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Quasi-interior ideals and fuzzy quasi-interior ideals of Γ-semigroups

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Abstract. In this paper, as a further generalization of ideals, we introduce the notion of a quasi interior ideal as a generalization of ideals, quasi ideals, bi-ideals and interior ideals of Γ-semigroups and study their properties. We characterize the quasi-interior simple Γ-semigroup and regular Γ-semigroup using quasi-interior ideals of Γ-semigroups. We introduce the notion of fuzzy quasi-interior ideals of a Γ-semigroup and study fuzzy quasi-interior ideals properties of a Γ-semigroup.

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1. Introduction

As a generalization of a ring, the notion of a Γ-ring was introduced by Nobusawa [1] in 1964. The notion of a ternary ring was introduced by Lehmer in 1932. In 1995, Murali Krishna Rao [2] introduced the notion of a Γ-semiring as a generalization of Γ-ring, ring, ternary semiring and semiring. Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. In 1981, Sen [3] introduced the notion of a Γ-semigroup as a generalization of a semigroup.

Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory and the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. In 1952, the concept of bi-ideals was introduced by Good and Hughes [4] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [5]. Quasi-ideals are generalization of right ideals and left ideals whereas bi-ideals are generalization of quasi ideals. Steinfeld [6] first introduced the notion of quasi ideals for semigroups and then for rings. Iséki [7] introduced the concept of quasi ideal for semiring.

The fuzzy set theory was developed by Zadeh [8] in 1965. The fuzzification of algebraic structure was introduced by Rosenfeld [9] and he introduced the notion of fuzzy subgroups in 1971. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. Kuroki studied fuzzy interior ideals in semigroups. Swamy and Swamy [10] studied fuzzy prime ideals in rings in 1988. In 1982, Liu [11] defined and studied fuzzy subrings as well as fuzzy ideals in rings. The author studied bi-quasi ideals, fuzzy bi-quasi ideals, bi-interior ideals and fuzzy bi-interior ideals of Γ-semigroups and quasi-interior ideals and fuzzy quasi-interior ideals of semigroups[12, 13, 14, 15]. In this paper, we introduce the notions of quasi-interior ideal and fuzzy quasi-interior ideal of a Γ-semigroup and study their properties.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1 ([13])**.** Let M and Γ be non-empty subsets. Then we call M a Γ-*semigroup*, if there exists a mapping $M \times \Gamma \times M \to M$ (the image of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x,\ y,\ z \in M$ and $\alpha,\ \beta \in \Gamma$.

Definition 2.2 ([3])**.** Let M be a Γ-semigroup an A be a non empty subset of M. AΓA=$\{x\alpha y | x, y \in A, \alpha \in \Gamma\}$ and AΓMΓA=$\{x\alpha y\beta z | x, z \in A, y \in M, \alpha, \beta \in \Gamma\}$.

Definition 2.3 ([13])**.** A non-empty subset A of Γ-semigroup M is called:

(i) a Γ-*subsemigroup* of M, if $A\Gamma A \subseteq A$,
(ii) a *quasi ideal* of M, if $A\Gamma M \cap M\Gamma A \subseteq A$,
(iii) a *bi-ideal* of M, if $A\Gamma M\Gamma A \subseteq A$,
(iv) an *interior ideal* of M, if $M\Gamma A\Gamma M \subseteq A$,
(v) a *left* (*right*) *ideal* of M, if $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$,
(vi) an *ideal* of M, if $A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Definition 2.4 ([13])**.** Let M be a Γ-semigroup. An element $1 \in M$ is said to be *unity*, if for each $x \in M$, there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.5 ([13])**.** A Γ-semigroup M is said to be *left* (*right*) *singular*, if for each $a \in M$, there exists $\alpha \in \Gamma$ such that $a\alpha b = a(a\alpha b = b)$ for all $b \in M$.

Definition 2.6 ([13])**.** A Γ-semigroup M is said to be *commutative*, if it satisfies the following conditions:

$$a\alpha b = b\alpha a \text{ for all } a,\ b \in M \text{ and all } \alpha \in \Gamma.$$

Definition 2.7 ([13])**.** Let M be a Γ-semigroup. An element $a \in M$ is said to be an *idempotent* of M, if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$, then a is called an α-idempotent.

Definition 2.8 ([13])**.** Let M be a Γ-semigroup. If every element of M is an idempotent of M, then M is said to be *band*.

Definition 2.9 ([13])**.** Let M be a Γ-semigroup. An element $a \in M$ is said to be a *regular element* of M, if there exist $x \in M,\ \alpha,\ \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.10 ([13])**.** Let M be a Γ-semigroup. Every element of M is a regular element of M, then M is said to be a *regular* Γ*-semigroup*.

Definition 2.11 ([15])**.** Let M be a non-empty set. A mapping $\mu : M \to [0,1]$ is called a *fuzzy subset* of M.

Definition 2.12 ([15])**.** Let μ be a fuzzy subset of M and $t \in [0,1]$. Then the set

$$\mu_t = \{x \in M \mid \mu(x) \geq t\}$$

is called a *level subset* of M with respect to μ.

Definition 2.13 ([15])**.** A fuzzy subset μ is called a non-empty fuzzy subset if μ is not a constant function.

Definition 2.14 ([15])**.** For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$.

3. Quasi-interior ideals of Γ-semigroups

In this section, we introduce the notion of a quasi-interior ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of a Γ- semigroup and study the properties of quasi-interior ideal of a Γ-semigroup.

Definition 3.1. A non-empty subset B of a Γ-semigroup M is said to be *left quasi-interior ideal* of M, if B is a Γ-sub semigroup of M and $M\Gamma B\Gamma M\Gamma B \subseteq B$.

Definition 3.2. A non-empty subset B of a Γ-semigroup M is said to be a *right quasi-interior ideal* of M, if B is a Γ-sub semigroup of M and $B\Gamma M\Gamma B\Gamma M \subseteq B$.

Definition 3.3. A non-empty subset B of a Γ-semigroup M is said to be a *quasi-interior ideal* of M, if B is a Γ-sub semigroup of M and B is a left and right quasi-interior ideal of M.

Remark 3.4. Every quasi-interior ideal of a Γ-semigroup M need not be quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of M.

Example 3.5. Let N be a the set of all natural numbers and $\Gamma = \mathcal{N}$ be additive abelian semigroups. Ternary operation is defined as $(x, \alpha, y) \to x + \alpha + y$, where $+$ is the usual addition of integers. Then N is an Γ-semigroup. Let I be the set of all odd natural numbers. Then I is a quasi-interior ideal of N but not quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of N.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 3.6. *Let M be a Γ-semigroup.*

(1) *Every left ideal is a left quasi-interior ideal of M.*

(2) *Every right ideal is a right quasi-interior ideal of M.*

(3) *Every quasi ideal is a quasi-interior ideal of M.*

(4) *Every ideal is a quasi-interior ideal of M.*

(5) *The intersection of a right ideal and a left ideal of M is a quasi-interior ideal of M.*

(6) *If L is a left ideal and R is a right ideal of Γ-semigroup M, then $B = R\Gamma L$ is a quasi-interior ideal of M.*

(7) *If B is a quasi-interior ideal and T is a Γ-subsemigroup of M, then $B \cap T$ is a quasi-interior ideal of M.*

(8) *If B is a Γ-subsemigroup of M and $M\Gamma M\Gamma M\Gamma B \subseteq B$, then B is a left quasi-interior ideal of M.*

(9) *If B is a Γ-subsemigroup of M and $M\Gamma M\Gamma M\Gamma B \subseteq B$ and $B\Gamma M\Gamma M\Gamma M \subseteq B$, then B is a quasi-interior ideal of M.*

(10) *The intersection of a right quasi-interior ideal and a left quasi-interior ideal of M is a quasi-interior ideal of M.*

(11) *If L is a left ideal and R is a right ideal of M, then $B = R \cap L$ is a quasi-interior ideal of M.*

Theorem 3.7. *If B be a left quasi-interior ideal of a Γ-semigroup M, then B is a left bi-quasi ideal of M.*

Proof. Suppose B is a left quasi-interior ideal of M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Thus $B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B$. So we get

$$M\Gamma B \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B.$$

Hence B is a left bi-quasi ideal of M. □

Theorem 3.8. *If B is a left quasi-interior ideal of a Γ-semigroup M, then B is a bi-interior ideal of M.*

Proof. Suppose B is a left quasi-interior ideal of M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Thus we have

$$M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B.$$

Hence, B is a bi-interior ideal of M. □

Theorem 3.9. *Every left quasi-interior ideal of a Γ-semigroup M is a bi-ideal of M.*

Proof. Let B be a left quasi-interior ideal of M. Then $B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$. Thus $B\Gamma M\Gamma B \subseteq B$. Hence, B is a bi-ideal of M. □

Theorem 3.10. *Every left quasi-interior ideal of a Γ-semigroup M is a bi-quasi interior ideal of M.*

Proof. Let B be a left quasi-interior ideal of M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Thus $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$. Hence, B is a bi-quasi interior ideal of M. □

Theorem 3.11. *Every interior ideal of a Γ-semigroup M is a left quasi-interior ideal of M.*

Proof. Let I be an interior ideal of M. Then $M\Gamma I\Gamma M\Gamma I \subseteq M\Gamma I\Gamma M \subseteq I$. Thus I is a left quasi-interior ideal of M. □

Theorem 3.12. *Let M be a Γ-semigroup and B be a Γ-subsemigroup of M. Then B is a quasi-interior ideal of M if and only if there exist left ideals L and R such that $R\Gamma L \subseteq B \subseteq R \cap L$.*

Proof. Suppose B is a quasi-interior ideal of M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Let $R = M\Gamma B$ and $L = M\Gamma B$. Then L and R are left ideals of M. Thus $R\Gamma L \subseteq B \subseteq R\cap L$.

Conversely, suppose that there exist L and R are left ideals of M such that $R\Gamma L \subseteq B \subseteq R \cap L$. Then we have

$$\begin{aligned} M\Gamma B\Gamma M\Gamma B &\subseteq M\Gamma(R\cap L)\Gamma M\Gamma(R\cap L) \\ &\subseteq M\Gamma(R)\Gamma M\Gamma(L) \\ &\subseteq R\Gamma L \\ &\subseteq B. \end{aligned}$$

Thus B is a left quasi-interior ideal of M. □

Theorem 3.13. *The intersection of a left quasi-interior ideal B of Γ-semigroup M and a left ideal A of M is always a left quasi-interior ideal of M.*

Proof. Let B be a left quasi interior ideal, A be a left ideal of M and C=B ∩ A.
$M\Gamma C\Gamma M\Gamma C \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$ and $M\Gamma C\Gamma M\Gamma C \subseteq M\Gamma A\Gamma M\Gamma A \subseteq A$.
Then, $M\Gamma C\Gamma M\Gamma C \subseteq B \cap A = C$. Thus C is a left quasi-interior ideal of M. □

Theorem 3.14. *Let M be a Γ-semigroup and T be a non-empty subset of M. If B is a Γ-subsemigroup of M containing $M\Gamma T\Gamma M\Gamma T$ and $B \subseteq T$, then is a left quasi-interior ideal of M.*

Proof. Suppose B is a Γ-subsemigroup of M containing $M\Gamma T\Gamma M\Gamma T$. Then

$$\begin{aligned} M\Gamma B\Gamma M\Gamma B &\subseteq M\Gamma T\Gamma M\Gamma T \\ &\subseteq B. \end{aligned}$$

Thus B is a left quasi-interior ideal of M. □

Theorem 3.15. *B is a left quasi-interior ideal of a Γ-semigroup M if and only if B is a left ideal of some ideal of M.*

Proof. Suppose B is a left ideal of some ideal R of M. Then $R\Gamma B \subseteq B, M\Gamma R \subseteq R$. Thus $M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma R\Gamma M\Gamma B \subseteq R\Gamma M\Gamma B \subseteq R\Gamma B \subseteq B$. So B is a left quasi-interior ideal of M.

Conversely, suppose that B is a left quasi-interior ideal of M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Thus B is a left ideal of an ideal $M\Gamma B\Gamma M$ of M. □

Theorem 3.16. *If B is a left quasi-interior ideal of a Γ-semigroup M and I is an interior ideal of M, then $B \cap I$ is a left quasi-interior ideal of M.*

Proof. Suppose B is a bi-ideal of M and I is an interior ideal of M. Obviously, $B \cap I$ is Γ-sub semigroup of M. Then

$$M\Gamma(B \cap I)\Gamma M\Gamma(B \cap I) \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B,$$
$$M\Gamma(B \cap I)\Gamma M\Gamma(B \cap I) \subseteq M\Gamma I\Gamma M\Gamma I \subseteq I.$$

Thus $M\Gamma(B\cap I)\Gamma M\Gamma(B\cap I) \subseteq B\cap I$. So $B\cap I$ is a left quasi-interior ideal of M. □

Theorem 3.17. *Let M be a Γ-semigroup and T be a Γ-sub semigroup of M. Then every Γ-sub semigroup of T containing $T\Gamma M\Gamma T\Gamma M\Gamma T$ is a left quasi-interior ideal of M.*

Proof. Let C be a Γ-sub semigroup of T containing $T\Gamma M\Gamma T\Gamma M\Gamma T$. Then

$$M\Gamma C\Gamma M\Gamma C \subseteq M\Gamma T\Gamma M\Gamma T \subseteq C.$$

Thus C is a quasi-interior ideal of M. □

Theorem 3.18. *Let $\{B_\lambda \mid \lambda \in \Lambda\}$ be a collection of left quasi-interior ideals of a Γ-semigroup M. Then $\bigcap_{\lambda\in\Lambda} B_\lambda$ is a left quasi-interior ideal of M, where Λ is an index set.*

Proof. Let $B = \bigcap\limits_{\lambda\in\Lambda} B_\lambda$. For each B_λ, $B_\lambda\Gamma B_\lambda \subseteq B_\lambda$, then B is a $\Gamma-$sub semigroup of M. Since each B_λ is a left quasi-interior ideal of M, we have

$$\begin{aligned} &M\Gamma B_\lambda\Gamma M\Gamma B_\lambda \subseteq B_\lambda \\ \Rightarrow &M\Gamma \cap B_\lambda\Gamma M \cap B_\lambda) \subseteq \cap B_\lambda \\ \Rightarrow &M\Gamma B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Thus B is a left quasi-interior ideal of M. □

Theorem 3.19. *Let B be a left quasi-interior ideal of Γ-semigroup M, $e \in B$ and e be β-idempotent. Then $e\Gamma B$ is a left quasi-interior ideal of M.*

Proof. Let B be a left quasi-interior ideal of M and $x \in B \cap e\Gamma M$. Then $x \in B$ and $x = e\alpha y, \alpha \in \Gamma, y \in M$. On the other hand, we have

$$\begin{aligned} x &= e\alpha y \\ &= e\beta e\alpha y \\ &= e\beta(e\alpha y) \\ &= e\beta x \in e\Gamma B. \end{aligned}$$

Thus we get

$$\begin{aligned} &B \cap e\Gamma M \subseteq e\Gamma B \\ &e\Gamma B \subseteq B \text{ and } e\Gamma B \subseteq e\Gamma M \\ \Rightarrow &e\Gamma B \subseteq B \cap e\Gamma M \\ \Rightarrow &e\Gamma B = B \cap e\Gamma M. \end{aligned}$$

So $e\Gamma B$ is a left quasi-interior ideal of M. □

Theorem 3.20. *Let M be a Γ-semigroup. If $M = M\Gamma a$ for all $a \in M$, then every left quasi-interior ideal of M is a quasi ideal of M.*

Proof. Let B be a left quasi-interior ideal of M and $a \in B$. Then

$$\begin{aligned}
&M\Gamma B\Gamma M\Gamma B \subseteq B\\
\Rightarrow &M\Gamma a \subseteq M\Gamma B,\\
\Rightarrow &M \subseteq M\Gamma B \subseteq M\\
\Rightarrow &M\Gamma B = M\\
\Rightarrow &B\Gamma M = B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B\\
\Rightarrow &M\Gamma B \cap B\Gamma M \subseteq M \cap B\Gamma M \subseteq B\Gamma M \subseteq B.
\end{aligned}$$

Thus B is a left quasi ideal of M. So the proof is complete. □

Definition 3.21. A Γ-semigroup M is called a *left* (*right*) *simple* Γ*-semigroup*, if M has no proper left (right) ideals of M.

Definition 3.22. A Γ-semigroup M is said to be a *simple* Γ*-semigroup*, if M has no proper ideals of M.

Definition 3.23. A Γ-semigroup M is said to be a *left quasi-interior simple* Γ*-semigroup*, if M has no left quasi-interior ideal other than M itself.

Theorem 3.24. *If M is a Γ-group, then M is a left quasi-interior simple Γ-semigroup.*

Proof. Suppose M is a Γ-group and B is a proper left quasi-interior ideal of M, $x \in M$ and $0 \neq a \in B$. Since M is a Γ-group, there exist $b \in M$, $\alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exist $\beta \in \Gamma$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Thus $x \in B\Gamma M$ and $M \subseteq B\Gamma M$, i.e., $B\Gamma M \subseteq M$. So $M = B\Gamma M$. Similarly, we can prove $M\Gamma B = M$. On the other hand, we get

$$\begin{aligned}
M &= M\Gamma B = M\Gamma B\Gamma B = M\Gamma B\Gamma B\Gamma B\\
&\subseteq M\Gamma B\Gamma M\Gamma B \subseteq B.
\end{aligned}$$

Since $B \subseteq M$, $M = B$. Hence M has no proper left-quasi-interior ideals. □

Theorem 3.25. *Let M be a simple Γ-semigroup. Every left quasi-interior ideal is a left ideal of M.*

Proof. Let M be a simple Γ-semigroup and B be a left quasi-interior ideal of M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$ and $M\Gamma B\Gamma M$ is an ideal of M. Since M is a simple Γ-semigroup, we have $M\Gamma B\Gamma M = M$. Thus $M\Gamma B\Gamma M\Gamma B \subseteq B$. So $M\Gamma B \subseteq B$. Hence B is a left ideal of M. □

Theorem 3.26. *Let M be a Γ-semigroup. Then M is a left quasi-interior simple Γ-semigroup if and only if $< a >= M$ for all $a \in M$, where $< a >$ is the smallest left quasi-interior ideal generated by a.*

Proof. Suppose M is a left quasi-interior simple Γ-semigroup, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. By Theorem 3.6, B is a left quasi-interior ideal of M. Thus $M\Gamma a \subseteq< a >\subseteq M$. So $M \subseteq< a >\subseteq M$. Hence $M =< a >$.

Conversely, suppose that $< a >$ is the smallest left quasi interior ideal of M generated by a and $< a > = M$, A is a left quasi interior ideal of M and $a \in A$. Then $< a > \subseteq A \subseteq M$. Thus $M \subseteq A \subseteq M$. So $A = M$. Hence M is a left quasi interior simple Γ-semigroup. □

Theorem 3.27. *Let M be a Γ-semigroup. Then M is a left quasi-interior simple Γ-semrgroup if and only if $M\Gamma a\Gamma M\Gamma a = M$ for all $a \in M$.*

Proof. Suppose M is left-quasi interior simple Γ-semigroup and $a \in M$. Then $M\Gamma a\Gamma M\Gamma a$ is a quasi-interior ideal of M. Thus $M\Gamma a\Gamma M\Gamma a = M$ for all $a \in M$.

Conversely, suppose that $M\Gamma a\Gamma M\Gamma a = M$ for all $a \in M$. Let B be a left quasi-interior ideal of Γ-semigroup M and $a \in B$. Then we have

$$M = M\Gamma a\Gamma M\Gamma a \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B.$$

Thus $M = B$. So M is a left quasi-interior simple Γ-semigroup. □

Theorem 3.28. *If a Γ-semigroup M is a left simple Γ-semigroup, then every left quasi-interior ideal of M is a right ideal of M.*

Proof. Let B be a left quasi-interior ideal of left simple Γ-semigroup. Then $M\Gamma B$ is a left ideal of M and $M\Gamma B \subseteq M$. Thus $M\Gamma B = M$. So we have

$$\begin{aligned} &\Rightarrow B\Gamma M = B\Gamma M\Gamma B \\ &\Rightarrow \ \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B \\ &\Rightarrow B\Gamma M \subseteq B. \end{aligned}$$

Hence B is a right ideal of M. □

Theorem 3.29. *Let M be a Γ-semigroup and B be a left quasi-interior ideal of M. Then B is a minimal left quasi-interior ideal of M if and only if B is a left quasi-interior simple Γ-sub semigroup of M.*

Proof. Suppose B is a minimal left quasi-interior ideal of M and C is a left quasi-interior ideal of B. Then $B\Gamma C\Gamma B\Gamma C \subseteq C$ and $B\Gamma C\Gamma B\Gamma C$ is a left quasi-interior ideal of M. Since C is a quasi-interior ideal of B,

$$\begin{aligned} &B\Gamma C\Gamma B\Gamma C = B \\ &\Rightarrow B = B\Gamma C\Gamma B\Gamma C \subseteq C \\ &\Rightarrow B = C. \end{aligned}$$

Thus B is a left quasi-interior simple Γ-sub semigroup of M.

Conversely, suppose that B is a left quasi-interior simple Γ-sub semigroup of M. Let C be a left quasi-interior ideal of M and $C \subseteq B$. Then B is a minimal left quasi-interior ideal of M. □

Theorem 3.30. *Let M be a Γ-semigroup and $B = L\Gamma L$, where L is a minimal left ideal of M. Then B is a minimal left quasi-interior ideal of M.*

Proof. Obviously, $B = L\Gamma L$ is a left quasi-interior ideal of M. Let A be a left quasi-interior ideal of M such that $A \subseteq B$. Then clearly, $M\Gamma A$ is a left ideal of M. Since L is a left ideal of M, we have

$$M\Gamma A \subseteq M\Gamma B = M\Gamma L\Gamma L \subseteq L.$$

Thus $M\Gamma A = L$. So $B = M\Gamma A\Gamma M\Gamma A \subseteq A$, i.e., $B = A$. Hence B is a minimal left quasi-interior ideal of M. Therefore $M\Gamma A = L$. □

4. Left quasi-interior ideals of regular Γ-semigroup

In this section, we characterize regular Γ-semigroup using left quasi-interior ideals of a Γ-semigroup.

Theorem 4.1. *Let M be a regular Γ-semigroup. Every left quasi-interior ideal of M is a left ideal of M.*

Proof. Let B be a left quasi interior ideal of a regular Γ-semigroup M. Then M $\subseteq$ M Γ B Γ M. Thus MΓB $\subseteq$ MΓB Γ M Γ B $\subseteq$ B. So B is a left ideal of M □

Theorem 4.2. *Let M be a Γ-semigroup. M is a regular Γ-semigroup if and only if $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of M.*

Theorem 4.3. *Let M be a Γ-semigroup. Then M* is regular *if and only if $B\Gamma M\Gamma B\Gamma M = B$ and $M\Gamma B\Gamma M\Gamma B = B$ for all quasi-interior ideals B of M.*

Proof. Suppose M is regular and let B be a quasi-interior ideal of M and $x \in B$. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$ and there exist $y \in M,\ \alpha, \beta \in$ such that $x = x\alpha y\beta x\alpha y\beta x \in M\Gamma B\Gamma M\Gamma B$. Thus $x \in M\Gamma B\Gamma M\Gamma B$. So $M\Gamma B\Gamma M\Gamma B = B$. Similarly. we can prove $B\Gamma M\Gamma B\Gamma M = B$.

Conversely, suppose that $B\Gamma M\Gamma B\Gamma M = B$ and $M\Gamma B\Gamma M\Gamma B = B$ for all quasi-interior ideals B of M. Let $B = R \cap L$, and $C = R\Gamma L$, where R is a right ideal and L is a left ideal of M. Then B and C are quasi-interior ideals of M. Thus we get

$$(R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M = R \cap L.$$

On the other hand, we have

$$\begin{aligned} R \cap L &= (R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M \\ &\subseteq R\Gamma M\Gamma L\Gamma M \\ &\subseteq R\Gamma L\Gamma M, \\ R \cap L &= (R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M \\ &\subseteq R\Gamma L\Gamma M\Gamma R\Gamma L\Gamma M \\ &\subseteq R\Gamma L \\ &\subseteq R \cap L \text{ (Since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R). \end{aligned}$$

So $R \cap L = R\Gamma L$. Hence M is a regular Γ-semigroup. □

5. Fuzzy quasi interior ideals of Γ-semigroup M

In this section, we introduce the notion of fuzzy right (left) quasi-interior ideal as a generalization of fuzzy bi-ideal of a Γ-semigroup and study the properties of fuzzy right (left) quasi-interior ideals.

Definition 5.1. (i) A fuzzy subset μ of a Γ-semigroup M is called a *fuzzy left* (*right*) *quasi-interior ideal* of M, if $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$ $\big(\mu \circ \chi_M \circ \mu \circ \chi_M \subseteq \mu\big)$.

(ii) A fuzzy subset μ of a Γ-semigroup M is called a *fuzzy quasi-interior ideal* of M, if it is both left and right quasi-interior ideal of M.

Example 5.2. Let Q be the set of all rational numbers and $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$. Then M is a Γ-semigroup with respect to usual addition of matrices and ternary operation is defined as the usual matrix multiplication. If $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}$, then A is a right quasi-interior ideal but not a bi-ideal of M. Define $\mu : M \to [0,1]$ such that $\mu(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$
Then μ is a fuzzy right quasi-interior ideal of M.

Theorem 5.3. *Every fuzzy right ideal of a Γ-semigroup M is a fuzzy right quasi-interior ideal of M.*

Proof. Let μ be a fuzzy right ideal of M and $x \in M$. Then we have

$$\begin{aligned} \mu \circ \chi_M(x) &= \sup_{x=a\alpha b} \min\{\mu(a), \chi_M(b)\} \ a,\ b \in M,\ \alpha \in \Gamma \\ &= \sup_{x=a\alpha b} \mu(a) \\ &\leq \sup_{x=a\alpha b} \mu(a\alpha b) \\ &= \mu(x). \end{aligned}$$

Thus $\mu \circ \chi_M(x) \leq \mu(x)$. On the other hand, we get

$$\begin{aligned} \mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=u\alpha v\beta s} \min\{\mu \circ \chi_M(u\alpha v), \mu \circ \chi_M(s)\} \\ &\leq \sup_{x=u\alpha v\beta s} \min\{\mu(u\alpha v), \mu(s)\} \\ &= \mu(x). \end{aligned}$$

So μ is a fuzzy right quasi-interior ideal of M. □

Corollary 5.4. *Every fuzzy left ideal of a Γ-semigroup M is a fuzzy left quasi-interior ideal of M.*

Corollary 5.5. *Every fuzzy ideal of a Γ-semigroup M is a fuzzy quasi-interior ideal of M.*

Theorem 5.6. *Let M be a Γ-semigroup and μ be a non-empty fuzzy subset of M. Then μ is a fuzzy left quasi-interior ideal of M if and only if the level subset μ_t of μ is a left quasi-interior ideal of M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Proof. Suppose μ is a fuzzy left quasi-interior ideal of M and let $\mu_t \neq \phi, t \in [0,1]$ and $a, b \in \mu_t$. Let $x \in M\Gamma\mu_t\Gamma M\Gamma\mu_t$. Then $x = b\alpha a\beta d\gamma c$, where $b, d \in M, a, c \in \mu_t, \alpha, \beta, \gamma \in \Gamma$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t$. Thus $\mu(x) \geq \chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t$. So $x \in \mu_t$. Hence μ_t is a left quasi-interior ideal of M.

Conversely, suppose that μ_t is a left quasi-interior ideal of M for all $t \in Im(\mu)$. Let $x,\ y \in M,\ \alpha \in \Gamma,\ \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x,\ y \in \mu_{t_2}$. Thus we have $M\Gamma\mu_l\Gamma M\Gamma\mu_l \subseteq \mu_t$ for all $l \in Im(\mu)$. Now let $t = \min\{Im(\mu)\}$. Then $M\Gamma\mu_t M\Gamma\mu_t\Gamma \subseteq \mu_t$. Thus $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. So μ is a fuzzy left quasi-interior ideal of M. □

Corollary 5.7. *Let M be a Γ-semigroup and μ be a non-empty fuzzy subset of M. Then μ is a fuzzy right quasi-interior ideal of M if and only if the level subset μ_t of μ is a right quasi-interior ideal of M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Theorem 5.8. *Let I be a non-empty subset of a Γ-semigroup M. Then I is a right quasi-interior ideal of M if and only if χ_I is a fuzzy right quasi-interior ideal of M.*

Proof. Suppose I is a right quasi-interior ideal of M. Obviously, χ_I is a fuzzy Γ-sub semigroup of M. Moreover, we have $I\Gamma M\Gamma I\Gamma M \subseteq I$. Then we get

$$\chi_I \circ \chi_M \circ \chi_I \circ \chi_M = \chi_{I\Gamma M\Gamma I\Gamma M} = \chi_{I\Gamma M\Gamma I\Gamma M} \subseteq \chi_I.$$

Thus χ_I is a fuzzy right quasi-interior ideal of M.

Conversely, suppose that χ_I is a fuzzy right quasi-interior ideal of M. Then I is a Γ-sub semigroup of M. Then $\chi_I \circ \chi_M \circ \chi_I \circ \chi_M \subseteq \chi_I$. Thus $\chi_{I\Gamma M\Gamma I\Gamma M} \subseteq \chi_I$. So $I\Gamma M\Gamma I\Gamma M \subseteq I$. Hence I is a right quasi-interior ideal of M. □

Theorem 5.9. *If μ and λ are fuzzy left quasi-interior ideals of a Γ-semigroup M, then $\mu \cap \lambda$ is a fuzzy left quasi-interior ideal of M.*

Proof. Let μ and λ be fuzzy left quasi ideals of the Γ-semigroup M. Let $x,\ y \in M$ and $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned}
\chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b} \min\{\chi_M(a), \mu \cap \lambda(b)\} \\
&= \sup_{x=a\alpha b} \min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\} \\
&= \sup_{x=a\alpha b} \min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\sup_{x=a\alpha b} \min\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b} \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\chi_M \circ \mu(x). \chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \cap \chi_M \circ \lambda(x).
\end{aligned}$$

Thus $\chi_M \circ (\mu \cap \lambda) = \chi_M \circ \mu \cap \chi_M \circ \lambda$. On the other hand, we get

$$\begin{aligned}
&\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda(x) \\
&\quad = \sup_{x=a\alpha b} \min\{\chi_M \circ \mu \cap \lambda(a), \chi_M \circ \mu \cap \lambda(b)\} \\
&\quad = \sup_{x=a\alpha b} \min\{\min\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}, \\
&\qquad\qquad \min\{\chi_M \circ \mu(b), \chi_M \circ \lambda(b)\}\}\} \\
&\quad = \sup_{x=a\alpha b} \min\{\min\{\chi_M \circ \mu(a), \chi_M \circ \mu(b)\}\}, \\
&\qquad\qquad \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(b)\}\}\} \\
&\quad = \min\{\sup_{x=a\alpha b} \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(b)\}, \\
&\qquad\qquad \sup_{x=a\alpha b} \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(b)\}\} \\
&\quad = \min\{\chi_M \circ \mu \circ \chi_M \circ \mu(x), \chi_M \circ \lambda \circ \chi_M \circ \lambda(x)\} \\
&\quad = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

So $\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda$. Hence we have

$$\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cap \lambda.$$

Therefore $\mu \cap \lambda$ is a left fuzzy quasi-interior ideal of M.

□

Corollary 5.10. *If μ and λ are fuzzy right quasi-interior ideals of a Γ-semigroup M, then $\mu \cap \lambda$ is a fuzzy right quasi-interior ideal of M.*

Corollary 5.11. *Let μ and λ be fuzzy right ideal and fuzzy left ideal of a Γ-semigroup M respectively. Then $\mu \cap \lambda$ is a fuzzy quasi-interior ideal of M.*

Theorem 5.12. *Let M be a regular Γ-semigroup. μ is a fuzzy left quasi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Proof. Suppose μ is a fuzzy left quasi-interior ideal of M and let $x \in M$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Assume that $\chi_M \circ \mu(x) > \mu(x)$ and $\mu \circ \chi_M(x) > \mu(x)$. Since M is regular, there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Then we get

$$\begin{aligned} \mu \circ \chi_M(x) &= \sup_{x=x\alpha y\beta x} \min\{\mu(x), \chi_M(y\beta x)\} \\ &= \sup_{x=x\alpha y\beta x} \min\{\mu(x), 1\} \\ &= \sup_{x=x\alpha y\beta x} \mu(x) \\ &> \mu(x), \end{aligned}$$

$$\begin{aligned} \mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=x\alpha y\beta x} \min\{\mu \circ \chi_M(x), \mu \circ \chi_M(y\beta x)\}) \\ &> \sup_{x=x\alpha y\beta x} \min\{\mu(x), \mu(y\beta x)\} \\ &= \mu(x). \end{aligned}$$

Which is a contradiction. Thus μ is a fuzzy quasi ideal of M.

The converse is true by Theorem 5.3. □

Corollary 5.13. *Let M be a regular Γ-semigroup. Then μ is a fuzzy right quasi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Theorem 5.14. *Let M be a Γ-semigroup. Then M is regular if and only if $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$ for any fuzzy left qussi-interior ideal μ of M.*

Proof. Suppose M is regular and μ be a fuzzy left quasi-interior ideal of M and $x,\ y \in M,\ \alpha,\ \beta \in \Gamma$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Thus

$$\begin{aligned} \chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=x\alpha y\beta x} \{\min\{\chi_M \circ \mu(x), \chi_M \circ \mu(y\beta x)\}\} \\ &\geq \sup_{x=x\alpha y\beta x} \{\min\{\mu(x), \mu(x)\}\} \\ &= \mu(x). \end{aligned}$$

So $\mu \subseteq \chi_M \circ \mu \circ \chi_M \circ \mu$. Hence $\chi_M \circ \mu \circ \chi_M \circ \mu = \mu$.

Conversely, suppose $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$ for any fuzzy quasi-interior ideal μ of M. Let B be a quasi-interior ideal of M. Hence,χ_B is a fuzzy quasi-interior ideal of M. Thus $\chi_B = \chi_M \circ \chi_B \circ \chi_M \circ \chi_B = \chi_{M\Gamma B\Gamma M\Gamma B}$. So $B = M\Gamma B\Gamma M\Gamma B$. Hence M is regular. □

Theorem 5.15. *Let M be a Γ-semigroup. Then M is regular if and only if $\mu \cap \gamma \subseteq \gamma \circ \mu \circ \gamma \cap \mu \circ \gamma \circ \mu$ for every fuzzy left quasi-interior ideal γ and every fuzzy ideal μ of M.*

Proof. Suppose M is regular and let $x \in M$. Then there exist $y \in M,\ \alpha,\ \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Thus we get

$$\begin{aligned}\mu \circ \gamma \circ \mu \circ \gamma(x) &= \sup_{x=x\alpha y\beta x} \{\min\{\mu \circ \gamma(x\alpha y), \mu \circ \gamma(x)\}\} \\ &= \min\{\sup_{x\alpha y=x\alpha y\beta x\alpha y} \{\min\{\mu(x), \gamma(y\beta x\alpha y)\}, \\ &\qquad \sup_{x\alpha y=x\alpha y\beta x\alpha y} \{\min\{\mu(x), \gamma(y\beta x\alpha y)\}\}\} \\ &\geq \min\{\min\{\mu(x), \gamma(x)\}, min\{\mu(x), \gamma(x)\}\} \\ &= \min\{\mu(x), \gamma(x)\} = \mu \cap \gamma(x).\end{aligned}$$

So $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu \circ \gamma$.

Conversely, suppose that the condition holds. Let μ be a fuzzy left quasi-interior ideal of M. Then $\mu \cap \chi_M \subseteq \chi_M \circ \mu \circ \chi_M \cap \mu$. Thus $\mu \subseteq \chi_M \circ \mu \circ \chi_M \cap \mu$. So M is regular. □

Corollary 5.16. *Let M be a Γ-semigroup. Then M is regular if and only if $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu \circ \gamma$, for every fuzzy right quasi-interior ideal γ and every fuzzy ideal μ of M.*

6. Conclusion

As a further generalization of ideals, we introduced the notion of a quasi-interior ideal of a Γ-semigroup as a generalization of ideals, left ideals, right ideals, quasi ideals, bi-ideals and interior ideals of a Γ-semigroup and studied some of their properties. We introduced the notion of a quasi-interior simple Γ-semigroup and characterized the quasi-interior simple Γ-semigroup, regular Γ-semigroup using quasi-interior ideal ideals of a Γ-semigroup and studied some of the properties. We introduced the notion of fuzzy right (left) quasi-interior ideal of a Γ-semigroup and characterized the regular Γ-semigroup in terms of fuzzy right (left) quasi-interior ideals of a Γ-semigroup and studied some of the algebraic properties. One can extend this work by studying the other (ordered) algebraic structures.

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