{ij} \leq 0) \geq \alpha_j \quad j = 1, 2, \cdots, n, \\
& \qquad u_r \geq 0 \qquad r = 1, 2, \cdots, s, \\
& \qquad v_i \geq 0 \qquad i = 1, 2, \cdots, m.
\end{aligned}
$$

In the above models, β_0, α_0, and α_j are the minimum possibility level for the first, second, and third constraints of the models 5.3 and 5.4, respectively. The interpretation of the PosCCR model is that the objective function $\overline{\phi}$ must be maximized so that the $\sum_{r=1}^{s} u_r \widetilde{y}_{ro}$ function can reach the minimum level of possibility β_o or higher so that the second and third constraints have the minimum level of possibility α_o, and α_j, respectively. The interpretation of the NesCCR model is also the same.

Because of the dual relationship between the possibility and necessity measures, we can solve the PosCCR and NesCCR models by implementing the same methods. Suppose that all of the fuzzy inputs and outputs are determined by $\widetilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij})$ and $\widetilde{y}_{ij} = (y^1_{ij}, y^2_{ij}, y^3_{ij})$ that for $i = 1, \cdots, m$ and $r = 1, \cdots, s, x^1_{ij} \geq 0$ and $y^1_{ij} \geq 0$. By using Lemmas 2.10 and 2.11 and given that the fuzzy inputs and outputs are the triangular fuzzy numbers, PosCCR and NesCCR models convert to the following linear programming models.

(**PosCCR DEA**)

(5.5)

$$
\begin{aligned}
& \min \quad \theta \\
& s.t. \quad \theta x^2_{io} \sum_{j=1}^{n} \lambda_j x^2_{ij} \leq (\beta_i - 1)(\sum_{j=1}^{n} \lambda_j (x^2_{ij} - x^1_{ij}) + \theta(x^3_{io} - x^2_{io}) \qquad i = 1, 2, \cdots, m, \\
& \qquad \sum_{j=1}^{n} \lambda_j y^2_{rj} - y^2_{ro} \geq (\alpha_r - 1)((y^3_{ro} - y^2_{ro}) + \sum_{j=1}^{n} \lambda_j (y^3_{rj} - y^2_{rj})) \quad r = 1, 2, \cdots, s, \\
& \qquad \lambda_j \geq 0 \qquad j = 1, 2, \cdots, n, \\
& \qquad \theta \quad \text{free.}
\end{aligned}
$$

(**PosCCR DEA**)

(5.6)

$$
\begin{aligned}
& \min \quad \theta \\
& s.t. \quad , \sum_{j=1}^{n} \lambda_j x^1_{ij} - \theta x^1_{io} + \beta_i (\sum_{j=1}^{n} \lambda_j (x^2_{ij} - x^1_{ij}) - \theta(x^2_{io} - x^1_{io}) \leq 0 \quad i = 1, 2, \cdots, m, \\
& \qquad \sum_{j=1}^{n} \lambda_j y^3_{rj} - y^3_{ro} - \alpha_r (\sum_{j=1}^{n} \lambda_j (y^3_{rj} - y^2_{rj}) - (y^3_{ro} - y^2_{ro}) \geq 0 \qquad r = 1, 2, \cdots, s, \\
& \qquad \lambda_j \geq 0 \qquad j = 1, 2, \cdots, n, \\
& \qquad \theta \quad \text{free.}
\end{aligned}
$$

(**NesCCR DEA**)

(5.7)

$$\begin{array}{lll} \min & \theta & \\ s.t. & ,\sum_{j=1}^{n}\lambda_j x_{ij}^2 - \theta x_{io}^2 + \beta_i(\sum_{j=1}^{n}\lambda_j(x_{ij}^3 - x_{ij}^2) - \theta(x_{io}^3 - x_{io}^2) \leq 0 & i = 1,2,\cdots,m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj}^2 - y_{ro}^2 - \alpha_r(\sum_{j=1}^{n}\lambda_j(y_{rj}^2 - y_{rj}^1) - (y_{ro}^2 - y_{ro}^1) \geq 0 & r = 1,2,\cdots,s, \\ & \lambda_j \geq 0 & j = 1,2,\cdots,n, \\ & \theta \quad \text{free.} & \end{array}$$

For the dual CCR, we have the above approach:

(**PosCCR DEA**)

$$(5.8)\quad \begin{array}{ll} \max & \overline{\phi} \\ s.t. & \sum_{r=1}^{s} u_r y_{ro}^3 - \beta_o(\sum_{r=1}^{s} u_r(y_{ro}^3 - y_{ro}^2)) \leq \overline{\phi} \\ & \sum_{i=1}^{m} v_i x_{io}^1 + \alpha_o(\sum_{i=1}^{m} v_i(x_{io}^2 - x_{io}^1)) \leq 1 \ i = 1,2,\cdots,m, \\ & \sum_{i=1}^{m} v_i x_{io}^3 - \alpha_o(\sum_{i=1}^{m} v_i(x_{io}^3 - x_{io}^2)) \geq 1 \ i = 1,2,\cdots,m, \\ & \sum_{r=1}^{s} u_r y_{rj}^1 - \sum_{i=1}^{m} v_i x_{ij}^1 + \alpha_j(\sum_{r=1}^{s} u_r(y_{rj}^2 - y_{rj}^1) - \sum_{i=1}^{m} v_i(x_{ij}^2 - x_{ij}^1)) \leq 0 \\ & u_r \geq 0 \ r = 1,2,\cdots,s, \\ & v_i \geq 0 \ i = 1,2,\cdots,m, \end{array}$$

(**NesCCR DEA**)

$$(5.9)\quad \begin{array}{ll} \max & \overline{\phi} \\ s.t. & \sum_{r=1}^{s} u_r y_{ro}^2 - \beta_o(\sum_{r=1}^{s} u_r(y_{ro}^2 - y_{ro}^1)) \leq \overline{\phi} \\ & \sum_{i=1}^{m} v_i x_{io}^2 + \alpha_o(\sum_{i=1}^{m} v_i(x_{io}^3 - x_{io}^2)) \leq 1 \ i = 1,2,\cdots,m, \\ & \sum_{i=1}^{m} v_i x_{io}^2 - \alpha_o(\sum_{i=1}^{m} v_i(x_{io}^2 - x_{io}^1)) \geq 1 \ i = 1,2,\cdots,m, \\ & \sum_{r=1}^{s} u_r y_{rj}^2 - \sum_{i=1}^{m} v_i x_{ij}^2 + \alpha_j(\sum_{r=1}^{s} u_r(y_{rj}^3 - y_{rj}^2) - \sum_{i=1}^{m} v_i(x_{ij}^3 - x_{ij}^2)) \leq 0 \\ & u_r \geq 0 \ r = 1,2,\cdots,s, \\ & v_i \geq 0 \ i = 1,2,\cdots,m, \end{array}$$

This article introduces a new approach to solving fuzzy Data Envelopment Analysis (DEA) models by combining fuzzy DEA models with the theory of possibility. This combination enhances realism and improves the performance and efficiency of fuzzy DEA models. The proposed approach addresses the uncertainty associated with fuzzy goals and constraints through the use of possibility and necessity measures. The necessity and possibility measures are duals of each other, providing a comprehensive framework for dealing with fuzzy events. By employing the concept of possibility and necessity, the traditional fuzzy CCR model is transformed into a possibility and necessity CCR model.

The article presents linear programming models for the possibility and necessity CCR models, both for primal and dual CCR. These models involve minimum possibility levels (β) and minimum necessity levels (α) as constraints, which ensure that the fuzzy constraints are satisfied at the specified levels. The objective is to optimize the objective function (θ orϕ) while meeting the given constraints.

Moreover, the article demonstrates that the possibility and necessity CCR models can be solved using the same methods due to the dual relationship between the possibility and necessity measures. They provide specific linear programming models for solving PosCCR and NesCCR models when the fuzzy inputs and outputs are represented by triangular fuzzy numbers.

Overall, this article introduces a novel approach that incorporates the theory of possibility into fuzzy DEA models, leading to more realistic and efficient solutions. By addressing the uncertainty inherent in fuzzy goals and constraints, this approach offers a valuable contribution to the field of DEA. Researchers and practitioners interested in enhancing the performance and efficiency of fuzzy DEA models should consider exploring the possibilities offered by this innovative approach.

5.1. **Numerical example.** We explain the proposed Method with a numerical example. Data are presented in Table 4.8 [28], which includes five DMU_s with two fuzzy inputs and two fuzzy outputs. The fuzzy inputs and outputs are considered symmetric triangular fuzzy numbers, a special form of triangular fuzzy numbers. Now, we want to examine the solution of the DEA model. In this example, all fuzzy constraints in primal and dual models are considered with the same level of possibility and necessity at five levels $(0, 0.25, 0.5, 0.75, 1)$. The results can be interpreted as follows:

Table 12. Tanaka and Gue data (2001)

DMU	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
$Input1$	$(3.5, 4.0, 4.5)$	$(2.9, 2.9, 2.9)$	$(4.4, 4.9, 5.4)$	$(3.4, 4.1, 4.8)$	$(5.9, 6.5, 7.1)$
$Input2$	$(1.9, 2.1, 2.3)$	$(1.4, 1.5, 1.6)$	$(2.2, 2.6, 3.0)$	$(2.2, 2.3, 2.4)$	$(3.6, 4.1, 4.6)$
$Output1$	$(2.4, 2.6, 2.8)$	$(2.2, 2.2, 2.2)$	$(2.7, 3.2, 3.7)$	$(2.5, 2.9, 3.3)$	$(4.4, 5.1, 5.8)$
$Output2$	$(3.8, 4.1, 4.4)$	$(3.3, 3.5, 3.7)$	$(4.3, 5.1, 5.9)$	$(5.5, 5.7, 5.9$	$(6.5, 7.4, 8.3)$

Table 5.1 presents the efficiency results of five DMU with the model (5.1) or the possibility of multiple model at different levels. At a possibility level of 0.75, the efficiency values of DMU1's and DMU2's possibility are 0.9024 and 1.0214, respectively. This means that with the possibility level of 0.75, DMU1 is an inefficient DMU, while DMU2 is an efficient DMU. In Table 5.1, DMU2, DMU4, and DMU5 are efficient DMUs, while DMU1 and DMU3 are inefficient DMUs at some of the possibility levels.

Table 5.1 shows the efficiency results for the possibility envelopment model (model (5.1)). The efficiency results of this section reveal that DMU2, DMU4, and DMU5, like for the possibility of multiple model, are efficient DMUs at all levels. It shows which DMUs are efficient at different possibility levels.

Table 13. Results of efficiency values with the possibility of multiple model

α	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
0	1.0580	1.0994	1.1256	1.2961	1.2958
0.25	1.0035	1.0716	1.0467	1.2053	1.2256
0.5	0.9518	1.0455	0.9828	1.1257	1.1474
0.75	0.9024	1.0214	0.9212	1.0570	1.0711
1	0.8556	1.0000	0.8613	1.0000	1.0000

Table 14. Results of efficiency values with possibility envelopment model

α	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
0	0.9030	1.0000	1.0000	1.0000	1.0000
0.25	0.9216	1.0000	1.0000	1.0000	1.0000
0.5	0.9126	1.0000	0.9587	1.0000	1.0000
0.75	0.8892	1.0000	0.9094	1.0000	1.0000
1	0.8556	1.0000	0.8613	1.0000	1.0000

Due to the dual relationship between possibility and necessity measures, we solved the NesPCCR and NesDCCR models. The efficiency of DMUs can be defined by the same Method using the possibility approach. Since the tips are repeated here, we have removed the details. Table 5.1 shows the efficiency results of the necessity multiple model (model (5.1)) using the data from Table 5.1. In this Table, DMU2, DMU4, and DMU5 have necessity efficiency at some of the necessity levels, while DMU1 and DMU3 do not have any necessity efficiency at all levels. Table 5.1 shows the efficiency results for the necessity envelopment model (model 5.2) at different levels. The results of the necessity approach are the same as the possibility approach. That is, as the necessity level increases, the values of DMUs decrease. Also, the efficiency value for the necessity approach is less than it for the possibility approach.

Table 15. Results of efficiency values with necessity multiple model

α	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
0	0.8556	1.0000	0.8613	1.0000	1.0000
0.25	0.8514	1.0000	0.8398	0.9839	0.9369
0.5	0.8215	1.0000	0.8190	0.9668	0.8740
0.75	0.8090	0.9934	0.7901	0.9497	0.8370
1	0.7818	0.9645	0.7304	0.9329	0.7644

The motivation for the reader lies in understanding and interpreting the efficiency results of the DMUs using the proposed Method. By studying these results, readers can gain insights into the efficiency levels of the DMUs and observe how the values change across different levels of possibility and necessity. This analysis provides a deeper understanding of the proposed Method's capabilities and its potential application in evaluating the efficiency of DMUs in various contexts.

Table 16. Results of efficiency values with necessity envelopment model

α	$DMU1$	$DMU2$	$DMU3$	$DMU4$	$DMU5$
0	0.8556	1.0000	0.8613	1.0000	1.0000
0.25	0.8258	1.0000	0.8138	1.0000	0.9762
0.5	0.8129	1.0000	0.7722	1.0000	0.9208
0.75	0.8003	1.0000	0.73424	1.0000	0.8677
1	0.7878	1.0000	0.6989	1.0000	0.8169

The decision on which DEA models based on possibility and necessity approaches to use depends on several factors, including the specific objectives of the analysis, the nature of the decision problem, and the preferences of the decision-makers. Here's a breakdown of considerations for managers.

1. Possibility Approach:

· Use the possibility-based DEA models when the focus is on identifying opportunities for improvement or efficiency enhancement.

· This approach is suitable for situations where decision-makers are more concerned about maximizing potential performance or exploring the boundaries of what is achievable.

· Managers might prefer this approach when they have flexibility to explore various scenarios and strategies for performance enhancement.

2. Necessity Approach:

· Employ necessity-based DEA models when the emphasis is on identifying essential constraints or minimum requirements for performance.

· This approach is appropriate for situations where decision-makers need to ensure compliance with certain standards or regulations, or where resources are limited and need to be allocated efficiently.

· Managers might favor this approach when they need to identify critical factors that must be addressed to avoid inefficiencies or failures.

3. Combined Use:

· Consider using both possibility and necessity-based DEA models in tandem to gain a comprehensive understanding of performance and resource allocation.

· The combined use of these approaches can provide a balanced perspective, highlighting both opportunities for improvement and critical constraints.

· Managers might opt for this approach when they need a holistic view of performance that takes into account both aspirational goals and practical constraints.

Ultimately, the choice between possibility and necessity-based DEA models, or their combination, should be guided by the specific context and objectives of the managerial decision-making problem. Managers should carefully evaluate the trade-offs and implications of each approach and select the most appropriate model or combination of models to support their decision-making process.

6. Conclusion

The current article is presented with a suggested approach to complete the previous articles and to obtain more detailed information or a different perspective on the issues and results. Due to the lack of complete knowledge and information, precise mathematics is not enough for modeling a complex system. In the current article, the DEA model with fuzzy coefficients is taken into account. We propose an approach to transform this problem into an interval linear programming model based on the approximation of the nearest interval. Therefore, the fuzzy DEA model is transformed into two conventional DEA models according to the technique used, which shows the efficiency as the interval efficiency. The proposed approach demonstrates its effectiveness through a comparison with other models, showcasing its robustness and practicality. Also, we compared the proposed approach with the other models, and a symptomatic interval-based ranking method is presented, which makes the approach to be more robust and practical.

Additionally, the article explores the possibility approach for solving fuzzy DEA, defining fuzzy events through possibility and necessity measures. The comparison between the possibility and necessity approaches reveals their distinct aspects, with the necessity approach offering a more pessimistic efficiency value, while the possibility approach presents a more optimistic perspective for each DMU. This dual relationship between possibility and necessity measures enhances our understanding of efficiency in fuzzy DEA models. In this study, when the fuzzy inputs and outputs data are fuzzy L-R and trapezoidal numbers, the fuzzy DEA models are converted to linear programming models with the minimum level of possibility and necessity. We compare the results of the possibility approach for primal-CCR and dual-CCR models with the necessity approach for primal-CCR and dual-CCR models. As the results show, some DMU is efficient with the possibility approach, while others are inefficient with the necessity approach. This reveals that the necessity approach offers the efficiency value, which is the pessimistic aspect for each DMU, and the possibility approach offers the efficiency value, which is the optimistic aspect for each DMU. In other words, due to the dual relationship between the possibility and necessity measures, the efficiency value for the necessity approach is less than for the possibility approach.

By considering both the technical transformations and the philosophical implications of the proposed approaches, this article paves the way for future research in the field of modeling complex systems. It motivates readers to embrace a multidimensional perspective that encompasses uncertainty, imprecision, and the inherent duality of possibility and necessity. Ultimately, this article encourages researchers and practitioners to adopt innovative methodologies that can address the challenges posed by complex systems, leading to more accurate and comprehensive analyses in various domains.

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