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Abstract. In this paper, we introduce the notion of $IPS$ ($DPS$)-soft sets based on the soft bitopological ordered space $(X, \tau_1, \tau_2, E, \lesssim)$ and study some of its properties. Based on this notion we introduce the notions of $IP$ ($DP$)- soft connected (disconnected) spaces and study some of their characterizations and properties. Also, we study the connected of $IP$ ($DP$)-soft sets by using the soft space $(X, \tau_{12}, E, \lesssim)$. Some examples have given to support these concepts.

2020 AMS Classification: 54A05, 54A10, 54B05, 54C55 54D10, 54E55.

Keywords: Soft bitopological ordered spaces, $IPS(DPS)$-soft sets, $IP(DP)$-soft connected (disconnected) spaces, $IP(DP)$-connected (disconnected) soft sets.

Corresponding Author: Salama Hussien Ali Shalil (`slamma_elarabi@yahoo.com`)

## 1. Introduction

In 1965, Nachbin [1] proposed the concept of topological ordered spaces which add partial order relations to topological structures as a generalization of topological spaces. McCartan [2] went on to utilize monotone neighborhoods in order to study ordered separation axioms. In order to deal with the vagueness and uncertainty of real-life problems, various mathematical tools have been developed such as fuzzy sets, intuitionistic fuzzy sets, rough sets, and vague sets. One such tool, soft sets, was introduced by Molodtsov [3] in 1999 and has since been developed and applied to decision-making problems, algebraic structures, and topological spaces.

Şenel [4] presented the soft topology generated by $L$-soft sets. Additionally, in 2016, Şenel [5] proposed a new approach to Hausdorff space theory via the soft sets. Şenel and Çağman [6] introduced soft topological subspaces. Also they [7] explored soft closed sets on soft bitopological space. In 2020, Şenel et al. [8] investigated distance and similarity measures for octahedron sets proposed by Lee et al. [9].

El-Sheikh et al. [10] extended the idea of soft topological spaces by introducing supra soft topological spaces. In a similar vein, Ittanagi [11] proposed the concept of soft bitopological spaces, which are defined over an initial universal set and incorporate a fixed set of parameters. Kandil et al. [12, 13] provided some structures on soft bitopological spaces and defined some basic notions such as pairwise open and closed soft sets, pairwise soft closure, interior, kernel operators, and related topics. They also studied pairwise soft continuous mappings and open and closed soft mappings between two soft bitopological spaces. Additionally, they studied the concept of soft connectedness in soft bitopological spaces, the concepts of pairwise separated soft sets, pairwise soft connected (disconnected) spaces, and pairwise connected soft sets.

El-Shafei et al. [14, 15] proposed two innovative forms of soft relations, established the concepts of monotone soft sets and increasing (descending) soft operators, revealing crucial insights into their fundamental properties. Moreover, they introduced the notion of soft topological ordered spaces and formulated ordered soft separation axioms.

Additionally, El-Sheikh et al. [16] introduced the concept of soft bitopological ordered spaces, which includes increasing (decreasing) pairwise open (closed) soft sets, as well as the notions of increasing (decreasing) total (partial) pairwise soft neighborhoods and increasing (decreasing) pairwise open soft neighborhoods. They also studied the relationships between these concepts, including the increasing (decreasing) pairwise soft closure (interior).

The purpose of this article is to introduce and study the concept of soft connectedness in soft bitopological ordered spaces. We study the concepts of increasing (decreasing) pairwise separated soft sets, increasing (decreasing) pairwise soft connected (disconnected) spaces and increasing (decreasing) pairwise connected soft sets. The rest of this paper is organized as follows. In Section 2, we introduced briefly the notions of soft set, soft topology, soft bitopological spaces, soft bitopological ordered spaces, soft mapping and some related topics. In Section 3, we introduce the notion of increasing (decreasing) pairwise separated soft sets and give some characterizations of these soft sets. In Section 4, we introduce the notions of increasing (decreasing) pairwise soft connected (disconnected) spaces and investigate some of their properties. In Section 5, we give the concept of $IP(DP)$-connected soft sets and some related properties are studied.

## 2. Preliminaries

This section provides a brief overview of key concepts and relevant results from the fields of soft sets, soft topological spaces, soft bitopological spaces, and soft topological ordered spaces, which will be used in this paper. For more detailed information on these topics, please refer to [10, 12, 13, 14, 16, 17, 18, 19, 20].

**Definition 2.1** ([3])**.** Let $X$ be a universe set and let $E$ be a fixed set of parameters. If $G_E : E \to 2^X$ is a function, then an ordered pair $(G, E)$ is called a *soft set*, where $2^X$ is the power set of $X$. The set of all soft sets over $X$ is denoted by $P(X)^E$.

**Definition 2.2** ([21])**.** Let $F_E,\ G_E \in P(X)^E$.

(i) $F_E$ is said to be a *null soft set*, denoted by $\Phi$, if $F(e) = \varnothing\ \forall e \in E$.

(ii) $F_E$ is called an *absolute soft set*, denoted by $X_E$, if $F(e) = X\ \forall e \in E$.

(iii) $F_E$ is called a *soft subset* of $G_E$, denoted by $F_E \sqsubseteq G_E$, if $F(e) \subseteq G(e)\ \forall e \in E$.

(iv) $F_E$ and $G_E$ are said to be *equal*, denoted by $F_E = G_E$, if $F_E \sqsubseteq G_E$ and $G_E \sqsubseteq F_E$.

(v) The *union* of $F_E$ and $G_E$ is a soft set $H_E$, defined as: $H(e) = F(e) \cup G(e)\ \forall e \in E$. We write $H_E = F_E \sqcup G_E$.

(vi) The *intersection* of $F_E$ and $G_E$ is a soft set $H_E$, defined as: $H(e) = F(e) \cap G(e)\ \forall e \in E$. We write $H_E = F_E \sqcap G_E$.

(vii) The *difference* of $F_E$ and $G_E$ is a soft set $H_E$, defined as: $H(e) = F(e) - G(e)\ \forall e \in E$. We write $H_E = F_E - G_E$.

(viii) The *complement* of $F_E$, denoted by $F^c_E$, defined by: $F^c(e) = (F(e))^c\ \forall e \in E$.

**Definition 2.3** ([22, 23]). A soft set $H_E : E \to 2^X$ defined as $H(\alpha) = \{x\}$ if $e = \alpha$ and $H(e) = \varnothing$ if $e \in E - \{\alpha\}$ is called a *soft point* and denoted by $x^\alpha$. The collection of all soft points over $X$ is denoted by $Sp(X)^E$. A soft point $x^\alpha$ is said to *belong to a soft set* $G_E$, denoted by $x^\alpha \widehat{\in} G_E$, if $x^\alpha(\alpha) \subseteq G(\alpha)$ for each $\alpha \in E$.

**Definition 2.4** ([22]). Let $\phi : X \to Y$ and $\psi : E \to K$ be two mappings. Then the mapping $\phi_\psi : P(X)^E \to P(Y)^K$ is called a *soft mapping from* $X$ *to* $Y$. Let $G_E \in P(X)^E$ and let $F_K \in P(Y)^K$.

(i) The *soft image of* $G_E \in P(X)^E$ *under* $\phi_\psi$, denoted by $\phi_\psi(G_E)$, is a soft set over $Y$ defined as follows: for each $k \in K$,

$$\phi_\psi(G_E)(k) = \bigcup_{\alpha \in \psi^{-1}(k)} G(\alpha) \text{ if } \psi^{-1}(k) \neq \varnothing \text{ and } \phi_\psi(G_E)(k) = \varnothing \text{ otherwise.}$$

(ii) The *soft inverse image of* $F_K$ *under* $\phi_\psi$, denoted by $\phi^{-1}_\psi(F_K)$, is a soft set over $X$ defined as follows: for each $e \in E$,

$$\phi^{-1}_\psi(F_K)(e) = \phi^{-1}(F(\psi(e))).$$

**Definition 2.5** ([24]). A soft mapping $\phi_\psi : P(X)^E \to P(Y)^K$ is called a *soft surjective* (resp. *injective*) *mapping*, if $\phi$ and $\psi$ are surjective (resp. injective) mappings, respectively.

**Proposition 2.6** ([14]). *The following two results hold for a soft mapping* $\phi_\psi : P(X)^E \to P(Y)^K$.

(1) *The image of each soft point is soft point.*

(2) *If* $\phi_\psi$ *is bijective, then the inverse image of each soft point is soft point.*

**Theorem 2.7** ([22]). *Let* $G^i_E \in P(X)^E$ *and* $H^i_K \in P(Y)^K$ *for all* $i \in J$, *where* $J$ *is an index set. Then, for a soft mapping* $\phi_\psi : P(X)^E \to P(Y)^K$, *the following conditions are satisfied:*

(1) *if* $G^1_E \sqsubseteq G^2_E$, *then* $\phi_\psi(G^1_E) \sqsubseteq \phi_\psi(G^2_E)$,

(2) *if* $H^1_K \sqsubseteq H^2_K$, *then* $\phi^{-1}_\psi(H^1_K) \sqsubseteq \phi^{-1}_\psi(H^2_K)$,

(3) $\phi_\psi(\sqcup_{i\in J}(G^i_E)) = \sqcup_{i\in J}(\phi_\psi(G^i_E))$,

(4) $\phi^{-1}_\psi(\sqcup_{i\in J}(H^i_K)) = \sqcup_{i\in J}(\phi^{-1}_\psi(H^i_K))$,

(5) $\phi^{-1}_\psi(\sqcap_{i\in J}(H^i_K)) = \sqcap_{i\in J}(\phi^{-1}_\psi(H^i_K))$,

(6) $\phi_\psi(\phi^{-1}_\psi(H_K)) \sqsubseteq H_K$,

(7) $\phi_\psi^{-1}(Y_K) = X_E,\ \phi_\psi^{-1}(\widehat{\phi}_K) = \widehat{\phi}_E$ *and* $\phi_\psi(\widehat{\phi}_E) = \widehat{\phi}_K$.

**Proposition 2.8** ([22])**.** *Let* $\phi_\psi : P(X)^E \to P(Y)^K$ *be a soft map and let* $G_E \in P(X)^E$, $F_K \in P(X)^E$. *Then we have the following results:*

(1) $G_E \sqsubseteq \phi_\psi^{-1}(\phi_\psi(G_E))$, *and the equality holds if* $\phi_\psi$ *is injective.*
(2) $\phi_\psi(\phi_\psi^{-1}(F_K)) \sqsubseteq F_K$, *and the equality holds if* $\phi_\psi$ *is surjective.*
(3) $\phi_\psi^{-1}(F_K^c) = (\phi_\psi^{-1}(F_K))^c$ *for any* $F_K \in P(Y)^K$.

**Definition 2.9** ([3, 14])**.** For a soft set $H_E$ over $X$ and an element $x \in X$, we say $x \in H_E$ if $x \in H(e)$ for every $e \in E$ and $x \notin H_E$ if $x \notin H(e)$ for some $e \in E$. We say $x \Subset H_E$ if $x \in H(e)$ for some $e \in E$ and $x \not\Subset H_E$ if $x \notin H(e)$ for every $e \in E$. The notations $\in, \notin, \Subset$ and $\not\Subset$ are respectively read as belong, non-belong, partial belong and total non-belong relations.

**Definition 2.10** ([25])**.** A *soft topology* $\tau$ on $X$ is a collection of soft sets over $X$ under $E$ that satisfy the following conditions:
(i) the null soft set and the absolute soft set are included in $\tau$,
(ii) the union of any collection of soft sets in $\tau$ is also in $\tau$,
(iii) the intersection of any two soft sets in $\tau$ is also in $\tau$.
The triple $(X, \tau, E)$ is called a *soft topological space over* $X$. Each member of $\tau$ is referred to as a *soft open set* and its relative complement is called a *soft closed set.*

**Definition 2.11** ([24])**.** A soft subset $N_E$ of a soft topological space $(X, \tau, E)$ is called a *soft neighborhood* of $x \in X$, if there exists a soft open set $G_E$ such that $x \in G_E \sqsubseteq N_E$.

**Definition 2.12** ([22])**.** A soft map $\phi_\psi : (X, \tau, E) \to (Y, \eta, K)$ is said to be *soft continuous*, if the inverse image of each soft open subset of $(Y, \eta, K)$ is a soft open subset of $(X, \tau, E)$.

**Definition 2.13** ([11, 12])**.** A quadrable system $(X, \eta_1, \eta_2, E)$ is called a *soft bitopological space*, where $\eta_1$ and $\eta_2$ represent soft topologies on the set $X$ with a fixed set of parameters $E$. Let $H_E$ be a soft set over a soft bitopological space $(X, \eta_1, \eta_2, E)$. Then $H_E$ is called a:

(i) *pairwise open soft* (briefly, $PO$-soft) *set*, if there exists an $\eta_1$-open soft set $H_E^1$ and an $\eta_2$-open soft set $H_E^2$ such that $H_E = H_E^1 \sqcup H_E^2$,

(ii) *pairwise closed soft* (briefly, $PC$-soft) *set*, if $H_E^c$ is a $PO$-soft set.

Furthermore, the family of all $PO$-soft sets, denoted by $\eta_{12}$, forms a supra soft topological space associated with the soft bitopological space $(X, \eta_1, \eta_2, E)$.

**Definition 2.14** ([13])**.** Let $(X, \tau_1, \tau_2, E)$ be a soft bitopological space and let $G_E,\ H_E$ are non-null soft sets over $X$. Then $G_E$ and $H_E$ are said to be *pairwise separated* (briefly, $P$-separated) soft sets, if $Scl_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Scl_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$.

**Definition 2.15** ([13])**.** Let $(X, \tau_1, \tau_2, E)$ be a soft bitopological space. Then $P$-separated soft sets $G_E$ and $H_E$ in $(X, \tau_1, \tau_2, E)$ are said to be a *pairwise soft separation* of $X$ (briefly, $P$-soft separation), if $X_E = G_E \sqcup H_E$. In this case, we say that $X_E$ *has an* $P$*-soft separation.*

**Definition 2.16** ([13])**.** A soft bitopological space $(X, \tau_1, \tau_2, E)$ is said to be a:

(i) *pairwise soft disconnected space* (briefly, $P$-soft disconnected), if $X_E$ has a $P$-soft separation.

(ii) *pairwise soft connected space* (briefly, $P$-soft connected), if it is not $P$-soft disconnected, i.e., $X_E$ has not an $P$-soft separation.

**Definition 2.17** ([26])**.** A binary relation $\lesssim$ on a set $X$ is called a *partial order relation* on $X$, if it is reflexive, anti-symmetric, and transitive. The *equality relation* on $X$, denoted by $\blacktriangle$, is defined as $\{(x,x) : x \in X\}$.

**Definition 2.18** ([1])**.** A triple $(X, \tau, \lesssim)$ is called a *topological ordered space*, if $(X, \tau)$ is a topological space and $(X, \lesssim)$ is a partially ordered set.

**Definition 2.19** ([14])**.** A triple $(X, E, \lesssim)$ is called a *partially ordered soft space*, if $\lesssim$ is a partial order relation on the set $X$.

(i) An *increasing soft operator* $i : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ is defined as follows: for each $H_E \in P(X)^E$,

$$i(H_E)(\alpha) = iH(\alpha) = \{x \in X : \delta \lesssim x \text{ for some } \delta \in H(\alpha)\}.$$

(ii) A *decreasing soft operator* $d : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ is defined as follows: for each $H_E \in P(X)^E$,

$$d(H_E)(\alpha) = dH(\alpha) = \{x \in X : x \lesssim \delta \text{ for some } \delta \in H(\alpha)\}.$$

(iii) A soft subset $H_E$ of the partially ordered soft space $(X, E, \lesssim)$ is said to be *increasing* (resp. *decreasing*), if $H_E = i(H_E)$ (resp. $H_E = d(H_E)$).

**Proposition 2.20** ([14])**.** *Let $i : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ and $d : (P(X)^E, \lesssim) \to (P(X)^E, \lesssim)$ be increasing and decreasing soft operators, and let $H_E$ and $G_E$ be two soft sets in $(X, E, \lesssim)$. Then*

(1) $i(\widehat{\phi}_E) = \widehat{\phi}_E$ *and* $d(\widehat{\phi}_E) = \widehat{\phi}_E$,
(2) $H_E \sqsubseteq i(H_E)$ *and* $H_E \sqsubseteq d(H_E)$,
(3) $i(i(H_E)) = i(H_E)$ *and* $d(d(H_E)) = d(H_E)$,
(4) $i[H_E \sqcup G_E] = i(H_E) \sqcup i(G_E)$ *and* $d[H_E \sqcup G_E] = d(H_E) \sqcup d(G_E)$.

**Definition 2.21** ([14])**.** (i) A quadrable system $(X, \tau, E, \lesssim)$ called a *soft topological ordered space* (briefly, $STOS$), if $(X, \tau, E)$ is a soft topological space and $(X, E, \lesssim)$ is a partially ordered soft space.

(ii) A soft set $H_E$ in a soft topological ordered space $(X, \tau, E, \lesssim)$ is called an *increasing* (resp. *decreasing*) *open soft set*, if it is soft open and increasing (resp. decreasing).

**Definition 2.22** ([14])**.** A soft subset $N_E$ of an STOS $(X, \tau, E, \lesssim)$ is called an *increasing* (resp. a *decreasing*) *soft neighborhood* of $x \in X$, if $N_E$ is a soft neighborhood of $x$ and increasing (resp. decreasing).

**Definition 2.23** ([20])**.** A quadrable system A $(X, \tau_1, \tau_2, \lesssim)$ is called a *bitopological ordered space* (briefly, *bto*), if $(X, \lesssim)$ is a partially ordered space and $(X, \tau_1, \tau_2)$ is a *bts*.

**Definition 2.24** ([16]). A quinary system $(X, \tau_1, \tau_2, E, \lesssim)$ is called a *soft bitopological ordered space* (briefly, SBTOS), if the following conditions hold:
(i) $(X, \tau_1, \tau_2, E)$ is a soft bitopological space,
(ii) $(X, E, \lesssim)$ is a partially ordered soft space.

**Definition 2.25** ([16]). Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$. A soft set $M_E$ over $X$ is said to be:
(i) an *increasing pairwise open soft* (briefly, $IPO$-soft) *set*, if $M_E = M_E^1 \sqcup M_E^2$, $M_E^\beta \in \tau_\beta$ and $M_E^\beta$ is increasing, $\beta = 1,\ 2$,
(ii) a *decreasing pairwise open soft* (briefly, $DPO$-soft) *set*, if $M_E = M_E^1 \sqcup M_E^2$, $M_E^\beta \in \tau_\beta$ and $M_E^\beta$ is decreasing, $\beta = 1,\ 2$,
(iii) an *increasing pairwise closed soft* (briefly, $IPC$-soft) *set*, if $M_E = M_E^1 \sqcap M_E^2$, $M_E^\beta \in \tau_\beta^c$ and $M_E^\beta$ is increasing, $\beta = 1,\ 2$,
(iv) a decreasing pairwise closed soft (briefly, $DPO$-soft) *set*, if $M_E = M_E^1 \sqcap M_E^2$, $M_E^\beta \in \tau_\beta^c$ and $M_E^\beta$ is decreasing, $\beta = 1,\ 2$.

**Definition 2.26** ([16]). Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and $G_E \in P(X)^E$.
(i) The *increasing pairwise soft closure* of $G_E$, denoted by $Icl_{12}^s(G_E)$, is the intersection of all increasing pairwise closed soft sets including $G_E$, i.e.,
$$Icl_{12}^s(G_E) = \sqcap\{F_E : F_E \text{ is } IPC\text{-soft set, } G_E \sqsubseteq F_E\}.$$
(ii) The *decreasing pairwise soft closure* of $G_E$, denoted by $Dcl_{12}^s(G_E))$, is the intersection of all decreasing pairwise closed soft sets including $G_E$, i.e.,
$$Dcl_{12}^s(G_E) = \sqcap\{K_E : K_E \text{ is } DPC\text{-soft set, } G_E \sqsubseteq K_E\}.$$
It is clear that $Icl_{12}^s(G_E)(Dcl_{12}^s(G_E))$ is the smallest $IPC$ (resp. $DPC$)-soft set including $G_E$.
(iii) The *increasing pairwise soft interior* of $G_E$, denoted by $Iint_{12}^s(G_E))$, is the union of all increasing pairwise open soft sets embodied in $G_E$, i.e.,
$$Iint_{12}^s(G_E) = \sqcup\{O_E : O_E \text{ is } IPO\text{-soft set, } O_E \sqsubseteq G_E\}.$$
(vi) The*decreasing pairwise soft interior* of $G_E$, denoted by $Dint_{12}^s(G_E))$, is the union of all decreasing pairwise open soft sets embodied in $G_E$, i.e.,
$$Dint_{12}^s(G_E) = \sqcup\{M_E : M_E \text{ is } DPO\text{-soft set, } M_E \sqsubseteq G_E\}.$$
It is obvious that $Iint_{12}^s(G_E)(Dint_{12}^s(G_E))$ is the largest $IPO$ (resp. $DPO$)-soft set embodied in $G_E$.

**Corollary 2.27** ([16]). *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and $G_E \in P(X)^E$. Then $Icl_{\tau_{12}}^s(G_E) = Icl_{\tau_1}^s(G_E) \sqcap Icl_{\tau_2}^s(G_E)$.*

**Theorem 2.28** ([16]). *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \lesssim_1) \to (Y, \eta_1, \eta_2, K, \lesssim_2)$ be a soft mapping. The following statements are equivalent:*
(1) *$\phi_\psi$ is $ISP$-continuous,*
(2) *$\phi_\psi(Icl_{12}^s(G_E)) \sqsubseteq cl_{12}^s(\phi_\psi(G_E))$ for any $G_E \in P(X)^E$,*
(3) *$Icl_{12}^s(\phi_\psi^{-1}(F_K)) \sqsubseteq \phi_\psi^{-1}(cl_{12}^s(F_K))$ for any $F_K \in P(Y)^K$,*
(4) *for any $PC$-soft subset $M_K$ of $(Y, \eta_1, \eta_2, K, \lesssim_2)$, $\phi_\psi^{-1}(M_K)$ is $DPC$-soft subset of $(X, \tau_1, \tau_2, E, \lesssim_1)$.*

**Definition 2.29** ([16]). A soft set $G_E$ in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be a *total pairwise soft neighborhood* of $x \in X$, if there is a $PO-$soft set $H_E$ such that $x \in H_E \sqsubseteq G_E$.

**Definition 2.30** ([16])**.** A soft set $W_E$ in an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is called an *increasing pairwise soft neighborhood* (briefly, $IPS$-nbd) (resp. a *decreasing pairwise soft neighborhood* (briefly, $DPS$-nbd )) of $x^e \widehat{\in} X_E$, if there exists a $PO-$soft set $H_E$ such that $x^e \widehat{\in} H_E \sqsubseteq W_E$ and $W_E$ is increasing (resp. decreasing).

**Definition 2.31** ([16])**.** A soft set $G_E$ in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim_1)$ is called:

(i) an *increasing total pairwise soft neighborhood* (briefly, $ITPS$-nbd) of $x \in X$, if $G_E$ is a total pairwise soft neighborhood of $x$ and increasing,

(ii) a *decreasing total pairwise soft neighborhood* (briefly, $DTPS$-nbd) of $x \in X$, if $G_E$ is a total pairwise soft neighborhood of $x$ and decreasing.

**Definition 2.32** ([16])**.** An $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be:

(i) *lower pairwise soft $T_1^{\bullet}$-ordered* (briefly, $LPST_1^{\bullet}$-ordered), if for any distinct points $x$ and $y$ in $X$ such that $x \not\lesssim y$, there exists an $ITPS$-nbd $G_E$ of $x$ such that $y \notin G_E$,

(ii)*upper pairwise soft $T_1^{\bullet}$-ordered* (briefly, $UPST_1^{\bullet}$-ordered), if for any distinct points $x$ and $y$ in $X$ such that $x \not\lesssim y$ ,there exists a $DTPS$-nbd $G_E$ of $y$ such that $x \notin G_E$,

(iii) *lower pairwise soft $SST_1$-ordered* (briefly, $LPSST_1$-ordered), if for every pair of soft points $x^{e_1},\ y^{e_2}$ such that $x^{e_1} \not\lesssim y^{e_2}$, there exists an $IPS$-nbd $W_E$ of $x^{e_1}$ such that $y^{e_2} \widehat{\notin} W_E$,

(iv) *upper pairwise soft $SST_1$-ordered* (briefly, $UPSST_1$-ordered), if for every pair of soft points $x^{e_1},\ y^{e_2}$ such that $x^{e_1} \not\lesssim y^{e_2}$ , there exists a $DPS$-nbd $W_E$ of $y^{e_2}$ such that $x^{e_1} \widehat{\notin} W_E$.

**Corollary 2.33** ([16])**.** *For an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$, the family of all IPO-soft and DPO-soft sets forms an increasing supra soft topology, denoted by $\tau_{12}^{IP}$, and decreasing supra soft topology, denoted by $\tau_{12}^{DP}$, respectively on X. It is also mentioned that the decreasing supra soft topology of complements of sets in $\tau_{12}^{IP}$ is equivalent to the increasing supra soft topology of complements of sets in $\tau_{12}^{DP}$ and vice versa. In fact,*

$$\tau_{12}^{IP} = \{M_E : M_E = M_E^1 \sqcup M_E^2, M_E^{\beta} \in \tau_{\beta} \text{ and increasing, } \beta = 1, 2\},$$
$$\tau_{12}^{DP} = \{N_E : N_E = N_E^1 \sqcup N_E^2, N_E^{\beta} \in \tau_{\beta} \text{ and decreasing, } \beta = 1, 2\}.$$

*However,* $\tau_{12}^{cIP} = \{H_E^c : H_E \in \tau_{12}^{DP}\},\ \tau_{12}^{cDP} = \{O_E^c : O_E \in \tau_{12}^{IP}\}$.

**Definition 2.34** ([16])**.** Let $Y \subseteq X$ and $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$. Then $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim_Y)$ is called soft $bi-$ordered subspace of $(X, \tau_1, \tau_2, E, \lesssim)$, provided that $(Y, \tau_{1Y}, \tau_{2Y}, E)$ is soft bitopological subspace of $(X, \tau_1, \tau_2, E)$, where $\lesssim_Y$ is a partial order relation on $Y$.

## 3. Increasing (Decreasing) pairwise separated soft sets

**Definition 3.1.** Let $(X, \tau_1, \tau_2, \lesssim)$ be a *bto*. A subset $A$ of $X$ is said to be:

(i) an *increasing pairwise open set* (briefly, $IPO$-set), if $A = A^1 \cup A^2, A^{\beta} \in \tau_{\beta}$ and $A^{\beta}$ is increasing, $\beta = 1,\ 2$,

(ii) a *decreasing pairwise open set* (briefly, $DPO$-set), if $A = A^1 \cup A^2, A^{\beta} \in \tau_{\beta}$ and $A^{\beta}$ is decreasing, $\beta = 1,\ 2$,

(iii) an *increasing pairwise closed set* (briefly, $IPC$-set), if $A = A^1 \cup A^2, A^{\beta} \in \tau_{\beta}^c$ and $A^{\beta}$ is increasing, $\beta = 1,\ 2$,

(iv) a *decreasing pairwise closed set* (briefly, $DPC$-set), if $A = A^1 \cup A^2, A^\beta \in \tau_\beta^c$ and $A^\beta$ is decreasing, $\beta = 1,\ 2$.

**Definition 3.2.** Let $(X, \tau_1, \tau_2, \lesssim)$ be a *bto* and let $A \in 2^X$.

(i) The *increasing pairwise closure* of $A$, denoted by $Icl_{12}(A)$, is the intersection of all $IPC$-sets containing $A$, i.e., $Icl_{12}(A) = \cap\{B : B \text{ is } IPC\text{-set}, A \subseteq B\}$.

(ii) The *decreasing pairwise closure* of $A$, denoted by $Dcl_{12}(A)$, is the intersection of all $DPC$-sets containing $A$ i.e., $Dcl_{12}(A) = \cap\{K : K \text{ is } DPC\text{-set}, A \subseteq K\}$.
It is obvious that $Icl_{12}(A)$ (resp. $Dcl_{12}(A)$) is the smallest $IPC$ (resp. $DPC$)-sets containing $A$.

(iii) The *increasing pairwise soft interior* of $A$, denoted by $Iint_{12}(A)$, is the union of all $IPO$-sets contained in $A$, i.e., $Iint_{12}(A) = \cup\{O : O \text{ is } IPO\text{-set}, O \subseteq A\}$.

(iv) The *decreasing pairwise soft interior* of $A$, denoted by $Dint_{12}(A)$, is the union of all $DPO$-soft sets contained in $A$, i.e., $Dint_{12}(A) = \cup\{G : G \text{ is } DPO\text{-set}, G \subseteq A\}$.
It is clear that $Iint_{12}(A)$ (resp. $Dint_{12}(A)$) is the largest $IPO$ (resp. $DPO$)-sets contained in $A$

**Definition 3.3.** Let $(X, \tau_1, \tau_2, \lesssim)$ be a *bto* and let $A,\ B$ be non-null subsets of $X$. Then $A$ and $B$ are said to be:

(i) an *increasing pairwise separated sets* (briefly, $IPS$-sets), if $Icl_{12}(A) \cap B = \varnothing$ and $Icl_{12}(B) \cap A = \varnothing$,

(ii) a *decreasing pairwise separated sets* (briefly, $DPS$-sets), if $Dcl_{12}(A) \cap B = \varnothing$ and $Dcl_{12}(B) \cap A = \varnothing$.

**Definition 3.4.** Let $(X, \tau_1, \tau_2, \lesssim)$ be a *bto* and let $A,\ B$ be $IPS$ (resp. $DPS$)-sets in $X$. Then $A$ ans $B$ are said to be an *increasing* (resp. *decreasing*) *pairwise separation* of $X$ (briefly, $IP$ (resp. $DP$)-separation), if $X = A \cup B$. In this case, we say that $X$ *has an* $IP$ (resp. $DP$)*-separation.*

**Definition 3.5.** A *bto* $(X, \tau_1, \tau_2, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise disconnected space* (briefly, $IP$ (resp. $DP$)-disconnected), if $X$ has an $IP$ (resp. $DP$)-separation. Otherwise, $(X, \tau_1, \tau_2, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise connected space* (briefly, $IP$ (resp. $DP$)-connected), i. e., A *sbo* $(X, \tau_1, \tau_2, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise connected*, if $X$ has not an $IP$ (resp. $DP$)-separation.

**Definition 3.6.** Let $(X, \tau, E, \lesssim)$ be an $STOS$ and let $G_E,\ H_E$ be non-null soft sets over $X$. Then $G_E$ and $H_E$ are said to be:

(i) *increasing soft separated sets* (briefly, $ISS$-sets), if $Icl^s_\tau(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_\tau(H_E) \sqcap G_E = \widehat{\phi}_E$,

(ii) *decreasing soft separated sets* (briefly, $DSS$-sets), if $Dcl^s_\tau(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Dcl^s_\tau(H_E) \sqcap G_E = \widehat{\phi}_E$.

**Definition 3.7.** Let $(X, \tau, E, \lesssim)$ be an $STOS$. Then $ISS$ (resp. $DSS$)-sets $G_E$ and $H_E$ in $X$ are said to be an *increasing* (resp. a *decreasing*) *soft separation* (briefly, $IS$ (resp. $DS$)-separation) of $X$, if $X_E = G_E \sqcup H_E$. In this case, we say that $X$ *has an* $IS$ (resp. a $DS$)*-separation.*

**Definition 3.8.** An $STOS$ $(X, \tau, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *soft disconnected space* (briefly, $IS$ (resp. $DS$)-disconnected), if $X$ has an $IS$

(resp. a $DS$)-separation. Otherwise, $(X, \tau, E, \lesssim)$ is said to be an *increasing* (resp. *decreasing*) *soft connected space* (briefly, $IS$ (resp. $DS$)-connected), i.e., an STOS $(X, \tau, E, \lesssim)$ is said to be *increasing* (resp. *decreasing*) *soft connected*, if $X$ has not an $IS$ (resp. a $DS$)-separation.

**Definition 3.9.** Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and let $G_E$, $H_E$ are non-null soft sets over $X$. Then $G_E$ and $H_E$ are said to be:

(i) *increasing pairwise separated soft sets* (briefly, $IPS$-soft sets), if $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$,

(ii) *decreasing pairwise separated soft sets* (briefly, $DPS$-soft sets), if $Dcl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Dcl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$.

**Proposition 3.10.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$. Then every $IPS$* (resp. *$DPS$*)*-soft sets are disjoint soft sets.*

*Proof.* Straightforward. □

**Remark 3.11.** The converse of Proposition 3.10 may not be true as shown by the following example.

**Example 3.12.** Let $E = \{e_1, e_2\}$ be a set of parameters, $\lesssim = \blacktriangle \cup \{(x, w)\}$ be a partial order relation on $X = \{x, y, z, w\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E, H_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, M_E, N_E\}$, where

$G_E = \{(e_1, \{x, w\}), (e_2, \{z, w\})\}$, $H_E = \{(e_1, \{y, z\}), (e_2, \{x, y\})\}$,
$M_E = \{(e_1, \{w\}), (e_2, \{y, w\})\}$, $N_E = \{(e_1, \{x, w\}), (e_2, X)\}$.

It is easy to verify that:

$$\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H_E, M_E, N_E, P^1_E, P^2_E\},$$

where $P^1_E = \{(e_1, \{x, w\}), (e_2, \{y, z, w\})\}$, $P^2_E = \{(e_1, \{y, z, w\}), (e_2, \{x, y, w\})\}$.
Then clearly, $\tau^c_{12} = \{\widehat{\phi}_E, X_E, G^c_E, H^c_E, M^c_E, N^c_E, P^{1c}_E, P^{2c}_E\}$,
where $G^c_E = \{(e_1, \{y, z\}), (e_2, \{x, y\})\}$, $H^c_E = \{(e_1, \{x, w\}), (e_2, \{z, w\})\}$,
$M^c_E = \{(e_1, \{x, y, z\}), (e_2, \{x, z\})\}$, $N^c_E = \{(e_1, \{y, z\}), (e_2, \varnothing)\}$,
$P^{1c}_E = \{(e_1, \{y, z\}), (e_2, \{x\})\}$, $P^{2c}_E = \{(e_1, \{x\}), (e_2, \{z\})\}$.

Thus we have

(1) The family of all $IPC-$soft sets are $H^c_E$ and $N^c_E$,
(2) The family of all $DPC-$soft sets are $G^c_E$, $M^c_E$, $N^c_E$, $P^{1c}_E$ and $P^{2c}_E$.

Now, consider soft sets $F^\alpha_E$, $\alpha = 1,\ 2,\ 3$ given by:

$$F^1_E = \{(e_1, \{x\}), (e_2, \{x\})\},\ F^2_E = \{(e_1, \{y\}), (e_2, \varnothing)\},\ F^3_E = \{(e_1, \varnothing), (e_2, \{z\})\}.$$

It is clear that $F^2_E$, $F^3_E$ are $IPS$ (resp. $DPS$)-soft sets. Although the soft sets $F^1_E$ and $F^3_E$ are disjoint, we find that they are not $IPS$-soft sets because

$$Icl^s_{12}(F^3_E) \sqcap F^1_E = \{(e_1, \{x\}), (e_2, \varnothing)\} \neq \widehat{\phi}_E.$$

Also they are not $DPS$-soft sets because

$$Dcl^s_{12}(F^3_E) \sqcap F^1_E = \{(e_1, \{x\}), (e_2, \varnothing)\} \neq \widehat{\phi}_E.$$

**Proposition 3.13.** *Every $IPS$* (resp. *$DPS$*)*-soft sets are $P$-separated soft sets.*

*Proof.* The proof is given from the fact $cl^s_{12}(G_E) \sqsubseteq Icl^s_{12}(G_E)$. □

The converse of the above Proposition is not true in general.

**Example 3.14.** From Example 4.3, let $F^4_E = \{(e_1, \{y\}), (e_2, \{y\})\}$, $F^5_E = \{(e_1, \{x\}), (e_2, \{w\})\}$. Although the soft sets $F^4_E$ and $F^5_E$ are $P$-separated soft sets, we find that their are not $IPS$-soft sets because $Icl^s_{12}(F^4_E) \sqcap F^5_E = \{(e_1, \{x\}), (e_2, \{w\})\} \neq \widehat{\phi}_E$. Also their are not $DPS$-soft sets because $Dcl^s_{12}(F^5_E) \sqcap F^4_E = \{(e_1, \{y\}), (e_2, \{y\})\} \neq \widehat{\phi}_E$.

**Proposition 3.15.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and let $G_E$, $H_E$ be non-null soft sets over $X$.*

1. *If $Icl^s_{12}(G_E) \sqcap Icl^s_{12}(H_E) = \widehat{\phi}_E$, then $G_E$ and $H_E$ are $IPS-$soft sets.*
2. *If $Dcl^s_{12}(G_E) \sqcap Dcl^s_{12}(H_E) = \widehat{\phi}_E$, then $G_E$ and $H_E$ are $DPS-$soft sets.*

*Proof.* Straightforward. □

**Note:** From Proposition 3.10, 3.13, 3.15, we deduce that the concept of $IPS$ (resp. $DPS$)-soft sets is a weaker than of the condition of disjoint increasing (resp. decreasing) pairwise soft closure of soft sets, but it is a stronger than of the concept of $P$-separated soft sets and disjoint soft sets.

**Remark 3.16.** The converse of Proposition 3.15 may not be true as shown by the following example.

**Example 3.17.** Let $E = \{e_1, e_2\}$ be the set of parameters, $\lesssim = \blacktriangle \cup \{(y, w)\}$ be a partial order relation on $X = \{x, y, z, w\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E, H^1_E, H^2_E, H^3_E, H^4_E, H^5_E, H^6_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, M_E, N_E\}$,
where $G_E = \{(e_1, \{x, w\}), (e_2, \{z, w\})\}$.
$H^1_E = \{(e_1, \{y, z\}), (e_2, \{x, y\})\}$, $H^2_E = \{(e_1, \{x\}), (e_2, \{x\})\}$,
$H^3_E = \{(e_1, \{x\}), (e_2, \varnothing)\}$, $H^4_E = \{(e_1, \{x, w\}), (e_2, \{x, z, w\})\}$,
$H^5_E = \{(e_1, \varnothing), (e_2, \{x\})\}$, $H^6_E = \{(e_1, \{x, y, z\}), (e_2, \{x, y\})\}$,
$M_E = \{(e_1, \{w\}), (e_2, \{y, w\})\}$, $N_E = \{(e_1, \{x, w\}), (e_2, X)\}$.
It is easy to verify that:

$$\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H^1_E, H^2_E, H^3_E, H^4_E, H^5_E, H^6_E, M_E, N_E, P^1_E, P^2_E, P^3_E, P^4_E, P^5_E, P^6_E\},$$

where $P^1_E = \{(e_1, \{x, w\}), (e_2, \{y, z, w\})\}$, $P^2_E = \{(e_1, \{y, z, w\}), (e_2, \{x, y, w\})\}$
$P^3_E = \{(e_1, \{x, w\}), (e_2, \{x, y, w\})\}$, $P^4_E = \{(e_1, X), (e_2, \{x, y, w\})\}$,
$P^5_E = \{(e_1, \{x, w\}), (e_2, \{y, w\})\}$, $P^6_E = \{(e_1, \{w\}), (e_2, \{x, y, w\})\}$.
Then we have

$$\tau^c_{12} = \{\widehat{\phi}_E, X_E, G^c_E, H^{1c}_E, H^{2c}_E, H^{3c}_E, H^{4c}_E, H^{5c}_E, H^{6c}_E, M^c_E, N^c_E, P^{1c}_E, P^{2c}_E, P^{3c}_E, P^{4c}_E, P^{5c}_E, P^{6c}_E\},$$

where $G^c_E = \{(e_1, \{y, z\}), (e_2, \{x, y\})\}$,
$H^{1c}_E = \{(e_1, \{x, w\}), (e_2, \{z, w\})\}$, $H^{2c}_E = \{(e_1, \{y, z, w\}), (e_2, \{y, z, w\})\}$,
$H^{3c}_E = \{(e_1, \{y, z, w\}), (e_2, X)\}$, $H^{4c}_E = \{(e_1, \{y, z\}), (e_2, \{y\})\}$,
$H^{5c}_E = \{(e_1, X), (e_2, \{y, z, w\})\}$, $H^{6c}_E = \{(e_1, \{w\}), (e_2, \{z, w\})\}$,
$M^c_E = \{(e_1, \{x, y, z\}), (e_2, \{x, z\})\}$, $N^c_E = \{(e_1, \{y, z\}), (e_2, \varnothing)\}$,
$P^{1c}_E = \{(e_1, \{y, z\}), (e_2, \{x\})\}$, $P^{2c}_E = \{(e_1, \{x\}), (e_2, \{z\})\}$,
$P^{3c}_E = \{(e_1, \{y, z\}), (e_2, \{z\})\}$, $P^{4c}_E = \{(e_1, \varnothing), (e_2, \{z\})\}$,
$P^{5c}_E = \{(e_1, \{y, z\}), (e_2, \{x, z\})\}$, $P^{6c}_E = \{(e_1, \{x, y, z\}), (e_2, \{z\})\}$.
Thus we have

(1) The family of all $IPC$-soft sets are $H^{1c}_E$, $H^{2c}_E$, $H^{3c}_E$, $H^{5c}_E$, $H^{6c}_E$, $P^{2c}_E$ and $P^{4c}_E$,
(2) The family of all $DPC$-soft sets are $G^c_E$, $H^{2c}_E$, $H^{3c}_E$, $H^{5c}_E$, $M^c_E$, $N^c_E$, $P^{1c}_E$, $P^{2c}_E$, $P^{3c}_E$, $P^{4c}_E$, $P^{5c}_E$ and $P^{6c}_E$.

Now, let $F^1_E = \{(e_1, \{y, z, w\}), (e_2, \varnothing)\},\ F^2_E = \{(e_1, \{x\}), (e_2, \varnothing)\}$. Then we have

$$Icl^s_{12}(F^1_E) = Dcl^s_{12}(F^1_E) = H^{2c}_E = \{(e_1, \{y, z, w\}), (e_2, \{y, z, w\})\}$$

and

$$Icl^s_{12}(F^2_E) = Dcl^s_{12}(F^2_E) = P^{2c}_E = \{(e_1, \{x\}), (e_2, \{z\})\}.$$

It is clear that

$$Icl^s_{12}(F^1_E) \sqcap F^2_E = \widehat{\phi}_E,\ Icl^s_{12}(F^2_E) \sqcap F^1_E = \widehat{\phi}_E$$

and

$$Dcl^s_{12}(F^1_E) \sqcap F^2_E = \widehat{\phi}_E,\ Dcl^s_{12}(F^2_E) \sqcap F^1_E = \widehat{\phi}_E.$$

Thus $F^1_E$, $F^2_E$ are $IPS$ (resp. $DPS$)-soft sets. But we get

$$Icl^s_{12}(F^1_E) \sqcap Icl^s_{12}(F^2_E) = \{(e_1, \varnothing), (e_2, \{z\})\} \neq \widehat{\phi}_E$$

and

$$Dcl^s_{12}(F^1_E) \sqcap Dcl^s_{12}(F^2_E) = \{(e_1, \varnothing), (e_2, \{z\})\} \neq \widehat{\phi}_E.$$

**Proposition 3.18.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and let $G_E$, $H_E$ be $IPO$* (resp. *$DPO$*)*-soft sets. Then $G_E$, $H_E$ are $IPS$* (resp. *$DPS$*)*-soft sets if and only if $G_E$ and $H_E$ are disjoint soft sets.*

*Proof.* The proof of necessary condition is obvious from Proposition 3.10.

Suppose that $G_E$ and $H_E$ are disjoint $IPO$-soft sets. Then clearly, $G_E \sqsubseteq H^c_E$, $H^c_E \in \tau^{IPc}_{12}$. It follows that $Icl^s_{12}(G_E) \sqsubseteq H^c_E$ implies $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$. By similar way, we can show that $Icl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$. □

**Proposition 3.19.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and let $F_E$, $M_E$ be $IPC$* (resp. *$DPC$*)*-soft sets. Then $F_E$, $M_E$ are $IPS$* (resp. *$DPS$*)*-soft sets if and only if $F_E$ and $M_E$ are disjoint soft sets.*

*Proof.* Straightforward. □

**Theorem 3.20.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS, $Y \subseteq X$ and let $G_E$, $H_E \sqsubseteq Y_E \sqsubseteq X_E$. If $G_E$ and $H_E$ are $IPS$* (resp. *$DPS$*)*-soft sets in $(X, \tau_1, \tau_2, E, \lesssim)$, then they are $IPS$* (resp. *$DPS$*)*-soft sets in $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim_Y)$.*

*Proof.* Suppose $G_E$ and $H_E$ are $IPS$-soft sets over $X$. Then we get

$$\begin{aligned} Icl^s_{12Y}(G_E) &= \sqcap\{F_E : F_E \in \tau^{IPc}_{12Y} : G_E \sqsubseteq F_E\} \\ &= \sqcap\{Y_E \sqcap M_E : M_E \in \tau^{IPc}_{12X} : G_E \sqsubseteq M_E\} \\ &= Y_E \sqcap [\sqcap\{M_E : M_E \in \tau^{IPc}_{12X} : G_E \sqsubseteq M_E\}] \\ &= Y_E \sqcap Icl^s_{12X}(G_E). \end{aligned}$$

Thus $Icl^s_{12Y}(G_E) \sqcap H_E = Y_E \sqcap H_E \sqcap Icl^s_{12X}(G_E)$. So we have

$$Icl^s_{12Y}(G_E) \sqcap H_E \sqsubseteq H_E \sqcap Icl^s_{12X}(G_E) = \widehat{\phi}_E.$$

Hence $Icl^s_{12Y}(G_E) \sqcap H_E = \widehat{\phi}_E$. By similar way, we can prove that $Icl^s_{12Y}(H_E) \sqcap G_E = \widehat{\phi}_E$. Therefore $G_E$, $H_E$ are $IPS$-soft sets in $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim_Y)$. □

**Theorem 3.21.** *Let* $(X, \tau_1, \tau_2, E, \lesssim)$ *be* $LPST_1^{\bullet}$ (resp. $UPST_1^{\bullet}$)*-ordered and let* $G_E$, $H_E$ *be two finite and disjoint increasing* (resp. *decreasing*) *soft sets. Then* $G_E$ *and* $H_E$ *are* $IPS$ (resp. $DPS$)*-soft sets.*

*Proof.* Since $(X, \tau_1, \tau_2, E, \lesssim)$ is $LPST_1^{\bullet}-$ordered, every crisp point is a $PC$-soft set. Since $G_E$ and $H_E$ are finite soft sets, $G_E$ and $H_E$ are $IPC$-soft sets. It follows by Proposition 3.19 that $G_E$ and $H_E$ are $IPS$-soft sets. □

**Theorem 3.22.** *Let* $(X, \tau_1, \tau_2, E, \lesssim)$ *be* $UPSST_1$ (resp. $LPSST_1$)*-ordered and let* $G_E$, $H_E$ *be two finite and disjoint increasing* (resp. *decreasing*) *soft sets. Then* $G_E$ *and* $H_E$ *are* $IPS$ (resp. $DPS$)*-soft sets.*

*Proof.* Since $(X, \tau_1, \tau_2, E, \lesssim)$ is $UPSST_1$-ordered, every soft point is a $PC$-soft set. Since $G_E$ and $H_E$ are finite soft sets, $G_E$ and $H_E$ are $IPC$-soft sets. It follows by Proposition 3.19 that $G_E$ and $H_E$ are $IPS$-soft sets. □

**Theorem 3.23.** *Let* $\phi_\psi : (X, \tau_1, \tau_2, E, \lesssim_1) \to (Y, \eta_1, \eta_2, K, \lesssim_2)$ *be an* $ISP$ (resp. a $DSP$)*-continuous and soft surjective mapping. If* $M_E$ *and* $N_E$ *are* $IPS$ (resp. $DPS$)*-soft sets in* $(Y, \eta_1, \eta_2, K, \lesssim_2)$*, then* $\phi_\psi^{-1}(M_E)$ *and* $\phi_\psi^{-1}(N_E)$ *are* $IPS$ (resp. $DPS$)*-soft sets in* $(X, \tau_1, \tau_2, E, \lesssim_1)$.

*Proof.* Suppose $M_K$ and $N_K$ are $IPS$-soft sets in $(Y, \eta_1, \eta_2, K, \lesssim_2)$. Then we have

$$Icl_{12}^s(M_K) \sqcap N_K = \widehat{\phi}_K,\ Icl_{12}^s(N_K) \sqcap M_K = \widehat{\phi}_K.$$

Since $\phi_\psi$ is an $ISP$-continuous mapping, by Theorem 2.28, we get

$$Icl_{12}^s[\phi_\psi^{-1}(M_K)] \sqsubseteq \phi_\psi^{-1}[Icl_{12}^s(M_K)].$$

Thus by Theorem 2.7, we have

$$\begin{aligned} Icl_{12}^s[\phi_\psi^{-1}(M_K)] \sqcap \phi_\psi^{-1}(N_K) &\sqsubseteq \phi_\psi^{-1}[Icl_{12}^s(M_K)] \sqcap \phi_\psi^{-1}(N_K) \\ &= \phi_\psi^{-1}[Icl_{12}^s(M_K) \sqcap N_K] \\ &= \phi_\psi^{-1}(\widehat{\phi}_K) \\ &= \widehat{\phi}_E. \end{aligned}$$

So $Icl_{12}^s[\phi_\psi^{-1}(M_K)] \sqcap \phi_\psi^{-1}(N_K) = \widehat{\phi}_E$. Similarly, we can prove that

$$Icl_{12}^s[\phi_\psi^{-1}(N_K)] \sqcap \phi_\psi^{-1}(M_K) = \widehat{\phi}_E.$$

Since $\phi_\psi$ is soft surjective mapping, $\phi_\psi^{-1}(M_K) \neq \widehat{\phi}_E$ and $\phi_\psi^{-1}(N_K) \neq \widehat{\phi}_E$. Hence $\phi_\psi^{-1}(M_K)$ and $\phi_\psi^{-1}(N_K)$ are $IPS$-soft sets in $(X, \tau_1, \tau_2, E, \lesssim_1)$. □

In Proposition 3.18 and Theorems 3.20, 3.21, 3.22, 3.23, a similar proof can be given for the case between parentheses.

**Theorem 3.24.** *Let* $(X, \tau_1, \tau_2, E, \lesssim)$ *be an SBTOS. If* $G_E$ *and* $H_E$ *are* $IPS$ (resp. $DPS$)*-soft sets in* $(X, \tau_1, \tau_2, E, \lesssim)$*, then* $G(e)$ *and* $H(e)$ *are* $IPS$ (resp. $DPS$)*-sets in* $(X, \tau_1^e, \tau_2^e, \lesssim)$ $\forall e \in E$.

*Proof.* Suppose $G_E$ and $H_E$ are $IPS$-soft sets in $X$ and let $e \in E$. Then we have

$$\tau_{12}^{IPe} = \tau_{12}^{IP}(e) = \{G(e) : G_E \in \tau_{12}^{IP}\}.$$

Since $Icl^s_{12}(G_E) = \bigcap\{F_E : F_E \in \tau_{12}^{IPc} : G_E \sqsubseteq F_E\}$, we get

$$Icl^s_{12}(G_E)(e) = \bigcap\{F(e) : F(e) \in \tau_{12}^{IPc}(e) : G(e) \subseteq F(e)\}.$$

Thus $Icl^s_{12}(G_E)(e) = Icl^s_{12}(G(e))$. Now, since $G_E$ and $H_E$ are $IPS$-soft sets in $(X, \tau_1, \tau_2, E, \lesssim)$, we have

$$Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E \text{ and } Icl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E.$$

So we get

$$[Icl^s_{12}(G_E) \sqcap H_E](e) = \varnothing \text{ and } [Icl^s_{12}(H_E) \sqcap G_E](e) = \varnothing.$$

Hence we have

$$Icl^s_{12}(G(e)) \cap H(e) = \varnothing \text{ and } Icl^s_{12}(H(e)) \cap G(e) = \varnothing.$$

It follows by Definition 3.3 that, $G(e)$ and $H(e)$ are $IPS$ (resp. $DPS$)-sets in $(X, \tau_1^e, \tau_2^e, \lesssim)$. □

## 4. Ordered pairwise soft disconnected (connected) spaces

**Definition 4.1.** Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$, and let $G_E$ and $H_E$ be $IPS$ (resp. $DPS$)-soft sets over $X$. Then $G_E$ and $H_E$ are said to be an *increasing* (resp. *decreasing*) *pairwise soft separation* of $X$ (briefly, $IP$ (resp. $DP$)-soft separation), if $X_E = G_E \sqcup H_E$. In this case, we say that $X_E$ *has an* $IP$ (resp. *a* $DP$)*-soft separation.*

**Definition 4.2.** An $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise soft disconnected space* (briefly, $IP$ (resp. $DP$)-soft disconnected), if $X_E$ has an $IP$ (resp. $DP$)- soft separation. Otherwise, $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise soft connected space* (briefly, $IP$ (resp. $DP$)-soft connected), i.e., an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. a *decreasing*) *pairwise soft connected*, if $X_E$ has not an $IP$ (resp. a $DP$)-soft separation.

**Example 4.3.** Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(x, w)\}$ be a partial order relation on $X = \{x, y, z, w\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E, H_E\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, M_E\}$,
where $G_E = \{(e_1, \{x, w\}), (e_2, \{y, z\})\}$, $H_E = \{(e_1, \{y, z\}), (e_2, \{x, w\})\}$,
$M_E = \{(e_1, \varnothing), (e_2, \{x\})\}$.
It is easy to verify that:
$\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H_E, M_E, N_E\}$, where $N_E = \{(e_1, \{x, w\}), (e_2, \{x, y, z\})\}$.
It is clear that $G_E$ and $H_E$ are form an $IP$-soft separation and $DP$-soft separation of $X_E$. Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$-soft disconnected and $DP$-soft disconnected space.

**Proposition 4.4.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$.*

(1) *If $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E\}$, then $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$* (resp. $DP$)*-soft connected.*

(2) *If $\tau_{12} = P(X)^E$, $\mid X \mid > 1$, then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$* (resp. $DP$)*-soft disconnected.*

(3) *If $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E, G_E\}$, then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$* (resp. $DP$)*-soft connected.*

*Proof.* (1) Suppose $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E\}$ and let $G_E$ and $H_E$ be soft sets over $X$. Then clearly, we have

$$Icl^s_{12}(G_E) = Icl^s_{12}(H_E) = X_E \text{ and } Dcl^s_{12}(G_E) = Dcl^s_{12}(H_E) = X_E.$$

Thus we cannot represented $X_E$ as a union of two $IPS$ (resp. $DPS$)-soft sets in $(X, \tau_1, \tau_2, E, \lesssim)$. So $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$ (resp. $DP$)-soft connected.

(2) Suppose $\tau_{12} = P(X)^E$, $\mid X \mid > 1$. Then every soft set is an $IP$ (resp. $DP$)-closed soft set. It follows that for every soft point $x^e \widehat{\in} X_E$, we have

$$Icl^s_{12}(\{x^e\}) = \{x^e\},\ Icl^s_{12}(\{x^e\}^c) = \{x^e\}^c$$

and

$$Dcl^s_{12}(\{x^e\}) = \{x^e\},\ Dcl^s_{12}(\{x^e\}^c) = \{x^e\}^c.$$

Thus $\{x^e\}$ and $\{x^e\}^c$ are $IPS$ (resp. $DPS$)-soft sets and $X_E = \{x^e\} \sqcup \{x^e\}^c$. So $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$ (resp. $DP$)-soft disconnected.

(3) Suppose $\tau_1 = \tau_2 = \{\widehat{\phi}_E, X_E, G_E\}$ and assume that $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$-soft disconnected. Then there exist two non-null soft sets $M_E$ and $N_E$ such that

$$Icl^s_{12}(M_E) \sqcap N_E = Icl^s_{12}(N_E) \sqcap M_E = \widehat{\phi}_E \text{ and } X_E = M_E \sqcup N_E.$$

Thus we have two cases: either $G^c_E$ is an $IPC$-soft set or not an $IPC$-soft set.

Case 1. Suppose $G^c_E$ is an $IPC$-soft set. Then we have three cases.

(i) If $[Icl^s_{12}(M_E) = G^c_E$ and $Icl^s_{12}(N_E) = X_E]$ or $[Icl^s_{12}(N_E) = G^c_E$ and $Icl^s_{12}(M_E) = X_E]$, without loss of generalization, we assume that $Icl^s_{12}(M_E) = G^c_E$ and $Icl^s_{12}(N_E) = X_E$, which a contradiction with disjointness between $Icl^s_{12}(N_E)$ and $M_E$.

(ii) If $Icl^s_{12}(M_E) = Icl^s_{12}(N_E) = G^c_E$, then it follows that $M_E \sqsubseteq G^c_E$ and $N_E \sqsubseteq G^c_E$ implies $M_E \sqcup N_E \sqsubseteq G^c_E$. Thus $X_E = G^c_E$, a contradicts with that $G_E \neq \widehat{\phi}_E$.

(iii) If $Icl^s_{12}(M_E) \neq G^c_E$ and $Icl^s_{12}(N_E) \neq G^c_E$, then $Icl^s_{12}(M_E) = Icl^s_{12}(N_E) = X_E$, which a contradiction with disjointness between $Icl^s_{12}(M_E)$ and $N_E$.

Case 2. Suppose $G^c_E$ is not an $IPC$-soft set. Then $Icl^s_{12}(M_E) = Icl^s_{12}(N_E) = X_E$, which a contradiction with disjointness between $Icl^s_{12}(M_E)$ and $N_E$.

So in either cases, $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$-soft connected. $(X, \tau_1, \tau_2, E, \lesssim)$ is a $DP$-soft connected in a similar way. □

**Theorem 4.5.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. Then the following are equivalent:*

(1) $(X, \tau_1, \tau_2, E, \lesssim)$ *is $IP$ (resp. $DP$)-soft connected,*

(2) $X_E$ *cannot represented as a union of two non-null disjoint $IPO$ (resp. $DPO$)-soft sets,*

(3) $X_E$ *cannot represented as a union of two non-null disjoint $IPC$ (resp $DPC$)-soft sets,*

(4) $X_E$ *has no proper soft subset which is both $IPO$ (resp. $DPO$)- and $IPC$ (resp. $DPC$)-soft set.*

*Proof.* (1) ⇒ (2): Suppose (1) holds and assume that there exist two non-null $IPO-$soft sets $G_E$ and $H_E$ such that $G_E \sqcap H_E = \widehat{\phi}_E$ and $X_E = G_E \sqcup H_E$. Since $G_E \sqcap H_E = \widehat{\phi}_E$, $G_E \sqsubseteq H^c_E$, $H_E \sqsubseteq G^c_E$. Thus $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_{12}(H_E) \sqcap$

$G_E = \widehat{\phi}_E$. It follows that $X_E$ has an $IP$-soft separation, i.e., $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$-soft disconnected which contradicts with (1).

(2) ⇒ (3): Suppose (2) holds and assume that there exist two non-null $IPC-$soft sets $F_E$ and $M_E$ such that $F_E \sqcap M_E = \widehat{\phi}_E$ and $X_E = F_E \sqcup M_E$. Then $F_E^c$ and $M_E^c$ are non-null $DPO$-soft sets and $F_E^c \sqcup M_E^c = X_E$, which contradicts with (2).

(3) ⇒ (4): Suppose (3) holds and assume that there exists $G_E \sqsubseteq X_E$, $G_E \neq X_E$ and $N_E \neq \widehat{\phi}_E$ such that $N_E$ is both $IPO$ and $IPC$-soft set. Then $N_E$ and $N_E^c$ are non-null disjoint $IPC$-soft set and $X_E = N_E \sqcup N_E^c$, which contradicts with (3).

(4) ⇒ (1): Suppose (4) holds and assume that $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$ (resp. $DP$)-soft disconnected. Then there exist two non-null $IPS$-soft sets $G_E$ and $H_E$ such that $X_E = G_E \sqcup H_E$. Thus $G_E^c \sqcap H_E^c = \widehat{\phi}_E$ implies $G_E^c \sqsubseteq H_E$, $H_E^c \sqsubseteq G_E$. Since $Icl_{12}^s(G_E) \sqcap H_E = \widehat{\phi}_E$, $Icl_{12}^s(G_E) \sqsubseteq H_E^c \sqsubseteq G_E$. So $G_E$ is an $IPC$-soft set. Similarly, $H_E$ is $IPC$-soft set. On the other hand, by Proposition 3.10, we deduce that $G_E \sqsubseteq H_E^c$. Hence $G_E = H_E^c$. It follows that $H_E^c$ is an $IPC$- soft set. Therefore $IPO$ and $IPC$-soft set, which contradicts with (4). The proof is similar in case of $(X, \tau_1, \tau_2, E, \lesssim)$ is a $DP$-soft connected. □

**Corollary 4.6.** *An SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$* (resp. *$DP$*)*-soft connected space if and only if the only soft sets over $X$ which are $IPO$* (resp. *$DPO$*) *and $IPC$* (resp. *$DPC$*)*-soft sets are $X_E$ and $\widehat{\phi}_E$.*

**Example 4.7.** Let $E = \{e_1, e_2, e_3\}, \lesssim = \blacktriangle \cup \{(a, b)\}$ be a partial order relation on $X = \{a, b, c, d\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E^1, G_E^2, G_E^3, G_E^4\}$, $\tau_2 = \{\widehat{\phi}_E, X_E, H_E^1, H_E^2\}$, where

$G_E^1 = \{(e_1, \{a, c\}), (e_2, \{a, b, c\}), (e_3, \{c, d\})\}$, $G_E^2 = \{(e_1, \varnothing), (e_2, \{a, c\}), (e_3, \{d\})\}$,
$G_E^3 = \{(e_1, \{c\}), (e_2, \{b\}), (e_3, \varnothing)\}$, $G_E^4 = \{(e_1, \{c\}), (e_2, \{a, b, c\}), (e_3, \{d\})\}$,
$H_E^1 = \{(e_1, \{a, b\}), (e_2, \{a, c\}), (e_3, \{a, d\})\}$, $H_E^2 = \{(e_1, \{b\}), (e_2, \{c\}), (e_3, \{a, d\})\}$.

Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $SBTOS$. Thus we have

$$\tau_{12} = \{\widehat{\phi}_E, X_E, G_E^1, G_E^2, G_E^3, G_E^4, H_E^1, H_E^2, P_E^1, P_E^2, P_E^3, P_E^4, P_E^5\},$$

where $P_E^1 = \{(e_1, \{a, b, c\}), (e_2, \{a, b, c\}), (e_3, \{a, c, d\})\}$,
$P_E^2 = \{(e_1, \{b\}), (e_2, \{a, c\}), (e_3, \{a, d\})\}$,
$P_E^3 = \{(e_1, \{a, b, c\}), (e_2, \{a, b, c\}), (e_3, \{a, d\})\}$,
$P_E^4 = \{(e_1, \{b, c\}), (e_2, \{b, c\}), (e_3, \{a, d\})\}$,
$P_E^5 = \{(e_1, \{b, c\}), (e_2, \{a, b, c\}), \{a, d\})\}$.

It is easy to see that the only soft sets over $X$ which are $IPO$ (resp. $DPO$) and $IPC$ (resp. $DPC$) -soft sets are $X_E$ and $\widehat{\phi}_E$. So by Corollary 4.6, we deduce that $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$ (resp. a $DP$)-soft connected space.

**Theorem 4.8.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. Then the following are equivalent:*

(1) *$(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$* (resp. *$DP$*)*-soft disconnected,*

(2)*$X_E$ can represented as a union of two non-null disjoint $IPO$* (resp. *$DPO$*)*-soft sets,*

(3) *$X_E$ can represented as a union of two non-null disjoint $IPC$* (resp. *$DPC$*)*-soft sets,*

(4) $X_E$ *has a proper soft subset which is both* $IPO$ (resp. $DPO$) *and* $IPC$ (resp. $DPC$)*-soft set.*

*Proof.* The proof is similar as Theorem 4.5. □

**Remark 4.9.** Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $IP$ (resp. a $DP$)- soft connected space and let $e \in E$. Then $(X, \tau_1^e, \tau_2^e, \lesssim)$ may not be an $IP$ (resp. a $DP$)-connected space as shown in the following example.

**Example 4.10.** Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(b, c)\}$ be a partial order relation on $X = \{a, b, c\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, G_E\},\ \tau_2 = \{\widehat{\phi}_E, X_E, H_E\}$,
where $G_E = \{(e_1, \{a\}), (e_2, \{b, c\})\},\ H_E = \{(e_1, \{b, c\}), (e_2, \{a, c\})\}$.
Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $SBTOS$. It is clear that $\tau_{12} = \{\widehat{\phi}_E, X_E, G_E, H_E\}$. Thus $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$-soft connected and $DP$-soft connected because we cannot represented $X_E$ as a union of two non-null disjoint $IPO$-soft sets and $DPO$-soft sets, respectively. On the other hand, $\tau_{12}^{IPe_1} = \tau_{12}^{DPe_1} = \{\varnothing, X, \{a\}, \{b, c\}\}$. So $(X, \tau_1^{e_1}, \tau_2^{e_1}, \lesssim)$ is an $IP$-disconnected and a $DP$-disconnected space because $\{a\}$ is both an $IPO$ (resp. a $DPO$) and an $IPC$ (resp. a $DPC$)-soft set.

**Remark 4.11.** Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $IP$ (resp. a $DP$)-soft disconnected space and let $e \in E$. Then $(X, \tau_1^e, \tau_2^e, \lesssim)$ may not be an $IP$ (resp. a $DP$)-disconnected space as shown in the following example.

**Example 4.12.** Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(a, c)\}$ be a partial order relation on $X = \{a, b, c\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, M_E, N_E\},\ \tau_2 = \{\widehat{\phi}_E, X_E, K_E\}$, where
$M_E = \{(e_1, \{b\}), (e_2, \{b\})\},\ N_E = \{(e_1, X), (e_2, \{b\})\},\ K_E = \{(e_1, \varnothing), (e_2, \{a, c\})\}$.
Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $SBTOS$. It is clear that $\tau_{12} = \{\widehat{\phi}_E, X_E, M_E, N_E, K_E, P_E\}$, where $P_E = \{(e_1, \{b\}), (e_2, X)\}$. Since, $\{(e_1, X), (e_2, \{b\})\}$ is both an $IPO$ (resp. a $DPO$) and an $IPC$ (resp. a $DPC$)-soft set, by Theorem 4.8 (4), $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$-soft disconnected and $DP$-soft disconnected. Now, $\tau_1^{e_1} = \{\varnothing, X, \{b\}\}$ and $\tau_2^{e_1} = \{\varnothing, X\}$. Thus $\tau_{12}^{IPe_1} = \tau_{12}^{DPe_1} = \{\varnothing, X, \{b\}\}$. Obvious that $(X, \tau_1^{e_1}, \tau_2^{e_1}, \lesssim)$ is an $IP$-connected and a $DP$-connected space. We can show that $(X, \tau_1^{e_2}, \tau_2^{e_2}, \lesssim)$ is an $IP$-disconnected and a $DP$-disconnected space.

**Theorem 4.13.** *Let* $(X, \tau_1, \tau_2, E, \lesssim)$ *be an* $SBTOS$ *and let* $Y \subseteq X$. *Then* $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ *is an* $IP$ (resp. *a* $DP$)*-soft disconnected if and only if there exist two* $IPS$ (resp. $DPS$)*-soft sets* $F_E^Y$ *and* $K_E^Y$ *in* $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ *such that* $Y_E = F_E^Y \sqcup K_E^Y$, *where* $F_E^Y = Y_E \sqcap F_E,\ K_E^Y = Y_E \sqcap K_E,\ F_E,\ K_E \in \tau_{12}$.

*Proof.* Straightforward. □

**Definition 4.14.** A property $P$ of an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is called *hereditary property*, if every soft bitopological ordered subspace $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ of $(X, \tau_1, \tau_2, E, \lesssim)$ is also has the property $P$.

**Remark 4.15.** The increasing (resp. decreasing) soft connectedness does not hereditary property as shown in the following example.

**Example 4.16.** From Example 4.5 in [13], if $\lesssim = \blacktriangle \cup \{(y, w)\}$, then we have $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$-soft connected and a $DP$-soft connected space. But $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ is an $IP$-soft disconnected and a $DP$-soft disconnected space. Thus the increasing (resp. decreasing) soft connectedness does not hereditary property.

**Theorem 4.17.** *Let $(X, \delta_1, \delta_2, E, \lesssim)$ be an SBTOS finer than an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$.*

(1) *If $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$* (resp. *a $DP$*)*-soft disconnected space, then $(X, \delta_1, \delta_2, E, \lesssim)$ is $IP$* (resp. *$DP$*)*-soft disconnected.*

(2) *If $(X, \delta_1, \delta_2, E, \lesssim)$ is an $IP$* (resp. a *$DP$*)*-soft connected space, then $(X, \tau_1, \tau_2, E, \lesssim)$ is $IP$* (resp. *$DP$*)*-soft connected.*

*Proof.* (1) Suppose $(X, \delta_1, \delta_2, E, \lesssim)$ is an $IP$-soft disconnected space. Then there exist $G_E,\ H_E \in P(X)^E$ such that $Icl^s_{12}(G_E) \sqcap H_E = \widehat{\phi}_E,\ Icl^s_{12}(H_E) \sqcap G_E = \widehat{\phi}_E$ and $G_E \sqcup H_E = X_E$. Since $(X, \delta_1, \delta_2, E, \lesssim)$ is finer than of $(X, \tau_1, \tau_2, E, \lesssim)$, $\tau_{12} \subseteq \delta_{12}$. It follows that for any soft set $G_E$, we have $Icl^s_{\delta_{12}}(G_E) \sqsubseteq Icl^s_{\tau_{12}}(G_E)$. Thus we have

$$Icl^s_{\delta_{12}}(G_E) \sqcap H_E = \widehat{\phi}_E,\ Icl^s_{\delta_{12}}(H_E) \sqcap G_E = \widehat{\phi}_E \text{ and } G_E \sqcup H_E = X_E.$$

So $(X, \delta_1, \delta_2, E, \lesssim)$ is an $IP$-soft disconnected space.

(2) Suppose $(X, \delta_1, \delta_2, E, \lesssim)$ is an $IP$-soft connected space. Assume that $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$-soft disconnected space. Then by (1), $(X, \delta_1, \delta_2, E, \lesssim)$ is an $IP$-soft disconnected space, a contradiction.

The proof is similar in case of $(X, \tau_1, \tau_2, E, \lesssim)$ is a $DP$-soft connected. □

**Theorem 4.18.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. If $(X, \tau_1, E, \lesssim)$ or $(X, \tau_2, E, \lesssim)$ is an increasing* (resp. *a* decreasing) *soft disconnected space, then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$* (resp. *a $DP$*)*-soft disconnected space.*

*Proof.* It is immediate from the fact that: $Icl^s_{\tau_{12}}(G_E) = Icl^s_{\tau_1}(G_E) \sqcap Icl^s_{\tau_2}(G_E)$. □

**Remark 4.19.** Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an *SBTOS*. If $(X, \tau_1, E, \lesssim)$ and $(X, \tau_2, E, \lesssim)$ are both increasing (resp. decreasing) soft connected spaces, then $(X, \tau_1, \tau_2, E, \lesssim)$ may not be an $IP$ (resp. a $DP$)-soft connected space as shown in the following example.

**Example 4.20.** Let $E = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(a, c)\}$ be a partial order relation on $X = \{a, b, c, d\}$ and $\tau_1 = \{\widehat{\phi}_E, X_E, M_E\},\ \tau_2 = \{\widehat{\phi}_E, X_E, N_E\}$,
where $M_E = \{(e_1, \{a, c\}), (e_2, \{b, d\})\},\ N_E = \{(e_1, \{b, d\}), (e_2, \{a, c\})\}$.
Then $(X, \tau_1, E, \lesssim)$ and $(X, \tau_2, E, \lesssim)$ are both increasing (resp. decreasing) soft connected spaces. Obvious that $(X, \tau_1, \tau_2, E, \lesssim)$ is an *SBTOS*. Moreover, $\tau_{12} = \{\widehat{\phi}_E, X_E, M_E, N_E\}$. Since $M_E$ and $N_E$ are non-null disjoint $IPO$ (resp. $DPO$)-soft sets, $M_E \sqcup N_E = X_E$. Then by Theorem 4.8, we deduce that $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$ (resp. a $DP$)-soft disconnected space.

**Theorem 4.21.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and let $\varnothing \neq Y \subseteq X$ and let $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ be an $IP$* (resp. *a $DP$*)*-soft connected space. If $G_E$ and $H_E$ are an $IP$* (resp. *a $DP$*)*-soft separation of $X_E$, then $Y_E \sqsubseteq G_E$ or $Y_E \sqsubseteq H_E$.*

*Proof.* Suppose $G_E$ and $H_E$ are an $IP$-soft separation of $X_E$ and assume that $Y_E \not\sqsubseteq G_E$ and $Y_E \not\sqsubseteq H_E$. Then $Y_E \sqsubseteq X_E = G_E \sqcup H_E$ implies $Y_E \sqcap [G_E \sqcup H_E] = Y_E$. It follows that $[Y_E \sqcap G_E] \sqcup [Y_E \sqcap H_E] = Y_E$. On the other hand, since $Y_E \not\sqsubseteq G_E, Y_E \not\sqsubseteq H_E$ and $Y_E \sqsubseteq [G_E \sqcup H_E]$, $\widehat{\phi}_E \neq Y_E \sqcap G_E \neq Y_E$ and $\widehat{\phi}_E \neq Y_E \sqcap H_E \neq Y_E$. Since $G_E \sqcap H_E = \widehat{\phi}_E$ and $Icl^s_{12Y}(G_E) = Y_E \sqcap Icl^s_{12X}(G_E)$, we have

$$Icl^s_{12Y}[Y_E \sqcap G_E] \sqcap [Y_E \sqcap H_E] = \widehat{\phi}_E$$

and

$$Icl^s_{12Y}[Y_E \sqcap H_E] \sqcap [Y_E \sqcap G_E] = \widehat{\phi}_E.$$

Thus $Y_E \sqcap G_E$ and $Y_E \sqcap H_E$ are an $IP$-soft separation of $Y_E$, which contradicts with that $(Y, \tau_{1Y}, \tau_{2Y}, E, \lesssim)$ is an $IP$-soft connected space. So our assumption is not true. Hence $Y_E \sqsubseteq G_E$ or $Y_E \sqsubseteq H_E$.

The proof is similar in case of $G_E$ and $H_E$ are a $DP$-soft separation. □

## 5. Increasing (decreasing) pairwise soft connected (disconnected) soft sets

**Definition 5.1.** A soft set $G_E$ in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ is said to be an *increasing* (resp. *decreasing*) *pairwise disconnected soft set* (briefly, $IP$ (resp. $DP$)-disconnected soft set), if there exist two non-null $IPO$ (resp. $DPO$)-soft sets $O^1_E$, $O^1_E$ such that

$$G_E \sqcap O^1_E \neq \widehat{\phi}_E,\ G_E \sqcap O^2_E \neq \widehat{\phi}_E,\ G_E \sqsubseteq O^1_E \sqcup O^2_E \text{ and } O^1_E \sqcap O^2_E \sqsubseteq G^c_E.$$

In this case, we say that $O^1_E \sqcup O^2_E$ is $IP$ (resp. $DP$)-*soft disconnected* of $G_E$. A soft set $G_E$ is called an $IP$ (resp. a $DP$)-connected soft set, if has no $IP$ (resp. $DP$)-soft disconnected set.

**Example 5.2.** From Example 3.17, let $F_E = \{(e_1, \{w\}), (e_2, \{x\})\}$. Take $O^1_E = G_E$, $O^2_E = H^5_E$. It is clear that $G_E$, $H^5_E$ are $IP$-soft sets, and $F_E \sqcap O^1_E \neq \widehat{\phi}_E$, $F_E \sqcap O^2_E \neq \widehat{\phi}_E$, $F_E \sqsubseteq O^1_E \sqcup O^2_E = \{(e_1, \{x, w\}), (e_2, \{x, z, w\})\}$ and $O^1_E \sqcap O^2_E = \widehat{\phi}_E \sqsubseteq F^c_E$. Then $F_E$ is an $IP$-disconnected soft set. If we take $\lesssim = \blacktriangle \cup \{(w, y)\}$, then we can show that $O^1_E = G_E$, $O^2_E = H^5_E$ are $DP$-soft disconnected of $F_E$.

**Theorem 5.3.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and $G_E \in P(X)^E$. Then $x^e \widehat{\in} Icl^s_{12}$ $(G_E)$ if and only if $G_E \sqcap O^{x^e}_E \neq \widehat{\phi}_E$, $\forall O^{x^e}_E \in \tau^{DP}_{12}(x^e)$, where $O^{x^e}_E$ is any $DPO$-soft set contains $x^e$ and $\tau^{DP}_{12}(x^e)$ is the family of all $IPO$-soft sets contains $x^e$.*

*Proof.* Suppose $x^e \widehat{\in} Icl^s_{12}(G_E)$ and assume that there exists $O^{x^e}_E \in \tau^{DP}_{12}(x^e)$ such that $G_E \sqcap O^{x^e}_E = \widehat{\phi}_E$. Then $G_E \sqsubseteq O^{x^e c}_E$. Thus $Icl^s_{12}(G_E) \sqsubseteq Icl^s_{12}(O^{x^e c}_E) = O^{x^e c}_E$ which implies $Icl^s_{12}(G_E) \sqcap O^{x^e}_E = \widehat{\phi}_E$, a contradiction. Conversely, suppose the necessary condition holds assume that $x^e \widehat{\notin} Icl^s_{12}(G_E)$. Then $x^e \widehat{\in} [Icl^s_{12}(G_E)]^c$. Thus $[Icl^s_{12}(G_E)]^c \in \tau^{DP}_{12}(x^e)$. So by the hypothesis, $[Icl^s_{12}(G_E)]^c \sqcap G_E \neq \widehat{\phi}_E$, a contradiction. □

**Theorem 5.4.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an $SBTOS$ and $G_E \in P(X)^E$. Then $x^e \widehat{\in} Dcl^s_{12}$ $(G_E)$ if and only if $G_E \sqcap O^{x^e}_E \neq \widehat{\phi}_E$ $\forall O^{x^e}_E \in \tau^{IP}_{12}(x^e)$., where $O^{x^e}_E$ is any $IPO$-soft set contains $x^e$ and $\tau^{IP}_{12}(x^e)$ is the family of all $IPO$-soft sets contains $x^e$.*

*Proof.* Straightforward. □

**Lemma 5.5.** *If $O^1_E \sqcup O^2_E$ is an $IP$* (resp. *$DP$*)*-soft disconnected of $G_E$ in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$, then $G_E \sqcap O^1_E$ and $G_E \sqcap O^2_E$ are $DPS$* (resp. *$IPS$*)*-soft sets.*

*Proof.* Suppose $O^1_E \sqcup O^2_E$ is $IP$-soft disconnected of $G_E$. Then we have

$$G_E \sqcap O^1_E \neq \widehat{\phi}_E,\ G_E \sqcap O^2_E \neq \widehat{\phi}_E,\ G_E \sqsubseteq O^1_E \sqcup O^2_E \text{ and } O^1_E \sqcap O^2_E \sqsubseteq G^c_E.$$

We shall prove that $G_E \sqcap O^1_E$ and $G_E \sqcap O^2_E$ are $DPS$-soft sets. Let $x^e \widehat{\in} Dcl^s_{12}(O^1_E \sqcap G_E)$. Then by Theorem 5.3, $(O^1_E \sqcap G_E) \sqcap O^{x^e}_E \neq \widehat{\phi}_E\ \forall O^{x^e}_E \widehat{\in} \tau^{IP}_{12}(x^e)$. Now, assume that $x^e \widehat{\in} (O^2_E \sqcap G_E)$. It follows that $x^e \widehat{\in} O^2_E$. Then $O^2_E \widehat{\in} \tau^{IP}_{12}(x^e)$. Thus $(O^1_E \sqcap G_E) \sqcap O^2_E \neq \widehat{\phi}_E$, which a contradicts with the given $O^1_E \sqcap O^2_E \sqsubseteq G^c_E$. So $x^e\ \widehat{\not\in} (O^2_E \sqcap G_E)$. Hence $Dcl^s_{12}(O^1_E \sqcap G_E) \sqcap (O^2_E \sqcap G_E) = \widehat{\phi}_E$. Similarly, $Dcl^s_{12}(O^2_E \sqcap G_E) \sqcap (O^1_E \sqcap G_E) = \widehat{\phi}_E$. Therefore $G_E \sqcap O^1_E$ and $G_E \sqcap O^2_E$ are $DPS$-soft sets.

The proof is similar in case of $O^1_E \sqcup O^2_E$ is a $DP$-soft disconnected sets of $G_E$. □

**Theorem 5.6.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS and $G_E \in P(X)^E$. Then $x^e \widehat{\in} Dcl^s_{12}(G_E)$ if and only if $G_E \sqcap O^{x^e}_E \neq \widehat{\phi}_E\ \forall O^{x^e}_E \in \tau^{DP}_{12}(x^e)$, where $O^{x^e}_E$ is any DPO-soft set contains $x^e$ and $\tau^{DP}_{12}(x^e)$ is the family of all DPO-soft sets contains $x^e$.*

*Proof.* Straightforward. □

**Theorem 5.7.** *A soft set $G_E$ in an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is a $DP$* (resp. $IP$)*-disconnected soft set if and only if there exist two $IP$* (resp. $DP$)*-separated soft sets $S^1_E$, $S^2_E$ such that $G_E = S^1_E \sqcup S^2_E$.*

*Proof.* Suppose that $G_E$ is a $DP$-disconnected soft set in $(X, \tau_1, \tau_2, E, \lesssim)$. Then $G_E$ has a $DP$-soft disconnection, say $O^1_E \sqcup O^2_E$, i.e., there exist two non-null $DPO$-soft sets $O^1_E$, $O^2_E$ such that $G_E \sqcap O^1_E \neq \widehat{\phi}_E$, $G_E \sqcap O^2_E \neq \widehat{\phi}_E$, $G_E \sqsubseteq O^1_E \sqcup O^2_E$ and $O^1_E \sqcap O^2_E \sqsubseteq G^c_E$. Then by lemma 5.5, it follows that $G_E \sqcap O^1_E$, $G_E \sqcap O^2_E$ are $IPS$-soft sets. Since $G_E \sqsubseteq O^1_E \sqcup O^2_E$, $G_E \sqcap (O^1_E \sqcup O^2_E) = G_E$ implies $(G_E \sqcap O^1_E) \sqcup (G_E \sqcap O^2_E) = G_E$. Take $S^1_E = G_E \sqcap O^1_E$ and $S^2_E = G_E \sqcap O^2_E$.

Conversely, let $S^1_E$, $S^2_E$ be two $IP$-soft sets and let $G_E \in P(X)^E$ such that $G_E = S^1_E \sqcup S^2_E$. Then $Icl^s_{12}(S^1_E) \sqcap S^2_E = \widehat{\phi}_E$ and $Icl^s_{12}(S^2_E) \sqcap S^1_E = \widehat{\phi}_E$. Take $O^1_E = [Icl^s_{12}(S^1_E)]^c$ and $O^2_E = [Icl^s_{12}(S^2_E)]^c$. Then $O^1_E$, $O^2_E$ are non-null $DPO$-soft sets. Since $Icl^s_{12}(S^1_E) \sqcap S^2_E = \widehat{\phi}_E$, $S^1_E \sqsubseteq [Icl^s_{12}(S^2_E)]^c = O^2_E$. By similar, we also have $S^2_E \sqsubseteq O^1_E$. It follows that $G_E \sqsubseteq O^1_E \sqcup O^2_E$. Since $[Icl^s_{12}(S^1_E)]^c \sqsubseteq S^{1c}_E$, $[Icl^s_{12}(S^2_E)]^c \sqsubseteq S^{2c}_E$, $O^1_E \sqcap O^2_E \sqsubseteq G^c_E$. Furthermore, since $S^1_E$, $S^2_E \sqsubseteq G_E$ and $S^2_E \sqsubseteq O^1_E$, $S^1_E \sqsubseteq O^2_E$, $S^2_E \sqsubseteq G_E \sqcap O^1_E$ and $S^1_E \sqsubseteq G_E \sqcap O^2_E$. But $S^1_E \neq \widehat{\phi}_E$, $S^2_E \neq \widehat{\phi}_E$. Thus $G_E \sqcap O^1_E \neq \widehat{\phi}_E$, $G_E \sqcap O^2_E \neq \widehat{\phi}_E$. So $G_E$ is a $DP$-disconnected soft set.

The proof is similar in case of $G_E$ is an $IP$-disconnected soft set. □

**Corollary 5.8.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS. If $S^1_E$, $S^2_E$ are two IPS* (resp. *DPS*)*-soft sets, then $S^1_E \sqcup S^2_E$ is a DP* (resp. *IP*)*-disconnected soft set.*

**Corollary 5.9.** *A soft set $G_E$ in an SBTOS $(X, \tau_1, \tau_2, E, \lesssim)$ is IP* (resp. *DP*)*-connected soft set if and only if it cannot expressed as a union of two IDS* (resp. *IPS*)*-soft sets.*

**Proposition 5.10.** *Let $(X, \tau_1, \tau_2, E, \lesssim)$ be an SBTOS.*

(1) *Every soft point is an IP* (resp. *a DP*)*-connected soft set.*
(2) *The null soft set is an IP* (resp. *a DP*)*-connected soft set.*

*Proof.* (1) Let $x^e \widehat{\in} X_E$. Then for any two non-null $IPO$-soft sets $O_E^1$, $O_E^2$ such that $\{x^e\} \sqcap O_E^1 \neq \widehat{\phi}_E$, $\{x^e\} \sqcap O_E^2 \neq \widehat{\phi}_E$, we have $x^e \widehat{\in} O_E^1 \sqcap O_E^2$. It follows that $O_E^1 \sqcap O_E^2 \not\sqsubseteq \{x^e\}^c$. Thus $x^e$ is an $IP$-connected soft set.

The proof is similar in case of a $DP$-connected soft set.

(2) Obvious. □

**Theorem 5.11.** *Let $F_E$ be an $IP$ (resp. $DP$)-connected soft set in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ and let $F_E \sqsubseteq M_E \sqsubseteq Icl_{12}^s(F_E)$. Then $M_E$, $Icl_{12}^s(F_E)$ are also $IP$ (resp. $DP$)-connected soft sets.*

*Proof.* Let $F_E$ be an $IP$-connected soft set in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$ and assume that $M_E$ is an $IP$-disconnected soft set in an $(X, \tau_1, \tau_2, E, \lesssim)$. Then there exist two non-null $IPO$-soft sets $O_E^1$, $O_E^2$ such that

$$M_E \sqcap O_E^1 \neq \widehat{\phi}_E,\ M_E \sqcap O_E^2 \neq \widehat{\phi}_E,\ M_E \sqsubseteq O_E^1 \sqcup O_E^2$$

and

$$O_E^1 \sqcap O_E^2 \sqsubseteq M_E^c.$$

Since $F_E \sqsubseteq M_E$, $F_E \sqsubseteq O_E^1 \sqcup O_E^2$ and $O_E^1 \sqcap O_E^2 \sqsubseteq F_E$. Since $F_E$ is an $IP$-connected soft set, either $F_E \sqcap O_E^1 = \widehat{\phi}_E$ or $F_E \sqcap O_E^2 = \widehat{\phi}_E$. If we claim that $F_E \sqcap O_E^1 = \widehat{\phi}_E$, then $O_E^{1c}$ is a $DPO$-soft set contains $F_E$. It follows that $Icl_{12}^s(F_E) \sqsubseteq O_E^{1c}$ which implies that $M_E \sqcap O_E^1 = \widehat{\phi}_E$, a contradicts with our assumption. Thus our assumption is false. So $M_E$ is an $IP$-connected soft set. In particular, put $M_E = Icl_{12}^s(F_E)$. Then $Icl_{12}^s(F_E)$ is also an $IP$-connected soft set.

The proof is similar in case of $F_E$ is a $DP$-disconnected soft set. □

**Remark 5.12.** The soft subset of an $IP$ (resp. $DP$)-soft connected space need not be an $IP$ (resp. $DP$)-connected soft set as seen in the following example.

**Example 5.13.** Consider Example 4.7. Let $\lesssim = \blacktriangle \cup \{(a, c)\}$. Then $(X, \tau_1, \tau_2, E, \lesssim)$ is an $IP$-soft connected space. Now, let $F_E = \{(e_1, \{c\}), (e_2, \varnothing), (e_3, \{d\})\}$. Take $O_E^1 = G_E^2$, $O_E^2 = G_E^3$. Then we have

$$F_E \sqcap O_E^1 = \{(e_1, \varnothing), (e_2, \varnothing), (e_2, \{d\})\}, F_E \sqcap O_E^2 = \{(e_1, \{c\}), (e_2, \varnothing), (e_2, \varnothing)\},$$

$$F_E \sqsubseteq O_E^1 \sqcup O_E^2 = \{(e_1, \{c\}), (e_2, \{a, c\}), (e_2, \{d\})\} \text{ and } O_E^1 \sqcap O_E^2 = \widehat{\phi}_E \sqsubseteq F_E^c.$$

Thus $F_E$ is an $IP$-disconnected soft subset of $(X, \tau_1, \tau_2, E, \lesssim)$. If we take $\lesssim = \blacktriangle \cup \{(c, a)\}$, then we have $F_E$ is a $DP$-disconnected soft subset of $(X, \tau_1, \tau_2, E, \lesssim)$.

**Remark 5.14.** The union of two $IP$ (resp. $DP$)-connected soft sets need not be an $IP$ (resp. $DP$)-connected soft set as seen in the following example.

**Example 5.15.** From Example 4.7 and Example 5.13, it is clear by Proposition 5.10 (1) that $c^{e_1}$, $d^{e_3}$ are $IP$ (resp. $DP$)-connected soft sets. Nevertheless, $\{c^{e_2}\} \sqcup \{d^{e_3}\} = \{(e_1, \{c\}), (e_2, \varnothing), (e_2, \{d\})\} = F_E$ is an $IP$-disconnected soft set and a $DP$-disconnected soft set.

**Theorem 5.16.** *Let $G_E$, $H_E$ be two $IP$ (resp. $DP$)-connected soft sets in an $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim)$. If $G_E \sqcap H_E \neq \widehat{\phi}_E$, then $G_E \sqcup H_E$ is an $IP$ (resp. $DP$)-connected soft set.*

*Proof.* Let $G_E$ and $H_E$ be $IP$ (resp. $DP$)-connected soft sets and suppose $G_E \sqcap H_E \neq \widehat{\phi}_E$. Assume that $G_E \sqcup H_E$ is an $IP$-disconnected soft set. Then there exist two non-null $IPO$-soft sets $O_E^1$, $O_E^2$ such that $[G_E \sqcup H_E] \sqcap O_E^1 \neq \widehat{\phi}_E$, $[G_E \sqcup H_E] \sqcap O_E^2 \neq \widehat{\phi}_E$, $[G_E \sqcup H_E] \sqsubseteq O_E^1 \sqcup O_E^2$ and $[O_E^1 \sqcap O_E^2] \sqsubseteq [G_E \sqcup H_E]^c$. Since $G_E \sqsubseteq G_E \sqcup H_E$, $G_E[\sqsubseteq O_E^1 \sqcup O_E^2]$ and $O_E^1 \sqcap O_E^2 \sqsubseteq G_E^c$. Since $G_E$ is an $IP$-connected soft set, $G_E \sqcap O_E^1 = \widehat{\phi}_E$ or $G_E \sqcap O_E^2 = \widehat{\phi}_E$. Thus $G_E \sqsubseteq O_E^1$ or $G_E \sqsubseteq O_E^2$ for $[G_E \sqsubseteq O_E^1 \sqcup O_E^2]$. Similarly, $H_E \sqsubseteq O_E^1$ or $H_E \sqsubseteq O_E^2$. Thus, if $G_E \sqsubseteq O_E^1$ and $H_E \sqsubseteq O_E^2$, then $G_E \sqcap H_E \sqsubseteq O_E^1 \sqcap O_E^2 \sqsubseteq G_E^c \sqcap H_E^c$ which implies that $G_E \sqcap H_E = \widehat{\phi}_E$, a contradiction. Similarly, when $G_E \sqsubseteq O_E^2$ and $H_E \sqsubseteq O_E^1$, we have a contradiction. So our assumption is not true. Hence $G_E \sqcup H_E$ is an $IP$-connected soft set. □

**Theorem 5.17.** *Let $\phi_\psi$ be an $ISP$ (resp. a $DSP$)-continuous and injective mapping from $SBTOS$ $(X, \tau_1, \tau_2, E, \lesssim_1)$ in to an $SBTOS$ $(Y, \eta_1, \eta_2, K, \lesssim_2)$. If $G_E$ is an $IP$ (resp $DP$)-connected soft set in $(X, \tau_1, \tau_2, E, \lesssim_1)$, then $\phi_\psi(G_E)$ is an $IP$ (resp. a $DP$)-connected soft set in $(Y, \eta_1, \eta_2, K, \lesssim_2)$.*

*Proof.* Suppose $G_E$ is an $IP$-connected soft set in $(X, \tau_1, \tau_2, E, \lesssim_1)$. Assume that $\phi_\psi(G_E)$ is not an $IP$-connected soft set in $(Y, \eta_1, \eta_2, K, \lesssim_2)$. Then there exist two non-null $IPO$-soft sets $O_K^1$, $O_K^2$ such that $\phi_\psi(G_E) \sqcap O_K^1 \neq \widehat{\phi}_E$, $\phi_\psi(G_E) \sqcap O_K^2 \neq \widehat{\phi}_E$, $\phi_\psi(G_E) \sqsubseteq O_K^1 \sqcup O_K^2$ and $[O_K^1 \sqcap O_K^2] \sqsubseteq Y_K - \phi_\psi(G_E)$. Then by Theorem 4.8 and Proposition 2.8, it follows that $G_E \sqcap \phi_\psi^{-1}(O_K^1) \neq \widehat{\phi}_E$, $G_E \sqcap \phi_\psi^{-1}(O_K^2) \neq \widehat{\phi}_E$, $G_E \sqsubseteq \phi_\psi^{-1}(O_K^1) \sqcup \phi_\psi^{-1}(O_K^2)$ and $[\phi_\psi^{-1}(O_K^1) \sqcap \phi_\psi^{-1}(O_K^2)] \sqsubseteq \phi_\psi^{-1}(Y_K - \phi_\psi(G_E)) = X_K - G_E$. Since $\phi_\psi$ is an $ISP$-continuous, $\phi_\psi^{-1}(O_K^1)$, $\phi_\psi^{-1}(O_K^2)$ are $IPO$-soft sets in $(X, \tau_1, \tau_2, E, \lesssim_1)$. Thus $\phi_\psi^{-1}(O_K^1) \sqcup \phi_\psi^{-1}(O_K^2)$ form an $IP$-soft disconnection of $G_E$ which contrary to the fact that $G_E$ is an $IP$-connected soft set in $(X, \tau_1, \tau_2, E, \lesssim_1)$. So $\phi_\psi(G_E)$ is an $IP$-connected soft set in $(Y, \eta_1, \eta_2, K, \lesssim_2)$. □

In Theorems 5.16 and 5.17, a similar proof can be given for the case between parentheses.

**Corollary 5.18.** *Let $\phi_\psi$ be an $ISP$ (resp. a $DSP$)-continuous and injective mapping from an $IP$ (resp. a $DP$)-connected soft space $(X, \tau_1, \tau_2, E, \lesssim_1)$ on to an $SBTOS$ $(Y, \eta_1, \eta_2, K, \lesssim_2)$, then $(Y, \eta_1, \eta_2, K, \lesssim_2)$ is an $IP$ (resp. a $DP$)-soft connected space.*

## 6. Conclusion

In 1965, Nachbin [1] introduced the concept of topological ordered space, which combines the properties of partial order relations and topological spaces. Later, in 1999, Molodtsov [3] proposed the idea of "soft sets" to address issues related to uncertainty, vagueness, imprecision, and incomplete data.Ittanagi [11] introduced the notion of a soft bitopological space. Building upon these concepts, El-Sheikh et al. [16] introduced the concept of soft bitopological ordered spaces

In this paper, we introduced and studied the notion of $IPS$ (resp. $DPS$)-soft sets. Based on this notion, we defined and studied some properties and characterizations of $IP$ (resp. $DP$)-soft connected spaces and $IP$ (resp. $DP$)-connected soft sets in soft bitopological ordered spaces. Some properties of such notions are obtained. We expect that the findings in this paper can be promoted to the further study on soft bitopology ordered to carry out general framework for the practical life applications.

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S. H. Shalil (slamma_elarabi@yahoo.com)
Mathematics Department, Faculty of science, Helwan University, Cairo, Egypt

S. A. El-Sheikh (elsheikh33@hotmail.com)
Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt

S. A. Kandil (shehab_ali@cic-cairo.com)
Mathematics Department, Canadian International college, Cairo, Egypt