Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 28, No. 1, (August 2024) pp. 85–104
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2024.28.1.85

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 28, No. 1, August 2024

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Received 30 December 2023; Revised 11 March 2024; Accepted 30 April 2024

Abstract. In this paper, we relate a bounded semihoop A to an abelian ℓ-group G to study G-states. Firstly, the notion of Bosbach G-states on bounded semihoops is presented, which reflects a certain characteristic of energy conservation. We also obtain some important conclusions of Bosbach G-state as follows: (1) Let s be a Bosbach G-state on a bounded semihoop A and H be a down-set subgroup of G. Then $s^{-1}(H)$ is an ideal of A. (2) Every bounded dual perfect semihoop has a non-zero Bosbach G-state. Secondly, we propose the notion of Riečan G-states on bounded semihoops and obtain some properties. We prove that every Bosbach G-state is a Riečan G-state on a bounded semihoop A but a Riečan G-state may not be a Bosbach G-state unless A has the DNP property. Finally, we introduce the concept of G-state morphisms and obtain some characterizations of G-state morphisms.

2020 AMS Classification: 03E72, 08A72

Keywords: Bounded semihoop, Abelian ℓ-group, Bosbach G-state, Riečan G-state, G-state morphism.

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1. Introduction

It is well-established that non-classical logic has become a valuable tool in computer science for addressing uncertain and fuzzy information. Various logic algebras have been proposed as semantic systems for non-classical logic systems, for instance, MV-algebras [1], BL-algebras [2], MTL-algebras [3], R_0-algebras [4], hoops [5] and residuated lattices [6]. Among all logic algebras, semihoops is the most basic residual structure that contain all logical algebras that satisfy the residuated law. The semihoops are a generalization of hoops that were first proposed by Bosbach under the name of complementary semigroups. In recent years, semihoops have received increasing attention from scholars and has achieved many important results. For example, Borzooei and Kologani [7] studied the relationships between various filters on semihoops in 2015. In 2019, Niu [8] introduced tense operators on bounded semihoops. In 2020, Niu and Xin [9] established the ideal theory on bounded semihoops. Therefore, as the most basic fuzzy structure, semihoops play an important role in the study of fuzzy logic and its related algebraic structures.

Probability theory is a common and effective tool for dealing with the concept of randomness and information. The idea of introducing probability into propositional logic to reflect the degree to which logical formulas are true has been around for a long time. Many scholars have worked on this topic and have achieved rich research results. For instance, Novák and Pavelka [10] devised all the basic concepts and reasoning processes in propositional logic and constructed a complete theoretical system in propositional logic system Łukasiewicz. American scholar Hailpertin [11] proposed probabilistic logic. State theory is also an effective way to combine probability theory with logic. In 1995, states on MV-algebras were first proposed by Mundici [12]. The purpose was to seek some average of the individual truth values of formulas in propositional logic of Łukasiewicz. Since then, scholars have successively extended the concept of state to other different logical algebraic structures, for example, BL-algebras [13], EQ-algebras [14], pseudo MV-algebras [15] and residuated lattices [16]. In particular, in reference [17], states and internal states on bounded semihoop were studied and a number of valuable conclusions were obtained. In [18], the existence of states based on Glivenko semihoops were studied.

However, in the study of state theory, scholars have taken the standard MV interval [0,1] as the assignment domain, ignoring the more general structures. One example is the group structure, which holds fundamental importance in abstract algebra. Many algebraic structures, including rings, domains, and modules, can be seen as formed by adding new operations and axioms to the basis of groups. The notion of group appears in many branches of mathematics, and the approach to group theory has had a significant influence on other branches of abstract algebra. Therefore, in this paper, we studied G-states in connection with a bounded semihoop and an abelian ℓ-group [19, 20]. The idea is to replace the interval [0,1] by the group, which is a generalisation of states on bounded semihoop. We know that everything in the world conforms to the law of conservation of energy. We can think of the bounded semihoop as an energy field and a G-states as energy measures, so that everything in life that satisfies the law of conservation of energy can be characterised by a G-state.

The structure of this paper is as follows: In Section 2, we review some of the basic definitions and properties used in this paper. In Section 3, we introduce the notion of Bosbach G-states on bounded semihoops and get some propositics and equivalent descriptions. In Section 4, we propose the concept of Riečan G-states on bounded semihoops and some properties are investigated. In particular, we discuss the relationship between Bosbach G-states and Riečan G-states. In Section 5, we prove $\Gamma(G,u)$ with some operations is a bounded semihoop and then investigate G-states morphisms.

2. Preliminaries

In this section, we review some definitions and conclusions that will be used in the following sections.

Definition 2.1 ([5]). An algebra $(A, \odot, \to, \wedge, 1)$ of type $(2, 2, 2, 0)$ is called a *semihoop*, if it satisfies: for for any $\alpha, \beta, \theta \in A$,
(i) $(A, \wedge, 1)$ is a $\wedge$-semilattice and it has an upper bound 1,
(ii) $(A, \odot, 1)$ is a commutative monoid,
(iii) $(\alpha \odot \beta) \to \theta = \alpha \to (\beta \to \theta)$.

A semihoop $(A, \odot, \to, \wedge, 1)$ is called a *bounded semihoop*, if there exists an element $0 \in A$ such that $0 \leq \alpha$ for all $\alpha \in A$. We denote a bounded semihoop $(A, \odot, \to, \wedge, 0, 1)$ by A.

In a semihoop $(A, \odot, \to, \wedge, 1)$, we define $\alpha \leq \beta$ if and only if $\alpha \to \beta = 1$ for any $\alpha, \beta \in A$. It is easy to check that $\leq$ is a partial order relation on A and we get $\alpha \leq 1$ for all $\alpha \in A$.

Proposition 2.2 ([5]). *Let A be a semihoop. Then the following properties hold: for every $\alpha, \beta, \theta \in A$,*
(1) $\alpha \odot \beta \leq \theta$ *if and only if* $\alpha \leq \beta \to \theta$,
(2) $\alpha \odot \beta \leq \alpha \wedge \beta$, $\alpha \leq \beta \to \alpha$,
(3) $1 \to \alpha = \alpha, \alpha \to 1 = 1$,
(4) $\alpha^n \leq \alpha$, *for every* $\alpha \in A$, $n \in \mathbb{N}^+$,
(5) $\alpha \odot (\alpha \to \beta) \leq \beta$,
(6) $\alpha \leq \beta$ *implies* $\alpha \odot \theta \leq \beta \odot \theta$, $\beta \to \theta \leq \alpha \to \theta$ *and* $\theta \to \alpha \leq \theta \to \beta$,
(7) $\alpha \leq (\alpha \to \beta) \to \beta$,
(8) $\alpha \to (\beta \to \theta) = \beta \to (\alpha \to \theta)$,
(9) $((\alpha \to \beta) \to \beta) \to \beta = \alpha \to \beta$,
(10 $\alpha \to \beta \leq (\theta \to \alpha) \to (\theta \to \beta)$, $\alpha \to \beta \leq (\beta \to \theta) \to (\alpha \to \theta)$,
(11) $\alpha \to (\alpha \wedge \beta) = \alpha \to \beta$
(12) $\alpha \odot \beta = \alpha \odot (\alpha \to (\alpha \odot \beta))$.

In a bounded semihoop A, we define $\star$: $\alpha^\star = \alpha \to 0$ for any $\alpha \in A$. A bounded semihoop is said to have the Double Negation Property or (DNP) for short if it satisfies $\alpha^{\star\star} = \alpha$ for all $\alpha \in A$.

Proposition 2.3 ([7]). *Let A be a bounded semihoop. Then the following statements hold: for any $\alpha, \beta \in A$,*
(1) $1^\star = 0$, $0^\star = 1$,
(2) $\alpha \leq \alpha^{\star\star}$,
(3) $\alpha^{\star\star\star} = \alpha^\star$,
(4) $\alpha \odot \alpha^\star = 0$,
(5) $\beta^\star \leq \beta \to \alpha$,
(6) $\alpha \leq \beta$ *implies* $\beta^\star \leq \alpha^\star$,
(7) $\alpha \to \beta \leq \beta^\star \to \alpha^\star$,
(8) $(\alpha \to \beta^{\star\star})^{\star\star} = \alpha \to \beta^{\star\star}$,
(9) $\alpha^{\star\star} \odot \beta^{\star\star} \leq (\alpha \odot \beta)^{\star\star}$,
(10 $(\alpha^{\star\star} \odot \beta)^\star = (\alpha \odot \beta)^\star$.

Definition 2.4 ([9]). Let A be a bounded semihoop. A non-empty subset I of A is called an *ideal*, if it satisfies the following conditions:
(I_1) for any $\alpha, \beta \in A$, $\alpha \leq \beta$ and $\beta \in I$ imply $\alpha \in I$,
(I_2) for any $\alpha, \beta \in I$, $\alpha \oplus \beta \in I$, where $\alpha \oplus \beta = \alpha^\star \to \beta$.

Definition 2.5 ([19]). A *lattice ordered abelian group* (briefly, ℓ-group) is a structure $(G, +, \leq)$ such that the following conditions hold:
(i) $(G, +)$ is an abelian group,
(ii) $(G, \leq)$ is a lattice,
(iii) $x \leq y$ implies $x + z \leq y + z$ for any $x, y, z \in G$.

The infimum and supremum of two elements $x, y \in G$ are denoted by $x \wedge_G y$ and $x \vee_G y$, respectively.

Lemma 2.6 ([19]). *Let $(G, +, \leq)$ be an ℓ-group. Then the following statements hold: for any $w, x, y, z \in G$,*
(1) $w + (x \wedge_G y) + z = (w + x + z) \wedge_G (w + y + z)$,
(2) $w + (x \vee_G y) + z = (w + x + z) \vee_G (w + y + z)$.

Proposition 2.7 ([7]). *Let A be a bounded semihoop. We define:*

$$\alpha \vee \beta = [(\alpha \to \beta) \to \beta] \wedge [(\beta \to \alpha) \to \alpha] \text{ for any } \alpha,\ \beta \in A.$$

Then the following conditions are equivalent: for all $\alpha, \beta, \theta \in A$,
(1) $\vee$ *is an associative operation on A,*
(2) $\alpha \leq \beta$ *implies* $\alpha \vee \theta \leq \beta \vee \theta$,
(3) $\alpha \vee (\beta \wedge \theta) \leq (\alpha \vee \beta) \wedge (\alpha \vee \theta)$,
(4) $\vee$ *is the join operation on A.*

Definition 2.8 ([7]). A bounded semihoop is called a *bounded $\vee$-semihoop*, if it satisfies one of the equivalent conditions of Proposition 2.7.

It is easy to prove that if A is a bounded $\vee$-semihoop, then $(A, \wedge, \vee)$ is a distributive lattice (See [12]).

Definition 2.9 ([17]). Let A be a bounded semihoop. A *Bosbach state* on A is a function s: $A \to [0, 1]$ such that the following conditions hold: for all $\alpha, \beta \in A$,
(B_1) $s(0) = 0$, $s(1) = 1$,
(B_2) $s(\alpha) + s(\alpha \to \beta) = s(\beta) + s(\beta \to \alpha)$.

Definition 2.10 ([17]). Let A be a bounded semihoop. Two elements $\alpha, \beta \in A$ are said to be *orthogonal*, denoted by $\alpha \perp \beta$, if $\beta^{\star\star} \leq \alpha^{\star}$.

Definition 2.11 ([17]). Let A be a bounded semihoop. A *Riečan state* on A is a function s: $A \to [0, 1]$ such that the following conditions hold: for any $\alpha, \beta \in A$,
(R_1) $s(1) = 1$,
(R_2) if $\alpha \perp \beta$, then $s(\alpha + \beta) = s(\alpha) + s(\beta)$, where $\alpha + \beta = \alpha^{\star} \to \beta^{\star\star}$.

Definition 2.12 ([9]). Let A be a bounded semihoop and I be a proper ideal of A. I is called a *maximal ideal* of A, if it is not properly contained in the any other proper ideal of A.

We denote the set of all maximal ideal of bounded semihoops A by $M(A)$. The radical of A is defined by $Rad(A) = \bigcap\{M | M \in Max(A)\}$. Also, we can define that

$$Rad(A)^{\star} = \{\alpha^{\star} | \alpha \in Rad(A)\}.$$

Definition 2.13 ([7]). Let A be a bounded semihoop. A nonempty subset F of A is called a *filter*, if it satisfies:
(F_1) for any $\alpha, \beta \in A$, $\alpha \leq \beta$ and $\alpha \in F$ imply $\beta \in F$,
(F_2) for any $\alpha, \beta \in F$, $\alpha \odot \beta \in F$.

Definition 2.14 ([19]). A ring with a partial order $\leq$ is called a *lattice ordered abelian ring* (briefly, ℓ-ring), if it satisfies the following conditions: for any $x, y, z \in R$,
(i) $x \leq y$ implies $x + z \leq y + z$,
(ii) $0 \leq x$ and $0 \leq y$ imply $0 \leq x \cdot y$.

3. Bosbach G-states

In this section we focus on Bosbach G-state theories on bounded semihoop and their related properties.

Definition 3.1. Let $(A, \odot, \rightarrow, \wedge, 0, 1)$ be a bounded semihoop and $(G, +, \leq)$ be an abelian ℓ-group. A *Bosbach G-state* on A is a function s: $A \rightarrow G$ such that the following conditions hold: for all $\alpha, \beta \in A$,
(BG_1) $s(0) = 0_G$,
(BG_2) $s(1)$ is the largest element of $Im(s)$,
(BG_3) $s(\alpha) + s(\alpha \rightarrow \beta) = s(\beta) + s(\beta \rightarrow \alpha)$.

Unless otherwise specified, all G appearing in this article is an abelian ℓ-group.

Remark 3.2. If we consider a bounded semihoop as an energy field and a Bosbach G-state as a measure of energy, then Definition 3.1 corresponds to the following observations:
(1) $s(0) = 0_G$ shows that energy is not generated out of thin air,
(2) $s(1)$ reflects the largest energy produced under the largest conditions,
(3) $s(\alpha) + s(\alpha \rightarrow \beta) = s(\beta) + s(\beta \rightarrow \alpha)$ shows that energy will only be converted from one form to another, while the total energy before and after the conversion remains the same.

Example 3.3 ([17]). Let $A = \{0, a, b, c, 1\}$ be a chain, where $0 < a < b < c < 1$. Define operations $\odot$ and $\rightarrow$ on A as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(A, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop. Let $G = (Z, +, \leq)$ be an ablian ℓ-group. For $n \in Z^+$, we define a map s_n: $A \rightarrow G$,

$$s_n(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, a \\ n & \text{if } \alpha = b, c, 1 \end{cases}$$

for all $\alpha \in A$. Then we can easily check that s_n is a Bosbach G-state on A.

By Example 3.3, we can find that unlike Bosbach state on the interval $[0, 1]$, Bosbach G-state may not be $s(1) = 1_G$, which is the largest element of $Im(s)$.

Proposition 3.4. *Let A be a bounded semihoop and s be a Bosbach G-state on A. Then the following properties hold: for any $\alpha, \beta, \gamma \in A$,*

(1) $s(\alpha) + s(\alpha^\star) = s(1)$ *and* $s(\alpha^\star) = s(1) - s(\alpha)$,
(2) $s(\alpha^{\star\star}) = s(\alpha)$,
(3) $\alpha \leq \beta$ *implies* $s(\alpha) \leq s(\beta)$,
(4) $s(\alpha \odot \beta) + s(\alpha \to \beta^\star) = s(1)$,
(5) $s(\alpha) + s(\beta) = s(\alpha \odot \beta) + s(\beta^\star \to \alpha)$,
(6) $s(\alpha^{\star\star} \to \alpha) = s(1)$,
(7) $s(\alpha \oplus \beta) \leq s(\alpha) + s(\beta)$,
(8) $s(\alpha \to \beta) = s(\beta \to \alpha)$ *if and only if* $s(\alpha) = s(\beta)$,
(9) $s(\alpha \to \beta^{\star\star}) = s(\alpha^{\star\star} \to \beta) = s(\alpha \to \beta)$,
(10) $s(\alpha^\star \to \beta^\star) = s(\alpha \to \beta)$,
(11) $s(\alpha \to (\beta^\star \to \gamma^\star)) = s(\alpha \to (\gamma \to \beta))$,
(12) $\alpha \leq \beta$ *implies* $s(\beta \ominus \alpha) = s(\beta) - s(\alpha)$, *where* $\alpha \ominus \beta = \alpha \odot \beta^\star$.

Proof. (1) Since s is a Bosbach G-state on A, we have

$$s(\alpha) + s(\alpha^\star) = s(\alpha) + s(\alpha \to 0) = s(0) + s(0 \to \alpha) = 0 + s(1) = s(1).$$

(2) Since $s(\alpha^\star)+s(\alpha^{\star\star}) = s(\alpha^\star)+s(\alpha^\star \to 0) = s(0)+s(0 \to \alpha^\star) = 0+s(1) = s(1)$, $s(1) - s(\alpha) + s(\alpha^{\star\star}) = s(1)$. Then $s(\alpha^{\star\star}) = s(\alpha)$.

(3) From $\alpha \leq \beta$, $\alpha \to \beta = 1$. Since $s(\alpha) + s(\alpha \to \beta) = s(\beta) + s(\beta \to \alpha)$, $s(\alpha) + s(1) = s(\beta) + s(\beta \to \alpha)$. Then $s(\alpha) - s(\beta) = s(\beta \to \alpha) - s(1) \leq 0$. Thus $s(\alpha) \leq s(\beta)$.

(4) From Definition 2.1 (3), we have

$$\begin{aligned} s(\alpha \odot \beta) + s(\alpha \to \beta^\star) &= s(\alpha \odot \beta) + s(\alpha \to (\beta \to 0)) \\ &= s(\alpha \odot \beta) + s((\alpha \odot \beta) \to 0)) \\ &= s(\alpha \odot \beta) + s((\alpha \odot \beta)^\star) \\ &= s(1). \end{aligned}$$

(5) Since $s(\alpha) + s(\alpha \to \beta^\star) = s(\beta^\star) + s(\beta^\star \to \alpha)$, from (1) and (4), we have

$$s(\alpha) + s(1) - s(\alpha \odot \beta) = s(1) - s(\beta) + s(\beta^\star \to \alpha).$$

Then $s(\alpha) + s(\beta) = s(\alpha \odot \beta) + s(\beta^\star \to \alpha)$.

(6) From $\alpha \leq \alpha^{\star\star}$, $\alpha \to \alpha^{\star\star} = 1$. Since $s(\alpha^{\star\star}) + s(\alpha^{\star\star} \to \alpha) = s(\alpha) + s(\alpha \to \alpha^{\star\star})$, by (2), $s(\alpha) + s(\alpha^{\star\star} \to \alpha) = s(\alpha) + s(1)$. Then $s(\alpha^{\star\star} \to \alpha) = s(1)$.

(7) Since $s(\alpha^\star) + s(\alpha \oplus \beta) = s(\alpha^\star) + s(\alpha^\star \to \beta) = s(\beta) + s(\beta \to \alpha^\star)$, we get

$$s(1) - s(\alpha) + s(\alpha \oplus \beta) = s(\beta) + s(\beta \to \alpha^\star).$$

Then $s(\alpha \oplus \beta) - (s(\alpha) + s(\beta)) = s(\beta \to \alpha^\star) - s(1) \leq 0$. Thus $s(\alpha \oplus \beta) \leq s(\alpha) + s(\beta)$.

(8) $-$ (11) These cases prove similarly as Proposition 3.3 in [13].

(12) From $\alpha \leq \beta$, $\alpha \to \beta = 1$. Then we have

$$\begin{aligned} s(\beta \ominus \alpha) &= s(\beta \odot \alpha^\star) \\ &= s(1) - s(\beta \to \alpha^{\star\star}) \\ &= s(1) - s(\beta \to \alpha) \\ &= s(1) - (s(\alpha) + s(\alpha \to \beta) - s(\beta)) \\ &= s(1) - s(\alpha) - s(1) + s(\beta) = s(\beta) - s(\alpha). \end{aligned}$$

□

Corollary 3.5. *Let A be a bounded semihoop and G be an ablian ℓ-group. Then a constant function $s\colon A \to G$, defined by $s(\alpha) = 0_G$ for all $\alpha \in A$, is a Bosbach G-state on A.*

Remark 3.6. The following example will indicate not every bounded semihoop has a non-zero Bosbach G-state.

Example 3.7 ([5])**.** Let $A = \{0, a, b, 1\}$ be a chain with $0 < a < b < 1$ and G be an abelian ℓ-group. Define operations $\odot$ and $\to$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	b	b	1	1
1	0	a	b	1

Then $(A, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop.

Let s be a Bosbach G-state on A. Then $s(0) = 0_G$. Let $s(a) = x, s(b) = y$. Then $s(a) + s(a \to 0) = s(0) + s(0 \to a)$, $s(b) + s(b \to 0) = s(0) + s(0 \to b)$. Thus $x + s(b) = x + y = 0 + s(1) = s(1)$, $y + s(b) = y + y = 0 + s(1) = s(1)$. So $y = s(b) = \frac{1}{2}s(1)$, $x = s(a) = \frac{1}{2}s(1)$. Moreover, $s(a) + s(a \to b) = s(b) + s(b \to a)$. Hence $\frac{1}{2}s(1) + s(1) = \frac{1}{2}s(1) + \frac{1}{2}s(1)$, i.e., $s(1) = 0$. This is a contradiction. Therefore the bounded semihoop A has no any non-zero Bosbach G-state.

Example 3.3 shows that every bounded semihoop has a Bosbach G-state. However, Example 3.7 shows that not every bounded semihoop has a non-zero Bosbach G-state. Therefore, this paper primarily concerned with the existence of non-zero Bosbach G-state on a bounded semihoop.

Proposition 3.8. *Let A a bounded semihoop and G be an abelian ℓ-group. If $s : A \to G$ is a map such that $s(0) = 0_G$ and $s(1)$ is the largest element of $Im(s)$, then the following are equivalent:*

(1) *s is a Bosbach G-state on A,*
(2) *$\alpha \le \beta$ implies $s(\beta) + s(\beta \to \alpha) = s(\alpha) + s(1)$ for any $\alpha, \beta \in A$,*
(3) *$s(\alpha) + s(\alpha \to \beta) = s(\alpha \wedge \beta) + s(1)$ for any $\alpha, \beta \in A$.*

Proof. (1) $\Rightarrow$ (2): Suppose s is a Bosbach G-state on A and $\alpha \le \beta$ for any $\alpha, \beta \in A$. Then $\alpha \to \beta = 1$. Since s is a Bosbach G-state on A, we get

$$s(\beta) + s(\beta \to \alpha) = s(\alpha) + s(\alpha \to \beta) = s(\alpha) + s(1).$$

(2) $\Rightarrow$ (3): Suppose (2) holds. From $\alpha \wedge \beta \le \alpha$, $s(\alpha) + s(\alpha \to (\alpha \wedge \beta)) = s(\alpha \wedge \beta) + s(1)$. By Proposition2.4 (11), we have $s(\alpha) + s(\alpha \to \beta) = s(\alpha \wedge \beta) + s(1)$.

(3) $\Rightarrow$ (1): Suppose (3) holds. Then we get

$$s(\alpha) + s(\alpha \to \beta) = s(\alpha \wedge \beta) + s(1) = s(\beta \wedge \alpha) + s(1) = s(\beta) + s(\beta \to \alpha).$$

Thus s is a Bosbach G-state on A. □

Proposition 3.9. *Let A be a bounded semihoop and G be an abelian ℓ-group. If $s : A \to G$ is a map such that $s(1)$ is the largest element of $Im(s)$ and $s' = s(1) - s$,*

then s is a Bosbach G-state if and only if the following conditions hold: for any $\alpha, \beta \in A$,
(1) $s'(0) = s(1)$,
(2) $\alpha \leq \beta$ *implies* $s'(\beta \to \alpha) = s'(\alpha) - s'(\beta)$.

Proof. ($\Rightarrow$): Suppose s be a Bosbach G-state. Then $s'(0) = s(1) - s(0) = s(1) - 0 = s(1)$. Thus (1) holds.

Let $\alpha \leq \beta$ for any $\alpha, \beta \in A$. Then by Proposition 3.8 (2), we have

$$s(\beta) + s(\beta \to \alpha) = s(\alpha) + s(1).$$

Thus we get

$$\begin{aligned} s'(\beta \to \alpha) &= s(1) - s(\beta \to \alpha) \\ &= s(\beta) - s(\alpha) \\ &= (s(1) - s'(\beta)) - (s(1) - s'(\alpha)) \\ &= s'(\alpha) - s'(\beta). \end{aligned}$$

So (2) holds.

($\Leftarrow$): Suppose the conditions (1) and (2)hold. From $s'(0) = s(1) - s(0) = s(1)$, $s(0) = 0_G$. Let $\alpha \leq \beta$for any $\alpha, \beta \in A$. Then by (2), we have

$$s(1) - s(\beta \to \alpha) = s(1) - s(\alpha) - (s(1) - s(\beta)).$$

Thus $s(\beta \to \alpha) + s(\beta) = s(\alpha) + s(1)$. So from Proposition 3.8 (2), s is a Bosbach G-state on A. □

Proposition 3.10. *Let s be a Bosbach G-state on a bounded $\vee$-semihoop A. Then the properties hold: for any $\alpha, \beta \in A$,*
(1) $s(\alpha \vee \beta) + s(\alpha^\star \to \beta^\star) = s(\alpha) + s(1)$,
(2) $s(\alpha^\star \vee \beta^\star) + s(\alpha) + s(\beta) = s(\alpha \vee \beta) + s(1)$,
(3) $s(\alpha^{\star\star} \vee \beta^{\star\star}) = s(\alpha \vee \beta)$,
(4) $s(\alpha) + s(\beta) = s(\alpha \wedge \beta) + s(\alpha \vee \beta)$.

Proof. (1) Since s is a Bosbach G-state, by Proposition 3.8 (3), we get

$$s(\alpha^\star) + s(\alpha^\star \to \beta^\star) = s(\alpha^\star \wedge \beta^\star) + s(1) = s((\alpha \vee \beta)^\star) + s(1).$$

Then $s(1) - s(\alpha) + s(\alpha^\star \to \beta^\star) = s(1) - s(\alpha \vee \beta) + s(1)$. Thus we have

$$s(\alpha \vee \beta) + s(\alpha^\star \to \beta^\star) = s(\alpha) + s(1).$$

(2) Since $\beta^\star \leq \alpha^\star \vee \beta^\star$, by Proposition 3.8 (2), we have

$$s(\alpha^\star \vee \beta^\star) + s((\alpha^\star \vee \beta^\star) \to \beta^\star) = s(\beta^\star) + s(\beta^\star \to (\alpha^\star \vee \beta^\star)).$$

Then $s(\alpha^\star \vee \beta^\star) + s(\alpha^\star \to \beta^\star) = s(\beta^\star) + s(1)$. Thus we get

$$s(\alpha^\star \vee \beta^\star) + s(\alpha \vee \beta) + s(\alpha^\star \to \beta^\star) = s(1) - s(\beta) + s(\alpha \vee \beta) + s(1).$$

From (1), $s(\alpha^\star \vee \beta^\star) + s(\alpha) + s(1) + s(\beta) = s(1) + s(1) + s(\alpha \vee \beta)$. So we have

$$s(\alpha^\star \vee \beta^\star) + s(\alpha) + s(\beta) = s(\alpha \vee \beta) + s(1).$$

(3) Since $s((\alpha^{\star\star} \vee \beta^{\star\star})^\star) = s(\alpha^{\star\star\star} \wedge \beta^{\star\star\star}) = s(\alpha^\star \wedge \beta^\star) = s((\alpha \vee \beta)^\star)$, we get

$$\begin{aligned} s(1) &= s(\alpha^{\star\star} \vee \beta^{\star\star}) + s((\alpha^{\star\star} \vee \beta^{\star\star})^\star) \\ &= s(\alpha^{\star\star} \vee \beta^{\star\star}) + s((\alpha \vee \beta)^\star) \\ &= s(\alpha^{\star\star} \vee \beta^{\star\star}) + s(1) - s(\alpha \vee \beta). \end{aligned}$$

Then $s(\alpha^{\star\star} \vee \beta^{\star\star}) = s(\alpha \vee \beta)$.

(4) From (2), we have $s(\alpha^\star \vee \beta^\star) + s(\alpha) + s(\beta) = s(\alpha \vee \beta) + s(1)$. Then we get

$$s((\alpha \wedge \beta)^\star) + s(\alpha) + s(\beta) = s(\alpha \vee \beta) + s(1).$$

Thus $s(1) - s(\alpha \wedge \beta) + s(\alpha) + s(\beta) = s(\alpha \vee \beta) + s(1)$. So we have

$$s(\alpha) + s(\beta) = s(\alpha \wedge \beta) + s(\alpha \vee \beta).$$

□

Proposition 3.11. *Let s be a Bosbach G-state on a bounded $\vee$-semihoop A. Then the following are equivalent: for any $\alpha, \beta \in A$,*
(1) $s(\alpha \wedge \beta) = s(\alpha \vee \beta)$,
(2) $s(\alpha) = s(\beta) = s(\alpha \vee \beta)$,
(3) $s(\alpha) = s(\beta) = s(\alpha \wedge \beta)$.

Proof. (1) ⇒ (2): Suppose (1) holds. From $s(\alpha \wedge \beta) \leq s(\alpha)$, $s(\beta) \leq s(\alpha \vee \beta)$. Since $s(\alpha \wedge \beta) = s(\alpha \vee \beta)$, $s(\alpha) = s(\beta) = s(\alpha \vee \beta)$.

(2) ⇒ (3): Suppose (2) holds. Then from Proposition 3.10 (4), we have

$$s(\alpha) + s(\beta) = s(\alpha \wedge \beta) + s(\alpha \vee \beta).$$

Since $s(\alpha) = s(\beta) = s(\alpha \vee \beta)$, $s(\alpha) + s(\alpha \vee \beta) = s(\alpha \wedge \beta) + s(\alpha \vee \beta)$. Thus $s(\alpha) = s(\alpha \wedge \beta)$. Similarly, we obtain $s(\beta) = s(\alpha \wedge \beta)$. So $s(\alpha) = s(\beta) = s(\alpha \wedge \beta)$.

(3) ⇒ (1): Suppose (3) holds. Then from Proposition 3.10 (4), we have $s(\alpha) + s(\beta) = s(\alpha \wedge \beta) + s(\alpha \vee \beta)$. Since $s(\alpha) = s(\beta) = s(\alpha \wedge \beta)$, we get

$$s(\alpha) + s(\alpha \wedge \beta) = s(\alpha \wedge \beta) + s(\alpha \vee \beta).$$

Thus $s(\alpha) = s(\alpha \vee \beta)$. Similarly, we obtain $s(\beta) = s(\alpha \vee \beta)$. So we have

$$s(\alpha) = s(\beta) = s(\alpha \vee \beta).$$

□

Proposition 3.12. *Let s be a non-zero Bosbach G-state on a bounded semihoop A. Then*
(1) $ker(s) := \{\alpha \in A | s(\alpha) = s(1)\}$ *is a proper filter of A,*
(2) $Coker(s) := \{\alpha \in A | s(\alpha) = 0_G\}$ *is a proper ideal of A.*

Proof. (1) We have $1 \in ker(s)$. Let $\alpha, \alpha \to \beta \in ker(s)$. Then $s(\alpha) = s(1)$ and $s(\alpha \to \beta) = s(1)$. Since s is a Bosbach G-state, $s(\alpha) + s(\alpha \to \beta) = s(\beta) + s(\beta \to \alpha)$. Thus $s(\beta) = s(1) + s(1) - s(\beta \to \alpha) \geq s(1)$. Thus $s(\beta) = s(1)$, $\beta \in ker(s)$. so $ker(s)$ is a filter of A. From $0 \notin ker(s)$, $ker(s)$ is a proper filter of A.

(2) For any $\alpha, \beta \in A$, let $\alpha \leq \beta$ and $\beta \in Coker(s)$. Then $s(\beta) = 0$ and $s(\alpha) \leq s(\beta) = 0$. Thus $s(\alpha) = 0$, $\alpha \in Coker(s)$. Let $\alpha, \beta \in Coker(s)$. Then $s(\alpha) = s(\beta) = 0$. From Proposition 3.4 (7), we have $s(\alpha \oplus \beta) \leq s(\alpha) + s(\beta) = 0$. Then $s(\alpha \oplus \beta) = 0$. Thus $\alpha \oplus \beta \in Coker(s)$. So $Coker(s)$ is an ideal of A. From $1 \notin Coker(s)$, $Coker(s)$ is a proper ideal of A. □

Example 3.13. In Example 3.3, for $0 \neq x \in G$, we define a map s_x: $A \to G$,

$$s_x(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0 \\ x & \text{if } \alpha = a, b, c, 1 \end{cases}$$

for all $\alpha \in A$. Then s_x is not a Bosbach G-state since $s(0) + s(0 \to a) = 0 + s(1) = 0 + x = x \neq 2x = s(a) + s(b) = s(a) + s(a \to 0)$.

By Example 3.13, we find that s_x is not a Bosbach G-state. In the following proposition, we will give a characterization for s_x being a Bosbach G-state.

Proposition 3.14. *Let A be a bounded hoop and G be an abelian ℓ-group. For $0 \neq x \in G$, we define s_x by*

$$s_x(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0 \\ x & \text{if } \alpha \neq 0 \end{cases}$$

for any $\alpha \in A$. Then s_x is a Bosbach G-state on A if and only if $(A - \{0\})^\star = \{0\}$.

Proof. ($\Rightarrow$): Suppose s_x is a Bosbach G-state and let $\alpha \in (A - \{0\})$. Then we have

$$s(0) + s(0 \to \alpha) = s(\alpha) + s(\alpha \to 0) = s(\alpha) + s(\alpha^\star).$$

Thus $x = x + s(\alpha^\star)$. So $s(\alpha^\star) = 0$. Hence $\alpha^\star = 0$. Since α is arbitrary, $(A - \{0\})^\star = \{0\}$.

($\Leftarrow$): Suppose $(A - \{0\})^\star = \{0\}$ and let $\alpha, \beta \in A$. We will discuss the following cases.

(i) If $\alpha = 0, \beta = 0$, then $s(0) + s(0 \to 0) = s(1) = x = s(1) = s(0) + s(0 \to 0)$.

(ii) If $\alpha \neq 0, \beta = 0$, then $s(\alpha) + s(\alpha \to 0) = s(\alpha) = x = s(1) = s(0) + s(0 \to \alpha)$.

(iii) If $\alpha = 0, \beta \neq 0$, then $s(0) + s(0 \to \beta) = s(1) = x = s(\beta) = s(\beta) + s(\beta \to 0)$.

(iv) If $\alpha \neq 0, \beta \neq 0, \alpha \wedge \beta = 0$, then $\alpha \odot (\alpha \to \beta) = 0$. from $\alpha \neq 0$, $\alpha \to \beta = 0$. Thus $s(\alpha) + s(\alpha \to \beta) = x = s(\alpha \wedge \beta) + s(1)$. Moreover, from $\alpha \neq 0$, $\alpha \to \beta \neq 0$. Also, from $\beta \neq 0$, $\beta \to \alpha \neq 0$. So $s(\alpha) + s(\alpha \to \beta) = x + x = s(\beta) + s(\beta \to \alpha)$.

(v) If $\alpha \neq 0, \beta \neq 0, \alpha \wedge \beta \neq 0$, then $\alpha \odot (\alpha \to \beta) \neq 0$. From $\alpha \neq 0$, $\alpha \to \beta \neq 0$. Thus $s(\alpha) + s(\alpha \to \beta) = x + x = s(\alpha \wedge \beta) + s(1)$.

Hence by Definition 3.1 and Proposition 3.8 (3), s_x is a Bosbach G-state on A. □

Definition 3.15. Let $(G, +, \cdot, \leq)$ be an ℓ-ring and s be a Bosbach G-state on a bounded semihoop A. We define $k \cdot s$ by $(k \cdot s)(\alpha) = k \cdot (s(\alpha))$ for any $k \in G$.

Proposition 3.16. *Let $(G, +, \cdot, \leq)$ be an ℓ-ring and s_1, s_2 be two Bosbach G-states on a bounded semihoop A. Then $s = k_1 \cdot s_1 + k_2 \cdot s_2$ is also a Bosbach G-state on A, for any $0 \leq k_1, k_2 \in G$.*

Proof. Since s_1, s_2 are Bosbach G-states on A, we get

$$s(0) = (k_1 \cdot s_1)(0) + (k_2 \cdot s_2)(0) = k_1 \cdot (s_1(0)) + k_2 \cdot (s_2(0)) = 0.$$

Moreover, $s_1(1)$ and $s_2(1)$ are the largest elements of $Im(s_1)$ and $Im(s_2)$, respectively. Then $s(1) = (k_1 \cdot s_1)(1) + (k_2 \cdot s_2)(1) = k_1 \cdot (s_1(1)) + k_2 \cdot (s_2(1))$ is the largest elements of $Im(s)$. From $s_1(\alpha) + s_1(\alpha \to \beta) = s_1(\beta) + s_1(\beta \to \alpha)$ and

$s_2(\alpha) + s_2(\alpha \to \beta) = s_2(\beta) + s_2(\beta \to \alpha)$, we have

$$\begin{aligned} s(\alpha) + s(\alpha \to \beta) &= (k_1 \cdot s_1)(\alpha) + (k_2 \cdot s_2)(\alpha) + (k_1 \cdot s_1)(\alpha \to \beta) + (k_2 \cdot s_2)(\alpha \to \beta) \\ &= k_1 \cdot (s_1(\alpha)) + k_2 \cdot (s_2(\alpha)) + k_1 \cdot (s_1(\alpha \to \beta)) + k_2 \cdot (s_2(\alpha \to \beta)) \\ &= k_1 \cdot (s_1(\alpha) + s_1(\alpha \to \beta)) + k_2 \cdot (s_2(\alpha) + s_2(\alpha \to \beta)) \\ &= k_1 \cdot (s_1(\beta) + s_1(\beta \to \alpha)) + k_2 \cdot (s_2(\beta) + s_2(\beta \to \alpha)) \\ &= (k_1 \cdot s_1)(\beta) + (k_2 \cdot s_2)(\beta) + (k_1 \cdot s_1)(\beta \to \alpha) + (k_2 \cdot s_2)(\beta \to \alpha) \\ &= s(\beta) + s(\beta \to \alpha). \end{aligned}$$

Thus $s = k_1 \cdot s_1 + k_2 \cdot s_2$ is also a Bosbach G-state on A. □

In Proposition 3.16, unlike a Bosbach state on the interval $[0, 1]$, where $0 \leq k_1, k_2$ can take any elements from the ℓ-ring and do not need to satisfy the convex combination.

Proposition 3.17. *Let s be a Bosbach G-state on a bounded semihoop A and H be a down-set subgroup of G. Then $s^{-1}(H)$ is an ideal of A.*

Proof. We will prove the proposition in the following parts.

(i) Since H is a subgroup of G, $0_G \in H$. Since s is a Bosbach G-state on A, $s(0) = 0_G$. Then $0 \in s^{-1}(H)$. Thus $s^{-1}(H)$ is a non-empty set of A.

(ii) Let $\alpha \leq \beta$ and $\beta \in s^{-1}(H)$. Then $s(\beta) \in H$ and $s(\alpha) \leq s(\beta)$. Since H is a down-set of G, $s(\alpha) \in H$. Thus $\alpha \in s^{-1}(H)$.

(iii) For any $\alpha, \beta \in s^{-1}(H)$, $s(\alpha), s(\beta) \in H$. Since H is a subgroup of G, $s(\alpha) + s(\beta) \in H$. Then From Proposition 3.4 (7), $s(\alpha \oplus \beta) \leq s(\alpha) + s(\beta)$. Since H is a down-set of G, $s(\alpha \oplus \beta) \in H$. Thus $\alpha \oplus \beta \in s^{-1}(H)$.

So $s^{-1}(H)$ is an ideal of A. □

Let A and B be two semihoops. A mapping h: $A \to B$ is called a *homomorphism*, if $h(\alpha \to \beta) = h(\alpha) \to h(\beta)$, $h(\alpha \odot \beta) = h(\alpha) \odot h(\beta)$, $h(\alpha \wedge \beta) = h(\alpha) \wedge h(\beta)$, $h(0) = 0$ for any $\alpha, \beta \in A$. It is easy to verify that $h(1) = 1$ and $h(\alpha^\star) = (h(\alpha))^\star$ for all $\alpha \in A$.

Let G and K be two abelian ℓ-groups. A mapping g: $G \to K$ is called a *group-homomorphism*, if $g(x + y) = g(x) + g(y)$, $g(0) = 0$ for any $x, y \in G$.

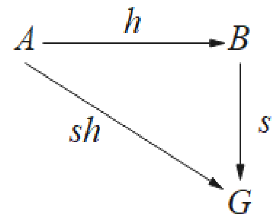
Proposition 3.18. *Let A and B be two bounded semihoops, h: $A \to B$ be a homomorphism and s be a Bosbach G-state of B. Then sh is also a Bosbach G-state of A.*

Proof. We have $(sh)(0) = s(h(0)) = s(0) = 0$. From $(sh)(1) = s(h(1)) = s(1)$, $(sh)(1)$ is the greatest element of $Im(sh)$. Moreover,

$$\begin{aligned} (sh)(\alpha) + (sh)(\alpha \to \beta) &= s(h(\alpha)) + s(h(\alpha \to \beta)) \\ &= s(h(\alpha)) + s(h(\alpha) \to h(\beta)) \\ &= s(h(\beta)) + s(h(\beta) \to h(\alpha)) \\ &= s(h(\beta)) + s(h(\beta \to \alpha)) \\ &= (sh)(\beta) + (sh)(\beta \to \alpha). \end{aligned}$$

Then sh is a Bosbach G-state of A.

□

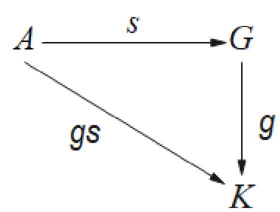


Proposition 3.19. *Let A be a bounded semihoop, G and K be two abelian ℓ-groups, g: $G \to K$ be a group-homomorphism and s be a Bosbach G-state of A. Then gs is a Bosbach K-state of A.*

Proof. We have $(gs)(0) = g(s(0)) = g(0) = 0$. From $(gs)(1) = g(s(1))$ and g is an order-preserving mapping, $(gs)(1)$ is the greatest element of $Im(gs)$. Moreover,

$$\begin{aligned}(gs)(\alpha) + (gs)(\alpha \to \beta) &= g(s(\alpha)) + g(s(\alpha \to \beta)) \\ &= g(s(\alpha) + s(\alpha \to \beta)) \\ &= g(s(\beta) + s(\beta \to \alpha)) \\ &= g(s(\beta)) + g(s(\beta \to \alpha)) \\ &= (gs)(\beta) + (gs)(\beta \to \alpha).\end{aligned}$$

Then gs is a Bosbach K-state of A.



□

Proposition 3.20. *Let A be a bounded semihoop, G and K be two abelian ℓ-groups, s: $A \to G$ be a map, g: $G \to K$ be a bijective group-homomorphism and gs be a Bosbach K-state of A. Then s is a Bosbach G-state of A.*

Proof. Since gs be a Bosbach K-state of A, $(gs)(0) = g(s(0)) = 0$ and $(gs)(1) = g(s(1))$ is the largest element of $Im(gs)$. Since g is a bijective group-homomorphism, $s(0) = 0_G$ and $s(1)$ is the largest element of $Im(s)$. Moreover, we have

$$(gs)(\alpha) + (gs)(\alpha \to \beta) = g(s(\alpha)) + g(s(\alpha \to \beta)) = g(s(\alpha) + s(\alpha \to \beta)),$$

$$(gs)(\beta) + (gs)(\beta \to \alpha) = g(s(\beta)) + g(s(\beta \to \alpha)) = g(s(\beta) + s(\beta \to \alpha)).$$

Then $s(\alpha) + s(\alpha \to \beta) = s(\beta) + s(\beta \to \alpha)$. Thus s is a Bosbach G-state of A. □

Proposition 3.21. *Let A be a bounded semihoop. Then $Rad(A) \bigcap Rad(A)^{\star} = \varnothing$.*

Proof. Assume that there exists $\alpha \in A$ such that $\alpha \in Rad(A)$ and $\alpha \in Rad(A)^\star$. Then $\alpha^\star \in Rad(A)$. Since $Rad(A)$ is an ideal of A, $\alpha \oplus \alpha^\star = 1 \in Rad(A)$. Thus $Rad(A) = A$, which is a contradiction. So $Rad(A) \bigcap Rad(A)^\star = \varnothing$. □

Lemma 3.22. *Let A be a bounded semihoop. Then $Rad(A)^\star$ is an up-set.*

Proof. Let $\alpha \leq \beta$ and $\alpha \in Rad(A)^\star$ for any $\alpha, \beta \in A$. From $\beta^\star \leq \alpha^\star$ and $\alpha^\star \in Rad(A)$, $\beta^\star \in Rad(A)$. Then $\beta \in Rad(A)^\star$. Thus $Rad(A)^\star$ is an up-set. □

Definition 3.23. A bounded semihoop A is called a *bounded dual perfect semihoop*, if $A = Rad(A) \bigcup Rad(A)^\star$.

Example 3.24. Let Z be an additive ℓ-group, $Z^+ = \{x \in Z | x \geq 0\}$ be the positive cone of Z and $Z \times Z$ be the lexicographic product of Z with Z. Give the order unit $(1, 0)$ of $Z \times Z$. Let $A := \Gamma(Z \times Z, (1, 0))$. Define operations $\odot$, $\rightarrow$ and $\wedge$ on A as follows: for any $(a, b), (c, d) \in A$,

$$(a, b) \odot (c, d) = max\{(a + c - 1, b + d), (0, 0)\},$$
$$(a, b) \rightarrow (c, d) = min\{(1 - a + c, -b + d), (1, 0)\},$$
$$(a, b) \wedge (c, d) = min\{(a, b), (c, d)\}.$$

We will verify the conditions in Definition 2.1 in the following parts.

(1) Clearly, $(A, \wedge, (1, 0))$ is a $\wedge$-semilattice and it has a upper bounded $(1, 0)$.

(2) (i) $(a, b) \odot (c, d) = max\{(a+c-1, b+d), (0, 0)\} = max\{(c+a-1, d+b), (0, 0)\} = (c, d) \odot (a, b)$ for any $(a, b), (c, d) \in A$.

(ii) $(a, b) \odot (1, 0) = max\{(a+1-1, b+0), (0, 0)\} = max\{(1+a-1, 0+b), (0, 0)\} = (1, 0) \odot (a, b) = (a, b)$ for any $(a, b) \in A$.

(iii) $((a, b) \odot (c, d)) \odot (e, f) = max\{(a + c - 1, b + d), (0, 0)\} \odot (e, f)$

$$= \begin{cases} (a + c - 1, b + d) \odot (e, f) & \text{if } a + c > 1 \\ (0, b + d) \odot (e, f) & \text{if } a + c = 1, b + d > 0 \\ (0, 0) \odot (e, f) & \text{if } a + c = 1, b + d = 0 \text{ or } a + c < 1 \end{cases}$$

$$= \begin{cases} max\{(a + c + e - 2, b + d + f), (0, 0)\} & \text{if } a + c > 1 \\ max\{(e - 1, b + d + f), (0, 0)\} & \text{if } a + c = 1, b + d > 0 \\ max\{(e - 1, f), (0, 0)\} & \text{if } a + c = 1, b + d = 0 \text{ or } a + c < 1 \end{cases}$$

$= (a, b) \odot ((c, d) \odot (e, f))$ for any $(a, b), (c, d), (e, f) \in A$.

Then $(A, \odot, (1, 0))$ is a commutative monoid.

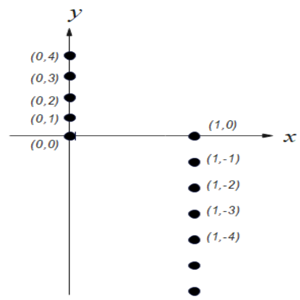
(3) $((a, b) \odot (c, d)) \rightarrow (e, f) = max\{(a + c - 1, b + d), (0, 0)\} \rightarrow (e, f)$

$$= \begin{cases} (a + c - 1, b + d) \rightarrow (e, f) & \text{if } a + c > 1 \\ (0, b + d) \rightarrow (e, f) & \text{if } a + c = 1, b + d > 0 \\ (0, 0) \rightarrow (e, f) & \text{if } a + c = 1, b + d = 0 \text{ or } a + c < 1 \end{cases}$$

$$= \begin{cases} min\{(2 - a - c + e, -b - d + f), (0, 0)\} & \text{if } a + c > 1 \\ min\{(1 - e, -b - d + f), (0, 0)\} & \text{if } a + c = 1, b + d > 0 \\ min\{(1 - e, f), (0, 0)\} & \text{if } a + c = 1, b + d = 0 \text{ or } a + c < 1 \end{cases}$$

$= (a, b) \rightarrow ((c, d) \odot (e, f))$ for any $(a, b), (c, d), (e, f) \in A$.

Thus, in summary, $A = \Gamma(Z \times Z, (1, 0)) = ([(0, 0), (1, 0)], \odot, \rightarrow, \wedge, (0, 0), (1, 0))$ is a bounded semihoop.



From $A = \{(0,a)|a \in Z^+\} \bigcup \{(1,-a)|a \in Z^+\}$. Represent A in a right-angled coordinate system as follows:

By verification, we get that $I = \{(0,a)|a \in Z^+\}$ is an ideal of A. Let $J = \langle I \bigcup \{(1,-b)\}\rangle$, where $(1,-b) \in \{(1,-a)|a \in Z^+\}$. For any $(1,-c) \in \{(1,-a)|a \in Z^+\}$, since $(1,-b) \oplus (1,-b) = (1,-b)^\star \to (1,-b) = ((1,-b) \to (0,0)) \to (1,-b) = (1,0) \geq (1,-c)$, $(1-c) \in J$. Since $(1,-c)$ is arbitrary, $J = A$. Then I is the only maximal ideal of A. Thus $Rad(A) = I = \{(0,a)|a \in Z^+\}$, $Rad(A)^\star = \{(0,a)^\star|a \in Z^+\} = \{(1,-a)|a \in Z^+\}$. So $A = Rad(A) \bigcup Rad(A)^\star$. Hence A is a bounded dual perfect semihoop.

Proposition 3.25. *Every bounded dual perfect semihoop has a non-zero Bosbach G-state.*

Proof. Let A be a bounded dual perfect semihoop and G be an abelian ℓ-group. Then $A = Rad(A) \bigcup Rad(A)^\star$. For $0 \neq y \in G$, we define s by

$$s(\alpha) = \begin{cases} 0 & \text{if } \alpha \in Rad(A) \\ y & \text{if } \alpha \in Rad(A)^\star \end{cases}$$

for any $\alpha \in A$. From $0 \in Rad(A)$, $s(0) = 0$. Since $Rad(A)$ is a proper ideal of A, $1 \notin Rad(A)$. Then $1 \in Rad(A)^\star$. Thus $s(1) = y$ and $s(1)$ is the largest element of $Im(s)$. For any $\alpha, \beta \in A$, To check the condition (BG_3) in Definition 3.1 or Proposition 3.8 (3), we consider the following cases.

(i) Suppose $\alpha, \beta \in Rad(A)$. Then $s(\alpha) = s(\beta) = 0$. Since $Rad(A)$ is an ideal, $\alpha \wedge \beta \in Rad(A)$. Thus $s(\alpha \wedge \beta) = 0$. Assume that $\alpha \to \beta \in Rad(A)$. from $\alpha \leq \alpha^{\star\star}$, $\alpha^{\star\star} \to \beta \leq \alpha \to \beta$. Then $\alpha^{\star\star} \to \beta \in Rad(A)$. Since $\alpha^\star \leq \alpha^{\star\star} \to \beta$, $\alpha^\star \in Rad(A)$. Thus $\alpha \oplus \alpha^\star = 1 \in Rad(A)$, which is a contradiction. So $\alpha \to \beta \in Rad(A)^\star$. Hence $s(\alpha \to \beta) = y$. Therefore $s(\alpha) + s(\alpha \to \beta) = 0 + y = s(\alpha \wedge \beta) + s(1)$.

(ii) Suppose $\alpha, \beta \in Rad(A)^\star$. Then $s(\alpha) = s(\beta) = y$. Since $\alpha \leq \beta \to \alpha$, $\beta \leq \alpha \to \beta$. By Lemma 3.20, we have $\beta \to \alpha,\ \alpha \to \beta \in Rad(A)^\star$. Thus $s(\beta \to \alpha) = s(\alpha \to \beta) = y$. So $s(\alpha) + s(\alpha \to \beta) = y + y = s(\beta) + s(\beta \to \alpha)$.

(iii) Suppose $\alpha \in Rad(A), \beta \in Rad(A)^\star$. Then $s(\alpha) = 0$, $s(\beta) = y$. Since $\alpha \wedge \beta \leq \alpha$, $\beta \leq \alpha \to \beta$. Thus $\alpha \wedge \beta \in Rad(A)$. By Lemma 3.20, we have $\alpha \to \beta \in Rad(A)^\star$. So $s(\alpha \wedge \beta) = 0$, $s(\alpha \to \beta) = y$. Hence $s(\alpha) + s(\alpha \to \beta) = 0 + y = s(\alpha \wedge \beta) + s(1)$.

Therefore s is a Bosbach G-state on A, i.e., a dual perfect semihoop admits a non-zero Bosbach G-state. □

In Proposition 3.25, we obtain that every bounded dual perfect semihoop has a non-zero Bosbach G-state but it is not unique because $y \neq 0$ is an arbitrary element of G.

Example 3.26. Let A be a bounded dual perfect semihoop in Example 3.22 and $G = (Z, +, \leq)$ be an abelian ℓ-group. For $n \in Z^+$, we define a map $s_n\colon A \to G$,

$$s_n(\alpha) = \begin{cases} 0 & \text{if } \alpha \in Rad(A) \\ n & \text{if } \alpha \in Rad(A)^\star \end{cases}$$

for all $\alpha \in A$. Then we can easily check that s_n is a Bosbach G-state on A.

4. Riečan G-states

In this section, we introduce the concept of Riečan G-states on a bounded semihoop, study its relevant properties and discuss the relationship between Bosbach G-state and Riečan G-state.

Definition 4.1. Let A be a bounded semihoop. Two elements $\alpha, \beta \in A$ said to be *orthogonal*, denoted by $\alpha \top \beta$, if $\alpha^{\star\star} \leq \beta^\star$.

For two orthogonal elements $\alpha, \beta \in A$, we define a binary operation $\uplus$ on A by $\alpha \uplus \beta := \beta^\star \to \alpha^{\star\star}$.

Example 4.2. In Example 3.7, we have $a \top b$ since $a^{\star\star} = b = b^\star$ and $a \uplus b = b^\star \to a^{\star\star} = b \to b = 1$.

Proposition 4.3. *Let A be a bounded semihoop. Then the following properties hold: for any $\alpha, \beta \in A$,*

(1) *$\alpha \top \beta$ if and only if $\beta \top \alpha$,*
(2) *$\alpha \top \alpha^\star$ and $\alpha \uplus \alpha^\star = 1$,*
(3) *$\alpha \top 0$ and $\alpha \uplus 0 = \alpha^{\star\star}$,*
(4) *if $\alpha \leq \beta$, then $\beta^\star \top \alpha$.*

Proof. (1) $\alpha \top \beta$ if and only if $\alpha^{\star\star} \leq \beta^\star$ if and only if $\beta^{\star\star} \leq \alpha^{\star\star\star} = \alpha^\star$ if and only if $\beta \top \alpha$.

(2) From $\alpha^{\star\star} \leq \alpha^{\star\star}$, $\alpha \top \alpha^\star$. It follows that $\alpha \uplus \alpha^\star = \alpha^{\star\star} \to \alpha^{\star\star} = 1$.

(3) From $\alpha^{\star\star} \leq 1 = 0^\star$, $\alpha \top 0$. It follows that $\alpha \uplus 0 = 0^\star \to \alpha^{\star\star} = 1 \to \alpha^{\star\star} = \alpha^{\star\star}$.

(4) From $\alpha \leq \beta$, $\beta^{\star\star\star} = \beta^\star \leq \alpha^\star$. Then $\beta^\star \top \alpha$. □

Definition 4.4. Let $(A, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop and $(G, +, \leq)$ be an abelian ℓ-group. A *Riečan G-state* on A is a function $s\colon A \to G$ such that the following conditions hold:

(RG_1) $s(1)$ is the largest element of $Im(s)$,
(RG_2) $\alpha \top \beta$ implies $s(\alpha \uplus \beta) = s(\alpha) + s(\beta)$ for any $\alpha, \beta \in A$.

Example 4.5 ([17])**.** Let $A = \{0, a, b, c, 1\}$ be a chain with $0 < a < b < c < 1$ and G be an abelian ℓ-group in Example 3.3. Define operations $\odot$ and $\to$ on A as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	a	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

Then $(A, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop. We define a map $s\colon A \rightarrow G$ by

$$s(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha = a, b \\ 2 & \text{if } \alpha = c \\ 3 & \text{if } \alpha = 1 \end{cases}$$

for all $\alpha \in A$. One can check that s is a Riečan G-state on A.

Proposition 4.6. *Let s be a Riečan G-state on a bounded semihoop A. Then the following properties hold: for any $\alpha, \beta \in A$,*
(1) $s(0) = 0_G$,
(2) $s(\alpha) + s(\alpha^\star) = s(1)$,
(3) $s(\alpha^{\star\star}) = s(\alpha)$,
(4) $\alpha \leq \beta$ *implies* $s(\alpha) \leq s(\beta)$.

Proof. (1) From $0 \top 0$, $s(0 \uplus 0) = s(0) = s(0) + s(0)$. Then $s(0) = 0$.
(2) From $\alpha \top \alpha^\star$, $s(\alpha \uplus \alpha^\star) = s(\alpha^{\star\star} \rightarrow \alpha^{\star\star}) = s(1) = s(\alpha) + s(\alpha^\star)$.
(3) From $\alpha \top 0$, we have

$$s(\alpha \uplus 0) = s(0^\star \rightarrow \alpha^{\star\star}) = s(1 \rightarrow \alpha^{\star\star}) = s(\alpha^{\star\star}) = s(\alpha) + s(0) = s(\alpha) + 0 = s(\alpha).$$

(4) Suppose $\alpha \leq \beta$ Then from Proposition 4.2 (4), $\beta^\star \top \alpha$. Thus we get

$$s(\beta^\star \uplus \alpha) = s(\alpha^\star \rightarrow \beta^{\star\star\star}) = s(\alpha^\star \rightarrow \beta^\star) = s(\beta^\star) + s(\alpha).$$

From (2), we have $s(\alpha^\star \rightarrow \beta^\star) = s(1) - s(\beta) + s(\alpha)$. So we get

$$s(\alpha) - s(\beta) = s(\alpha^\star \rightarrow \beta^\star) - s(1) \leq 0.$$

Hence $s(\alpha) \leq s(\beta)$. □

Theorem 4.7. *Let A be a bounded semihoop. Then each Bosbach G-state on A is a Riečan G-state.*

Proof. Let s be a Bosbach G-state on A. Then $s(1)$ is the largest element of $Im(s)$. Let $\alpha \top \beta$, i.e., $\alpha^{\star\star} \leq \beta^\star$. Then $\alpha^{\star\star} \rightarrow \beta^\star = 1$. Since $s(\beta^\star) + s(\beta^\star \rightarrow \alpha^{\star\star}) = s(\alpha^{\star\star}) + s(\alpha^{\star\star} \rightarrow \beta^\star)$, $s(\beta^\star) + s(\alpha \uplus \beta) = s(\alpha) + s(1)$. Thus $s(1) - s(\beta) + s(\alpha \uplus \beta) = s(\alpha) + s(1)$. So $s(\alpha \uplus \beta) = s(\alpha) + s(\beta)$. Hence s is a Riečan G-state on A. □

Example 4.8. In Example 4.5, s is a Riečan G-state but it is not a Bosbach G-state since $s(a) + s(a \rightarrow b) = 4 \neq 3 = s(b) + s(b \rightarrow a)$.

Example 4.8 shows that in a Riečan G-state is not necessarily a Bosbach G-state. The following theorem will give the conditions under which a Riečan G-state is a Bosbach G-state.

Theorem 4.9. *Let A be a bounded semihoop with DNP. Then a Riečan G-state on A is a Bosbach G-state.*

Proof. Let s be a Riečan G-state on A. Then $s(1)$ is the largest element of $Im(s)$. From Proposition 4.6 (1), we have $s(0) = 0_G$. Suppose $\alpha \top \beta$. Then $\alpha = \alpha^{\star\star} \leq \beta^{\star}$. Thus $s(\alpha \uplus \beta) = s(\beta^{\star} \to \alpha^{\star\star}) = s(\beta^{\star} \to \alpha) = s(\alpha) + s(\beta)$. So we have

$$s(\beta^{\star}) + s(\beta^{\star} \to \alpha) = s(\alpha) + s(\beta) + s(\beta^{\star}) = s(\alpha) + s(1).$$

Hence from Proposition 3.8 (2), s is a Bosbach G-state. □

5. G-states morphism

In this section, we study the G-states morphism on a bounded semihoop, give the definition of a G-states morphism and study its properties.

Definition 5.1 ([20]). A *strong unit* of an abelian ℓ-group G is an element $u \in G$ such that for any $x \in G$, there is $n \in N^{+}$ with $x \leq nu$.

Let G be an abelian ℓ-group with a strong unit u. We define operations $\odot_G$ and $\to_G$ as follows: for any $x, y \in G$,

$$x \odot_G y = 0 \vee_G (x + y - u),\ x \to_G y = u \wedge_G (u - x + y).$$

Proposition 5.2. *The algebra* $\Gamma(G, u) = ([0, u], \odot_G, \to_G, \wedge_G, u)$ *is a bounded semihoop.*

Proof. The proof is similar to [21, Example 5.1]. □

According to the Proposition 5.2 and some related knowledge, we can find that semihoop-algebras and MV-algebras are the positive cone of an abelian ℓ-group and ℓ-algebras are the negative cone of an abelian ℓ-group.

Definition 5.3. Let $(A, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop. A *G-state morphism* on A is a function s: $A \to \Gamma(G, u)$ such that: for any $\alpha, \beta \in A$,
(GSM$_1$) $s(0) = 0_G$,
(GSM$_2$) $s(\alpha \to \beta) = s(\alpha) \to_G s(\beta)$.

Example 5.4. In Example 3.3, putting $n = u$, then s_u is a G-state morphism.

Proposition 5.5. *Let s be a G-state morphism on a bounded semihoop A. Then* $s(1) = u$.

Proof. Since s is a G-state morphism, thus $s(0) = 0_G$. From $s(1) = s(0 \to \alpha) = s(0) \to_G s(\alpha) = u \wedge_G (u - s(0) + s(\alpha)) = u \wedge_G (u + s(\alpha)) = u$. □

Proposition 5.6. *Let A be a bounded semihoop. Then every G-state morphism on A is a Bosbach G-state.*

Proof. Suppose that s is an G-state morphism on A. Then $s(0) = 0_G$. Thus from Proposition 5.4, we have that $s(1)$ is the largest element of $Im(s)$. Moreover, from

Definition 5.3 and Lemma 2.8,

$$\begin{aligned}
s(\alpha)+s(\alpha\rightarrow\beta) &= s(\alpha)+(s(\alpha)\rightarrow_G s(\beta))\\
&= s(\alpha)+u\wedge_G (u-s(\alpha)+s(\beta))\\
&= (u+s(\alpha))\wedge_G (u+s(\beta))\\
&= s(\beta)+((u-s(\beta)+s(\alpha))\wedge_G (u+s(\beta)-s(\beta)))\\
&= s(\beta)+((u-s(\beta)+s(\alpha))\wedge_G u)\\
&= s(\beta)+(s(\beta)\rightarrow_G s(\alpha))\\
&= s(\beta)+s(\beta\rightarrow\alpha).
\end{aligned}$$

So s is a Bosbach G-state on A. □

Example 5.7. In Examlpe 3.3, putting $n = u + 1$. Then s is a Bosbach G-state but it is not a G-state morphism since $s(1) > u$.

In order to study G-state morphism through Bosbach G-state, we will give the following definition.

Definition 5.8. A Bosbach G-state s is said to be *regular*, if $s(1) = u$.

Let s be a regular Bosbach G-state. For any $\alpha \in A$, as $0 \leq \alpha \leq 1$, $0 = s(0) \leq s(\alpha) \leq s(1) = u$. Then a regular Bosbach G-state s is a mapping from A to $[0, u]$.

Proposition 5.9. *Let s be a regular Bosbach G-state on a bounded semihoop A. Then the following statements are equivalent: for any $\alpha, \beta \in A$,*
(1) *s is a G-state morphism,*
(2) $s(\alpha\wedge\beta) = s(\alpha)\wedge_G s(\beta)$,
(3) $s(\alpha\rightarrow\beta) = u\wedge_G (s(\alpha^\star)+s(\beta))$.

Proof. (1) ⇒ (2): Suppose s is a G-state morphism on A. Then by Proposition 3.8 (3) and Lemma2.8, we have

$$\begin{aligned}
s(\alpha\wedge\beta)+s(1) &= s(\alpha)+s(\alpha\rightarrow\beta)\\
&= s(\alpha)+(s(\alpha)\rightarrow_G s(\beta))\\
&= s(\alpha)+(u\wedge_G (u-s(\alpha)+s(\beta))\\
&= (u+s(\alpha))\wedge_G (u+s(\beta))\\
&= u+(s(\alpha)\wedge_G s(\beta))\\
&= s(1)+(s(\alpha)\wedge_G s(\beta)).
\end{aligned}$$

Thus $s(\alpha\wedge\beta) = s(\alpha)\wedge_G s(\beta)$.

(2) ⇒ (3): Suppose (2) holds. Since s is a regular Bosbach G-state on A, from Proposition 3.8 (3) and Lemma 2.8, we have

$$\begin{aligned} s(\alpha) + s(\alpha \to \beta) &= s(\alpha \wedge \beta) + s(1) \\ &= (s(\alpha) \wedge_G s(\beta)) + s(1) \\ &= (s(\alpha) + s(1)) \wedge_G (s(\beta) + s(1)) \\ &= s(\alpha) + ((s(1) + s(\alpha) - s(\alpha)) \wedge_G (s(1) - s(\alpha) + s(\beta))) \\ &= s(\alpha) + (s(1) \wedge_G (s(1) - s(\alpha) + s(\beta))) \\ &= s(\alpha) + (s(1) \wedge_G (s(\alpha^\star) + s(\beta))) \\ &= s(\alpha) + (u \wedge_G (s(\alpha^\star) + s(\beta))). \end{aligned}$$

Then $s(\alpha \to \beta) = u \wedge_G (s(\alpha^\star) + s(\beta))$.

(3) ⇒ (1): Suppose (3) holds. Since s is a regular Bosbach G-state on A, $s(0) = 0_G$. Then we have

$$\begin{aligned} s(\alpha \to \beta) &= u \wedge_G (s(\alpha^\star) + s(\beta)) \\ &= u \wedge_G (s(1) - s(\alpha) + s(\beta)) \\ &= u \wedge_G (u - s(\alpha) + s(\beta)) \\ &= s(\alpha) \to_G s(\beta). \end{aligned}$$

Thus s is a G-state morphism on A. □

6. Conclusion

In this paper, we associate a bounded semihoop A with an ℓ-group G and introduce G-states on A. In particular, we summarize some properties that are different from the states on $[0, 1]$.

• For a Bosbach state s on a bounded semihoop A, s satisfies $s(1) = 1$. However, a Bosbach G-state on A may not be $s(1) = 1$, which is the largest element of $Im(s)$. Thus we can use Bosbach G-states to characterize the largest energy produced under certain constraints.

• Let s_1, s_2 be two Bosbach G-states on a bounded semihoop A, where G is an ℓ-ring. Then $s = k_1 s_1 + k_2 s_2$ is also a Bosbach G-state on A, for any $0 \leq k_1, k_2 \in G$, which maintains any linear combination. Unlike Bosbach state on the interval $[0, 1]$, here $0 \leq k_1, k_2$ can take any element of G and do not need to satisfy the convex combination.

• Let s be a G-state morphism on a bounded semihoop A. Since the elements in abelian ℓ-group may not be comparable, some non-comparable physical quantities can be inscribed by G-state morphism.

This paper is the first to introduce the notion of G-states on a logic algebra and obtain some important conclusions. Since semihoops are fundamental residuated structures, these properties and conclusions in this article can be applied to other residuated structures.

Acknowledgement This research is supported by the following grants:

1. 2022 Research and Practice of Degree and Graduate Education Reform of Xi'an Engineering University (22yjzg05).

2. National Natural Science Foundation of China (11971384).
3. Foreign Expert Program of China (DL20230410021).

References

[1] C. C. Chang, Algebraic analysis of many valued logics, Transactions of the American Mathematical society 88 (2) (1958) 467–490.
[2] P. Hájek and Metamathematics of fuzzy logic, Kluwer Academic Publishers, Dordrecht 1998.
[3] F. Esteva, L. Godo and Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems 124 (3) (2001) 271–288.
[4] G. J. Wang, Non-classical mathematical logic and approximate reasoning, Science in China Press, Beijing 2000.
[5] F. Esteva, L. Godo, P. Hájek and F. Montagan, Hoops and fuzzy logic, Journal of Logic and Computation 13 (4) (2003) 532–555.
[6] M. Ward and R. P. Dilworth, Residuated lattices, Proceedings of the National Academy of Sciences 24 (3) (1938) 162–164.
[7] R. A. Borzooei and M. A. Kologani, Local and perfect semihoops, Journal of Intelligent and Fuzzy Systems 29 (1) (2015) 223–234.
[8] H. L. Niu and X. L. Xin, Tense operators on bounded semihoops, Pure and Applied Mathematics 35 (3) (2019) 325–335.
[9] H. L. Niu, X. L. Xin and J. T. Wang, Ideal theory on bounded semihoops, Italian Journal of Pure and Applied Mathematics 26 (2020) 911–925.
[10] V. Novák, I. Perfilieva and J. Mockor, Mathematical Principless of Fuzzy Logic, Boston: Kluwer Academic Publishers 1999.
[11] T. Hailperin, Sentential Probability Logic, London: Associated University Press 1996.
[12] D. Mundici, Averaging the truth-value in Lukasiewicz logic, Studia Logica 55 (1) (1995) 113–127.
[13] C. C. Lavinia, D. Anatolij and H. Marek, State BL-algebras, Soft Comput 15 (4) (2011) 619–634.
[14] R. A. Borzooei and B. G. Saffar, States on EQ-algebras, Journal of Intelligent and Fuzzy Systems 29 (1) (2015) 209–221.
[15] D. Anatolij, States on Pseudo MV-algebras, Studia Logica: An International Journal for Symbolic Logic 68 (3) (2001) 301–327.
[16] K. Michiro, States on bounded commutative residuated lattices, Mathematica Slovaca 64 (5) (2014) 1093–1104.
[17] P. F. He, B. Zhao and X. L. Xin, States and internal states on semihoops, Soft Computing 21 (11) (2017) 2941–2957.
[18] P. F. He, J. T. Wang and J. Yang, The existence of states based on Glivenko semihoops, Archive for Mathematical Logic 61 (2022) 1145–1170.
[19] A. Bigard, K. Keimel, S. Wolfenstein, Groupes et Anneaux Réticulés, Springer-Verlag 1977.
[20] W. Szmielew, Elementary properties of Abelian groups, Fundamenta Mathematicae 41 (1955) 203–271.
[21] G. Georgescu, L. Leustean1 and V. Preoteasa, Pseudo-hoops, Journal of Multiple-valued Logic and Soft Computing 11 (1-2) (2014) 153–184.

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