Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 28, No. 1, (August 2024) pp. 57–73
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2024.28.1.57

@FMI

SR-fuzzy deductive systems in equality algebras

Young Bae Jun

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 28, No. 1, August 2024

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 28, No. 1, (August 2024) pp. 57–73
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2024.28.1.57

@FMI

SR-fuzzy deductive systems in equality algebras

Young Bae Jun

Received 28 January 2024; Revised 8 March 2024; Accepted 14 April 2024

Abstract. For the purpose of studying the deductive system (or filter) of equality algebras by applying the SR-fuzzy set to equality algebras, the concepts of SR-fuzzy subalgebras and SR-fuzzy deductive systems are introduced and related properties are investigated. Conditions under which the SR-fuzzy set becomes an SR-fuzzy subalgebra are explored, and the characterizations of the SR-fuzzy deductive system are discussed. Conditions are found in which the SR-fuzzy set becomes an SR-fuzzy subalgebra, and the characterizations of the SR-fuzzy deductive system are discussed. The relationship between SR-fuzzy subalgebra and SR-fuzzy deductive system is identified, and an SR-fuzzy deductive system established in a given deductive system is formed.

2020 AMS Classification: 03G25, 06F35, 08A72.

Keywords: (Positive implicative) equality algebra, (SR-fuzzy) subalgebra, (SR-fuzzy) deductive system.

Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

Intuitionistic fuzzy sets are extensions of classical fuzzy sets, introduced by Atanasov in 1986. This adds an additional degree of freedom to the traditional concept of fuzzy sets, taking into account not only the degree of membership, but also the degree of non-membership and the degree of hesitation or uncertainty about each element. In the paper [1], Al-shami et al. have proposed a new extension of the intuitionistic fuzzy set called the SR-fuzzy set which is not obtained from the q-rung orthopair fuzzy set. And then they introduced novel types of weighted aggregation operators and discussed their main properties. Also they investigated a MCDM methods depending on these operators. Jenei [2] have introduced a new structure called an equality algebra, which has two connectives: a meet operation and an equivalence, and a constant 1. Several scientists have studied various types of (fuzzy) filters (or deductive systems) in equity algebras (See [3, 4, 5, 6]).

The purpose of this paper is to study deductive systems (or filters) of equality algebras by applying the SR-fuzzy set to equality algebras. We introduce the concepts of SR-fuzzy subalgebras and SR-fuzzy deductive systems, and investigate some of the relevant properties. We find the conditions under which the SR-fuzzy set becomes an SR-fuzzy subalgebra. We discuss the characterization of the SR-fuzzy deductive system. We establish the relationship between the SR-fuzzy subalgebra and the SR-fuzzy deductive system. We form the SR-fuzzy deductive system, which is established at a given deductive system.

2. Preliminaries

Definition 2.1 ([2])**.** By an *equality algebra*, we mean an algebra $(X, \wedge, \sim, 1)$ satisfying the following conditions:

- (E1) $(X, \wedge, 1)$ is a commutative idempotent integral monoid (i.e., meet semilattice with the top element 1),
- (E2) the operation "$\sim$" is commutative,
- (E3) $(\forall \mathfrak{a} \in X)(\mathfrak{a} \sim \mathfrak{a} = 1)$,
- (E4) $(\forall \mathfrak{a} \in X)(\mathfrak{a} \sim 1 = \mathfrak{a})$,
- (E5) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)(\mathfrak{a} \leq \mathfrak{b} \leq \mathfrak{c} \;\Rightarrow\; \mathfrak{a} \sim \mathfrak{c} \leq \mathfrak{b} \sim \mathfrak{c},\; \mathfrak{a} \sim \mathfrak{c} \leq \mathfrak{a} \sim \mathfrak{b})$,
- (E6) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)(\mathfrak{a} \sim \mathfrak{b} \leq (\mathfrak{a} \wedge \mathfrak{c}) \sim (\mathfrak{b} \wedge \mathfrak{c}))$,
- (E7) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)(\mathfrak{a} \sim \mathfrak{b} \leq (\mathfrak{a} \sim \mathfrak{c}) \sim (\mathfrak{b} \sim \mathfrak{c}))$,

where $\mathfrak{a} \leq \mathfrak{b}$ if and only if $\mathfrak{a} \wedge \mathfrak{b} = \mathfrak{a}$.

In an equality algebra $(X, \wedge, \sim, 1)$, we define two operations "$\rightarrow$" and "$\leftrightarrow$" on X as follows:

$$\mathfrak{a} \rightarrow \mathfrak{b} := \mathfrak{a} \sim (\mathfrak{a} \wedge \mathfrak{b}), \tag{2.1}$$

$$\mathfrak{a} \leftrightarrow \mathfrak{b} := (\mathfrak{a} \rightarrow \mathfrak{b}) \wedge (\mathfrak{b} \rightarrow \mathfrak{a}). \tag{2.2}$$

Proposition 2.2 ([2]). *Let* $(X, \wedge, \sim, 1)$ *be an equality algebra. Then the following assertions are valid: for all* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$,

$$\mathfrak{a} \to \mathfrak{b} = 1 \;\Leftrightarrow\; \mathfrak{a} \leq \mathfrak{b}, \tag{2.3}$$

$$\mathfrak{a} \to (\mathfrak{b} \to \mathfrak{c}) = \mathfrak{b} \to (\mathfrak{a} \to \mathfrak{c}), \tag{2.4}$$

$$1 \to \mathfrak{a} = \mathfrak{a},\; \mathfrak{a} \to 1 = 1,\; \mathfrak{a} \to \mathfrak{a} = 1, \tag{2.5}$$

$$\mathfrak{a} \leq \mathfrak{b} \to \mathfrak{c} \;\Leftrightarrow\; \mathfrak{b} \leq \mathfrak{a} \to \mathfrak{c}, \tag{2.6}$$

$$\mathfrak{a} \leq \mathfrak{b} \to \mathfrak{a}, \tag{2.7}$$

$$\mathfrak{a} \leq (\mathfrak{a} \to \mathfrak{b}) \to \mathfrak{b}, \tag{2.8}$$

$$\mathfrak{a} \to \mathfrak{b} \leq (\mathfrak{b} \to \mathfrak{c}) \to (\mathfrak{a} \to \mathfrak{c}), \tag{2.9}$$

$$\mathfrak{b} \leq \mathfrak{a} \;\Rightarrow\; \mathfrak{a} \leftrightarrow \mathfrak{b} = \mathfrak{a} \to \mathfrak{b} = \mathfrak{a} \sim \mathfrak{b}, \tag{2.10}$$

$$\mathfrak{a} \sim \mathfrak{b} \leq \mathfrak{a} \Leftrightarrow \mathfrak{b} \leq \mathfrak{a} \to \mathfrak{b}, \tag{2.11}$$

$$\mathfrak{a} \leq \mathfrak{b} \;\Rightarrow \begin{cases} \mathfrak{b} \to \mathfrak{c} \leq \mathfrak{a} \to \mathfrak{c}, \\ \mathfrak{c} \to \mathfrak{a} \leq \mathfrak{c} \to \mathfrak{b}, \end{cases} \tag{2.12}$$

$$\mathfrak{a} \leq (\mathfrak{a} \sim \mathfrak{b}) \sim \mathfrak{b}, \tag{2.13}$$

$$\mathfrak{a} \sim \mathfrak{b} \leq \mathfrak{a} \sim (\mathfrak{a} \wedge \mathfrak{b}). \tag{2.14}$$

An equality algebra $(X, \wedge, \sim, 1)$ is said to be *bounded*, if there exists an element $0 \in X$ such that $0 \leq \mathfrak{a}$ for all $\mathfrak{a} \in X$. In a bounded equality algebra $(X, \wedge, \sim, 1)$, we define the negation "$\neg$" on X by $\neg\mathfrak{a} = \mathfrak{a} \to 0 = \mathfrak{a} \sim 0$ for all $\mathfrak{a} \in X$.

An equality algebra $(X, \wedge, \sim, 1)$ is said to be *positive implicative* (See [3]), if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)(\mathfrak{a} \to (\mathfrak{b} \to \mathfrak{c}) = (\mathfrak{a} \to \mathfrak{b}) \to (\mathfrak{a} \to \mathfrak{c})). \tag{2.15}$$

A subset D of X is called a *deductive system* (or *filter*) of an equality algebra $(X, \wedge, \sim, 1)$ (See [7]), if it satisfies:

$$1 \in D, \tag{2.16}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in D, \mathfrak{a} \leq \mathfrak{b} \;\Rightarrow\; \mathfrak{b} \in D), \tag{2.17}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in D, \mathfrak{a} \sim \mathfrak{b} \in D \;\Rightarrow\; \mathfrak{b} \in D). \tag{2.18}$$

Lemma 2.3 ([8]). *Let* $(X, \wedge, \sim, 1)$ *be an equality algebra. A subset* D *of* X *is a deductive system of* X *if and only if it satisfies* (2.16) *and*

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in D, \mathfrak{a} \to \mathfrak{b} \in D \;\Rightarrow\; \mathfrak{b} \in D). \tag{2.19}$$

Definition 2.4. Let X be a universal set. Then the pair $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ of two fuzzy sets $\mu_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ in X is called

- an *intuitionistic fuzzy set* in X (See [9]), if it satisfies:

$$(\forall \mathfrak{a} \in X)(0 \leq \mu_{\mathcal{S}}(\mathfrak{a}) + \nu_{\mathcal{S}}(\mathfrak{a}) \leq 1), \tag{2.20}$$

- an *SR-fuzzy set* in X (See [1]), if it satisfies:

$$(\forall \mathfrak{a} \in X)(0 \leq (\mu_{\mathcal{S}}(\mathfrak{a}))^2 + \sqrt{\nu_{\mathcal{S}}(\mathfrak{a})} \leq 1). \tag{2.21}$$

3. SR-fuzzy subalgebras

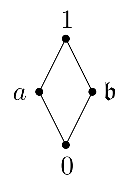
Definition 3.1. An SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ in X is called an *SR-fuzzy subalgebra* of an equality algebra $(X, \wedge, \sim, 1)$ if it satisfies:

$$(\forall x, y \in X)\left(\begin{array}{c} (\mu_{\mathcal{S}}(x \sim y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(x \sim y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\} \end{array}\right), \tag{3.1}$$

and

$$(\forall x, y \in X)\left(\begin{array}{c} (\mu_{\mathcal{S}}(x \wedge y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(x \wedge y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\} \end{array}\right). \tag{3.2}$$

Example 3.2. Let $X = \{0, a, \mathfrak{b}, 1\}$ be a set with the following Hasse diagram.



Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation "$\sim$" on X by the Table 1. Then $(X, \wedge, \sim, 1)$ is an equality algebra (See Example 3.13, [10]).

Table 1. Cayley table for the binary operation "$\sim$"

$\sim$	0	a	$\mathfrak{b}$	1
0	1	$\mathfrak{b}$	a	0
a	$\mathfrak{b}$	1	0	a
$\mathfrak{b}$	a	0	1	$\mathfrak{b}$
1	0	a	$\mathfrak{b}$	1

Let $\mu_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be fuzzy sets in X given as follows:

X	0	a	$\mathfrak{b}$	1
$\mu_{\mathcal{S}}(x)$	0.3	0.5	0.3	0.6
$\nu_{\mathcal{S}}(x)$	0.52	0.52	0.52	0.33

Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is given by the following table.

X	0	a	$\mathfrak{b}$	1
$(\mu_{\mathcal{S}}(x))^2$	0.09	0.25	0.09	0.36
$\sqrt{\nu_{\mathcal{S}}(x)}$	0.7211	0.7211	0.7211	0.5477

where $\sqrt{\nu_{\mathcal{S}}(x)}$ is calculated to four decimal places. By routine calculation, we can check that $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$.

Proposition 3.3. *In an equality algebra* $(X, \wedge, \sim, 1)$, *every SR-fuzzy subalgebra* $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ *satisfies:*

$$(\forall x \in X)\left((\mu_{\mathcal{S}}(1))^2 \geq (\mu_{\mathcal{S}}(x))^2),\ \sqrt{\nu_{\mathcal{S}}(1)} \leq \sqrt{\nu_{\mathcal{S}}(x)}\right). \tag{3.3}$$

$$(\forall x, y \in X)\left(\begin{array}{c} (\mu_{\mathcal{S}}(x \rightarrow y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(x \rightarrow y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\} \end{array}\right). \tag{3.4}$$

Proof. The result (3.3) is directly induced by (E3) and (3.1). Using (2.1), (3.1) and (3.2), we have

$$\begin{aligned}(\mu_{\mathcal{S}}(x \rightarrow y))^2 &= (\mu_{\mathcal{S}}(x \sim (x \wedge y)))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \wedge y))^2\right\} \\ &\geq \min\left\{(\mu_{\mathcal{S}}(x))^2, \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\}\right\} \\ &= \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\}\end{aligned}$$

and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(x \rightarrow y)} &= \sqrt{\nu_{\mathcal{S}}(x \sim (x \wedge y))} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \wedge y)}\right\} \\ &\leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\}\right\} \\ &= \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\}\end{aligned}$$

for all $x, y \in X$, which proves (3.4). □

We have a question: If an SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ satisfies (3.3) or (3.4), then is it an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$? The following example provides a negative response to that question.

Example 3.4. (1) Consider the equality algebra $(X, \wedge, \sim, 1)$ in Example 3.2, and let $\mu_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be fuzzy sets in X given as follows:

X	0	a	$\mathfrak{b}$	1
$\mu_{\mathcal{S}}(x)$	0.2	0.4	0.2	0.5
$\nu_{\mathcal{S}}(x)$	0.54	0.52	0.48	0.43

Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is given by the following table.

X	0	a	$\mathfrak{b}$	1
$(\mu_{\mathcal{S}}(x))^2$	0.04	0.16	0.04	0.25
$\sqrt{\nu_{\mathcal{S}}(x)}$	0.7348	0.7211	0.6928	0.6557

where $\sqrt{\nu_{\mathcal{S}}(x)}$ is calculated to four decimal places. Then $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ satisfies (3.3). But it is not an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$ since

$$\sqrt{\nu_{\mathcal{S}}(a \wedge \mathfrak{b})} = \sqrt{\nu_{\mathcal{S}}(0)} = 0.7348 \not\leq 0.7211 = \max\left\{\sqrt{\nu_{\mathcal{S}}(a)}, \sqrt{\nu_{\mathcal{S}}(\mathfrak{b})}\right\}.$$

(2) Let $(X, \wedge, \sim, 1)$ be the equality algebra in Example 3.2. Then the implication "$\rightarrow$" is given by the Table 2.
Let $\mu_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be fuzzy sets in X given as follows:

TABLE 2. Cayley table for the implication "→"

$\rightarrow$	0	a	$\mathfrak{b}$	1
0	1	1	1	1
a	$\mathfrak{b}$	1	$\mathfrak{b}$	1
$\mathfrak{b}$	a	a	1	1
1	0	a	$\mathfrak{b}$	1

X	0	a	$\mathfrak{b}$	1
$\mu_{\mathcal{S}}(x)$	0.33	0.44	0.39	0.66
$\nu_{\mathcal{S}}(x)$	0.32	0.32	0.32	0.26

Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is given by the following table.

X	0	a	$\mathfrak{b}$	1
$(\mu_{\mathcal{S}}(x))^2$	0.1089	0.1936	0.1521	0.4356
$\sqrt{\nu_{\mathcal{S}}(x)}$	0.5657	0.5657	0.5657	0.5099

where $\sqrt{\nu_{\mathcal{S}}(x)}$ is calculated to four decimal places. Then $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ satisfies (3.4). But it is not an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$ since

$$(\mu_{\mathcal{S}}(a \wedge \mathfrak{b}))^2 = (\mu_{\mathcal{S}}(0))^2 = 0.1089 \not\geq 0.1521 = \min\{(\mu_{\mathcal{S}}(a))^2, (\mu_{\mathcal{S}}(\mathfrak{b}))^2\}.$$

We search for a condition in which an SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ becomes an SR-fuzzy subalgebra if it satisfies the condition (3.4).

Theorem 3.5. *If an equality algebra $(X, \wedge, \sim, 1)$ is linearly ordered, then every SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ satisfying (3.4) is an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$.*

Proof. Let $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ be an SR-fuzzy set that satisfies the condition (3.4). For every $x, y \in X$ we have $x \leq y$ or $y \leq x$ because $(X, \wedge, \sim, 1)$ is linearly ordered. If $x \leq y$, then $x \wedge y = x$ and $x \sim y = y \sim x = y \rightarrow x$ by (E2) and (2.10). It follows from (3.4) that $(\mu_{\mathcal{S}}(x \sim y))^2 = (\mu_{\mathcal{S}}(y \rightarrow x))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\}$ and $\sqrt{\nu_{\mathcal{S}}(x \sim y)} = \sqrt{\nu_{\mathcal{S}}(y \rightarrow x)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\}$. Since $x \wedge y = x$, it is clear that $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ satisfies (3.2). Hence $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$. In the case of $y \leq x$, it is proved in the same way. □

4. SR-fuzzy deductive systems

Definition 4.1. Let $(X, \wedge, \sim, 1)$ be an equality algebra. An SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ in X is called an *SR-fuzzy deductive system* (or *SR-fuzzy filter*) of $(X, \wedge,$

$\sim$, 1), if it satisfies:

$$(\forall x \in X)\left((\mu_{\mathcal{S}}(1))^2 \geq (\mu_{\mathcal{S}}(x))^2,\ \sqrt{\nu_{\mathcal{S}}(1)} \leq \sqrt{\nu_{\mathcal{S}}(x)}\right), \tag{4.1}$$

$$(\forall x, y \in X)\left(\ x \leq y\ \Rightarrow \left\{ \begin{array}{l} (\mu_{\mathcal{S}}(x))^2 \leq (\mu_{\mathcal{S}}(y))^2 \\ \sqrt{\nu_{\mathcal{S}}(x)} \geq \sqrt{\nu_{\mathcal{S}}(y)} \end{array} \right.\ \right), \tag{4.2}$$

$$(\forall x, y \in X)\left(\begin{array}{l} (\mu_{\mathcal{S}}(y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \sim y))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \sim y)}\right\} \end{array} \right). \tag{4.3}$$

Example 4.2. Consider the equality algebra $(X, \wedge, \sim, 1)$ in Example 3.2, and let $\mu_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be fuzzy sets in X given as follows:

X	0	a	$\mathfrak{b}$	1
$\mu_{\mathcal{S}}(x)$	0.44	0.55	0.44	0.66
$\nu_{\mathcal{S}}(x)$	0.63	0.43	0.53	0.23

Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is given by the following table.

X	0	a	$\mathfrak{b}$	1
$(\mu_{\mathcal{S}}(x))^2$	0.1936	0.3025	0.1936	0.4356
$\sqrt{\nu_{\mathcal{S}}(x)}$	0.7937	0.6557	0.7280	0.4796

where $\sqrt{\nu_{\mathcal{S}}(x)}$ is calculated to four decimal places. By the routine calculations, we know that $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$.

Example 4.3. Let $X := [0, 1]$. Define two operations "$\wedge$" and "$\sim$" on X as follows:

$$(\forall x, y \in X)(x \wedge y = \min\{x, y\},\ x \sim y = 1 - |x - y|).$$

Then $(X, \wedge, \sim, 1)$ is an equality algebra. Let $\mu_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be fuzzy sets in X which are defined by

$$\mu_{\mathcal{S}}(x) := \left\{ \begin{array}{ll} t & \text{if } x = 1, \\ s & \text{if } x \neq 1, \end{array} \right. \qquad \nu_{\mathcal{S}}(x) := \left\{ \begin{array}{ll} \alpha & \text{if } x = 1, \\ \beta & \text{if } x \neq 1, \end{array} \right.$$

where $t, s, \alpha, \beta \in [0, 1]$ such that $t > s$, $\alpha < \beta$, $0 \leq t^2 + \sqrt{\alpha} \leq 1$ and $0 \leq s^2 + \sqrt{\beta} \leq 1$. Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is calculated as follows:

$$(\mu_{\mathcal{S}}(x))^2 := \left\{ \begin{array}{ll} t^2 & \text{if } x = 1, \\ s^2 & \text{if } x \neq 1, \end{array} \right. \qquad \sqrt{\nu_{\mathcal{S}}(x)} := \left\{ \begin{array}{ll} \sqrt{\alpha} & \text{if } x = 1, \\ \sqrt{\beta} & \text{if } x \neq 1. \end{array} \right.$$

It is routine to verify that $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$.

Proposition 4.4. *Every SR-fuzzy deductive system $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ of an equality algebra $(X, \wedge, \sim, 1)$ satisfies:*

$$(4.4)\qquad (\forall x, y \in X)\left(\begin{array}{c} \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\} = (\mu_{\mathcal{S}}(x \wedge y))^2 \\ \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\} \\ \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\} = \sqrt{\nu_{\mathcal{S}}(x \wedge y)} \\ \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\} \end{array}\right).$$

$$(4.5)\qquad (\forall x, y, z \in X)\left(\begin{array}{c} (\mu_{\mathcal{S}}(x \sim z))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x \sim y))^2, (\mu_{\mathcal{S}}(y \sim z))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(x \sim z)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x \sim y)}, \sqrt{\nu_{\mathcal{S}}(y \sim z)}\right\} \end{array}\right).$$

Proof. Let $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ be an SR-fuzzy deductive system of an equality algebra $(X, \wedge, \sim, 1)$. The combination of (2.1) and (4.3) derive to

$$\begin{aligned} \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\} &= \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \sim (x \wedge y))^2\right\} \\ &\leq (\mu_{\mathcal{S}}(x \wedge y))^2 \end{aligned}$$

and

$$\begin{aligned} \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\} &= \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \sim (x \wedge y))}\right\} \\ &\geq \sqrt{\nu_{\mathcal{S}}(x \wedge y)} \end{aligned}$$

for all $x, y \in X$. It follows from (2.7) and (4.2) that

$$(\mu_{\mathcal{S}}(x \wedge y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\} \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\}$$
$$\sqrt{\nu_{\mathcal{S}}(x \wedge y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\}$$

for all $x, y \in X$. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, we have

$$(\mu_{\mathcal{S}}(x \wedge y))^2 \leq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\}$$
$$\sqrt{\nu_{\mathcal{S}}(x \wedge y)} \geq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\}$$

for all $x, y \in X$ by (4.2). This proves (4.4). The result (4.5) comes from the combination of (E2), (E7), (4.2) and (4.3). □

Theorem 4.5. *Let $(X, \wedge, \sim, 1)$ be an equality algebra. An SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ in X is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$ if and only if it satisfies (4.1) and*

$$(4.6)\qquad (\forall x, y \in X)\left(\begin{array}{c} (\mu_{\mathcal{S}}(y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\} \end{array}\right).$$

Proof. Assume that $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$. For every $x, y \in X$, we have

$$\begin{aligned} (\mu_{\mathcal{S}}(y))^2 &\overset{(4.2)}{\geq} (\mu_{\mathcal{S}}(x \wedge y))^2 \overset{(4.3)}{\geq} \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \sim (x \wedge y)))^2\right\} \\ &\overset{(2.1)}{=} \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\} \end{aligned}$$

and

$$\sqrt{\nu_{\mathcal{S}}(y)} \overset{(4.2)}{\leq} \sqrt{\nu_{\mathcal{S}}(x \wedge y)} \overset{(4.3)}{\leq} \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \sim (x \wedge y))}\right\}$$
$$\overset{(2.1)}{=} \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\}.$$

Conversely, let $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ be an SR-fuzzy set in X that satisfies (4.1) and (4.6). Let $x, y \in X$, If $x \leq y$, then $x \to y = 1$ by (2.3), and so

$$\begin{aligned}(\mu_{\mathcal{S}}(x))^2 &= \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(1))^2\right\}\\ &= \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\}\\ &\leq (\mu_{\mathcal{S}}(y))^2\end{aligned}$$

and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(x)} &= \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(1)}\right\}\\ &= \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\}\\ &\geq \sqrt{\nu_{\mathcal{S}}(y)}\end{aligned}$$

by (4.1) and (4.6). It follows from (2.11) that $(\mu_{\mathcal{S}}(x \sim y))^2 \leq (\mu_{\mathcal{S}}(x \to y))^2$ and $\sqrt{\nu_{\mathcal{S}}(x \sim y)} \geq \sqrt{\nu_{\mathcal{S}}(x \to y)}$. Hence

$$(\mu_{\mathcal{S}}(y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\} \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \sim y))^2\right\}$$

and

$$\sqrt{\nu_{\mathcal{S}}(y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \sim y)}\right\}$$

for all $x, y \in X$. Therefore $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$. □

Corollary 4.6. *If $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of an equality algebra $(X, \wedge, \sim, 1)$, then $(\mu_{\mathcal{S}}(x))^2 \leq (\mu_{\mathcal{S}}(y))^2$ and $\sqrt{\nu_{\mathcal{S}}(x \to y)} \geq \sqrt{\nu_{\mathcal{S}}(1)}$ for all $x, y \in X$ with $(\mu_{\mathcal{S}}(x \to y))^2 = (\mu_{\mathcal{S}}(1))^2$ and $\sqrt{\nu_{\mathcal{S}}(x \to y)} = \sqrt{\nu_{\mathcal{S}}(1)}$.*

Proof. Let $x, y \in X$ be such that $(\mu_{\mathcal{S}}(x \to y))^2 = (\mu_{\mathcal{S}}(1))^2$ and $\sqrt{\nu_{\mathcal{S}}(x \to y)} = \sqrt{\nu_{\mathcal{S}}(1)}$. Then

$$\begin{aligned}(\mu_{\mathcal{S}}(x))^2 &= \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(1))^2\right\}\\ &= \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \to y))^2\right\}\\ &\leq (\mu_{\mathcal{S}}(y))^2\end{aligned}$$

and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(x)} &= \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(1)}\right\}\\ &= \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \to y)}\right\}\\ &\geq \sqrt{\nu_{\mathcal{S}}(y)}.\end{aligned}$$

This completes the proof. □

Proposition 4.7. *Every SR-fuzzy deductive system* $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ *of an equality algebra* $(X, \wedge, \sim, 1)$ *satisfies:*

$$\left\{\begin{array}{l}(\mu_{\mathcal{S}}(x \to z))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x \to y))^2, (\mu_{\mathcal{S}}(y \to z))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(x \to z)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x \to y)}, \sqrt{\nu_{\mathcal{S}}(y \to z)}\right\}\end{array}\right. \tag{4.7}$$

$$x \leq y \;\Rightarrow\; \left\{\begin{array}{l}(\mu_{\mathcal{S}}(x))^2 = \min\left\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(y \to x))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(x)} = \max\left\{\sqrt{\nu_{\mathcal{S}}(y)}, \sqrt{\nu_{\mathcal{S}}(y \to x)}\right\}\end{array}\right. \tag{4.8}$$

for all $x, y, z \in X$*.*

Proof. Let $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ be an SR-fuzzy deductive system of an equality algebra $(X, \wedge, \sim, 1)$. By the combination of (2.9), (4.2) and (4.6), we have

$$\begin{aligned}(\mu_{\mathcal{S}}(x \to z))^2 &\geq \min\left\{(\mu_{\mathcal{S}}((y \to z) \to (x \to z)))^2, (\mu_{\mathcal{S}}(y \to z))^2\right\} \\ &\geq \min\left\{(\mu_{\mathcal{S}}(x \to y))^2, (\mu_{\mathcal{S}}(y \to z))^2\right\}\end{aligned}$$

and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(x \to z)} &\leq \max\left\{\sqrt{\nu_{\mathcal{S}}((y \to z) \to (x \to z))}, \sqrt{\nu_{\mathcal{S}}(y \to z)}\right\} \\ &\leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x \to y)}, \sqrt{\nu_{\mathcal{S}}(y \to z)}\right\}\end{aligned}$$

for all $x, y, z \in X$. This proves (4.7). Let $x, y \in X$ be such that $x \leq y$. Then $(\mu_{\mathcal{S}}(x))^2 \leq (\mu_{\mathcal{S}}(y))^2$ and $\sqrt{\nu_{\mathcal{S}}(x)} \geq \sqrt{\nu_{\mathcal{S}}(y)}$ by (4.2). Also, we have $(\mu_{\mathcal{S}}(x))^2 \leq (\mu_{\mathcal{S}}(y \to x))^2$ and $\sqrt{\nu_{\mathcal{S}}(x)} \geq \sqrt{\nu_{\mathcal{S}}(y \to x)}$ by the combination of (2.7) and (4.2). It follows from Theorem 4.5 that

$$\begin{aligned}(\mu_{\mathcal{S}}(x))^2 &\geq \min\left\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(y \to x))^2\right\} \\ &\geq \min\left\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(x))^2\right\} \\ &= (\mu_{\mathcal{S}}(x))^2\end{aligned}$$

and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(x)} &\leq \max\left\{\sqrt{\nu_{\mathcal{S}}(y)}, \sqrt{\nu_{\mathcal{S}}(y \to x)}\right\} \\ &\leq \max\left\{\sqrt{\nu_{\mathcal{S}}(y)}, \sqrt{\nu_{\mathcal{S}}(x)}\right\} \\ &= \sqrt{\nu_{\mathcal{S}}(x)}.\end{aligned}$$

Hence $(\mu_{\mathcal{S}}(x))^2 = \min\left\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(y \to x))^2\right\}$ and

$$\sqrt{\nu_{\mathcal{S}}(x)} = \max\left\{\sqrt{\nu_{\mathcal{S}}(y)}, \sqrt{\nu_{\mathcal{S}}(y \to x)}\right\}$$

for all $x, y \in X$ with $x \leq y$. □

Theorem 4.8. *Let* $(X, \wedge, \sim, 1)$ *be an equality algebra. An SR-fuzzy set* $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ *in* X *is an SR-fuzzy deductive system of* $(X, \wedge, \sim, 1)$ *if and only if it satisfies:*

$$x \leq y \to z \;\Rightarrow\; \left\{\begin{array}{l}(\mu_{\mathcal{S}}(z))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(z)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\right\}\end{array}\right. \tag{4.9}$$

for all $x, y, z \in X$.

Proof. Assume that $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$. Let $x, y, z \in X$ be such that $x \leq y \to z$. Then $(\mu_\mathcal{S}(x))^2 \leq (\mu_\mathcal{S}(y \to z))^2$ and $\sqrt{\nu_\mathcal{S}(x)} \geq \sqrt{\nu_\mathcal{S}(y \to z)}$ by (4.2). It follows from (4.6) that

$$(\mu_\mathcal{S}(z))^2 \geq \min\left\{(\mu_\mathcal{S}(y))^2, (\mu_\mathcal{S}(y \to z))^2\right\} \geq \min\left\{(\mu_\mathcal{S}(y))^2, (\mu_\mathcal{S}(x))^2\right\}$$

and

$$\sqrt{\nu_\mathcal{S}(z)} \leq \max\left\{\sqrt{\nu_\mathcal{S}(y)}, \sqrt{\nu_\mathcal{S}(y \to z)}\right\} \leq \max\left\{\sqrt{\nu_\mathcal{S}(y)}, \sqrt{\nu_\mathcal{S}(x)}\right\}.$$

Conversely, let $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ be an SR-fuzzy set in X that satisfies (4.9). The combination of (2.3) and (2.5) induces $x \leq x \to 1$ and $x \to y \leq x \to y$ for all $x, y \in X$. Using (4.9), we have

$$\begin{aligned}
&(\mu_\mathcal{S}(1))^2 \geq \min\left\{(\mu_\mathcal{S}(x))^2, (\mu_\mathcal{S}(x))^2\right\} = (\mu_\mathcal{S}(x))^2,\\
&\sqrt{\nu_\mathcal{S}(1)} \leq \max\left\{\sqrt{\nu_\mathcal{S}(x)}, \sqrt{\nu_\mathcal{S}(x)}\right\} = \sqrt{\nu_\mathcal{S}(x)},\\
&(\mu_\mathcal{S}(y))^2 \geq \min\left\{(\mu_\mathcal{S}(x))^2, (\mu_\mathcal{S}(x \to y))^2\right\},\\
&\sqrt{\nu_\mathcal{S}(y)} \leq \max\left\{\sqrt{\nu_\mathcal{S}(x)}, \sqrt{\nu_\mathcal{S}(x \to y)}\right\},
\end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$ by Theorem 4.5. □

Corollary 4.9. *Every SR-fuzzy deductive system* $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ *of* $(X, \wedge, \sim, 1)$ *satisfies:*

$$(\forall x, y \in X)\left(\ x \sim y \leq x\ \Rightarrow \left\{\begin{array}{l}(\mu_\mathcal{S}(y))^2 \geq \min\left\{(\mu_\mathcal{S}(x))^2, (\mu_\mathcal{S}(y))^2\right\}\\ \sqrt{\nu_\mathcal{S}(y)} \leq \max\left\{\sqrt{\nu_\mathcal{S}(x)}, \sqrt{\nu_\mathcal{S}(y)}\right\}\end{array}\right.\ \right).$$

Proof. Let $x, y \in X$ be such that $x \sim y \leq x$. Then $y \leq x \to y$ by (2.11). Thus the required result comes from (4.9). □

Lemma 4.10 ([3]). *An equality algebra* $(X, \wedge, \sim, 1)$ *is positive implicative if and only if it satisfies:*

$$(\forall x, y \in X)(x \to (x \to y) = x \to y). \tag{4.10}$$

Theorem 4.11. *Let* $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ *be an SR-fuzzy set in a positive implicative equality algebra* $(X, \wedge, \sim, 1)$. *Then the following are equivalent:*

(1) $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ *is an SR-fuzzy deductive system of* $(X, \wedge, \sim, 1)$,

(2) $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ *satisfies:*

$$\left\{\begin{array}{l}(\mu_\mathcal{S}(y \to x))^2 \geq \min\left\{(\mu_\mathcal{S}(z \to (y \to (y \to x))))^2, (\mu_\mathcal{S}(z))^2\right\}\\ \sqrt{\nu_\mathcal{S}(y \to x)} \leq \max\left\{\sqrt{\nu_\mathcal{S}(z \to (y \to (y \to x)))}, \sqrt{\nu_\mathcal{S}(z)}\right\}\end{array}\right. \tag{4.11}$$

for all $x, y, z \in X$,

(3) $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ *satisfies:*

$$\left\{\begin{array}{l}(\mu_\mathcal{S}(x \to z))^2 \geq \min\left\{(\mu_\mathcal{S}(x \to y))^2, (\mu_\mathcal{S}(x \to (y \to z)))^2\right\}\\ \sqrt{\nu_\mathcal{S}(x \to z)} \leq \max\left\{\sqrt{\nu_\mathcal{S}(x \to y)}, \sqrt{\nu_\mathcal{S}(x \to (y \to z))}\right\}\end{array}\right. \tag{4.12}$$

for all $x, y, z \in X$.

Proof. Let $(X, \wedge, \sim, 1)$ be a positive implicative equality algebra. Suppose $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$. Then

$$\begin{aligned}(\mu_{\mathcal{S}}(y \to x))^2 &\overset{(4.10)}{=} (\mu_{\mathcal{S}}(y \to (y \to x)))^2 \\ &\overset{(4.6)}{\geq} \min\left\{(\mu_{\mathcal{S}}(z \to (y \to (y \to x))))^2, (\mu_{\mathcal{S}}(z))^2\right\}\end{aligned}$$

and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(y \to x)} &\overset{(4.10)}{=} \sqrt{\nu_{\mathcal{S}}(y \to (y \to x))} \\ &\overset{(4.6)}{\geq} \min\left\{\sqrt{\nu_{\mathcal{S}}(z \to (y \to (y \to x)))}, \sqrt{\nu_{\mathcal{S}}(z)}\right\}\end{aligned}$$

for all $x, y, z \in X$. Thus (4.11) is valid.

Suppose $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ satisfies (4.11). Then we have

$$\begin{aligned}&\min\left\{(\mu_{\mathcal{S}}(x \to y))^2, (\mu_{\mathcal{S}}(x \to (y \to z)))^2\right\} \\ &\overset{(2.15)}{=} \min\left\{(\mu_{\mathcal{S}}(x \to y))^2, (\mu_{\mathcal{S}}((x \to y) \to (x \to z)))^2\right\} \\ &\overset{(4.10)}{=} \min\left\{(\mu_{\mathcal{S}}(x \to y))^2, (\mu_{\mathcal{S}}((x \to y) \to (x \to (x \to z))))^2\right\} \\ &\overset{(4.11)}{\leq} (\mu_{\mathcal{S}}(x \to z))^2\end{aligned}$$

and

$$\begin{aligned}&\max\left\{\sqrt{\nu_{\mathcal{S}}(x \to y)}, \sqrt{\nu_{\mathcal{S}}(x \to (y \to z))}\right\} \\ &\overset{(2.15)}{=} \max\left\{\sqrt{\nu_{\mathcal{S}}(x \to y)}, \sqrt{\nu_{\mathcal{S}}((x \to y) \to (x \to z))}\right\} \\ &\overset{(4.10)}{=} \max\left\{\sqrt{\nu_{\mathcal{S}}(x \to y)}, \sqrt{\nu_{\mathcal{S}}((x \to y) \to (x \to (x \to z)))}\right\} \\ &\overset{(4.11)}{\geq} \sqrt{\nu_{\mathcal{S}}(x \to z)},\end{aligned}$$

which proves (4.12).

Suppose $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ satisfies (4.12). Putting $x = 1$ in (4.12) and using (2.5), we derive

$$\left\{\begin{array}{l}(\mu_{\mathcal{S}}(z))^2 \geq \min\left\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(y \to z))^2\right\} \\ \sqrt{\nu_{\mathcal{S}}(z)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(y)}, \sqrt{\nu_{\mathcal{S}}(y \to z)}\right\}\end{array}\right. \tag{4.13}$$

for all $y, z \in X$. Also, if we put $z = 1$ in (4.13) and use (2.5), then we have $(\mu_{\mathcal{S}}(1))^2 \geq (\mu_{\mathcal{S}}(y))^2$ and $\sqrt{\nu_{\mathcal{S}}(1)} \leq \sqrt{\nu_{\mathcal{S}}(y)}$ for all $y \in X$. Thus $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$ by Theorem 4.5. □

We establish the relationship between SR-fuzzy subalgebra and SR-fuzzy deductive system.

Theorem 4.12. *In an equality algebra* $(X, \wedge, \sim, 1)$, *every SR-fuzzy deductive system is an SR-fuzzy subalgebra.*

Proof. Let $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ be an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$. Let $x, y \in X$. Since $x \leq (x \sim y) \sim y$ by (2.13), we have

$$(\mu_{\mathcal{S}}(x))^2 \leq (\mu_{\mathcal{S}}((x \sim y) \sim y))^2 = (\mu_{\mathcal{S}}(y \sim (x \sim y)))^2$$

and

$$\sqrt{\nu_{\mathcal{S}}(x)} \geq \sqrt{\nu_{\mathcal{S}}((x \sim y) \sim y)} = \sqrt{\nu_{\mathcal{S}}(y \sim (x \sim y))}$$

by (E2) and (4.2). It follows from (4.3) that

$$\begin{aligned}(\mu_{\mathcal{S}}(x \sim y))^2 &\geq \min\left\{(\mu_{\mathcal{S}}(y))^2, (\mu_{\mathcal{S}}(y \sim (x \sim y)))^2\right\} \\ &\geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\}\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(x \sim y)} &\leq \max\left\{\sqrt{\nu_{\mathcal{S}}(y)}, \sqrt{\nu_{\mathcal{S}}(y \sim (x \sim y))}\right\} \\ &\leq \max\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\}.\end{aligned} \tag{4.15}$$

Since $x \sim y \leq x \sim (x \wedge y)$ by (2.14), we get

$$(\mu_{\mathcal{S}}(x \sim y))^2 \leq (\mu_{\mathcal{S}}(x \sim (x \wedge y)))^2 \text{ and } \sqrt{\nu_{\mathcal{S}}(x \sim y)} \geq \sqrt{\nu_{\mathcal{S}}(x \sim (x \wedge y))}$$

by (4.2). It follows from (4.3), (4.14) and (4.15) that

$$\begin{aligned}(\mu_{\mathcal{S}}(x \wedge y))^2 &\geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \sim (x \wedge y)))^2\right\} \\ &\geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \sim y))^2\} \\ &\geq \min\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(y))^2\}\end{aligned}$$

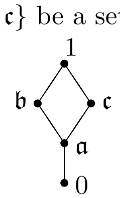
and

$$\begin{aligned}\sqrt{\nu_{\mathcal{S}}(x \wedge y)} &\leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \sim (x \wedge y))}\right\} \\ &\leq \max\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \sim y)}\} \\ &\leq \max\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(y)}\}.\end{aligned}$$

Therefore $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$. □

The example below shows that the convese of Theorem 4.12 may not be true.

Example 4.13. Let $X = \{0, 1, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ be a set with the following Hasse diagram.



and define a binary operation "$\sim$" on X by the Table 3.
Then $(X, \wedge, \sim, 1)$ is an equality algebra (See Example 4.2, CGASA7-33), and the implication "$\rightarrow$" is calculated by the Table 4.
Let $\mu_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be fuzzy sets in X given as follows:

Table 3. Cayley table for the binary operation "∼"

$\sim$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1
0	1	0	0	0	0
$\mathfrak{a}$	0	1	$\mathfrak{c}$	$\mathfrak{b}$	$\mathfrak{a}$
$\mathfrak{b}$	0	$\mathfrak{c}$	1	$\mathfrak{a}$	$\mathfrak{b}$
$\mathfrak{c}$	0	$\mathfrak{b}$	$\mathfrak{a}$	1	$\mathfrak{c}$
1	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1

Table 4. Cayley table for the implication "→"

$\rightarrow$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1
0	1	1	1	1	1
$\mathfrak{a}$	0	1	1	1	1
$\mathfrak{b}$	0	$\mathfrak{c}$	1	$\mathfrak{c}$	1
$\mathfrak{c}$	0	$\mathfrak{b}$	$\mathfrak{b}$	1	1
1	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1

X	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1
$\mu_{\mathcal{S}}(x)$	0.58	0.32	0.58	0.58	0.32
$\nu_{\mathcal{S}}(x)$	0.37	0.71	0.37	0.37	0.71

Then the SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is given by the following table.

X	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	1
$(\mu_{\mathcal{S}}(x))^2$	0.3364	0.1024	0.3364	0.3364	0.1024
$\sqrt{\nu_{\mathcal{S}}(x)}$	0.6083	0.8426	0.6083	0.6083	0.8426

where $\sqrt{\nu_{\mathcal{S}}(x)}$ is calculated to four decimal places. It is routine to verify that $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy subalgebra of $(X, \wedge, \sim, 1)$. But it is not an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$ since

$$(\mu_{\mathcal{S}}(\mathfrak{a}))^2 = 0.1024 \ngeq 0.3364 = \min\{(\mu_{\mathcal{S}}(\mathfrak{b}))^2, (\mu_{\mathcal{S}}(\mathfrak{b} \sim \mathfrak{a}))^2\}$$

and/or $\sqrt{\nu_{\mathcal{S}}(\mathfrak{a})} = 0.8426 \nleq 0.6083 = \max\{\sqrt{\nu_{\mathcal{S}}(\mathfrak{b})}, \sqrt{\nu_{\mathcal{S}}(\mathfrak{b} \sim \mathfrak{a})}\}$.

Let $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ be an SR-fuzzy set in X. For every $(t, s) \in [0, 1] \times [0, 1]$ we consider the next sets:

$$\mu_{\mathcal{S}}(\geq t) := \{x \in X \mid (\mu_{\mathcal{S}}(x))^2 \geq t\},$$
$$\nu_{\mathcal{S}}(\leq s) := \{x \in X \mid \sqrt{\nu_{\mathcal{S}}(x)} \leq s\},$$
$$\mathcal{S}_t^s := \mu_{\mathcal{S}}(\geq t) \cap \nu_{\mathcal{S}}(\leq s).$$

Theorem 4.14. *Let $(X, \wedge, \sim, 1)$ be an equality algebra. Then an SR-fuzzy set $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ in X is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$ if and only if the nonempty sets $\mu_{\mathcal{S}}(\geq t)$ and $\nu_{\mathcal{S}}(\leq s)$ are deductive systems of $(X, \wedge, \sim, 1)$ for all $(t, s) \in [0, 1] \times [0, 1]$.*

Proof. Suppose $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$ and let $(t, s) \in [0, 1] \times [0, 1]$ be such that $\mu_\mathcal{S}(\geq t) \neq \varnothing \neq \nu_\mathcal{S}(\leq s)$. Then there exist $x \in \mu_\mathcal{S}(\geq t)$ and $\mathfrak{a} \in \nu_\mathcal{S}(\leq s)$. Thus $(\mu_\mathcal{S}(1))^2 \geq (\mu_\mathcal{S}(x))^2 \geq t$ and $\sqrt{\nu_\mathcal{S}(1)} \leq \sqrt{\nu_\mathcal{S}(\mathfrak{a})} \leq s$. So $1 \in \mu_\mathcal{S}(\geq t) \cap \nu_\mathcal{S}(\leq s)$. Let $x, y, \mathfrak{a}, \mathfrak{b} \in X$ be such that $x, x \to y \in \mu_\mathcal{S}(\geq t)$ and $\mathfrak{a}, \mathfrak{a} \to \mathfrak{b} \in \nu_\mathcal{S}(\leq s)$. Using (4.6) induces

$$(\mu_\mathcal{S}(y))^2 \geq \min\left\{(\mu_\mathcal{S}(x))^2, (\mu_\mathcal{S}(x \to y))^2\right\} \geq t$$

and $\sqrt{\nu_\mathcal{S}(\mathfrak{b})} \leq \max\left\{\sqrt{\nu_\mathcal{S}(\mathfrak{a})}, \sqrt{\nu_\mathcal{S}(\mathfrak{a} \to \mathfrak{b})}\right\} \leq s$. Hence $y \in \mu_\mathcal{S}(\geq t)$ and $\mathfrak{b} \in \nu_\mathcal{S}(\leq s)$. Therefore $\mu_\mathcal{S}(\geq t)$ and $\nu_\mathcal{S}(\leq s)$ are deductive systems of $(X, \wedge, \sim, 1)$ by Lemma 2.3.

Conversely, suppose the nonempty sets $\mu_\mathcal{S}(\geq t)$ and $\nu_\mathcal{S}(\leq s)$ are deductive systems of $(X, \wedge, \sim, 1)$ for all $(t, s) \in [0, 1] \times [0, 1]$. If $(\mu_\mathcal{S}(1))^2 < (\mu_\mathcal{S}(x))^2$ or $\sqrt{\nu_\mathcal{S}(1)} > \sqrt{\nu_\mathcal{S}(x)}$ for some $x \in X$, then $(\mu_\mathcal{S}(1))^2 < t_x \leq (\mu_\mathcal{S}(x))^2$ or $\sqrt{\nu_\mathcal{S}(1)} > s_x \geq \sqrt{\nu_\mathcal{S}(x)}$ for some $(t_x, s_x) \in [0, 1] \times [0, 1]$. It follows that $x \in \mu_\mathcal{S}(\geq t_x)$ or $x \in \nu_\mathcal{S}(\leq s_x)$ but $1 \notin \mu_\mathcal{S}(\geq t_x) \cap \nu_\mathcal{S}(\leq s_x)$. This is a contradiction, and so $(\mu_\mathcal{S}(1))^2 \geq (\mu_\mathcal{S}(x))^2$ and $\sqrt{\nu_\mathcal{S}(1)} \leq \sqrt{\nu_\mathcal{S}(x)}$ for all $x \in X$. Assume that $(\mu_\mathcal{S}(y))^2 < \min\left\{(\mu_\mathcal{S}(x))^2, (\mu_\mathcal{S}(x \to y))^2\right\}$ or

$$\sqrt{\nu_\mathcal{S}(\mathfrak{b})} > \max\left\{\sqrt{\nu_\mathcal{S}(\mathfrak{a})}, \sqrt{\nu_\mathcal{S}(\mathfrak{a} \to \mathfrak{b})}\right\}$$

for some $x, y, \mathfrak{a}, \mathfrak{b} \in X$. Then $(\mu_\mathcal{S}(y))^2 < t_y \leq \min\left\{(\mu_\mathcal{S}(x))^2, (\mu_\mathcal{S}(x \to y))^2\right\}$ or

$$\sqrt{\nu_\mathcal{S}(\mathfrak{b})} > s_\mathfrak{b} \geq \max\left\{\sqrt{\nu_\mathcal{S}(\mathfrak{a})}, \sqrt{\nu_\mathcal{S}(\mathfrak{a} \to \mathfrak{b})}\right\}$$

for some $(t_y, s_\mathfrak{b}) \in [0, 1] \times [0, 1]$. It follows that Let $x, x \to y \in \mu_\mathcal{S}(\geq t_y)$ or $\mathfrak{a}, \mathfrak{a} \to \mathfrak{b} \in \nu_\mathcal{S}(\leq s_\mathfrak{b})$. Since $\mu_\mathcal{S}(\geq t_y)$ and $\nu_\mathcal{S}(\leq s_\mathfrak{b})$ are deductive systems of $(X, \wedge, \sim, 1)$, we have $y \in \mu_\mathcal{S}(\geq t_y)$ and $\mathfrak{b} \in \nu_\mathcal{S}(\leq s_\mathfrak{b})$. Then $(\mu_\mathcal{S}(y))^2 \geq t_y$ and $\sqrt{\nu_\mathcal{S}(\mathfrak{b})} \leq s_\mathfrak{b}$, which is a contradiction. Thus $(\mu_\mathcal{S}(y))^2 \geq \min\left\{(\mu_\mathcal{S}(x))^2, (\mu_\mathcal{S}(x \to y))^2\right\}$ and $\sqrt{\nu_\mathcal{S}(y)} \leq \max\left\{\sqrt{\nu_\mathcal{S}(x)}, \sqrt{\nu_\mathcal{S}(x \to y)}\right\}$ for all $x, y \in X$. It follows from Theorem 4.5 that $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$. □

Corollary 4.15. *Let $(X, \wedge, \sim, 1)$ be an equality algebra. If an SR-fuzzy set $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ in X is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$, then the sets $X_{\mu_\mathcal{S}} := \{x \in X \mid (\mu_\mathcal{S}(x))^2 = (\mu_\mathcal{S}(1))^2\}$ and $X_{\nu_\mathcal{S}} := \{x \in X \mid \sqrt{\nu_\mathcal{S}(x)} = \sqrt{\nu_\mathcal{S}(1)}\}$ are deductive systems of $(X, \wedge, \sim, 1)$.*

Theorem 4.16. *Let $(X, \wedge, \sim, 1)$ be an equality algebra. For a subset D of X, let $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ be an SR-fuzzy set in X which is defined as follows:*

$$(\mu_\mathcal{S}(x))^2 := \begin{cases} t \in (0, 1] & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases} \qquad \sqrt{\nu_\mathcal{S}(x)} := \begin{cases} s \in [0, 1) & \text{if } x \in D, \\ 1 & \text{otherwise.} \end{cases}$$

Then D is a deductive system of $(X, \wedge, \sim, 1)$ if and only if $\mathcal{S} := (\mu_\mathcal{S}, \nu_\mathcal{S})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$.

Proof. Suppose D is a deductive system of $(X, \wedge, \sim, 1)$. Since $1 \in D$, it is clear that $(\mu_\mathcal{S}(1))^2 = t \geq (\mu_\mathcal{S}(x))^2$ and $\sqrt{\nu_\mathcal{S}(1)} = s \leq \sqrt{\nu_\mathcal{S}(x)}$ for all $x \in X$. Let $x, y \in X$. If

$x \in D$ and $x \rightarrow y \in D$, then $y \in D$ and so

$$(\mu_{\mathcal{S}}(y))^2 = t = \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \rightarrow y))^2\right\}$$

and $\sqrt{\nu_{\mathcal{S}}(y)} = s = \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \rightarrow y)}\right\}$. If $x \notin D$ or $x \rightarrow y \notin D$, then $(\mu_{\mathcal{S}}(x))^2 = 0$ and $\sqrt{\nu_{\mathcal{S}}(x)} = 1$, or $(\mu_{\mathcal{S}}(x \rightarrow y))^2 = 0$ and $\sqrt{\nu_{\mathcal{S}}(x \rightarrow y)} = 1$. Thus $(\mu_{\mathcal{S}}(y))^2 \geq 0 = \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \rightarrow y))^2\right\}$ and

$$\sqrt{\nu_{\mathcal{S}}(y)} \leq 1 = \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \rightarrow y)}\right\}.$$

So $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$ by Theorem 4.5.

Conversely, suppose $\mathcal{S} := (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ is an SR-fuzzy deductive system of $(X, \wedge, \sim, 1)$. Then $\mu_{\mathcal{S}}(\geq t)$ and $\nu_{\mathcal{S}}(\leq s)$ are nonempty deductive systems of $(X, \wedge, \sim, 1)$ (See Theorem 4.14). For every $x \in \mu_{\mathcal{S}}(\geq t)$ and $\mathfrak{a} \in \nu_{\mathcal{S}}(\leq s)$, we have

$$(\mu_{\mathcal{S}}(1))^2 \geq (\mu_{\mathcal{S}}(x))^2 \geq t$$

and $\sqrt{\nu_{\mathcal{S}}(1)} \leq \sqrt{\nu_{\mathcal{S}}(\mathfrak{a})} \leq s$. Thus $1 \in D$. Let $x, y \in X$ be such that $x \in D$ and $x \rightarrow y \in D$. Then $(\mu_{\mathcal{S}}(x))^2 = (\mu_{\mathcal{S}}(x \rightarrow y))^2 = t$ and $\sqrt{\nu_{\mathcal{S}}(x)} = \sqrt{\nu_{\mathcal{S}}(x \rightarrow y)} = s$. Then $(\mu_{\mathcal{S}}(y))^2 \geq \min\left\{(\mu_{\mathcal{S}}(x))^2, (\mu_{\mathcal{S}}(x \rightarrow y))^2\right\} = t$ and

$$\sqrt{\nu_{\mathcal{S}}(y)} \leq \max\left\{\sqrt{\nu_{\mathcal{S}}(x)}, \sqrt{\nu_{\mathcal{S}}(x \rightarrow y)}\right\} = s.$$

It follows that $(\mu_{\mathcal{S}}(y))^2 = t$ and $\sqrt{\nu_{\mathcal{S}}(y)} = s$ which shows that $y \in D$. Thus D is a deductive system of $(X, \wedge, \sim, 1)$ by Lemma 2.3. □

5. Conclusion and future work

Intuitionistic fuzzy sets are a mathematical model that extends the concept of a fuzzy set by incorporating the idea of uncertainty or hesitation. In 2022, Al-shami et al. [1] presented a novel extension of intuitionistic fuzzy set called SR-fuzzy sets which is not obtained from q-rung orthopair fuzzy sets. And then, they compared it with the other types of fuzzy sets. Also, they introduced the concepts of weighted aggregated operators for SR-fuzzy sets, and described MADM problems under these operators. In this paper, we applied the SR-fuzzy set introduced by Al-shami et al. to the equality algebras. We looked at various properties, including interrelationships and characterizations, while discussing SR-fuzzification of subalgebras and deductive systems, which are substructures of equality algebra. In the future, we will study SR-fuzzification of substructures in various types of logical algebras, using the ideas or the results covered in this paper.

Acknowledgments

The author would like to thank the anonymous reviewers for their valuable suggestions.

References

[1] T. M. Al-shami, H. Z. Ibrahim, A. A. Azzam and A. I. EL-Maghrabi, SR-fuzzy sets and their weighted aggregated operators in application to decision-making, Journal of Function Spaces 2022 (2022) Article ID 3653225 14 pages. https://doi.org/10.1155/2022/3653225

[2] S. Jenei, Equality algebras, Studing Logics 100 (2012) 1201–1209.

[3] M. Aaly Kologani, X. L. Xin, R. A. Borzooei and Y. B. Jun, Positive implicative equality algebras and equality algebras with some types, Journal of Algebraic Hyperstructures and Logical Algebras 3 (2) (2022) 69–86. https://doi.org/10.52547/HATEF.JAHLA.3.2.6

[4] R. A. Borzooei, F. Zebardast and M. Aaly Kologani, Some types of filters in equality algebras, Categories and General Algebraic Structures with Applications 7 (2017) 33–55.

[5] X. Y. Yu, X. L. Xin and J. T. Wang, Fuzzy filters on equality algebras with applications, Journal of Intelligent & Fuzzy Systems 35 (2018) 3709–3719. DOI:10.3233/JIFS-18509

[6] F. Zebardast, R. A. Borzooei and M. Aaly Kologani, Results on equality algebras, Inform. Sci. 381 (2017) 270–282.

[7] S. Jenei and Kóródi, On the variety of equality algebras, Proceedings of the 7th conference 221 of the European Society for Fuzzy Logic and Technology (EUSFLAT-11) (2011) 153–155. DOI:10.2991/eusflat.2011.1

[8] S. Jenei, Equality algebras, Proceeding of CINTI 2010 conference-11th IEEE International Symposium on Computational Intelligence and Informatics, Budapest, Hungary 2010.

[9] K. T. Atanassov, Intuitionistic Fuzzy Sets-Theory and Applications, Springe Verlag, New York 1999.

[10] G. R. Rezaei, R. A. Borzooei, M. Aaly Kologhani and Y. B. Jun, Roughness of filters in equality algebras, Bulletin of the Section of Logic 52 (1) (2023) 1–18. https://doi.org/10.18778/0138-0680.2023.01

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea