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Abstract. In this paper, we introduce new soft hyperspaces and study their basic properties. Also, we deal with some relationships between separation axioms in a soft topological space and separation axioms in a soft hyperspace.

2020 AMS Classification: 54A40, 54B20, 54D10, 54D15

Keywords: Soft topological space, Soft T_i-space (i=0, 1,2, 3, 4), soft connectedness, soft compact space, Hyperspace.

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1. Introduction

In early 1920, the notion of hyperspaces was introduced by Vietories (See [1, 2]) and it has been extensively investigated. Kelley [3] studied the hyperspace of metric continua which has been applied in dynamical systems (See [4, 5, 6, 7, 8, 9, 10, 11] for further background). Recently, Macías and Nadler [12] discussed continua whose hyperspace of subcontinua is a cone.

In 1999, Molodtsov [13] introduced the concept of soft sets applied to several fields as a tool for dealing with uncertainties (See [14, 15, 16, 17, 18]). Shabir and Naz [19] defined a soft topology and separation axioms in a soft topological space, and studied their various properties (See [20, 21, 23, 24, 25]). Nazmul and Samanta [26] proposed the notion of soft metrics and dealt with its basic properties. Alcantud [27] dealt with the relationships between fuzzy soft toplogies and soft topologies. Bayramov and Aras [28] defined newly separation axioms in a soft topological space and investigated their some properties (See [29, 30, 31] for the further researches). In particular, Debnath and Tripathy [32] studied separation axioms in a soft bitopological space. Recently, Baek et al. [33] proposed separation axioms in an interval-valued topological space and discussed some of their properties and some relationships among them (See [34, 35]). Zorlutuna et al. [22] covered compactness in a soft topological space as well as basic properties related to soft topology (See [36, 37, 38]). Lin [39] introduced the concept of soft connectednes and soft paracompactness and studied some of their properties (See [38, 40, 41]).

In 2015, Akdağ and Erol [42] defined a soft Vietoris topology based on soft sets and gave the relationship between Vietoris continuity of soft multifunction and continuity of soft mapping (See [43] for further background). Shakir [44], independently from Akdağ and Erol, defined Vietoies soft hyperspace and obtained some basic properties of such a soft hyperspace. After that, Demir [45] discussed the Axiom of countabilty in Vietoies soft hyperspaces in the viewpoint of Akdağ and Erol. Özkan [46] studied various properties of the continuity of soft multifunction proposed by Akdağ and Erol. As a generalization of classical hyperspaces, it is necessary to obtain various properties for soft hyperspace. To do our research, first, we recall some basic notions and results related soft topological spaces. Next, we deal with basic properties of soft hyperspaces proposed by Shakir [44]. Finally, we discuss separation axioms in a soft hyperspace.

2. Preliminaries

In this section, we recall basic concepts and results needed in the next sections. Unless otherwise stated, let $X,\ Y,\ Z, \cdots$ denote non-empty universe sets, E a set of parameters and 2^X the power set of X.

Definition 2.1 ([13, 20])**.** Let $A \in 2^E$, Then an F_A is called a *soft set* over X, if $F_A : A \to 2^X$ is a mapping such that $F_A(e) = \varnothing$ for each $e \notin A$.

For each $e \in A$, $F_A(e)$ may be considered as the set of e-approximate elements of the soft set F_A.

Definition 2.2 ([14, 15])**.** Let $F_A \in SS(X)$. Then F_A is called:

(i) a *null soft set* or a *relative null soft set* (with respect to A), denoted by $\varnothing_A$, if $F_A(e) = \varnothing$ for each $e \in A$,

(ii) an *absolute soft set* or a *relative whole soft set* (with respect to A), denoted by X_A, if $F_A(e) = X$ for each $e \in A$.

The *empty* [resp.*whole*] *soft set* over X with respect to E, denoted by $\varnothing_E$ [resp. X_E], is a soft set over X defined by: for each $e \in E$,

$$\varnothing_E(e) = \varnothing \text{ [resp. } X_E(e) = X].$$

We will denote the set of all soft sets over X as $SS(X)$, while $SS_E(X)$ will denote the set of all soft sets over X with respect to a fixed E. Throughout this paper, members of $SS_E(X)$ will be written by $A,\ B,\ C$, etc.

Definition 2.3 (See [14, 19])**.** Let $A,\ B \in SS_E(X)$.

(i) We say that A *is a soft subset* B, denoted by $A \subset B$, if $A(e) \subset B(e)$.

(ii) The *intersection* of A and B, denoted by $A \cap B$, is a soft set over X defined by:

$$(A \cap B)(e) = A(e) \cap B(e) \text{ for each } e \in E.$$

(iii) The *union* of A and B, denoted by $A \cup B$, is a soft set over X defined by:

$$(A \cap B)(e) = A(e) \cup B(e) \text{ for each } e \in E.$$

(iv) The *complement* of A, denoted by A^c, is a soft set over X defined by:

$$A^c(x) = A(e)^c \text{ for each } e \in E.$$

(v) The *difference* of A and B, denoted by $A \setminus B$, is a soft set over X defined by:

$$(A \setminus B)(e) = A(e) \setminus B(e) \text{ for each } e \in E.$$

Definition 2.4 ([24, 26, 47])**.** Let $A \in SS_E(X)$. Then

(i) A is called a *soft point in* X *with the value* $x \in X$ *and the support* $e \in E$ or a *soft element*, denoted by e_x, if for each $f \in E$,

$$e_x(f) = \begin{cases} x & \text{if } f = e \\ \varnothing & \text{otherwise.} \end{cases}$$

(ii) we say that e_x *belongs to* A, denoted by $e_x \in A$, if $e_x(e) = x \in A(e)$.

We will denote the set of all soft points over X with respect to E by $SP_E(X)$.

Definition 2.5 ([26])**.** Let $e_x,\ f_y \in SP_E(X)$. Then we say that e_x and f_y are equal, denoted by $e_x = f_y$, if $e = f$ and $e_x(e) = f_y(f)$, i.e., $x = y$.

It is obvious that $e_x(e) \neq f_y(f)$ if and only if $x \neq y$ or $e \neq f$.

Result 2.6 (Proposition 3.5, [26])**.** *For each* $A \in SS_E(X)$, $A = \bigcup_{e_x \in A} e_x$.

Result 2.7 (Proposition 3.6, [26])**.** *Let* $A,\ B \in SS_E(X)$. *Then* $A \subset B$ *if and only if* $e_x \in B$ *for each* $e_x \in A$ *and thus* $A = B$ *if and only if* $e_x \in A \Leftrightarrow e_x \in B$.

Result 2.8 (Proposition 3.7, [26])**.** *Let* $A,\ B \in SS_E(X)$ *and* $e_x \in SP_E(X)$. *Then*

(1) $e_x \in A$ *if and only if* $e_x \notin A^c$,

(2) $e_x \in A \cup B$ *if and only if* $e_x \in A$ *or* $e_x \in B$,

(3) $e_x \in A \cap B$ *if and only if* $e_x \in A$ *and* $e_x \in B$.

Definition 2.9 ([19])**.** Let $\tau \subset SS_E(X)$. Then τ is called a *soft topology* on X, if it satisfies the following conditions:

(i) $\varnothing_E,\ X_E \in \tau$,

(ii) $A \cap B \in \tau$ for any $A,\ B \in \tau$,

(iii) $\bigcup_{j \in J} A_j \in \tau$ for each $(A_j)_{j \in J} \subset \tau$, where J denotes an index set.

The triple (X, τ, E) is called a *soft topological space* over X. Each member of τ is called a *soft open set* in X and a soft set A over X is called a *closed soft set* in X, if $A^c \in \tau$.

It is obvious that $\{\varnothing_E, X_E\}$ [resp. $SS_E(X)$] is a soft topology on X. In this case, $\{\varnothing_E, X_E\}$ [resp. $SS_E(X)$] is called the *soft indiscrete* [resp. *discrete*] *topology* on X and the triple $(X, \{\varnothing_E, X_E\}, E)$ [resp. $(X, SS_E(X), E)$] is called a *soft indiscrete* [resp. *discrete*] *space* (See[19]).

Result 2.10 (Proposition 4, [19])**.** *Let* (X, τ, E) *be a soft topological space,* τ^c *the set of all closed soft sets in* (X, τ, E). *Then*

(1) $\varnothing_E,\ X_E \in \tau^c$,

(2) $A \cup B \in \tau^c$ *for any* $A,\ B \in \tau^c$,

(3) $\bigcap_{j \in J} A_j \in \tau^c$ *for each* $(A_j)_{j \in J} \subset \tau^c$.

Result 2.11 (Proposition 5, [19]). *Let (X, τ, E) be a soft topological space. Then the collection of subsets of X*

$$\tau_e = \{A(e) \in 2^X : A \in \tau\} \text{ for each } e \in E$$

is a topology on X.

In this case, τ_e is called the *topology on X induced by E*.

Definition 2.12 ([47]). Let (X, τ, E) be a soft topological space, $e_x \in SP_E(X)$ and $A \in SS_E(X)$. Then e_x is called a *soft interior point* of A, if there is $U \in \tau$ such that $e_x \in U \subset A$.

Definition 2.13 ([20]). Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. Then the *soft interior* and the *soft closure* of A, denoted by $Sint(A)$ or A° and $Scl(A)$ or $\overline{A}$, are soft sets over X, respectively defined as follows:

$$Sint(A) = \bigcup\{U \in \tau : U \subset A\},\ Scl(A) = \bigcap\{F \in \tau^c : A \subset F\}.$$

Result 2.14 (Theorems 8 and 11, [20]). *Let (X, τ, E) be a soft topological space and $A,\ B \in SS_E(X)$. Then the followings hold:*
(1) $Scl(A)^c = Sint(A^c)$,
(2) $Sint(Sint(A)) = Sint(A)$, $Scl(Scl(A)) = Scl(A)$,
(3) *if $A \subset B$, then $Sint(A) \subset Sint(B)$, $Scl(A) \subset Scl(B)$,*
(4) $Sint(A) \cap Sint(B) = Sint(A \cap B)$, $Scl(A) \cap Scl(B) \subset Scl(A \cap B)$,
(5) $Sint(A) \cup Sint(B) \subset Sint(A \cup B)$, $Scl(A) \cup Scl(B) = Scl(A \cup B)$.

Definition 2.15 ([47]). Let (X, τ, E) be a soft topological space, $e_x \in SP_E(X)$ and $A \in SS_E(X)$. Then A is called a *soft neighborhood* (briefly, snbd) of e_x, if there is $U \in \tau$ such that $e_x \in U \subset A$, i.e., e_x is a soft interior point of A. The set of all soft nbds of e_x will be denoted by $\mathcal{N}(e_x)$, i.e.,

$$\mathcal{N}(e_x) = \{A \in SS_E(X) : \text{there is } U \in \tau \text{ such that } e_x \in U \subset A\}.$$

In particular, the family of all soft open nbds of e_x, denoted by $\mathcal{SN}(e_x)$,

$$\mathcal{SN}(e_x) = \{U \in \tau : e_x \in U\}$$

will be called the *system of soft open neighborhoods* of e_x.

Result 2.16 ([47]). *Let (X, τ, E) be a soft topological space, $e_x \in SP_E(X)$ and $A,\ B \in SS_E(X)$. Then the followings hold:*
(1) *if $A \in \mathcal{SN}(e_x)$, then $e_x \in A$,*
(2) *if $A,\ B \in \mathcal{SN}(e_x)$, then $A \cap B \in \mathcal{SN}(e_x)$,*
(3) *if $A \in \mathcal{SN}(e_x)$ and $A \subset B$, then $B \in \mathcal{SN}(e_x)$,*
(4) *if $A \in \mathcal{SN}(e_x)$, then there is $U \in \mathcal{SN}(f_y)$ such that $A \in \mathcal{SN}(e_x)$ for each $f_y \in SP_E(X)$ such that $f_y \in U$,*
(5) *$A \in \tau$ if and only if A contains a soft nbd of each of its points.*

Definition 2.17 (Proposition 3, [48]). Let (X, τ, E) be a soft topological space, $e_x \in SP_E(X)$ and $A \in SS_E(X)$. Then e_x is called a *soft limit point* of A, if $U \cap (A \setminus \{e_x\}) \neq \varnothing_E$ for each $U \in \mathcal{SN}(e_x)$. The set of all soft limit points of A is called the *derived soft set* over X and will be denoted by $Sd(A)$.

Result 2.18 (Theorems 13 and 15, [20]). *Let (X, τ, E) be a soft topological space and $A,\ B \in SS_E(X)$. Then the followings hold:*
(1) $A \cup Sd(A) = Scl(A)$,
(2) $Sd(A) \subset Scl(A)$,
(3) *if* $A \subset B$, *the* $Sd(A) \subset Sd(B)$,
(4) $Sd(A \cap B) \subset Sd(A) \cap Sd(B)$,
(5) $Sd(A \cup B) = Sd(A) \cup Sd(B)$,
(6) $A \in \tau^c$ *if and only if* $Sd(A) \subset A$.

Definition 2.19 ([28]). A soft topological space (X, τ, E) is called a:
(i) *soft T_0-space*, if for any $e_x,\ f_y \in SP_E(X)$ with $e_x \neq f_y$, there are a $U \in \mathcal{SN}(e_x),\ a\ V \in \mathcal{SN}(f_y)$ such that either $e_x \in U,\ f_y \notin U$ or $f_y \in V,\ e_x \notin V$,
(ii) *soft T_1-space*, if for any $e_x,\ f_y \in SP_E(X)$ with $e_x \neq f_y$, there are a $U \in \mathcal{SN}(e_x),\ a\ V \in \mathcal{SN}(f_y)$ such that $e_x \in U,\ f_y \notin U$ and $f_y \in V,\ e_x \notin V$,
(iii) *soft T_2-space*, if for any $e_x,\ f_y \in SP_E(X)$ with $e_x \neq f_y$, there are a $U \in \mathcal{SN}(e_x),\ a\ V \in \mathcal{SN}(f_y)$ such that $e_x \in U,\ f_y \in V$ and $U \cap V = \varnothing_E$,
(iv) *soft regular space*, if for each $A \in \tau^c$ with $e_x \notin A$, there are two $U,\ V \in \tau$ such that $e_x \in U,\ A \subset V$ and $U \cap V = \varnothing_E$,
(v) *soft T_3-space*, if it is both a soft regular and a soft T_1-space.

Result 2.20 (Theorems 4.1 and 4.4, [28]). *Let (X, τ, E) be a soft topological space. Then*
(1) *X is a soft T_1-space if and only if $e_x \in \tau^c$ for each $e_x \in SP_E(X)$,*
(2) *X is a soft T_3-space if and only if for each $e_x \in U \in \tau$, there is $V \in \tau$ such that $e_x \in V \subset Scl(V) \subset U$.*

Result 2.21 (Proposition 4.1 and Theorem 4.5, [28]). *Let (X, τ, E) be a soft topological space. If X is a soft T_i-space, then (X, τ_e) is a T_i-space for each $e \in E$, where $i \in \{0, 1, 2, 3\}$.*

Definition 2.22 ([19]). A soft topological space (X, τ, E) is called a:
(i) *soft normal space*, if for any $A,\ B \in \tau^c$ with $A \cap B = \varnothing_E$, there are $U,\ V \in \tau$ such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing_E$,
(ii) *soft T_4-space*, if it is both a soft normal-space and a soft T_1-space.

From Remarks 4.1, 4.2 and 4.3 in [28], the following implication hold:

$$soft\ T_4 \Rightarrow soft\ T_3 \Rightarrow soft\ T_2 \Rightarrow soft\ T_1 \Rightarrow soft\ T_0. \tag{2.1}$$

Result 2.23 (Theorem 4.6, [28]). *Let (X, τ, E) be a soft topological space. Then X is a soft T_4-space if and only if for each $A \in \tau^c$ and each $U \in \tau$ with $A \subset U$, there is a $V \in \tau$ such that $A \subset V \subset Scl(V) \subset U$,*

Definition 2.24 ([24]). Let (X, τ, E) be a soft topological space and $\beta \subset \tau$. Then β is called a *soft base* for τ, if every member of τ can be expressed as the union of some members of β.

Result 2.25 (Proposition 3.13, [24]). *Let (X, τ, E) be a soft topological space and $\beta \subset \tau$. Then β is a soft base for τ if and only if for each $A \in \tau$ and each $e_x \in A$, there is a $B \in \beta$ such that $e_x \in B \subset A$.*

Result 2.26 (Proposition 3.14, [24])**.** *Let $\beta \subset SS_E(X)$. Then β is a soft base for a soft topology on X if and only if it satisfies the following conditions:*
(1) $\varnothing_E \in \beta$,
(2) $X_E = \bigcup \beta$,
(3) *if $B_1,\ B_2 \in \beta$, then there is a $\beta' \subset \beta$ such that $B_1 \cap B_2 = \bigcup \beta'$, i.e., if $B_1,\ B_2 \in \beta$ and $e_x \in B_1 \cap B_2$, then $B \in \beta$ such that $e_x \in B \subset B_1 \cap B_2$.*

3. A soft Vietories topology and its basic properties

In this section, we define a soft hyperspace and study its various properties.

Notation 3.1. Let (X, τ, E) be a soft topological space, $A \in SS_E(X)$ and $e \in E$. Then
(i) $2^X_E = \{F \in \tau^c : F \neq \varnothing_E\}$,
(ii) $2^A_E = \{F \in 2^X_E : F \subset A\}$,
(iii) $2^X_e = \{F(e) \in \tau^c_e : F(e) \neq \varnothing\}$,
(iv) $2^{A(e)}_e = \{F(e) \in 2^X_e : F(e) \subset A(e)\}$.

Example 3.2. (1) (See Example 4, [19]) Let $X = \{x, y, z\}$, $E = \{e, f\}$ and τ be be the soft topology on X given by:

$$\tau = \{\varnothing_E, A_1, A_2, A_3, A_4, A_5, A_6, A_7, X_E\},$$

where $A_1(e) = A_1(f) = \{x, y\},\ A_2(e) = \{y\},\ A_2(f) = \{x, z\}$,
$A_3(e) = \{y, z\},\ A_3(f) = \{x\},\ A_4(e) = \{y\},\ A_4(f) = \{x\}$,
$A_5(e) = \{x, y\},\ A_5(f) = X,\ A_6(e) = X,\ A_6(f) = \{x, y\}$,
$A_7(e) = \{y, z\},\ A_7(f) = \{x, z\}$.
Then $\tau^c = \{\varnothing_E, A^c_1, A^c_2, A^c_3, A^c_4, A^c_5, A^c_6, A^c_7, X_E\}$,
where $A^c_1(e) = A^c_1(f) = \{z\},\ A^c_2(e) = \{x, z\},\ A^c_2(f) = \{y\}$,
$A^c_3(e) = \{x\},\ A^c_3(f) = \{y, z\},\ A^c_4(e) = \{x, z\},\ A^c_4(f) = \{y, z\}$,
$A^c_5(e) = \{z\},\ A^c_5(f) = \varnothing,\ A^c_6(e) = \varnothing,\ A^c_6(f) = \{z\}$,
$A^c_7(e) = \{x\},\ A^c_7(f) = \{y\}$.
Thus $2^X_E = \tau^c \setminus \{\varnothing_E\}$.

Now consider the soft set A over X defined by $A(e) = \{x, y\},\ A(f) = \{y, z\}$. Then clearly, $2^A_E = \{A^c_3, A^c_6, A^c_7\}$.

(2) (See Example 2.13, [24]) Let $X = \{a, b\}$ and $E = \{e, f\}$. Consider all possible soft sets over X given as follows:

$$\varnothing_E,\ A_1(e) = \varnothing,\ A_1(f) = \{a\},\ A_2(e) = \varnothing,\ A_2(f) = \{b\},$$

$$A_3(e) = \varnothing,\ A_3(f) = X,\ A_4(e) = \{a\},\ A_4(f) = \varnothing,\ A_5(e) = A_5(f) = \{a\},$$

$$A_6(e) = \{a\},\ A_6(f) = \{b\},\ A_7(e) = \{a\},\ A_7(f) = X,\ A_8(e) = \{b\},\ A_8(f) = \varnothing,$$

$$A_9(e) = \{b\},\ A_9(f) = \{a\},\ A_{10}(e) = A_{10}(f) = \{b\},\ A_{11}(e) = \{b\},\ A_{11}(f) = X,$$

$$A_{12}(e) = X,\ A_{12}(f) = \varnothing,\ A_{13}(e) = X,\ A_{13}(f) = \{a\},\ A_{14}(e) = X,\ A_{14}(f) = \{b\},\ X_E.$$

Let $\tau = \{\varnothing_E, A_2, A_6, A_{11}, X_E\}$. Then we can easily check that τ is a soft topology on X. Thus $\tau^c = \{\varnothing_E, A^c_2, A^c_6, A^c_{11}, X_E\} = \{\varnothing_E, A_{13}, A_9, A_4, X_E\}$. So we have

$$2^X_E = \{A_4, A_9, A_{13}, X\} = \tau^c \setminus \{\varnothing_E\},\ 2^{A_1}_E = \varnothing,\ 2^{A_7}_E = \{A_4\}.$$

On the other hand, $\tau_e = \{\varnothing, \{a\}, \{b\}, X\}$ and $\tau_f = \{\varnothing, \{b\}, X\}$. Hence we get

$$2_e^X = \{\{a\}, \{b\}, X\},\ 2_f^X = \{\{a\}, X\},\ 2_e^{\{a\}} = \{a\},\ 2_f^{\{a\}} = \varnothing.$$

Proposition 3.3. *Let (X, τ, E) be a soft topological space, $A,\ B \in SS_E(X)$ and $(A_j)_{j \in J} \subset SS_E(X)$. Then*

(1) $2_E^{A \cap B} = 2_E^A \cap 2_E^B$ *and generally,* $2_E^{\bigcap_{j \in J} A_j} = \bigcap_{j \in J} 2_E^{A_j}$,

(2) $A \subset B$ *if and only if* $2_E^A \subset 2_E^B$ *and thus* $A = B$ *if and only if* $2_E^A = 2_E^B$.

Proof. The proofs follows from Notation 3.1 and the definitions of the inclusion and the intersection of soft sets. □

Corollary 3.4 (See Exponential topology, [49]). *Let (X, τ, E) be a soft topological space, $e \in E$, $A,\ B \in SS_E(X)$ and $(A_j)_{j \in J} \subset SS_E(X)$. Then*

(1) $2_e^{A(e) \cap B(e)} = 2_e^{A(e)} \cap 2_e^{B(e)}$ *and generally,* $2_e^{\bigcap_{j \in J} A_j(e)} = \bigcap_{j \in J} 2_e^{A_j(e)}$,

(2) $A(e) \subset B(e)$ *if and only if* $2_e^{A(e)} \subset 2_e^{B(e)}$ *and thus* $A(e) = B(e)$ *if and only if* $2_e^{A(e)} = 2_e^{B(e)}$.

Proposition 3.5. *Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. Then*

$$2_E^X \setminus 2_E^{A^c} = \{F \in 2_E^X : F \cap A \neq \varnothing_E\}.$$

Corollary 3.6 (See Exponential topology, [49]). *Let (X, τ, E) be a soft topological space, $e \in E$ and $A \in SS_E(X)$. Then*

$$2_e^X \setminus 2_e^{A(e)^c} = \{F(e) \in 2_e^X : F(e) \cap A(e) \neq \varnothing\}.$$

Notation 3.7. Let (X, τ, E) be a soft topological space and $A_0, A_1, \cdots, A_n$ any finite system of arbitrary soft sets over X. Then $\mathbf{B}(A_0, A_1, \cdots, A_n)$ is defined as follows:

$$\mathbf{B}(A_0, A_1, \cdots, A_n) = 2_E^{A_0} \cap (2_E^X \setminus 2_E^{A_1^c}) \cap (2_E^X \setminus 2_E^{A_2^c}) \cap \cdots \cap (2_E^X \setminus 2_E^{A_n^c}).$$

It is obvious that

$$\mathbf{B}(A_0, A_1, \cdots, A_n) = \{F \in 2_E^X : F \subset A_0,\ F \cap A_i \neq \varnothing_E \ for\ each\ i = 1,\ \cdots,\ n\}.$$

In this case, $\mathbf{B}(A_0, A_1, \cdots, A_n)$ will be denoted by $\langle A_0, A_1, \cdots, A_n \rangle$.

Proposition 3.8. *Let (X, τ, E) be a soft topological space and $\mathcal{S}_{Se}$ the collection of all of 2_E^U and $2_E^X \setminus 2_E^{U^c}$, where $U \in \tau$. Let $\mathcal{B}_{Se}$ be the collection of all finite intersections of members of $\mathcal{S}_{Se}$. In fact, for each $\mathbf{B} \in \mathcal{B}_{Se}$, there are $U_0, U_1, \cdots, U_n \in \tau$ such that*

$$\mathbf{B} = \langle U_0, U_1, \cdots, U_n \rangle.$$

Then there is a unique topology $\mathcal{T}_{Se}$ on 2_E^X such that $\mathcal{S}_{Se}$ is a soft subbase for $\mathcal{T}_{Se}$ and $\mathcal{B}_{Se}$ a soft base for $\mathcal{T}_{Se}$.

In this case, $\mathcal{T}_{Se}$ is called the *soft exponential topology* on 2_E^X.

Proof. The proof is almost similar to Theorem 1.12 in [50]. □

Corollary 3.9 (See Exponetial topology, [49]). *Let (X, τ, E) be a soft topological space and $\mathcal{S}_e$ the collection of all of $2_e^{U(e)}$ and $2_e^X \setminus 2_e^{U(e)^c}$, where $U(e) \in \tau_e$ for each*

$e \in E$. Let $\mathcal{B}_e$ be the collection of all finite intersections of members of $\mathcal{S}_e$. In fact, for each $\mathbf{B} \in \mathcal{B}_e$, there are $U_0(e), U_1(e), \cdots, U_n(e) \in \tau_e$ such that

$$\mathbf{B} = \langle U_0(e), U_1(e), \cdots, U_n(e) \rangle .$$

Then there is a unique topology $\mathcal{T}_{ep,e}$ on 2^X_e such that $\mathcal{S}_e$ is a subbase for $\mathcal{T}_{ep,e}$ and $\mathcal{B}_e$ a base for $\mathcal{T}_{ep,e}$.

In this case, $\mathcal{T}_{ep,e}$ is called the *exponential topology* on 2^X_e for $e \in E$.

For a soft topological space (X, τ, E) and any soft sets $U_1,\ U_2,\ \cdots,\ U_n$ be over X, let $\langle U_1, U_2, \cdots, U_n \rangle$ be the collection of soft closed sets in X defined by:

$$\langle U_1, U_2, \cdots, U_n \rangle = \{F \in 2^X_E : F \subset \bigcup_{i=1}^{n} U_i,\ E \cap U_i \neq \varnothing_E \textit{ for each } i = 1,\ 2,\ \cdots, n\}.$$

Proposition 3.10. *Let (X, τ, E) be a soft topological space and $\mathcal{B}_{Sv}$ a family of the form $\langle U_1, U_2, \cdots, U_n \rangle$ such that $U_i \in \tau$ for each $i = 1,\ 2,\ \cdots, n$. Then $\mathcal{B}_{Sv}$ is a soft base for some topology $\mathcal{T}_{Sv}$ on 2^X_E. In fact,*

$$\mathcal{T}_{Sv} = \{\varnothing_E\} \cup \{U \in SS_E(X) : U = \bigcup \mathcal{B} \textit{ for some } \mathcal{B} \subset \mathcal{B}_{Sv}\}.$$

In this case, $\mathcal{T}_{Sv}$ is called the *soft Vietories (finite) topology* on 2^X_E. The pair $(2^X_E, \mathcal{T}_{Sv})$ is called a *soft hyperspace with soft Vietories topology* (briefly, soft hyperspace) (See Definition 3.2, [44]).

Proof. Since $2^X_E = \langle X_E \rangle$ and $\langle X_E \rangle \in \mathcal{B}_{Sv}$, $2^X_E = \bigcup \mathcal{B}_{Sv}$. Suppose $\mathcal{U} = \langle U_1, U_2, \cdots, U_n \rangle$, $\mathcal{V} = \langle V_1, V_2, \cdots, V_m \rangle \in \mathcal{B}_{Sv}$ and let $U = \bigcup_{i=1}^{n} U_i,\ V = \bigcup_{j=1}^{m} V_j$. Then clearly, $U,\ V \in \tau$. Moreover, we can easily prove that the following holds:

$$\mathcal{U} \cap \mathcal{V} = \langle U_1 \cap V, \cdots, U_n \cap V, U \cap V_1, \cdots, U \cap V_m \rangle .$$

Thus $\mathcal{B}_{Sv}$ is a base for $\mathcal{T}_{Sv}$. □

Remark 3.11. If E is a singleton set, then soft hyperspace of a soft topological space (X, τ, E) coincide with the classical hyperspace.

Corollary 3.12 (See Proposition 2.1, [51])**.** *Let (X, τ, E) be a soft topological space and $\mathcal{B}_{v,e}$ a family of the form $\langle U_1(e), U_2(e), \cdots, U_n(e) \rangle$ such that $U_i(e) \in \tau_e$ for each $i = 1,\ 2,\ \cdots, n$ and each $e \in E$. Then $\mathcal{B}_{v,e}$ is a base for some topology $\mathcal{T}_{v,e}$ on 2^X_e. In fact,*

$$\mathcal{T}_{v,e} = \{\varnothing\} \cup \{U(e) \in 2^X : U(e) = \bigcup \mathcal{B} \textit{ for some } \mathcal{B} \subset \mathcal{B}_{v,e}\}.$$

In this case, $\mathcal{T}_{v,e}$ is called the *Vietories (finite) topology* on 2^X_e for $e \in E$. The pair $(2^X_e, \mathcal{T}_{v,e})$ is called a *hyperspace with Vietories topology* (briefly, hyperspace) for $e \in E$.

Example 3.13. Let (X, τ, E) be the soft topological space given in Example 3.2 (2). Then we have

$$\langle A_2 \rangle = \langle A_6 \rangle = \langle A_2, A_6 \rangle = \varnothing,$$
$$\langle A_{11} \rangle = \langle A_2, A_{11} \rangle = \{A_4, A_9\},\ \langle A_6, A_{11} \rangle = \langle X_E \rangle = 2^X_X.$$

Thus $\mathcal{T}_{Sv} = \{\varnothing, \{A_4, A_9\}, 2^X_E\}$. Furthermore, we get

$$\mathcal{T}_{v,e} = \{\varnothing, \{a\}, \{b\}, 2^X_e\},\ \mathcal{T}_{v,f} = \{\varnothing, 2^X_f\}.$$

Proposition 3.14. *$\mathcal{B}_{Se}$ and $\mathcal{B}_{Sv}$ are equivalent and then $\mathcal{T}_{Se} = \mathcal{T}_{Sv}$.*

Proof. Let $\langle U_0, U_1, \cdots, U_n \rangle \in \mathcal{B}_{Se}$ and $F \in \langle U_0, U_1, \cdots, U_n \rangle$. Then clearly, $F \subset U_0$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, 2, \cdots, n\}$. Let $G_i = U_0 \cup U_i$ for each $i \in \{1, 2, \cdots, n\}$. Then $G_i \in \tau$, $F \cap G_i \neq \varnothing_E$ for each $i \in \{1, 2, \cdots, n\}$ and $F \subset \bigcup_{i=1}^n G_i$. Thus $F \in \langle G_1, G_2, \cdots, G_n \rangle \in \mathcal{B}_{Sv}$ and $\langle G_1, G_2, \cdots, G_n \rangle \subset \langle U_0, U_1, \cdots, U_n \rangle$.

Conversely, let $F \in \langle G_1, G_2, \cdots, G_n \rangle \in \mathcal{B}_{Sv}$. Then $F \subset \bigcup_{i=1}^n G_i$ and $F \cap G_i \neq \varnothing_E$ for each $i \in \{1, 2, \cdots, n\}$. Let $U_0 = \bigcup_{i=1}^n G_i$ and $U_i = U_0 \cap G_i$ for each $i \in \{1, 2, \cdots, n\}$. Then clearly, $F \subset U_0$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, 2, \cdots, n\}$. Thus $F \in \langle U_0, U_1, \cdots, U_n \rangle \in \mathcal{B}_{Se}$ and $\langle G_1, G_2, \cdots, G_n \rangle \subset \langle U_0, U_1, \cdots, U_n \rangle$. So $\mathcal{B}_{Se}$ and $\mathcal{B}_{Sv}$ are equivalent. □

Lemma 3.15. *Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. If $A \in \tau^c$, then $2^A_{_E},\ 2^X_{_E} \setminus 2^{A^c}_{_E} \in \mathcal{T}^c_{Sv}$, i.e.,*

$$\{F \in 2^X_{_E} : F \subset A\},\ \{F \in 2^X_{_E} : F \cap A \neq \varnothing_E\} \in \mathcal{T}^c_{Sv}.$$

Proof. Let $F \in d(2^A_{_E})$, where $d(2^A_{_E})$ denotes the derived set of $2^A_{_E}$ in $(2^X_{_E}, \mathcal{T}_{Sv})$ and assume that $F \notin 2^A_{_E}$. Then $F \not\subset A$, i.e., there are $e \in E$ and $x \in X$ such that $e_{_x} \in F \setminus A$. Since $A \in \tau^c$ and $e_{_x} \notin A$, by Result 2.18(6), $e_{_x} \notin Sd(A)$. Thus there is $U \in \mathcal{SN}(e_{_x})$ such that $U \cap A = \varnothing_E$. Since $U,\ X_E \in \tau$, $F \subset U \cup X_E$, $F \cap U \neq \varnothing_E$ and $F \cap X_E \neq \varnothing_E$, $F \in \langle U, X_E \rangle$. Thus $2^A_{_E} \cap \langle U, X_E \rangle = \varnothing$. So $F \notin d(2^A_{_E})$. This is a contradiction. Hence $2^A_{_E} \in \mathcal{T}^c_{Sv}$.

Now we prove the second part. Let $F \in d(2^X_{_E} \setminus 2^{A^c}_{_E})$ and assume that $F \notin 2^X_{_E} \setminus 2^{A^c}_{_E}$. Then clearly, $F \cap A = \varnothing_E$. Since $A \in \tau^c$, $e_{_x} \notin Sd(A)$ for each $e_{_x} \in A$. By Result 2.18(6), there is $U \in \tau$ such that $A \subset U$ and $A \cap U = \varnothing_E$. Thus $F \in \langle U \rangle$ and $(2^X_{_E} \setminus 2^{A^c}_{_E}) \cap \langle U \rangle = \varnothing$. So $F \notin d(2^X_{_E} \setminus 2^{A^c}_{_E})$. This is a contradiction. Hence $2^X_{_E} \setminus 2^{A^c}_{_E} \in \mathcal{T}^c_{Sv}$. □

Corollary 3.16 (See Lemma 2.2, [51])**.** *Let (X, τ, E) be a soft topological space, $A \in SS_E(X)$ and $e \in E$. If $A(e) \in \tau^c_{_e}$, then $2^{A(e)}_{_e},\ 2^X_{_e} \setminus 2^{A(e)^c}_{_e} \in \mathcal{T}^c_{v,e}$, i.e.,*

$$\{F(e) \in 2^X_{_e} : F \in 2^X_{_E},\ F(e) \subset A(e)\},\ \{F(e) \in 2^X_{_e} : F \in 2^X_{_E},\ F(e) \cap A(e) \neq \varnothing\} \in \mathcal{T}^c_{v,e}.$$

Lemma 3.17. *If (X, τ, E) is a soft T_1-space and $A \in SS_E(X)$, then*

$$cl(2^A_{_E}) = 2^{Scl(A)}_{_E},\ int(2^A_{_E}) = 2^{Sint(A)}_{_E},$$

where $cl(2^A_{_E})$ and $int(2^A_{_E})$ denote the closure and the interior of $2^A_{_E}$ in $(2^X_{_E}, \mathcal{T}_{Sv})$.

Proof. It is clear that $2^A_{_E} \subset 2^{Scl(A)}_{_E}$ and $2^{Scl(A)}_{_E} \subset 2^A_{_E}$. It remains to prove that

$$2^{Scl(A)}_{_E} \subset cl(2^A_{_E}), \tag{3.1}$$

$$int(2^A_{_E}) \subset 2^{Sint(A)}_{_E}. \tag{3.2}$$

Let $F \in 2^{Scl(A)}_{_E}$, i.e., $F \subset Scl(A)$. Let $\mathcal{U} = \langle U_1, \cdots, U_n \rangle$ be any basic open set in $2^X_{_E}$ such that $F \in \langle U_1, \cdots, U_n \rangle$ and $U = \bigcup_{i=1}^n U_i$. Then clearly, $F \subset U$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Since $F \subset Scl(A)$, by Result 2.18(1), $F \subset A \cup Sd(A)$. Then there is $e_{i_{x_i}} \in A \cap U_i$ for each $i \in \{1, \cdots, n\}$. Put $G = \bigcup_{i=1}^n e_{i_{x_i}}$. Since X is soft T_1, $G \in \tau^c$. Moreover, $G \subset A \cap U$ and $G \cap U_i \neq \varnothing_E$ for each

$i \in \{1, \cdots, n\}$. Thus $G \in 2^A_{_E} \cap \mathcal{U}$. So $F \in cl(2^A_{_E})$. Hence (3.1) holds. Therefore $cl(2^A_{_E}) = 2^{Scl(A)}_{_E}$.

Now let $F \notin 2^{Sint(A)}_{_E}$. By Result 2.14(1), $Scl(A^c) = Sint(A)^c$. Then $F \not\subset [Scl(A^c)]^c$, i.e., $F \cap Scl(A)^c \neq \varnothing_E$. Let $\mathcal{U} = \langle U_1, \cdots, U_n \rangle$ be any basic open set in $2^X_{_E}$ such that $F \in \langle U_1, \cdots, U_n \rangle$ and $U = \bigcup_{i=1}^n U_i$. Then $F \subset U$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Since $F \cap Scl(A)^c \neq \varnothing_E$, there is $e_{_x} \in U \setminus A$. Put $G = F \cup e_{_x}$. Since X is T_1, $e_{_x} \in \tau^c$. Thus $G \in \tau^c$, i.e., $G \in 2^X_{_E}$. Moreover, $G \not\subset A$, $G \subset U$ and $G \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$, i.e., $G \in \mathcal{U}$. So each neighborhood of F in $2^X_{_E}$ contains G such that $G \not\subset A$, i.e., $G \in 2^X_{_E} \setminus 2^A_{_E}$. It follows that $F \notin cl(2^X_{_E} \setminus 2^A_{_E})$. Hence $F \notin int(2^A_{_E})$, i.e., (3.2) holds. Therefore $2^{Scl(A)}_{_E} = int(2^A_{_E})$. □

Corollary 3.18 (See page 161, [49]). *If (X, τ, E) is a soft T_1-space and $e \in E$, then $cl(2^{A(e)}_{_e}) = 2^{cl(A(e))}_{_e}$, $int(2^{A(e)}_{_e}) = 2^{int(A(e))}_{_e}$ for each $A \in SS_E(X)$.*

Theorem 3.19. *Let (X, τ, E) be a soft T_1-space and $A \in SS_E(X)$. Then $2^A_{_E}$ and $2^X_{_E} \setminus 2^A_{_E}$ are closed* [resp. *open*] *in $2^X_{_E}$ if and only if A is soft closed* [resp. *open*] *in X.*

Proof. The proof follows from Lemma 3.17. □

Corollary 3.20 (See Theorem 17.III.1, [49]). *Let (X, τ, E) be a soft T_1-space, $e \in E$ and $A \in SS_E(X)$. Then $2^{A(e)}_{_e}$ and $2^X_{_e} \setminus 2^{A(e)}_{_e}$ are closed* [resp. *open*] *in $2^X_{_e}$ if and only if $A(e)$ is closed* [resp. *open*] *in $(X, \tau_{_e})$.*

Proposition 3.21. *If (X, τ, E) is a soft T_1-space and $A \in SS_E(X)$, then the set $\{F \in 2^X_{_E} : A \subset F\}$ is closed in $2^X_{_E}$.*

Proof. It is sufficient to show that the set $\{F \in 2^X_{_E} : A \not\subset F\}$ is open in $2^X_{_E}$. Note that $\{F \in 2^X_{_E} : A \not\subset F\} = \bigcup_{e_x \in A} \{F \in 2^X_{_E} : F \subset e^c_{_x}\} = \bigcup_{e_x \in A} 2^{e^c_x}_E$. Since X is soft T_1, $e_{_x} \in \tau^c$. Then $e^c_{_x} \in \tau$. Thus by Theorem 3.19, $2^{e^c_x}_E$ is open in $2^X_{_E}$. So $\{F \in F \in 2^X_{_E} : A \not\subset F\}$ is open in $2^X_{_E}$. □

Corollary 3.22 (See Theorem 17.III.2, [49]). *Let (X, τ, E) be a soft T_1-space, $e \in E$ and $A \in SS_E(X)$. Then the set $\{F(e) \in 2^X_{_e} : F \in 2^X_{_E},\ A(e) \subset F(e)\}$ is closed in $2^X_{_e}$.*

Proposition 3.23. *For a soft T_1-space (X, τ, E), the followings hold:*

(1) *$\langle U_1, \cdots, U_n \rangle \subset \langle V_1, \cdots, V_m \rangle$ if and only if $\bigcup_{i=1}^n U_i \subset \bigcup_{j=1}^m U_j$ and for each $j \in \{1, \cdots, m\}$, there is $i \in \{1, \cdots, n\}$ such that $U_i \subset V_j$,*

(2) *$cl(\langle U_1, \cdots, U_n \rangle) = \langle Scl(U_1), \cdots, Scl(U_n) \rangle$, where $cl(\langle U_1, \cdots, U_n \rangle)$ denotes the closure of $\langle U_1, \cdots, U_n \rangle$ in $(2^X_{_E}, \mathcal{T}_{Sv})$,*

(3) *if $\{U_\alpha\}_{\alpha \in \Lambda}$ is a soft neighborhood base at $e_{_x}$, then $\{\langle U_\alpha \rangle\}_{\alpha \in \Lambda}$ is a neighborhood base at $\{e_{_x}\}$ in $(2^X_{_E}, \mathcal{T}_{Sv})$,*

(4) *if $\mathcal{O} \in \mathcal{T}_{Sv}$, then $\bigcup \mathcal{O} \in \tau$.*

Proof. (1) Let $\mathcal{U} = \langle U_1, \cdots, U_n \rangle$ and $\mathcal{V} = \langle V_1, \cdots, U_m \rangle$. Suppose $\mathcal{U} \subset \mathcal{V}$ and assume that $\bigcup_{i=1}^n U_i \not\subset \bigcup_{j=1}^m V_j$, say $e_{i_{x_i}} \in U_i$ for each $i \in \{1, \cdots, n\}$ and $e_{n+1_{x_{n+1}}} \in \bigcup_{i=1}^n U_i$ but $e_{n+1_{x_{n+1}}} \notin \bigcup_{j=1}^m V_j$, where $e_i \in E$ and $x_i \in X$ for each $i \in \{1, \cdots, n, n+$

$1\}$. Since X is a soft T_1-space, by Result 2.20(1), $e_{i_{x_i}} \in \tau^c$. Thus $F = \bigcup_{i=1}^{n+1} e_{i_{x_i}} \in \tau^c$ and $F \in \mathcal{U} \setminus \mathcal{V}$. This is a contradiction. So $\bigcup_{i=1}^{n} U_i \subset \bigcup_{j=1}^{m} V_j$. Now assume that the second part is not true, say there is $j \in \{1, \cdots, m\}$ such that $U_i \setminus V_j \neq \varnothing_E$ for each each $i \in \{1, \cdots, n\}$. Then there are $e_i \in E$ and $x_i \in X$ such that $e_{i_{x_i}} \in U_i \setminus V_j$. Since X is a soft T_1-space, by Result 2.20(1), $e_{i_{x_i}} \in \tau^c$. Thus $A = \bigcup_{i=1}^{n} e_{i_{x_i}} \in \tau^c$ and $A \in \mathcal{U} \setminus \mathcal{V}$. This is a contradiction. So the second part holds.

Conversely, suppose the necessary conditions hold and let $F \in \mathcal{U}$. Then clearly, $F \subset \bigcup_{i=1}^{n} U_i$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Thus by the hypothesis, $F \subset \bigcup_{j=1}^{m} V_j$ and $F \cap V_j \neq \varnothing_E$ for each $j \in \{1, \cdots, m\}$. So $F \in \mathcal{V}$. Hence $\mathcal{U} \subset \mathcal{V}$.

(2) Let $F \in \langle Scl(U_1), \cdots, Scl(U_n)\rangle$, $\mathcal{V} = \langle V_1, \cdots, V_m\rangle$ be an open neighborhood of F in 2^X_E, $U = \bigcup_{i=1}^{n} U_i$ and $V = \bigcup_{j=1}^{m} V_j$. Since $V \in \mathcal{V}$, $F \subset V$ and $F \cap V_j \neq \varnothing_E$ for each $j \in \{1, \cdots, m\}$. Since $F \in \langle Scl(U_1), \cdots, Scl(U_n)\rangle$, $F \cap Scl(U_i) \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Moreover, $F \subset Scl(V)$ and $Scl(V) = \bigcup_{i=1}^{n} Scl(U_i)$ by Result 2.14(5). Then $V \cap Scl(U_i) \neq \varnothing_E \neq V_j \cap Scl(U)$ for each $i \in \{1, \cdots, n\}$ and each $j \in \{1, \cdots, m\}$. Thus $V \cap U_i \neq \varnothing_E \neq V_j \cap U$ for each $i \in \{1, \cdots, n\}$ and each $j \in \{1, \cdots, m\}$. Let us take $e_{i_{x_i}} \in V \cap U_i$ and $f_{j_{y_j}} \in V_j \cap U$ for each $i \in \{1, \cdots, n\}$ and each $j \in \{1, \cdots, m\}$. Since X is a soft T_1-space, by Result 2.20(1), $e_{i_{x_i}},\ f_{j_{y_j}} \in \tau^c$. Then $F = \left(\bigcup_{i=1}^{n} e_{i_{x_i}}\right) \cup \left(\bigcup_{j=1}^{m} f_{j_{y_j}}\right) \in \tau^c$. Thus $F \in \langle U_1, \cdots, U_n\rangle \cap \langle V_1, \cdots, V_m\rangle$. So $F \in d(\langle U_1, \cdots, U_n\rangle)$. Hence $\langle Scl(U_1), \cdots, Scl(U_n)\rangle \subset cl(\langle U_1, \cdots, U_n\rangle)$.

Now we show that the converse inclusion holds. It is clear that

$$\langle U_1, \cdots, U_n\rangle \subset \langle Scl(U_1), \cdots, Scl(U_n)\rangle \tag{3.3}$$

and

$$\langle Scl(U_1), \cdots, Scl(U_n)\rangle = \left(\bigcap_{i=1}^{n}\{F \in 2^X_E : F \cap Scl(U_i) \neq \varnothing_E\}\right) \cap \langle Scl(U)\rangle, \tag{3.4}$$

where $U = \bigcup_{i=1}^{n} U_i$. Since $Scl(U_i) \in \tau^c$ for each $i \in \{1, \cdots, n\}$ and $SclU) \in \tau^c$, by Lemma 3.15, $\left(\bigcap_{i=1}^{n}\{F \in 2^X_E : F \cap Scl(U_i) \neq \varnothing_E\}\right) \cap \langle Scl(U)\rangle$ is closed in 2^X_E. Thus $\langle Scl(U_1), \cdots, Scl(U_n)\rangle$ is closed in 2^X_E. So $cl(\langle U_1, \cdots, U_n\rangle) \subset \langle Scl(U_1), \cdots, Scl(U_n)\rangle$. Hence $cl(\langle U_1, \cdots, U_n\rangle) = \langle Scl(U_1), \cdots, Scl(U_n)\rangle$.

(3) The proof is straightforward.

(4) It is sufficient to show that $\bigcup \mathcal{U} \in \tau$ for each base element $\mathcal{U} = \langle U_1, \cdots, U_n\rangle$ in 2^X_E. Let $V = \bigcup \langle U_1, \cdots, U_n\rangle$ and $e_x \in V$. Let $W \in \mathcal{SN}(e_x)$ such that $W \subset \bigcup_{i=1}^{n} U_i$ and $f_y \in W$. Take $e_{i_{x_i}} \in U_i$ for each $i \in \{1, \cdots, n\}$ and let $F = \left(\bigcup_{i=1}^{n} e_{i_{x_i}}\right) \cup \{f_y\}$. Since X is soft T_1, $F \in \tau^c$. Moreover, $F \subset \bigcup_{i=1}^{n} U_i$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Then $F \in \mathcal{U}$. Thus $f_y \in V$. So $W \subset V$. Hence $V \in \tau$. □

The following is an immediate consequence of Proposition 3.23.

Corollary 3.24 (See Lemma 2.3, [51])**.** *Let (X, τ, E) be a soft topological space, $\langle U_1, \cdots, U_n\rangle$ and $\langle V_1, \cdots, V_m\rangle$ any base members in $(2^X_e, \mathcal{T}_{v,e})$ for each $e \in E$. If X is a soft T_1-space, then the followings hold:*

(1) $\langle U_1(e), \cdots, U_n(e)\rangle \subset \langle V_1(e), \cdots, V_m(e)\rangle$ if and only if $\bigcup_{i=1}^{n} U_i(e) \subset \bigcup_{j=1}^{m} U_j(e)$ and for each $j \in \{1, \cdots, m\}$, there is $i \in \{1, \cdots, n\}$ such that $U_i(e) \subset V_j(e)$,

(2) $Cl(\langle U_1(e), \cdots, U_n(e)\rangle) = \langle cl(U_1(e)), \cdots, cl(U_n(e))\rangle$, *where* $Cl(\langle U_1(e), \cdots, U_n(e)\rangle)$ *denotes the closure of* $\langle U_1(e), \cdots, U_n(e)\rangle$ *in* $(2^X_e, \mathcal{T}_{v,e})$,

(3) *if* $\{U_\alpha(e)\}_{\alpha\in\Lambda}$ *is a neighborhood base at* x *in* (X, τ_e), *then* $\{\langle U_\alpha(e)\rangle\}_{\alpha\in\Lambda}$ *is a neighborhood base at* $\{x\}$ *in* $(2^X_e, \mathcal{T}_{v,e})$,

(4) *if* $\mathcal{O} \in \mathcal{T}_{v,e}$, *then* $\bigcup \mathcal{O} \in \tau_e$.

Notation 3.25. Let (X, τ, E) be a soft topological space and $e \in E$. Then

(i) $\mathcal{SF}_n(X) = \{F \in 2^X_E : F$ *has at most n soft points*$\}$,
(ii) $\mathcal{SF}(X) = \{F \in 2^X_E : F$ *is finite*$\}$,
(iii) $\mathcal{SK}(X) = \{F \in 2^X_E : F$ *is soft compact*$\}$,
(iv) $SC(X) = \{F \in 2^X_E : F$ *is soft connected*$\}$,
(v) $SC_K(X) = \mathcal{SK}(X) \cap SC(X)$,
(Vi) $\mathcal{F}_{n,e}(X) = \{F(e) \in 2^X_e : F(e)$ *has at most n elements*$\}$,
(Vii) $\mathcal{F}_e(X) = \{F(e) \in 2^X_e : F(e)$ *is finite*$\}$,
(Viii) $\mathcal{K}_e(X) = \{F(e) \in 2^X_e : F(e)$ *is compact*$\}$,
(ix) $C_e(X) = \{F(e) \in 2^X_e : F(e)$ *is connected*$\}$,
(x) $C_{K_e}(X) = \mathcal{K}_e(X) \cap C_e(X)$.

The topology on $\mathcal{SK}(X)$ [resp. $\mathcal{SF}(X)$, $\mathcal{SF}_n(X)$, $SC(X)$ and $SC_K(X)$] is the subspace topology induced by $\mathcal{T}_{Sv}$. Also, the topology on $\mathcal{K}_e(X)$ [resp. $\mathcal{F}_e(X)$, $\mathcal{F}_{n,e}(X)$, $C_e(X)$ and $C_{K_e}(X)$] is the subspace topology induced by $\mathcal{T}_{v,e}$. Moreover, $\mathcal{SF}(X)$ [resp. $\mathcal{SF}_n(X)$ and $SC_K(X)$] is a subspace of $\mathcal{SK}(X)$ and $\mathcal{F}_e(X)$ [resp. $\mathcal{F}_{n,e}(X)$ and $C_{K_e}(X)$] is a subspace of $\mathcal{K}_e(X)$.

Proposition 3.26. *Let* (X, τ, E) *be a soft* T_1*-space.*

(1) *If* $\mathcal{U}$ *is an open set in the subspace* $\mathcal{SK}(X)$, *then* $\bigcup \mathcal{U} \in \tau$,
(3) *If* $\mathcal{U}$ *is an open set in the subspace* $\mathcal{SF}_n(X)$, *then* $\bigcup \mathcal{U} \in \tau$,
(3) *If* $\mathcal{U}$ *is an open set in the subspace* $\mathcal{SF}(X)$, *then* $\bigcup \mathcal{U} \in \tau$.

Proof. (1) Suppose $\mathcal{U}$ is an open set in $\mathcal{SK}(X)$. Without loss of generality, let $\mathcal{U} = \langle U_1, \cdots, U_n\rangle \cap \mathcal{SK}(X)$ and $U = \bigcup \mathcal{U}$. Let $e_x \in U$. Then there is $i \in \{1, \cdots, n\}$ such that $e_x \in U_i$. Choose $e_{j_{x_j}} \in U_j$ for each $j \neq i$. For each $f_y \in U_i$, let $F_{f_y} = \left(\bigcup_{j=1}^n e_{j_{x_j}}\right) \cup \{f_y\}$. Since X is a soft T_1-space, $F_{f_y} \in \tau^c$. Then $F_{f_y} \in \langle U_1, \cdots, U_n\rangle \cap \mathcal{SK}(X)$. Thus $f_y \in F_{f_y} \subset U$. So $U_i \subset U$. Hence $\bigcup \mathcal{U} \in \tau$.

(2) Suppose $\mathcal{U}$ is an open set in $\mathcal{SF}_n(X)$, let $U = \bigcup \mathcal{U}$ and $e_{1_{x_1}} \in U$. Then there is $F \in \mathcal{SF}_n(X)$ such that $e_{1_{x_1}} \in F \in \mathcal{U}$. Let $F = \bigcup_{i=1}^m e_{i_{x_i}}$ for $m \leq n$. By the hypothesis, there is a basic open set $\langle U_1, \cdots, U_k\rangle \cap \mathcal{SF}_n(X)$ in 2^X_E such that $F \in \langle U_1, \cdots, U_k\rangle \cap \mathcal{SF}_n(X) \subset \mathcal{U}$, where $k \leq n$. We may suppose $e_{1_{x_1}} \in U_1$ and let $\mathcal{F} = \{U_1, U_2, \cdots, U_k\}$. For each $e_{i_{x_i}} \in F$, let $\mathcal{F}_i = \{U_j \in \mathcal{F} : e_{i_{x_i}} \in U_j\}$ and $V_i = \bigcap_{j=1}^k \mathcal{F}_i$, where $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, k\}$. It is clear that $V_i \in \tau$ for each $i \in \{1, 2, \cdots, m\}$. Then by Proposition 3.23(1), we have

$$F \in \langle V_1, \cdots, V_m\rangle \cap \mathcal{F}_n(X) \subset \langle U_1, \cdots, U_k\rangle \cap \mathcal{F}_n(X).$$

Let $f_y \in V_1$ and $E_{f_y} = \{f_y\} \cup \left(\bigcup_{i=1}^m e_{i_{x_i}}\right)$. Then by Result 2.20(1), $E_{f_y} \in \tau^c$. Moreover, $E_{f_y} \in \langle V_1, \cdots, V_m\rangle \cap \mathcal{F}_n(X)$. Thus $E_{f_y} \in \mathcal{U}$. So $E_{f_y} \subset U$. Hence $e_{1_{x_1}}, f_y \in V_1 \subset U$. Therefore $U = \bigcup \mathcal{U} \in \tau$.

(3) The proof follows from (2). $\square$

Corollary 3.27. *Let (X, τ, E) be a soft T_1-space and $e \in E$.*
(1) *If $\mathcal{U}$ is an open set in the subspace $\mathcal{K}_e(X)$, then $\bigcup \mathcal{U} \in \tau_{_e}$,*
(3) *If $\mathcal{U}$ is an open set in the subspace $\mathcal{F}_{n,e}(X)$, then $\bigcup \mathcal{U} \in \tau_{_e}$,*
(3) *If $\mathcal{U}$ is an open set in the subspace $\mathcal{F}_e(X)$, then $\bigcup \mathcal{U} \in \tau_{_e}$.*

Proposition 3.28. *(X, τ, E) is a soft topological space.*
(1) *If X is soft T_1, then $\mathcal{SF}(X)$ is dense in $2^X_{_E}$.*
(2) *If X is soft T_2, then $\mathcal{SF}_n(X)$ is closed in $2^X_{_E}$ for each $n \in \mathbb{N}$.*

Proof. (1) Suppose X is soft T_1. Then clearly, $cl(\mathcal{SF}(X)) \subset 2^X_{_E}$. Let $F \in 2^X_{_E}$ and $\mathcal{U} = \langle U_1, \cdots, U_n \rangle$ be any basic open set in $2^X_{_E}$ such that $F \in \mathcal{U}$. Then $F \subset \bigcup_{i=1}^n U_i$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Let $e_{i_{x_i}} \in F \cap U_i$ for each $i \in \{1, \cdots, n\}$ and $G = \bigcup_{i=1}^n e_{i_{x_i}}$. Then $G \in \tau^c$, $G \subset \bigcup_{i=1}^n U_i$ and $G \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Thus $G \in \mathcal{U} \cap \mathcal{SF}(X)$. So $F \in cl(\mathcal{SF}(X))$. Hence $2^X_{_E} \subset cl(\mathcal{SF}(X))$, i.e., $2^X_{_E} = cl(\mathcal{SF}(X))$. Therefore $\mathcal{SF}(X)$ is dense in $2^X_{_E}$.

(2) Suppose X is soft T_2 and let $n \in \mathbb{N}$ be fixed. Assume that $d(\mathcal{SF}_n(X)) \not\subset \mathcal{SF}_n(X)$, say $F \in d(\mathcal{SF}_n(X)) \setminus \mathcal{SF}_n(X)$.

Case 1. Suppose F is a soft set over X having exactly $n+1$ distinct elements, say $F = \bigcup_{i=1}^{n+1} e_{i_{x_i}}$. Then by the hypothesis, for any $i \neq j \in \{1, \cdots, n+1\}$, there are $U_i \in \mathcal{SN}(e_{i_{x_i}})$ and $U_j \in \mathcal{SN}(e_{j_{x_j}})$ such that $U_i \cap U_j = \varnothing_E$. Note that $F \subset \bigcup_{i=1}^{n+1} U_i$ and $F \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n+1\}$. Thus $F \in \langle U_1, \cdots, U_{n+1} \rangle$. So $\langle U_1, \cdots, U_{n+1} \rangle \cap \mathcal{SF}_n(X) \neq \varnothing$. But $\langle U_1, \cdots, U_{n+1} \rangle \cap \mathcal{SF}_n(X) = \varnothing$. This is a contradiction.

Case 2. Suppose F is a soft set over X containing more than $n+1$ elements, say $V = X \setminus \bigcup_{i=1}^{n+1} e_{i_{x_i}}$. Since X is soft T_1, $\bigcup_{i=1}^{n+1} e_{i_{x_i}} \in \tau^c$. Then $V \in \tau$. Now let $\{U_i\}$ be the collection of soft open neighborhoods of $e_{i_{x_i}}$ such as Case 1. Then $F \in \langle U_1, \cdots, U_{n+1}, V \rangle$. Thus $\langle U_1, \cdots, U_{n+1}, V \rangle \cap \mathcal{SF}_n(X) \neq \varnothing$. But $\langle U_1, \cdots, U_{n+1}, V \rangle \cap \mathcal{SF}_n(X) = \varnothing$. This is a contradiction. So, in either case, $\mathcal{SF}_n(X)$ is closed in $2^X_{_E}$. $\square$

Corollary 3.29 (See Lemma 2.4, [51])**.** *Let (X, τ, E) be a soft topological space.*
(1) *If X is soft T_1, then $\mathcal{F}_e(X)$ is dense in $2^X_{_e}$ for each $e \in E$.*
(2) *If X is soft T_2, then $\mathcal{F}_{n,e}(X)$ is closed in $2^X_{_e}$ for each $n \in \mathbb{N}$ and each $e \in E$.*

Definition 3.30 ([22])**.** Let (X, τ, E) be a soft topological space, Ψ a family of soft sets over X and $A \in SS_E(X)$. Then Ψ is called a:
(i) *cover* of A, if $A \subset \bigcup \Psi$,
(ii) *soft open cover* of A, if it is a cover of A and $\Psi \subset \tau$.
If Ψ is a cover of A and $\Omega \subset \Psi$ is a cover of A, then Ω is called a *subcover* of Ψ.

Definition 3.31 ([22])**.** Let (X, τ, E) be a soft topological space and Ψ a family of soft sets over X. Then
(i) we say that Ψ *has the finite intersection property*, if $\bigcap \Omega \neq \varnothing_E$ for each finite $\Omega \subset \Psi$,
(ii) X is said to be *compact*, if each soft open cover of X_E has a finite subcover.

Result 3.32 (Theorem 7.4, [22]). *A soft topological space is compact if and only if* $\bigcap \Psi \neq \varnothing_E$ *for each family* Ψ *of soft closed sets over* X *with the finite intersection property.*

Proposition 3.33. (1) *If* (X, τ, E) *is a soft regular space, then* $\bigcup \mathcal{B} \in 2^X_E$ *for each* $\mathcal{B} \in \mathcal{SK}(X)$.

(2) *If* (X, τ, E) *is a soft topological space, then* $\bigcup \mathcal{B} \in \mathcal{SK}(X)$ *for each* $\mathcal{B} \in \mathcal{SK}(X)$.

Proof. (1) Suppose X is a soft regular space and let $\mathcal{B} \in \mathcal{SK}(2^X_E)$. Let $A = \bigcup \mathcal{B}$, $e_x \in Scl(A)$ and $N \in \tau^c$ with $e_x \in N$. Then by Lemma 3.15,

$$\{F \in 2^X_E : F \cap N \neq \varnothing_E\} \in \mathcal{T}^c_v.$$

Thus $\mathcal{B} \bigcap \{F \in 2^X_E : F \cap N \neq \varnothing_E\}$ is a closed subcollection of $\mathcal{B}$ and has the finite intersection property. Since $\mathcal{B}$ is compact in 2^X_E, there is $\mathcal{D} \subset \mathcal{B}$ such that $\mathcal{D} \neq \varnothing$. Since X is a soft regular space, $e_x \in B$ for each $B \in \mathcal{D}$. Since $D = \bigcup \mathcal{D} \subset A$, $e_x \in A$. So $Scl(A) \subset A$, i.e., $Scl(A) = A$. Hence $\bigcup \mathcal{B} \in 2^X_E$.

(2) Suppose X is a soft topological space and let $\mathcal{B} \in \mathcal{SK}(\mathcal{SK}(X))$. Let $A = \bigcup \mathcal{B}$ and $\mathcal{U}$ be a collection of soft open sets which covers A. Now let $F \in \mathcal{B}$. Since $\mathcal{B} \in \mathcal{SK}(\mathcal{SK}(X))$, F is a compact subset of X_E. Since $F \subset A$ and $\mathcal{U}$ is a soft open cover of A, $\mathcal{U}$ is a soft open cover of F. Then there is a finite subcollection $\{U_{F,1}, U_{F,2} \cdots, U_{F,n}\}$ of $\mathcal{U}$ which covers F and $U_{F,i} \cap F \neq \varnothing_E$ for each $i \in \{1, 2, \cdots, n\}$. Thus $\mathcal{U}_F = \langle U_{F,1}, U_{F,2} \cdots, U_{F,n}\rangle \in \mathcal{T}_v$ and $F \in \mathcal{U}_F$. So $\{\mathcal{U}_F\}_{F \in \mathcal{B}}$ is an open cover of $\mathcal{B}$. On the other hand, since $\mathcal{B}$ is compact in $\mathcal{SK}(X)$, there is a finite subcollection $\{F_1, F_2, \cdots, F_m\}$ of $\mathcal{B}$ such that $\{U_{F_1}, U_{F_2}, \cdots, U_{F_m}\}$ is a cover of $\mathcal{B}$. Hence $\{U_{F_i,j}\}_{i=1,2,\cdots,m; j=1,2,\cdots,n}$ is a finite subcollection of $\mathcal{U}$ which covers A. Therefore $\bigcup \mathcal{B} \in \mathcal{SK}(X)$. □

We obtain the following results from Result 2.23 and Theorem 2.5 in [51].

Corollary 3.34 (See Theorem 2.5, [51]). (1) *If* (X, τ, E) *is a soft regular space, then* $\bigcup \mathcal{B} \in 2^X_e$ *for each* $\mathcal{B} \in \mathcal{K}_e(X)$ *for each* $e \in E$.

(2) *If* (X, τ, E) *is a soft topological space, then* $\bigcup \mathcal{B} \in \mathcal{K}_e(X)$ *for each* $\mathcal{B} \in \mathcal{K}_e(\mathcal{K}_e(X))$ *for each* $e \in E$.

Definition 3.35 ([39]). Let (X, τ, E) be a soft topological space and A, $B \in SS_E(X)$. Then A and B are said to be *soft separated* in X, if $A \cap Scl(B) = \varnothing_E = Scl(A) \cap B$.

Definition 3.36 ([39]). Let (X, τ, E) be a soft topological space and A, $B \in SS_E(X)$. Then

(i) A and B are said to be a *soft division* or a *soft separation* of X, if A and B are soft separated in X such that $A \cup B = X_E$, $A \neq \varnothing_E$ and $B \neq \varnothing_E$,

(ii) X is said to be *soft disconnected*, if it has a soft division,

(iii) X is said to be *soft connected*, if it has no a soft division, i.e., it cannot be expressed as the union of two nonempy disjoint soft open sets.

Result 3.37 (Theorem 4.5, [39]). *Let* (X, τ, E) *be a soft topological space. Then the followings are equivalent:*

(1) *X has a soft division.*

(2) *there are* A, $B \in \tau^c$ *such that* $A \cup B = X_E$ *and* $A \cap B = \varnothing_E$,

(3) *there are* $A,\ B \in \tau$ *such that* $A \cup B = X_E$ *and* $A \cap B = \varnothing_E$,
(4) X *has a proper soft open and soft closed set in* X.

Result 3.38 (Theorem 4.6, [39])**.** *Let* (X, τ, E) *be a soft topological space. Then the followings are equivalent:*
(1) X *is soft connected,*
(2) *there are not* $A,\ B \in \tau^c$ *such that* $A \cup B = X_E$ *and* $A \cap B = \varnothing_E$,
(3) *there are not* $A,\ B \in \tau$ *such that* $A \cup B = X_E$ *and* $A \cap B = \varnothing_E$,
(4) X *has at most two soft open and soft closed set in* X, *i.e.,* $\varnothing_E$ *and* X_E.

Proposition 3.39. *Let* (X, τ, E) *be a soft topological space. If* $\mathcal{B}$ *is a connected subset of* 2^X_E *containing at least one soft connected element, then* $\bigcup \mathcal{B}$ *is soft connected in* X.

Proof. Suppose $\mathcal{B}$ is a connected subset of 2^X_E containing at least one soft connected element, say $D \in \mathcal{B}$ and let $A = \bigcup \mathcal{B}$. Assume that A is not soft connected, i.e., A has a soft division. Then there are $A_1,\ A_2 \in SS_E(X)$ such that $A_1 \neq \varnothing_E,\ A_2 \neq \varnothing_E$, $A = A_1 \cup A_2$ and $Scl(A_1) \cup A_2 = \varnothing_E = A_1 \cup Scl(A_2)$. Suppose $D \subset A_1$ and let

$$\mathcal{A}_1 = \{F \in \mathcal{B} : F \subset A_1\},\ \mathcal{A}_2 = \{F \in \mathcal{B} : F \cap A_2 \neq \varnothing_E\}.$$

Then clearly, $D \in \mathcal{A}_1$ and $A \in \mathcal{A}_2$. Thus $\mathcal{A}_1 \neq \varnothing$ and $\mathcal{A}_2 \neq \varnothing$. Moreover, $\mathcal{B} = \mathcal{A}_1 \cup \mathcal{A}_2$. By Lemma 3.15, $\{F \in \mathcal{B} : F \subset Scl(A_1)\}$ and $\{F \in \mathcal{B} : F \cap A_2 \neq \varnothing_E\}$ are closed in $\mathcal{B}$ such that $\{F \in \mathcal{B} : F \subset Scl(A_1)\} \supset \mathcal{A}_1$ and $\{F \in \mathcal{B} : F \cap A_2 \neq \varnothing_E\} \supset \mathcal{A}_2$. So $\{F \in \mathcal{B} : F \subset Scl(A_1)\} = cl(\mathcal{A}_1)$ and $\{F \in \mathcal{B} : F \cap A_2 \neq \varnothing_E\} = cl(\mathcal{A}_2)$. Now assume that $cl(\mathcal{A}_1) \cap \mathcal{A}_2 \neq \varnothing$, say $F \in cl(\mathcal{A}_1) \cap \mathcal{A}_2$. Then clearly, $F \subset Scl(A_1)$ and $F \cap A_2 \neq \varnothing_E$. Since $Scl(A_1) \cup A_2 = \varnothing_E$, this is impossible. Thus $cl(\mathcal{A}_1) \cap \mathcal{A}_2 = \varnothing$. Similarly, $\mathcal{A}_1 \cap cl(\mathcal{A}_2) = \varnothing$. So $\mathcal{B}$ is not connected. This is a contradiction. Hence $\bigcup \mathcal{B}$ is soft connected in X. □

Corollary 3.40. *Let* (X, τ, E) *be a soft topological space. If* $\mathcal{B}$ *is a connected subset of* $\mathcal{SF}_n(X)$ [resp. $\mathcal{SF}(X)$ and $\mathcal{SK}(X)$] *containing at least one soft connected element, then* $\bigcup \mathcal{B}$ *is soft connected in* X. *In particular, if* $\mathcal{B}$ *is a connected subset of* $SC_K(X)$ *or* $SC(X)$, *then* $\bigcup \mathcal{B}$ *is soft connected in* X.

Proof. The proof follows from Proposition 3.39. □

Corollary 3.41 (See Theorem 2.8, [51])**.** *Let* (X, τ, E) *be a soft topological space and* $e \in E$. *If* $\mathcal{B}$ *is a connected subset of* 2^X_e *containing at least one connected element, then* $\bigcup \mathcal{B}$ *is connected in* (X, τ_e).

Corollary 3.42. *Let* (X, τ, E) *be a soft topological space and* $e \in E$. *If* $\mathcal{B}$ *is a connected subset of* $\mathcal{F}_{n,e}(X)$ [resp. $\mathcal{F}_e(X)$ and $\mathcal{K}_e(X)$] *containing at least one connected element, then* $\bigcup \mathcal{B}$ *is connected in* X. *In particular, if* $\mathcal{B}$ *is a connected subset of* $C_{K_e}(X)$ *or* $C_e(X)$, *then* $\bigcup \mathcal{B}$ *is connected in* X.

Proof. The proof follows from Corollary 3.40. □

4. Separation axioms in 2^X_E

In this section, we discuss some relationships between soft separation axioms in a soft topological space X and separation axioms in a soft hyperspace 2^X_E.

Proposition 4.1. *Let (X, τ, E) be a soft topological space. Then*
(1) 2^X_E *is always* T_0 (See Proposition 3.4(1), [44]),
(2) *if X is a soft T_1-space, then 2^X_E is T_1 but the converse is not true.*

Proof. (1) Let $A,\ B \in 2^X_E$ with $A \neq B$, i.e., $A \setminus B \neq \varnothing_E$. Let $x(e) \in (A \setminus B)(e) = A(e) \setminus B(e)$ for each $e \in E$ and $U = B^c$. Then $x \in A \setminus B$ and $U \in \tau$. Moreover, $A \subset X_E \cup U,\ A \cap X_E \neq \varnothing_E,\ A \cap U \neq \varnothing_E$ but $B \cap U = \varnothing_E$. Thus $A \in \langle X_E, U\rangle$, $B \notin \langle X_E, U\rangle$ and $\langle X_E, U\rangle \in \mathcal{T}_v$. So 2^X_E is T_0.

(2) Suppose X is a soft T_1-space and let $A,\ B \in 2^X_E$ such that $B \setminus A \neq \varnothing_E$, say $e_x \in B \setminus A$. Then $B \in \langle X_E, A^c\rangle,\ A \notin \langle X_E, A^c\rangle$ and $\langle X_E, A^c\rangle \in \mathcal{T}_v$. On the other hand, by Result 2.19(1), $e_x \in \tau^c$. Thus $e^c_x \in \tau$. Moreover, $A \in \left\langle e^c_x \right\rangle,\ B \notin \left\langle e^c_x \right\rangle$ and $\left\langle e^c_x \right\rangle \in \mathcal{T}_v$. So 2^X_E is T_1.

See Example 4.3 for the converse. □

Corollary 4.2 (See Theorems 4.9.1 and 4.9.2, [51])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. Then*
(1) 2^X_e *is always* T_0,
(2) *if X is a soft T_1-space, then 2^X_e is T_1 but the converse is not true.*

Example 4.3. Let X be a finite set containing more than two points and τ be the soft indiscrete topology on X. Then clearly, (X, τ, E) is not a soft T_1-space. On the other hand, $2^X_E = \{X_E\}$. Thus 2^X_E is T_1.

Result 4.4 (Theorem 3.2, [36])**.** *Let (X, τ, E) be a soft topological space. If X is T_2, A and B are soft compact subsets of X and $A \cap B = \varnothing_E$, then there are $U,\ V \in \tau$ such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing_E$.*

Lemma 4.5. *Let (X, τ, E) be a soft topological space. If X is soft T_1, then X is homeomorphic to the subspace $\mathcal{SF}_1(X)$.*

Proof. Let $f : X \to 2^X_E$ be the natural mapping defined by $f(x) = \{x\}$ for each $x \in X$. Let $e_x \in SP_E(X)$. Since X is T_1, $e_x \in \tau^c$. Then $\{e_x\} \in 2^X_E$. Let $\mathcal{V}$ be any neighborhood of $\{e_x\}$ in 2^X_E. Then $\mathcal{V}$ has the form $\langle V\rangle$, where $V \in \tau$. Thus $f(V) = \{V\} \subset \langle V\rangle$. So f is continuous. Now let $U \in \tau$ and $\{e_x\} \in f(U) = \{U\}$. Since $e_x \in U \in \tau$, there is U_{e_x} such that $e_x \in U_{e_x} \subset U$. Then $\{e_x\} \in \left\langle U_{e_x} \right\rangle = 2^{U_{e_x}}_E$ and $\bigcup_{e_x \in U} 2^{U_{e_x}}_E \subset U$, i.e., $\bigcup_{e_x \in U} 2^{U_{e_x}}_E = U$. Since $2^{U_{e_x}}_E$ is open in 2^X_E, $\bigcup_{e_x \in U} 2^{U_{e_x}}_E$ is open in 2^X_E. Furthermore, f is a homeomorphism between X and the subspace $\mathcal{SF}_1(X)$. Thus X is homeomorphic to $\mathcal{SF}_1(X)$. □

Corollary 4.6. *Let (X, τ, E) be a soft topological space and $e \in E$. If X is soft T_1, then (X, τ_e) is homeomorphic to the subspace $\mathcal{F}_{1,e}(X)$.*

Proof. The proof is similar to Lemma 4.5. □

Theorem 4.7. *Let (X, τ, E) be a soft topological space. Then X is soft T_2 if and only if $\mathcal{SK}(X)$ is T_2.*

Proof. Suppose X is soft T_2 and let $A,\ B \in \mathcal{SK}(X)$ such that $A \neq B$, say $B \setminus A \neq \varnothing_E$, i.e., $e_x \in B \setminus A$. Since $A \in \mathcal{SK}(X)$, A is a soft compact subset of X. Also,

$e_x \notin A$ is a soft compact subset of X. Then by Result 4.4, there are $U,\ V \in \tau$ such that $e_x \in U,\ A \subset V$ and $U \cap V = \varnothing_E$. Thus we have

$$A \in \langle V\rangle \cap \mathcal{SK}(X),\ B \in \langle X_E, U\rangle \cap \mathcal{SK}(X)$$

and

$$(\langle V\rangle \cap \mathcal{SK}(X)) \cap (\langle X_E, U\rangle \cap \mathcal{SK}(X)) = \varnothing.$$

Moreover, $\langle V\rangle \cap \mathcal{SK}(X)$ and $\langle X_E, U\rangle \cap \mathcal{SK}(X)$ are open in the subspace $\mathcal{SK}(X)$. So $\mathcal{SK}(X)$ is T_2.

Conversely, suppose $\mathcal{SK}(X)$ is T_2. Then by Lemma 4.5, X is homeomorphic to the subspace $\mathcal{SF}_1(X)$. Since $\mathcal{SF}_1(X) \subset \mathcal{SK}(X)$, $\mathcal{SF}_1(X)$ is T_2. Thus X is soft T_2. $\square$

Corollary 4.8 (See Theorem 4.9.8, [51])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. Then (X, τ_e) is T_2 if and only if $\mathcal{K}_e(X)$ is T_2.*

Proof. Suppose (X, τ_e) is T_2 and let $A(e),\ B(e) \in \mathcal{K}_e(X)$ such that $A(e) \neq B(e)$, i.e., $B(e) \setminus A(e) \neq \varnothing$, say $x \in B(e) \setminus A(e)$. Since $A(e)$ is compact set in (X, τ_e) and $x \notin A(e)$, there are $U(e),\ V(e) \in \tau_e$ such that $x \in U(e),\ A(e) \subset V(e)$ and $U(e) \cap V(e) = \varnothing$. Then we have

$$A(e) \in \langle V(e)\rangle = \langle V(e)\rangle \cap \mathcal{K}_e(X),\ B \in \langle X, U(e)\rangle = \langle X, U(e)\rangle \cap \mathcal{K}_e(X)$$

and

$$(\langle V(e)\rangle \cap \mathcal{K}_e(X)) \cap (\langle X, U(e)\rangle \cap \mathcal{K}_e(X)) = \varnothing.$$

Thus $\mathcal{K}_e(X)$ is T_2.

Conversely, suppose $\mathcal{K}_e(X)$ is T_2. Then by Corollary 4.6, X is homeomorphic to $\mathcal{F}_{1,e}(X)$. Since $\mathcal{F}_{1,e}(X) \subset \mathcal{K}_e(X)$, by the hypothesis, $\mathcal{F}_{1,e}(X)$ is T_2. Since the property of T_2 is a topological property, (X, τ_e) is T_2. $\square$

Theorem 4.9. *Let (X, τ, E) be a soft topological space. Then the following statements are equivalent:*

(1) *X is soft T_2,*
(2) *$\mathcal{K}(X)$ is T_2,*
(3) *$\mathcal{F}_n(X)$ is T_2,*
(4) *$\mathcal{F}(X)$ is T_2,*
(5) *$\mathcal{C}_K(X)$ is T_2.*

Proof. (1)⇔(2) The proof follows from Theorem 4.7.

The proofs of (2)⇒(3), (2)⇒(4) and (2)⇒(5) are straightforward.

(5)⇔(1) Suppose $\mathcal{C}_K(X)$ is T_2. Then clearly, $\mathcal{F}_1(X) \subset \mathcal{C}_K(X)$. By Lemma 4.5, $\mathcal{F}_1(X)$ is homeomorphic to X. Thus by the hypothesis, X is soft T_2. $\square$

Corollary 4.10. *Let (X, τ, E) be a soft topological space and $e \in E$. Then the following statements are equivalent:*

(1) *(X, τ_e) is T_2,*
(2) *$\mathcal{K}_e(X)$ is T_2,*
(3) *$\mathcal{F}_{n,e}(X)$ is T_2,*
(4) *$\mathcal{F}_e(X)$ is T_2,*
(5) *$\mathcal{C}_{K_e}(X)$ is T_2.*

Proposition 4.11. *Let (X, τ, E) be a soft topological space.*
(1) *If X is soft regular, then 2^X_E is T_2.*
(2) *If X is soft T_1 and 2^X_E is T_2, then X is soft regular.*

Proof. (1) Suppose X is soft regular and let $A,\ B \in 2^X_E$ such that $A \neq B$, say $e_x \in B \setminus A$. Then clearly, $A \in \tau^c$ such that $e_x \notin A$. Since X is soft regular, there are $U,\ V \in \tau$ such that $A \subset U,\ e_x \in V$ and $U \cap V = \varnothing_E$. Thus $A \in \langle U \rangle,\ B \in \langle X_E, V\rangle$ and $\langle U \rangle \cap \langle X_E, V\rangle = \varnothing$. So 2^X_E is T_2.

(2) Suppose X is soft T_1 and 2^X_E is T_2, let $F \in \tau^c$ and $e_x \in SP_E(X)$ such that $e_x \notin F$. Since X is soft T_1, $F \cup e_x \in \tau^c$. Then $F,\ F \cup e_x \in 2^X_E$ such that $F \neq F \cup e_x$. Since 2^X_E is T_2, there are basic open sets $\mathcal{U} = \langle U_1 \cdots, U_n\rangle,\ \mathcal{V} = \langle V_1 \cdots, V_m\rangle$ in 2^X_E such that

$$F \in \mathcal{U},\ F \cup e_x \in \mathcal{V},\ \mathcal{U} \cap \mathcal{V} = \varnothing. \tag{4.1}$$

Let $U = \bigcup_{i=1}^n U_i,\ V = \bigcup_{j=1}^m V_j$ and assume that if $e_x \in V_j$, then $U \cap V \cap V_j \neq \varnothing_E$, say ${e_j}_{x_j} \in U \cap V \cap V_j$ for each $j \in \{1, \cdots, m\}$. Let $K = \bigcup_{j=1}^m {e_j}_{x_j}$ and $A = F \cup K$. Then we can easily see that

$$A \subset U,\ A \cap U_i \neq \varnothing_E \text{ and } A \subset V,\ A \cap V_j \neq \varnothing_E$$

for each $i \in \{1, \cdots, n\}$ and each $j \in \{1, \cdots, m\}$. Thus $A \in \mathcal{U} \cap \mathcal{V}$. This is a contradiction. So there is $j \in \{1, \cdots, m\}$ such that $e_x \in V_j,\ U \cap V \cap V_j = \varnothing_E$. By (4.1), $F \subset U \cap V$ and $U \cap V \in \tau$. Hence X is soft regular. $\square$

Remark 4.12. If (X, τ, E) is soft T_1 but not soft regular, then Proposition 4.11 (1) may be not true in general (See Example 4.13).

Example 4.13. Let (X, τ, E) be a soft T_1-space which is not soft regular. Then there are $e_x \in SP_E(X)$ and $A \in \tau^c$ with $e_x \notin A$ such that e_x and A cannot be separated by soft open sets in X. It is obvious that $A,\ A \cup \{e_x\} \in 2^X_E$ such that $A \neq A \cup \{e_x\}$. Let $\mathcal{U} = \langle U_1, \cdots, U_n\rangle,\ \mathcal{V} = \langle V_1, \cdots, V_m\rangle \in \mathcal{T}_v$ such that $A \in \mathcal{U}$ and $A \cup \{e_x\} \in \mathcal{V}$. Let $U = \bigcup_{i=1}^n U_i$ and $W = \bigcap_{j=1}^m \{V_j : e_x \in V_j\}$. Then clearly, $U,\ W \in \tau$, $e_x \in W$ and $A \subset U$. Since e_x and A cannot be separated, there $f_y \in W \cap U$. Let $B = A \cup \{f_y\}$. Then clearly, $B \in \mathcal{U} \cap \mathcal{V}$. Thus 2^X_E is not T_2.

Corollary 4.14 (See Theorems 17.IV.3 and 17.IV.4, [49]; Theorem 4.9.3, [51]). *Let (X, τ, E) be a soft topological space and $e \in E$.*
(1) *If X is soft regular, then 2^X_e is T_2.*
(2) *If X is soft T_1 and 2^X_e is T_2, then (X, τ_e) is regular.*

Question 4.15. (1) Let (X, τ, E) be a soft topological space. If $SC(X)$ is T_2, then is X soft regular?

(2) Let (X, τ, E) be a soft T_1-space. Then X is soft regular if and only if $\mathcal{K}(X)$ is regular?

Theorem 4.16 (See Theorem 4.9.10, [51]). *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then (X, τ_e) is regular if and only if $\mathcal{K}_e(X)$ is regular.*

Proof. Suppose $\mathcal{K}_e(X)$ is regular. Then the subspace $\mathcal{F}_{1,e}(X)$ is regular. By Corollary 4.6, (X, τ_e) and $\mathcal{F}_{1,e}(X)$ are homeomorphic. Thus (X, τ_e) is regular.

Conversely, suppose (X, τ_e) is regular. Let $A(e) \in \mathcal{K}_e(X)$, $A(e) \in \langle U_1(e), \cdots, U_n(e)\rangle \cap \mathcal{K}_e(X)$ and $U = \bigcup_{i=1}^{n} U_i(e)$, where $\langle U_1(e), \cdots, U_n(e)\rangle \in \mathcal{T}_{v,e}$. Since $A(e) \subset U(e)$, $A(e)$ is compact and (X, τ_e) is regular, by Theorem 5.10 in [52], there is $V(e) \in \tau_e$ such that $A(e) \subset V(e) \subset cl(V(e)) \subset U(e)$. Let $x_i \in U_i(e) \cap V(e)$ for each $i \in \{1, 2, \cdots, n\}$. Since (X, τ_e) is regular, by Theorem 5.11 in [50], there is $V_i(e) \in \tau_e$ such that $x_i \in V_i(e) \subset cl(V_i(e)) \subset U_i(e)$ for each $i \in \{1, 2, \cdots, n\}$. Then $A(e) \in \langle V_1(e), \cdots, V_n(e), V\rangle \cap \mathcal{K}_e(X)$, where $\langle V_1(e), \cdots, V_n(e), V(e)\rangle \in \mathcal{T}_{v,e}$. Since (X, τ, E) is a soft T_1-space, by Corollary 3.24(2), we have

$$cl(\langle V_1(e), \cdots, V_n(e), V(e)\rangle) = \langle cl(V_1(e)), \cdots, cl(V_n(e)), cl(V(e))\rangle \subset \langle U_1(e), \cdots, U_n(e)\rangle .$$

Thus $\mathcal{K}_e(X)$ is regular. □

Theorem 4.17. *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then The followings are equivalent:*

(1) *(X, τ_e) is regular,*
(2) *$\mathcal{K}_e(X)$ is regular,*
(3) *$\mathcal{F}_e(X)$ is regular,*
(4) *$\mathcal{F}_{n,e}(X)$ is regular,*
(5) *$\mathcal{C}_{K_e}(X)$ is regular.*

Proposition 4.18. *If (X, τ, E) is a soft T_3-space, then the sets*

$$\{(K, L) \in 2^X_E \times 2^X_E : K \subset L\} \text{ and } \{(e_x, K) \in X \times 2^X_E : e_x \in K\}$$

are closed in $2^X_E \times 2^X_E$ and $X \times 2^X_E$, respectively.

Proof. Suppose X is soft T_3 and let $\mathcal{U} = \{(K, L) \in 2^X_E \times 2^X_E : K \subset L\}$. It is sufficient to prove that $\mathcal{U}^c$ is open in $2^X_E \times 2^X_E$. It is clear that

$$\mathcal{U}^c = \{(K, L) \in 2^X_E \times 2^X_E : K \not\subset L\}.$$

Since X is soft regular and $K \not\subset L$, there is $V \in \tau$ such that $K \cap V \neq \varnothing_E$ and $L \subset Scl(V)^c$. Then we have

$$\mathcal{U}^c = \bigcup_{V \in \tau} \left[\{K \in 2^X_E : K \cap V \neq \varnothing_E\} \times \{L \in 2^X_E : L \subset Scl(V)^c\}\right].$$

Since $V,\ Scl(V)^c \in \tau$, by Theorem 3.19, we get

$$\{K \in 2^X_E : K \cap V \neq \varnothing_E\},\ \{L \in 2^X_E : L \subset Scl(V)^c\} \in \mathcal{T}_{Sv}.$$

Thus $\mathcal{U}^c$ is open in $2^X_E \times 2^X_E$. So $\mathcal{U}$ is closed in $2^X_E \times 2^X_E$.

The proof of the second part is similar to the one of the first part. □

Corollary 4.19 (See Theorem 17.IV.1, [49])**.** *Let (X, τ, E) be a soft T_3-space and $e \in E$. Then the sets $\{(K(e), L(e)) \in 2^X_e \times 2^X_e : K,\ L \in 2^X_E,\ K \subset L\}$ and*

$$\{(x, K(e)) \in X \times 2^X_e : e_x \in X_E,\ K \in 2^X_E,\ e_x \in K\}$$

are closed in $2^X_e \times 2^X_e$ and $(X, \tau_e) \times (2^X_e, \mathcal{T}_{v,e})$, respectively.

Proposition 4.20. *Let (X, τ, E) be a soft T_1-space. If the set $\{(e_x, K) \in X \times 2^X_E : e_x \in K\}$ is closed in $X \times 2^X_E$, then X is soft regular.*

Proof. Suppose $\{(e_x, K) \in X \times 2^X_E : e_x \in K\}$ is closed in $X \times 2^X_E$, let $\mathcal{U} = \{(e_x, K) \in X \times 2^X_E : e_x \in K\}$ and $(e_x, K) \in \mathcal{U}^c$. Since $\mathcal{U}^c = \{(e_x, K) \in X \times 2^X_E : e_x \notin K\} \in \mathcal{T}_{Sv}$, there are $U,\ U_1,\ \cdots,\ U_n \in \tau$ such that

$$(e_x, K) \in U \times \langle U_1, \cdots, U_n\rangle \subset \mathcal{U}.$$

Then $e_x \in U$ and $K \in \langle U_1, \cdots, U_n\rangle$. Thus $K \subset \bigcup_{i=1}^n U_i = V$ and $K \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. It is obvious that $V \in \tau$.

Assume that $U \cap V \neq \varnothing_E$, say $e_x \in U \cap V$ and let $F = K \cup e_x$. Since X is T_1, $e_x \in \tau^c$. Then $F \in \tau^c$. Moreover, $e_x \in U$, $F \subset U$ and $F \cap U_i$ for each $i \in \{1, \cdots, n\}$. Thus $(e_x, F) \in \mathcal{U}^c$. So $e_x \notin F$. This is a contradiction. Hence $U \cap V = \varnothing_E$. Therefore X is soft regular. □

Corollary 4.21 (See Theorem 17.IV.2, [49])**.** *Let (X, τ, E) be a soft T_1-space and $e \in E$. Then the set*

$$\{(x, K(e)) \in X \times 2^X_e : e_x \in X_E,\ K \in 2^X_E,\ e_x \in K\}$$

is closed in $(X, \tau_e) \times (2^X_E, \mathcal{T}_{v,e})$.

Theorem 4.22. *If (X, τ, E) is a soft T_1-space, then followings are equivalent:*
(1) *X is soft regular,*
(2) *$\{(K, L) \in 2^X_E \times 2^X_E : K \subset L\}$ is closed in $2^X_E \times 2^X_E$,*
(3) *$\{(e_x, K) \in X \times 2^X_E : e_x \in K\}$ is closed in $X \times 2^X_E$,*
(4) *2^X_E is T_2.*

Proof. The proof follows from Propositions 4.18 and 4.20 □

Theorem 4.23 (See Corollary 17.IV.5, [49])**.** *Let (X, τ, E) be a soft T_3-space and $e \in E$. Then followings are equivalent:*
(1) *(X, τ_e) is regular,*
(2) *$\{(K(e), L(e)) \in 2^X_e \times 2^X_E : K,\ L \in 2^X_E,\ K \subset L\}$ is closed in $2^X_2 \times 2^X_e$,*
(3) *$\{(x, K(e)) \in X \times 2^X_e : e_x \in X_E,\ K \in 2^X_E,\ e_x \in K\}$ is closed in $(X, \tau_e) \times (2^X_e, \mathcal{T}_{v,e})$,*
(4) *2^X_e is T_2.*

Proposition 4.24. *If (X, τ, E) is a soft normal space, then $SC(X)$ is closed in 2^X_E.*

Proof. Suppose X is a soft normal space and assume that $F \in 2^X_E$ such that F is a limit point of $SC(X)$ and $F \in 2^X_E \setminus SC(X)$. Let $F \in \langle U_1, \cdots, U_n\rangle$ and $U = \bigcup_{i=1}^n U_i$. Since F is not soft disconnected and soft closed in X, there are $F_1,\ F_2 \in \tau^c$ such that $F = F_1 \cup F_2$ and $F_1 \cap F_2 = \varnothing_E$. Since X is soft normal, there are $W_1,\ W_2 \in \tau$ such that $F_1 \subset W_1,\ F_2 \subset W_2$, $W_1 \cup W_2 \subset U$ and $W_1 \cap W_2 = \varnothing_E$. Let $\{U^1_{i_1}, \cdots, U^1_{i_k}\}$ and $\{U^2_{i_1}, \cdots, U^2_{i_p}\}$ be the collection of all U_i such that $U_i \cap F_1 \neq \varnothing_E$ and $U_i \cap F_2 \neq \varnothing_E$, respectively for each $i \in \{1, \cdots, n\}$. Now let $V^1_j = W_1 \cap U^1_{i_j}$ for each $j \in \{1, \cdots, k\}$ and $V^2_l = W_2 \cap U^2_{i_l}$ for each $l \in \{1, \cdots, p\}$. Then we have

$$F_1 \subset \bigcup_{j=1}^k V^1_j = V^1,\ F_2 \subset \bigcup_{l=1}^p V^p_j = V^2 \text{ and } V^1 \cap V^1 = \varnothing_E.$$

We can easily prove that $F \in \left\langle V_1^1, \cdots, V_k^1, V_1^2, \cdots, V_p^2 \right\rangle \subset \langle U_1, \cdots, U_n \rangle$. Since F is a limit point of $SC(X)$, there is $C \in SC(X)$ such that $C \in \left\langle V_1^1, \cdots, V_k^1, V_1^2, \cdots, V_p^2 \right\rangle$. Thus $C \subset V^1 \cup V^2$ and $C \cap V^i \neq \varnothing_E$ for each $i \in \{1, 2\}$. So C is soft disconnected. Since $C \in SC(X)$, C is soft connected. This is a contradiction. Hence $F \in SC(X)$. Therefore $SC(X)$ is closed in $2^X_{_E}$. □

Corollary 4.25. *Let (X, τ, E) be a soft topological space and $e \in E$. If $(X, \tau_{_e})$ is normal, then $C_e(X)$ is closed in $2^X_{_e}$.*

Proof. The proof is similar to Proposition 4.24. □

Proposition 4.26. *If (X, τ, E) is a soft normal space, then the set*

$$\{(K, L) \in 2^X_{_E} \times 2^X_{_E} : K \cap L = \varnothing_E\}$$

is open in $2^X_{_E} \times 2^X_{_E}$.

Proof. Suppose X is soft normal and let $K,\ L \in \tau^c$ such that $K \cap L = \varnothing_E$. Then there are $U,\ V \in \tau$ such that $K \subset U,\ L \subset V,\ U \cap V = \varnothing_E$. Thus we have

$$\begin{aligned} &\{(K, L) \in 2^X_{_E} \times 2^X_{_E} : K \cap L = \varnothing_E\} \\ &= \textstyle\bigcup_{U,\ V \in \tau,\ U \cap V = \varnothing_E} \{(K, L) \in 2^X_{_E} \times 2^X_{_E} : K \subset U,\ L \subset V\} \\ &= \textstyle\bigcup_{U,\ V \in \tau,\ U \cap V = \varnothing_E} \left(2^U_{_E} \times 2^V_{_E}\right). \end{aligned}$$

Since $U,\ V \in \tau$, by Theorem 3.19, $2^U_{_E},\ 2^V_{_E} \in \mathcal{T}_{Sv}$. Thus $2^U_{_E} \times 2^V_{_E}$ is open in $2^X_{_E} \times 2^X_{_E}$. So the result holds. □

Corollary 4.27 (See Theorem 17.V.1, [49])**.** *Let (X, τ, E) be a soft topological space and $e \in E$. If $(X, \tau_{_e})$ is normal, then the set*

$$\{(K(e), L(e)) \in 2^X_{_e} \times 2^X_{_e} : K,\ L \in 2^X_{_E},\ K \cap L = \varnothing_E\}$$

is open in $2^X_{_e} \times 2^X_{_e}$.

Proposition 4.28. *If (X, τ, E) is soft T_1 and $\{(K, L) \in 2^X_{_E} \times 2^X_{_E} : K \cap L = \varnothing_E\}$ is open in $2^X_{_E} \times 2^X_{_E}$, then X is soft normal.*

Proof. Suppose X is soft T_1 and $\{(K, L) \in 2^X_{_E} \times 2^X_{_E} : K \cap L = \varnothing_E\}$ is open in $2^X_{_E} \times 2^X_{_E}$. Then there are basic open sets $\mathcal{U} = \langle U_1, \cdots, U_n \rangle,\ \mathcal{V} = \langle V_1, \cdots, V_m \rangle$ in $2^X_{_E}$ such that $K \in \mathcal{U},\ L \in \mathcal{V}$ and $(K, L) \in \mathcal{U} \times \mathcal{V} \Rightarrow K \cap L = \varnothing_E$. Thus we get

$$K \subset \bigcup_{i=1}^n U_i = U,\ K \cap U_i \neq \varnothing_E\ (i \leq n), \tag{4.2}$$

$$L \subset \bigcup_{j=1}^m V_j = V,\ L \cap V_j \neq \varnothing_E\ (j \leq m), \tag{4.3}$$

$$[(K \subset U)(K \cap U_i \neq \varnothing_E)(L \subset V)(L \cap V_j \neq \varnothing_E)] \Rightarrow K \cap L = \varnothing_E. \tag{4.4}$$

Assume that $U \cap V \neq \varnothing_E$, say $e_{_x} \in U \cap V$. Let us take $e_{i_{a_i}} \in K \cap U_i$ for $i \leq n$ and $e_{j_{b_j}} \in L \cap V_j$ for $j \leq m$. Let $F = e_{_x} \cup \bigcup_{i=1}^n e_{i_{a_i}}$ and $G = e_{_x} \cup \bigcup_{j=1}^m e_{j_{b_j}}$. Since X is T_1, $F,\ G \in \tau^c$. Moreover, $[(F \subset U)(F \cap U_i \neq \varnothing_E)(G \subset V)(G \cap V_j \neq \varnothing_E)]$. Then by (4.4), $F \cap G = \varnothing_E$. But $e_{_x} \in F \cap G$. This is a contradiction. Thus $U \cap V \neq \varnothing_E$. So X is soft regular. □

Corollary 4.29 (See Theorem 17.V.2, [49]). *Let* (X, τ, E) *be soft* T_1 *and* $e \in E$. *If* $\{(K(e), L(e)) \in 2^X_e \times 2^X_e : K,\ L \in 2^X_E,\ K \cap L = \varnothing_E\}$ *is open in* $2^X_e \times 2^X_e$, *then* (X, τ_e) *is normal.*

Proposition 4.30. *If* (X, τ, E) *is soft normal, then* 2^X_E *is regular.*

Proof. Suppose X is soft normal and let $\mathcal{A}$ be any basic open set in 2^X_E with $F \in \mathcal{A}$, i.e., $F \notin \mathcal{A}^c$, where $F \in 2^X_E$ and $\mathcal{A} = \langle A_1, \cdots, A_n \rangle$. To prove that 2^X_E is regular, let us show that the followings hold: there are $\mathcal{U},\ \mathcal{V} \in \mathcal{T}_{Sv}$ such that

$$F \in \mathcal{U},\ \mathcal{A}^c \subset \mathcal{V},\ \mathcal{U} \cap \mathcal{V} = \varnothing. \tag{4.5}$$

Since $F \in \mathcal{A}$, we have

$$F \subset \bigcup_{i=1}^{n} A_i = A, \tag{4.6}$$

$$F \cap A_i \neq \varnothing_E \text{ each } i \in \{1, \cdots, n\}. \tag{4.7}$$

Since $A_i \in \tau$ for each $i \in \{1, \cdots, n\}$, $A \in \tau$. Then $A^c \in \tau^c$. Moreover, by (4.6), $F \cap A^c = \varnothing_E$. Since X is soft normal, there are $U,\ V \in \tau$ such that

$$F \subset U,\ A^c \subset V,\ U \cap V = \varnothing_E. \tag{4.8}$$

According to (4.7), put $e_{i_{x_i}} \in F \cap A_i$ and $U_i,\ V_i \in \tau$ with $U_i \subset U,\ V_i \subset V$ such that

$$e_{i_{x_i}} \in U_i,\ A_i^c \subset V_i,\ U_i \cap V_i = \varnothing_E \tag{4.9}$$

for each $i \in \{1, \cdots, n\}$, where $U = \bigcup_{i=1}^n U_i$ and $V = \bigcup_{i=1}^n V_i$.

Now let us define $\mathcal{U},\ \mathcal{V}$ in (4.5) as follows:

$$\mathcal{U} = \langle U_1, \cdots, U_n \rangle,\ \mathcal{V} = \{G \in 2^X_E : \text{either } G \cap V \neq \varnothing_E \text{ or } G \subset V_i \text{ for some } i \leq n\}.$$

By the first parts of (4.8) and (4.9), $F \in \mathcal{U}$. Also, by the third parts of (4.8) and (4.9), $\mathcal{U} \cap \mathcal{V} = \varnothing$. Then the first and third parts of (4.5) hold. Let $F \in \mathcal{A}^c$, i.e., $F \notin \mathcal{A}$. Then either $F \not\subset A$ or $F \cap A_i = \varnothing_E$ for some $i \leq n$. Thus by the second parts of (4.8) and (4.9), either $F \cap V \neq \varnothing_E$ or $F \subset V_i$ for some $i \leq n$. So $F \in \mathcal{V}$, i.e., $\mathcal{A}^c \subset \mathcal{V}$. Hence the second part of (4.5) holds. Therefore 2^X_E is regular. □

Corollary 4.31 (See Theorem 17.V.3, [49]). *Let* (X, τ, E) *be a soft topological space and* $e \in E$. *If* (X, τ_e) *is normal, then* 2^X_e *is regular.*

Proposition 4.32. *If* (X, τ, E) *is soft* T_1 *and* 2^X_E *is regular, then* X *is soft normal.*

Proof. Suppose X is soft T_1 and 2^X_E is regular and let $A,\ B \in \tau^c$ such that $A \cap B = \varnothing_E$. In order to prove that X is soft normal, we will show that the following conditions hold:

(4.10) there is $U \in \tau$ such that $A \subset U \subset Scl(U) \subset B^c$, i.e., $Scl(U) \cap B = \varnothing_E$.

Since $A \cap B = \varnothing_E$, $A \subset B^c$. Then $A \in 2^{B^c}_E$. Since X is soft T_1 and $B^c \in \tau$, by Theorem 3.19, $2^{B^c}_E \in \mathcal{T}_{Sv}$. Since 2^X_E is regular, there is a basic open set $\mathcal{U} = \langle U_1, \cdots, U_n \rangle$ in 2^X_E such that

$$A \in \mathcal{U} \subset cl(\mathcal{U}) \subset 2^{B^c}_E. \tag{4.11}$$

Thus $A \subset U = \bigcup_{i=1}^{n} U_i$ and $A \cap U_i \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. Since X is soft T_1, by Proposition 3.23(2), $cl(\mathcal{U}) = \langle Scl(U_1), \cdots, Scl(U_n)\rangle \subset 2_E^{B^c}$. Since $\varnothing_E \neq U_i \subset U$ for each $i \in \{1, \cdots, n\}$, $Scl(U) \cap Scl(U_i) = Scl(U_i) \neq \varnothing_E$ for each $i \in \{1, \cdots, n\}$. So $Scl(U) \in \langle Scl(U_1), \cdots, Scl(U_n)\rangle = cl(\mathcal{U})$. By (4.10), $Scl(U) \in 2_E^{B^c}$, i.e., $Scl(U) \subset B^c$. Hence (4.11) holds. Therefore X is soft normal. □

Corollary 4.33 (See Theorem 17.V.4, [49]). *Let (X, τ, E) be soft T_1 and $e \in E$. If 2_e^X is regular, then (X, τ_e) is normal.*

We have the following consequence from Propositions 4.26, 4.28, 4.30 and 4.32.

Theorem 4.34. *Let (X, τ, E) be soft T_1. Then the followings are equivalent:*
(1) *X is soft normal,*
(2) *$\{(K, L) \in 2_E^X \times 2_E^X : K \cap L = \varnothing_E\}$ is open in $2_E^X \times 2_E^X$,*
(3) *2_E^X is regular.*

Corollary 4.35 (See Corollary 17.V.5, [49]). *Let (X, τ, E) be soft T_1 and $e \in E$. the followings are equivalent:*
(1) *(X, τ_e) is normal,*
(2) *$\{(K(e), L(e)) \in 2_e^X \times 2_e^X : K,\ L \in 2_E^X,\ K \cap L = \varnothing_E\}$ is open in $2_e^X \times 2_e^X$,*
(3) *2_e^X is regular.*

5. Conclusions

We obtained various basic properties in a soft hyperspace, for examples, Propositions 3.23, 3.28, 3.33 and 3.39. Also, we studied some relationships between separation axioms in a soft topological space and its soft hyperspace. In particular, in Theorem 4.7, it has been proven that X is soft T_2 and $\mathcal{SK}(X)$ is T_2 are equivalent. In Proposition 4.11, sufficient conditions were obtained for 2_E^X to be T_2 and X to be soft regular, respectively. Also, we had a sufficient condition for $SC(X)$ to be closed in 2_E^X (See Proposition 4.24). We got two sufficient conditions for X to be soft normal (See Propositions 4.28 and 4.32). Furthermore, we obtained a sufficient condition for 2_E^X to be regular (See Proposition 4.30). Results related to classical hyperspace were treated as Corollary whenever possible.

In the future, we intend to investigate compactness, local compactness, separability, first and second countability, connectedness and local connectedness in a soft hyperspace.

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