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Generalization of fuzzy ideals of semirings and metatheorem

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Abstract. We investigate fuzzy bi-interior ideals, fuzzy bi-quasi ideals, fuzzy bi-quasi-interior ideals and fuzzy quasi-interior ideals of a semiring as a generalization of fuzzy bi-ideals, fuzzy interior ideals and fuzzy quasi-ideals in a semiring by utilizing metatheorem formulated by Tom Head. We studied properties of fuzzy bi-interior ideals, fuzzy bi-quasi ideals, fuzzy bi-quasi-interior ideals and fuzzy quasi-interior ideals of a semiring, simple semiring and regular semiring. The classes of different types of fuzzy ideals are shown to be projection closed. Tom Head's metatheorem is used to provide proofs of several results pertaining to these fuzzy ideals which doesn't involve calculations.

2020 AMS Classification: 39B82, 39B52, 39B55, 81Q05

Keywords: Semiring, Fuzzy bi-interior ideals, Fuzzy bi-quasi ideals, Fuzzy quasi-interior ideals, Metatheorem, Projection closed.

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1. Introduction

Semiring is an algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by Vandiver [1] in 1934. Generalization of ideals of algebraic structures plays a very remarkable role and also necessary for further advance studies. Notably, the concept of a one-sided ideal, present in various algebraic structures, represents a generalization of the notion of an ideal. Further generalizations were established, where quasi-ideals are a generalization of left and right ideals and bi-ideals are generalization of quasi-ideals. During the period 1980-2016, no new generalization of ideals of algebraic structures can be seen. Then, the author in [2, 3, 4, 5] introduced and studied bi-quasi ideals, bi-interior ideals, bi-quasi-interior ideals and quasi-interior ideals of a semirings as a generalization of bi-ideals, quasi-ideals and interior ideals. Iséki [6] introduced quasi-ideals for semirings for semirings without zero. Good and Hughes [7] first introduced bi-ideals for semigroups, while Lajos and Szasz [8] did so for rings.

In the year 1995, Head [9] proposed his metatheorem, providing a framework for investigating the intrinsic behaviour of the fuzzy algebraic structures. Soon this metathoerem approach was adopted by [10, 11, 12] in fuzzy algebra.

In the present paper, we have thrown further light on the notions of fuzzy bi-interior ideals, fuzzy bi-quasi ideals, fuzzy bi-quasi-interior ideals and fuzzy quasi-interior ideals as a generalization of fuzzy bi-ideals, fuzzy interior ideals and fuzzy quasi-ideals in a semiring. Characterization of a semiring that is regular in terms of fuzzy left(right, two sided) ideals, fuzzy bi-interior ideals, fuzzy bi-quasi ideals, fuzzy bi-quasi-interior ideals and fuzzy quasi-interior ideals of S is provided by using metatheorem. Metatheorem is employed to offer alternative proofs for a range of results concerning these fuzzy ideals, avoiding the need for extensive calculations. The primary objective of this metatheorem was to derive fuzzy versions of classical outcomes. In essence, metatheorem establishes a straightforward conceptual framework for transferring results from crisp settings to fuzzy contexts, whenever such a transition is feasible. The visible lucidity and brevity of our proofs in comparison to the already existing proofs of some of these results clearly establishes the metatheorem approach as an effective tool for studying the fuzzy algebraic structures.

2. Preliminaries

Before coming to next section, we recall that a fuzzy subset of a non empty set X is a function from $f : X \to [0, 1]$.

We recall the following definitions for subsequent use.

Definition 2.1 ([5, 12])**.** A fuzzy set f of a semiring S is called a:

(i) *fuzzy subsemiring* of S, if $f(x + y) \geq \min\{f(x), f(y)\}$ and $f(xy) \geq \min\{f(x), f(y)\}\ \forall\ x,\ y \in S$,

(ii) *fuzzy left* (*right*) ideal of S, if $f(x + y) \geq \min\{f(x), f(y)\}$ and $f(xy) \geq f(y)(f(xy) \geq f(x))\ \forall\ x,\ y \in S$,

(iii) *fuzzy ideal* of S, if $f(x + y) \geq \min\{f(x), f(y)\}$, $f(xy) \geq f(y)$ and $f(xy) \geq f(x)\ \forall\ x, y \in S$,

(iv) *fuzzy bi-ideal* of S, if f is a fuzzy subsemiring of S and $f(xyz) \geq \min\{f(x), f(z)\}\ \forall\ x,\ y,\ z \in S$,

(v) *fuzzy interior-ideal* of S, if f is a fuzzy subsemiring of S and $f(xyz) \geq f(y)\ \forall\ x,\ y,\ z \in S$,

(vi) *fuzzy quasi-ideal* of S, if f is a fuzzy subsemiring of S and $f(z) \geq \min\{\sup\limits_{z=xy} f(x), \sup\limits_{z=xy} f(y)\}\ \forall\ z \in S$,

(vii) *fuzzy bi-interior ideal* of S, if f is a fuzzy subsemiring of S and $(S \circ f \circ S) \cap (f \circ S \circ f) \subseteq f$,

(viii) *fuzzy left bi-quasi ideal* of S, if f is a fuzzy subsemiring of S and $(S \circ f) \cap (f \circ S \circ f) \subseteq f$

(ix) *fuzzy right bi-quasi ideal* of S, if f is a fuzzy subsemiring of S and $(f \circ S) \cap (f \circ S \circ f) \subseteq f$,

(x) *fuzzy bi-quasi ideal* of S, if f is a fuzzy subsemiring of S and $(S\circ f)\cap(f\circ S\circ f)\subseteq f$ and $(f\circ S)\cap(f\circ S\circ f)\subseteq f$,

(xi) *fuzzy left quasi-interior ideal* of S, if f is a subsemiring of S and $S\circ f\circ S\circ f\subseteq f$,

(xi) *fuzzy right quasi-interior ideal* of S, if f is a subsemiring of S and $f\circ S\circ f\circ S\subseteq f$,

(xii) *fuzzy quasi-interior ideal* of S, if f is a subsemiring of S and $S\circ f\circ S\circ f\subseteq f$ and $f\circ S\circ f\circ S\subseteq f$,

(xiii) *fuzzy bi-quasi-interior ideal* of S, if f is a fuzzy subsemiring of S and $f\circ S\circ f\circ S\circ f\subseteq f$.

Let S be a semiring. Let $P(S)$, $C(S)$ and $F(S)$ denote respectively the set of all subsets of S, the set of all characteristic functions of S and the set of all fuzzy sets of S. It is well known that the mapping $\mathrm{Chi}: P(S)\to C(S)$ defined by $\mathrm{Chi}(A)=\chi_A$ is a bijection and provides an isomorphism of complete lattices.

Definition 2.2 ([9])**.** Let S be a semiring and $J=[0,1)$. The representation function $\mathrm{Rep}: F(S)\to C(S)^J$ is defined by

$$\mathrm{Rep}(f)(r)(x)=\begin{cases}1 & \text{if } f(x)>r\\ 0 & \text{if } f(x)\leq r,\end{cases}$$

where $f\in F(S)$ and $r\in J$.

Proposition 2.3 ([9])**.** Rep *is an injective function, commutes with infs of finite sets of fuzzy sets and with sups of arbitrary sets of fuzzy sets. Morover,* Rep *is an order isomorphism from $F(S)$ onto $I(S)$, where let $I(S)$ be the image of the* Rep *function.*

We define binary operations $+, o: F(S)\times F(S)\to F(S)$ as follows:

$$(f_1+f_2)(x)=\begin{cases}\sup\limits_{x=x_1+x_2}[\min\{f_1(x_1),f_2(x_2)\}] & \\ 0 & \text{if } x \text{ not expressed as } x=x_1+x_2\end{cases}$$

and

$$(f_1 o f_2)(x)=\begin{cases}\sup\limits_{x=x_1 o x_2}[\min\{f_1(x_1),f_2(x_2)\}] & \\ 0, & \text{if } x \text{ not expressed as } x=x_1 o x_2\end{cases}$$

The above binary operations '+' and 'o' on $F(S)$ are called by Head [9], the convolutional extension of the binary operations '+' and 'o' on S. Interestingly the convolutional extensions '+' and 'o' on $F(S)$ coincide with the sum and product of fuzzy sets introduced by Liu [13] in a groupoid or in any other algebraic system.

Proposition 2.4 ([9])**.** *For the binary operations '+' and 'o' on a semiring S, $C(S)$ is closed with respect to the convolutional extensions of '+' and 'o' to $F(S)$. Moreover,* $\mathrm{Chi}: P(S)\to C(S)$ *commutes with '+' and 'o' operations on $P(S)$ and $C(S)$.*

Proposition 2.5 ([9])**.** *For the binary operations + and o on a semiring S, the representation function* $\mathrm{Rep}: F(S)\to C(S)^J$ *commutes with the operations of '+' and 'o'.*

Definition 2.6 ([9])**.** Let $\mathcal{C}$ be a class of fuzzy sets in a semiring S. We say that $\mathcal{C}$ *is closed under projection*, if for each $f \in \mathcal{C}$ and $r \in J$, $\text{Rep}(f)(r) \in \mathcal{C}$.

Proposition 2.7 ([9])**.** *Let C, D be the classes of crisp subsets of a semiring S and $\mathcal{C}$, $\mathcal{D}$ be their corresponding fuzzy classes which are projection closed. Then*
(1) *$\mathcal{C} \subseteq \mathcal{D}$ if and only if $C \subseteq D$,*
(2) *$\mathcal{C} = \mathcal{D}$ if and only if $C = D$.*

Metatheorem 2.8 ([9])**.** *Let S be a semiring with binary operations '$+$' and '$\cdot$'. Consider the* $(\inf, \sup, o)$*-algebra $F(S)$. Let $Q(u_1, u_2, \cdots, u_m)$ and $R(u_1, u_2, \cdots, u_m)$ be two expressions on $P(S)$ over the variables set $\{u_1, u_2, \cdots, u_m\}$ and operations set* $\{\inf, \sup, +, \cdot\}$*. Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m$ be the classes of fuzzy sets of S that are projection closed. Then the inequality*

$$Q(f_1, f_2, \cdots, f_m) \text{ REL } R(f_1, f_2, \cdots, f_m)$$

holds for all fuzzy sets f_1 in $\mathcal{C}_1, \cdots, f_m$ in $\mathcal{C}_m$ if and only if it holds for all crisp sets f_1 in $\mathcal{C}_1, \cdots, f_m$ in $\mathcal{C}_m$ where REL *is anyone of the three relations $\leq$, $=$ or $\geq$.*

3. Metatheorem and generalized fuzzy ideals in semirings

Theorem 3.1 ([12])**.** *The classes $\mathcal{C}_{ss}$, $\mathcal{C}_l$ $(\mathcal{C}_r, \mathcal{C}_i, \mathcal{C}_b, \mathcal{C}_{in}, \mathcal{C}_q)$ of all fuzzy subsemirings, fuzzy left (right, two sided, bi-, interior, quasi-) ideals of a semiring S are projection closed.*

Theorem 3.2. *$\mathcal{C}_{bi}$, the class of all fuzzy bi-interior ideals of a semiring S is closed under projection.*

Proof. Let $f \in \mathcal{C}_{bi}$. Then f is fuzzy subsemiring of S and $(S \circ f \circ S) \cap (f \circ S \circ f) \subseteq f$. $\mathcal{C}_{ss}$ is closed under projection by Theorem 3.1. Now for $r \in J$, we have

$$\begin{aligned}
\text{Rep}(f)(r) &\geq \text{Rep}(S \circ f \circ S \cap f \circ S \circ f)(r) \\
&= \text{Rep}(S \circ f \circ S)(r) \cap \text{Rep}(f \circ S \circ f)(r) \\
&= \text{Rep}(S)(r) \circ \text{Rep}(f)(r) \circ \text{Rep}(S)(r) \cap \text{Rep}(f)(r) \circ \text{Rep}(S)(r) \circ \text{Rep}(f)(r) \\
&= S \circ \text{Rep}(f)(r) \circ S \cap \text{Rep}(f)(r) \circ S \circ \text{Rep}(f)(r).
\end{aligned}$$

Thus $\text{Rep}(f)(r) \in \mathcal{C}_{bi}$. □

Theorem 3.3. *$\mathcal{C}_{bql}$, the class of all fuzzy left bi-quasi ideals of a semiring S is closed under projection.*

Proof. Let $f \in \mathcal{C}_{bql}$. Then f is fuzzy subsemiring of S and $(S \circ f) \cap (f \circ S \circ f) \subseteq f$. $\mathcal{C}_{ss}$ is closed under projection by Theorem 3.1. Now for $r \in J$, we have

$$\begin{aligned}
\text{Rep}(f)(r) &\geq \text{Rep}(S \circ f \cap f \circ S \circ f)(r) \\
&= \text{Rep}(S \circ f)(r) \cap \text{Rep}(f \circ S \circ f)(r) \\
&= \text{Rep}(S)(r) \circ \text{Rep}(f)(r) \cap \text{Rep}(f)(r) \circ \text{Rep}(S)(r) \circ \text{Rep}(f)(r) \\
&= S \circ \text{Rep}(f)(r) \cap \text{Rep}(f)(r) \circ S \circ \text{Rep}(f)(r).
\end{aligned}$$

Thus $\text{Rep}(f)(r) \in \mathcal{C}_{bql}$. □

Theorem 3.4. *$\mathcal{C}_{bqr}$, the class of all fuzzy right bi-quasi ideals of a semiring S is closed under projection.*

Proof. Let $f \in \mathcal{C}_{bqr}$. Then f is fuzzy subsemiring of S and $(f \circ S) \cap (f \circ S \circ f) \subseteq f$. $\mathcal{C}_{ss}$ is closed under projection by Theorem 3.1. Now for $r \in J$, we have

$$\begin{aligned}\mathrm{Rep}(f)(r) &\geq \mathrm{Rep}(f \circ S \cap f \circ S \circ f)(r)\\ &= \mathrm{Rep}(f \circ S)(r) \cap \mathrm{Rep}(f \circ S \circ f)(r)\\ &= \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(S)(r) \cap \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(f)(r)\\ &= \mathrm{Rep}(f)(r) \circ S \cap \mathrm{Rep}(f)(r) \circ S \circ \mathrm{Rep}(f)(r).\end{aligned}$$

Thus, $\mathrm{Rep}(f)(r) \in \mathcal{C}_{bqr}$. □

Theorem 3.5. *$\mathcal{C}_{bq}$, the class of all fuzzy bi-quasi ideals of a semiring S is closed under projection.*

Theorem 3.6. *$\mathcal{C}_{qil}$, the class of all fuzzy left quasi-interior ideals of a semiring S is closed under projection.*

Proof. Let $f \in \mathcal{C}_{qil}$. Then f is fuzzy subsemiring of S and $S \circ f \circ S \circ f \subseteq f$. $\mathcal{C}_{ss}$ is closed under projection by Theorem 3.1. Now for $r \in J$, we have

$$\begin{aligned}\mathrm{Rep}(f)(r) &\geq \mathrm{Rep}(S \circ f \circ S \circ f)(r)\\ &= \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(f)(r)\\ &= S \circ \mathrm{Rep}(f)(r) \circ S \circ \mathrm{Rep}(f)(r).\end{aligned}$$

Thus $\mathrm{Rep}(f)(r) \in \mathcal{C}_{qil}$. □

Theorem 3.7. *$\mathcal{C}_{qir}$, the class of all fuzzy right quasi-interior ideals of a semiring S is closed under projection.*

Proof. Let $f \in \mathcal{C}_{qir}$. Then f is fuzzy subsemiring of S and $f \circ S \circ f \circ S \subseteq f$. $\mathcal{C}_{ss}$ is closed under projection by Theorem 3.1. Now for $r \in J$, we have

$$\begin{aligned}\mathrm{Rep}(f)(r) &\geq \mathrm{Rep}(f \circ S \circ f \circ S)(r)\\ &= \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(S)(r)\\ &= \mathrm{Rep}(f)(r) \circ S \cap \mathrm{Rep}(f)(r) \circ S \circ \mathrm{Rep}(f)(r).\end{aligned}$$

Thus $\mathrm{Rep}(f)(r) \in \mathcal{C}_{qir}$. □

Theorem 3.8. *$\mathcal{C}_{qi}$, the class of all fuzzy quasi-interior ideals of a semiring S is closed under projection.*

Theorem 3.9. *$\mathcal{C}_{bqi}$, the class of all fuzzy bi-quasi-interior ideals of a semiring S, is closed under projection.*

Proof. Let $f \in \mathcal{C}_{bqi}$. Then f is fuzzy subsemiring of S and $f \circ S \circ f \circ S \circ f \subseteq f$. $\mathcal{C}_{ss}$ is closed under projection by Theorem 3.1. Now for $r \in J$, we have

$$\begin{aligned}Rep(f)(r) &\geq Rep(f \circ S \circ f \circ S \circ f)(r)\\ &= \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(f)(r)\\ &= \mathrm{Rep}(f)(r) \circ S \cap \mathrm{Rep}(f)(r) \circ S \circ \mathrm{Rep}(f)(r).\end{aligned}$$

Thus $\mathrm{Rep}(f)(r) \in \mathcal{C}_{bqi}$. □

Theorem 3.10. *The following statements hold in a semiring S:*
(1) *every fuzzy left (right) ideal of S is a fuzzy bi-quasi ideal of S,*
(2) *every fuzzy two sided ideal of S is a fuzzy bi-quasi ideal of S,*
(3) *every fuzzy bi-ideal (quasi-ideal) of S is a fuzzy bi-quasi ideal of S.*

Proof. We establish (2). By Theorem 3.1 and 3.5, both the classes $\mathcal{C}_i$ and $\mathcal{C}_{bq}$ of fuzzy two sided ideals and fuzzy bi-quasi ideals are closed under projection. Then by Proposition 2.7, $\mathcal{C}_i \subseteq \mathcal{C}_{bq}$ if and only if $C_i \subseteq C_{bq}$, where C_i and C_{bq} are classes of all crisp two sided ideals and crisp bi-quasi ideals of S respectively. Thus by Corollary 3.1 of [2], every two sided ideal of a semiring is a bi-quasi ideal and Chi is an isomorphism from $P(S)$ to $C(S)$. So $C_i \subseteq C_{bq}$. Hence $\mathcal{C}_i \subseteq \mathcal{C}_{bq}$. □

Others are extension of Theorem 3.2 to 3.5 of [2] in fuzzy setting and can be proved similarly.

Theorem 3.11. *Every fuzzy left bi-quasi ideals of a regular semiring S is a fuzzy two sided ideal of S.*

Proof. By Theorem 3.3 and 3.1, both the classes $\mathcal{C}_{bql}$ and $\mathcal{C}_i$ of fuzzy fuzzy left bi-quasi ideals and fuzzy two sided ideals of a semiring S are closed under projection. Then by Proposition 2.7, $\mathcal{C}_{bql} \subseteq \mathcal{C}_i$ if and only if $C_{bql} \subseteq C_i$, where C_{bql} and C_i are classes of all crisp left bi-quasi ideals and crisp two sided ideals of S respectively. Thus by Theorem 3.18 of [2], every left bi-quasi ideals of a regular semiring is a two sided ideal and Chi is an isomorphism from $P(S)$ to $C(S)$. So we get $C_{bql} \subseteq C_i$. Hence $\mathcal{C}_{bql} \subseteq \mathcal{C}_i$. □

The following results are extension of Theorem 3.1, 3.2, 3.11 and Corollary 3.6 of [5] in fuzzy setting and can be proved similarly.

Theorem 3.12. *The following statements hold in a semiring S.*
(1) *Every fuzzy right $(left)$ ideal of a semiring S is a fuzzy right $(left)$ bi-quasi ideal of S.*
(2) *Let S is regular semiring. Then, f is a fuzzy left (right) bi-quasi ideal of S if and only if f is a fuzzy quasi-ideal of S.*
(3) *Let S is regular semiring. Then, f is a fuzzy bi-quasi ideal of S if and only if f is a fuzzy quasi-ideal of S.*

Theorem 3.13. *A semiring S is regular semiring if and only if $f = S \circ f \cap f \circ S \circ f$ $\forall\ f \in \mathcal{C}_{bql}$.*

Proof. Consider the $(\inf, \sup, o, +)$-algebra $F(S)$. Let $D(u) = Su \cap uSu$ and $E(u) = u$ be two expressions over the set of variable $\{u\}$ and set of operations $(\inf, \sup, \cdot, +)$. By Theorem 3.3, the class $\mathcal{C}_{bql}$ of fuzzy left bi-quasi ideals of S is projection closed. Then by Metatheorem, $D(f) = E(f)\ \forall\ f \in \mathcal{C}_{bql}$ if and only if $D(f) = E(f)\ \forall\ f \in C_{bql}$, where C_{bql} is the class of all crisp left bi-quasi ideals of S.

By Theorem 3.15 of [2], a semiring S is regular if and only if $SB \cap BSB = B$ for every left bi-quasi ideal B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $SB \cap BSB = B$ if and only if $\chi_{SB \cap BSB} = \chi_B$. Moreover, Chi commutes with the finite intersection and product of sets. Thus S is regular if and only if $\chi_{SB} \cap \chi_{BSB} = \chi_B$ for every left bi-quasi ideal B of S. That is, S is regular if and only if $\chi_S\chi_B \cap \chi_B\chi_S\chi_B = \chi_B$ for every left bi-quasi ideal B of S. That is, S is regular if and only if $S\chi_B \cap \chi_B S\chi_B = \chi_B$ for every left bi-quasi ideal B of S. That is, S is regular if and only if $D(f) = E(f)\ \forall\ f \in C_{bql}$. So S is regular if and only if $D(f) = E(f)\ \forall\ f \in \mathcal{C}_{bql}$. □

Similarly we can prove.

Theorem 3.14. *A semiring S is regular if and only if $f = f \circ S \cap f \circ S \circ f\ \forall\ f \in \mathcal{C}_{bqr}$.*

Theorem 3.15. *The following statements hold in a semiring S.*
(1) *If $f \in \mathcal{C}_l$ and $g \in \mathcal{C}_r$ then $f \cap g \in \mathcal{C}_{bq}$.*
(2) *If $f, g \in \mathcal{C}_{bql}(\mathcal{C}_{bqr})$ then $f \cap g \in \mathcal{C}_{bql}(\mathcal{C}_{bqr})$.*
(3) *If $f \in \mathcal{C}_l$ and $g \in \mathcal{C}_r$ then $f \cap g \in \mathcal{C}_{bqr}$.*

Proof. We establish (1). Others follow similarly. We define the classes $C_{l,r} = \{f_1 \cap f_2 : f_1 \in C_l,\ f_2 \in C_r\}$ and $\mathcal{C}_{l,r} = \{f \cap g : f \in \mathcal{C}_l,\ g \in \mathcal{C}_r\}$, where C_l and C_r are classes of all crisp left ideals and crisp right ideals of S respectively. First we show that $\mathcal{C}_{l,r}$ is closed under projection. Let $f \cap g \in \mathcal{C}_{l,r}$. Since Rep commutes with infs of finite sets of fuzzy subsets (Proposition 2.3), we have for all $f \in \mathcal{C}_r, g \in \mathcal{C}_l$, $\mathrm{Rep}(f \cap g)(r) = \mathrm{Rep}(f)(r) \cap \mathrm{Rep}(g)(r)\ \forall\ r \in J$. Since the classes $\mathcal{C}_l$ and $\mathcal{C}_r$ are closed under projection, we have $\mathrm{Rep}(f)(r) \in \mathcal{C}_l$ and $Rep(g)(r) \in \mathcal{C}_r$. Then $\mathrm{Rep}(f \cap g)(r) \in \mathcal{C}_{l,r}\ \forall\ r \in J$. Thus $\mathcal{C}_{l,r}$ is closed under projection. Also by Theorem 3.5, $\mathcal{C}_{bq}$, the class of all fuzzy bi-quasi ideals of S is closed under projection. By Corollary 3.2 of [2], the intersection of a left ideals and a right ideal of S is a bi-quasi ideal and Chi is an isomorphism from $P(R)$ to $C(R)$. So we have $C_{l,r} \subseteq C_{bq}$. Hence by Proposition 2.7, $\mathcal{C}_{l,r} \subseteq \mathcal{C}_{bq}$. □

Theorem 3.16. *A semiring S is regular semiring if and only if $f \cap g \subseteq g \circ f \cap f \circ g \circ f\ \forall\ f \in \mathcal{C}_{bql}$ and $\forall\ f \in \mathcal{C}_i$.*

Proof. Consider the $(\inf, \sup, o, +)$-algebra $F(S)$. Let $D(u, v) = u \cap v$ and $E(u, v) = vu \cap uvu$ be two expressions over the set of variables $\{u, v\}$ and set of operations $(\inf, \sup, \cdot, +)$. By Theorem 3.3 and 3.1, both the classes $\mathcal{C}_{bql}$ and $\mathcal{C}_i$ of fuzzy left bi-quasi ideals and fuzzy two sided ideals of S are projection closed. Then by Metatheorem, $D(f, g) \subseteq E(f, g)\ \forall\ f \in \mathcal{C}_{bql}$ and $\forall\ f \in \mathcal{C}_i$ if and only if $D(f, g) \subseteq E(f, g)\ \forall\ f \in C_{bql}$ and $\forall\ f \in C_i$, where C_{bql} and C_i are classes of all crisp left bi-quasi ideals and crisp two sided ideals S.

By Theorem 3.14 of [5], a semiring S is regular if and only if $A \cap B \subseteq BA \cap ABA$ for every left bi-quasi ideal A and every two-sided ideal B of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $A \cap B \subseteq BA \cap ABA$ if and only if $\chi_{A \cap B} \subseteq \chi_{BA \cap ABA}$. Moreover, Chi commutes with the finite intersection and product of sets. Thus S is regular if and only if $\chi_A \cap \chi_B \subseteq \chi_B\chi_A \cap \chi_A\chi_B\chi_A$ for every left bi-quasi ideal A and every two-sided ideal B of S. So S is regular if and only if $D(f, g) \subseteq E(f, g)\ \forall\ f \in C_{bql}$ and $\forall\ f \in C_i$. Hence S is regular if and only if $D(f, g) \subseteq E(f, g)\ \forall\ f \in \mathcal{C}_{bql}$ and $\forall\ f \in \mathcal{C}_i$. □

Similarly we can prove.

Corollary 3.17. *A semiring S is a regular if and only if $f \cap g \subseteq f \circ g \cap f \circ g \circ f\ \forall\ f \in \mathcal{C}_{bqr}$ and $\forall\ f \in \mathcal{C}_i$.*

Theorem 3.18. *The following statements hold in a semiring S.*

(1) *Every fuzzy left (right) quasi-interior ideal of S is a fuzzy bi-interior ideal of S.*

(2) *Every fuzzy quasi-interior ideal of S is a fuzzy bi-interior ideal of S.*

Proof. We establish (2). By Theorem 3.8 and 3.2, both the classes $\mathcal{C}_{qi}$ and $\mathcal{C}_{bi}$ of fuzzy quasi-interior ideals and fuzzy bi-interior ideals are closed under projection. Then by Proposition 2.7, $\mathcal{C}_{qi} \subseteq \mathcal{C}_{bi}$ if and only if $C_{qi} \subseteq C_{bi}$, where C_{qi} and C_{bi} are classes of all crisp quasi-interior ideals and crisp bi-interior ideals of S respectively. Thus by Corollary 3.4 of [4], every quasi-interior ideal of a semiring S is a bi-interior ideal and Chi is an isomorphism from $P(S)$ to $C(S)$. So we get $C_{qi} \subseteq C_{bi}$. Hence $\mathcal{C}_{qi} \subseteq \mathcal{C}_{bi}$.

Other results are extension of Theorem 3.3 and Corollary 3.3 of [4] in fuzzy setting and can be proved similarly. □

Theorem 3.19. *The following statements hold in a semiring S.*

(1) *Every fuzzy quasi-interior ideal of S is a fuzzy bi-quasi-interior ideal of S.*

(2) *Every fuzzy left ideal (right ideal, two sided ideal, interior ideai, bi-ideal, quasi-ideal) of S is a fuzzy bi-quasi-interior ideal of S.*

(3) *Every fuzzy left (right) quasi-interior ideal of S is a fuzzy bi-quasi-interior ideal of S.*

Proof. We establish (1). By Theorem 3.8 and 3.9, both the classes $\mathcal{C}_{qi}$ and $\mathcal{C}_{bqi}$ are closed under projection. Then by Proposition 2.7, $\mathcal{C}_{qi} \subseteq \mathcal{C}_{bqi}$ if and only if $C_{qi} \subseteq C_{bqi}$, where C_{qi} and C_{bqi} are classes of all crisp quasi-interior ideals and crisp bi-interior ideals of S respectively. Thus by Corollary 3.8 of [4], every quasi-interior ideal of a semiring S is a bi-quasi-interior ideal and Chi is an isomorphism from $P(S)$ to $C(S)$. So we get $C_{qi} \subseteq C_{bqi}$. Hence $\mathcal{C}_{qi} \subseteq \mathcal{C}_{bqi}$.

Other results are extension of Theorem 3.1, 3.5 and Corollary 3.7 of [4] in fuzzy setting and can be proved similarly. □

The following results are extension of Theorem 3.1, 3.2 and Corollary 3.1, 3.2 of [4] in fuzzy setting and can be proved similarly.

Theorem 3.20. *The following statements hold in a semiring S.*

(1) *Every fuzzy left (right) quasi-interior ideal of a semiring S is a fuzzy left (right) bi-quasi ideal of S.*

(2) *Every fuzzy quasi-interior ideal of S is a fuzzy bi-quasi ideal of S.*

(3) *Every fuzzy left (right) ideal of S is a fuzzy left (right) quasi-interior ideal of S.*

(4) *Every fuzzy two sided ideal (quasi-ideal) is a fuzzy quasi-interior ideal of S.*

The following results are extension of Theorem 3.4, 3.6, 4.2, 4.5, 4.8 and Corollary 3.5, 3.6, 4.3, 4.6, 4.7, 4.9, 4.10 of [4] in fuzzy setting and can be proved similarly.

Theorem 3.21. *The following statements hold in a semiring S.*

(1) *Every fuzzy left (right) quasi-interior ideal of S is a fuzzy bi-ideal of S.*

(2) *Every fuzzy quasi-interior ideal of S is a fuzzy bi-ideal of S.*

(3) *Every fuzzy interior ideal of S is a fuzzy left quasi-interior ideal of S.*

(4) *Every fuzzy left (right) quasi-interior ideal of a simple semiring S is a fuzzy left(right) ideal of S.*

(5) *Every fuzzy quasi-interior ideal of a simple semiring is a fuzzy ideal of S.*

(6) *Every fuzzy left (right) quasi-interior ideal of is a regular semiring S a fuzzy left(right) ideal of S.*

(7) *Every fuzzy quasi-interior ideal of a regular semiring S is a fuzzy ideal of S.*

(8) *Every fuzzy left (right) quasi-interior ideal of a left (right) simple semiring S is a fuzzy right (left)ideal of S.*

(9) *Every fuzzy left quasi-interior ideal of a left and right simple semiring S is a fuzzy ideal of S.*

Theorem 3.22. *A semriring S is a regular if and only if $f \cap g \cap h \subseteq f \circ g \circ h$ $\forall$ $f \in \mathcal{C}_{qil}$ $\forall$ $g \in \mathcal{C}_i$ and $\forall$ $h \in \mathcal{C}_l$.*

Proof. Consider the $(\inf, \sup, o, +)$-algebra $F(S)$. Let $D(u, v, w) = u \cap v \cap w$ and $E(u, v, w) = uvw$ be two expressions over the set of variables $\{u, v, w\}$ and set of operations $(\inf, \sup, \cdot, +)$. By Theorem 3.6 and 3.1, the classes $\mathcal{C}_{qil}$, $\mathcal{C}_i$ and $\mathcal{C}_l$ of fuzzy left quasi-interior ideals, fuzzy two sided ideals and fuzzy left ideals are projection closed. Then by Metatheorem, $D(f, g, h) \subseteq E(f, g, h)$ $\forall$ $f \in \mathcal{C}_{qil}$, $\forall$ $g \in \mathcal{C}_i$ and $\forall$ $h \in \mathcal{C}_l$ if and only if $D(f, g, h) \subseteq E(f, g, h)$ $\forall$ $f \in C_{qil}$, $\forall$ $g \in C_i$ and $\forall$ $h \in C_l$, where C_{qil}, C_i and C_l are classes of all crisp left quasi-interior ideals, crisp two sided ideals and crisp left ideals of S.

By Theorem 4.14 of [4], S is regular if and only if $A \cap B \cap C \subseteq ABC$ for every left quasi-interior ideal A, every two sided ideal B and every left ideal C of S. Since $P(S)$ is isomorphic to $C(S)$ under the isomorphism Chi, $A \cap B \cap C \subseteq ABC$ if and only if $\chi_{A\cap B\cap C} \subseteq \chi_{ABC}$. Moreover, Chi commutes with the finite intersection and product of sets. Thus S is regular if and only if $\chi_A \cap \chi_B \cap \chi_C \subseteq \chi_A\chi_B\chi_C$ for every left quasi-interior ideal A, every two sided ideal B and every left ideal C of S. That is, S is regular if and only if $D(f, g, h) \subseteq E(f, g, h)$ $\forall$ $f \in C_{qil}$ $\forall$ $g \in C_i$ and $\forall$ $h \in C_l$. So S is regular if and only if $D(f, g, h) \subseteq E(f, g, h)$ $\forall$ $f \in \mathcal{C}_{qil}$ $\forall$ $g \in \mathcal{C}_i$ and $\forall$ $h \in \mathcal{C}_l$. □

Theorem 3.23. *If $f \in \mathcal{C}_{qi}$ and $g \in \mathcal{C}_i$, then $f \cap g \in \mathcal{C}_{qi}$.*

Proof. We define the classes $C_{qi,i} = \{f_1 \cap f_2 : f_1 \in C_{qi}, f_2 \in C_i\}$ and $\mathcal{C}_{qi,i} = \{f \cap g : f \in \mathcal{C}_{qi}, g \in \mathcal{C}_i\}$, where C_{qi} and C_i is classes of all crisp quasi-interior ideals and crisp two sided ideals of S respectively. First we show that $\mathcal{C}_{qi,i}$ is closed under projection. Let $f \cap g \in \mathcal{C}_{qi,i}$. Since Rep commutes with infs of finite sets of fuzzy sets (Proposition 2.3), we have for $f \in \mathcal{C}_{qi}$ and $g \in \mathcal{C}_i$, $\text{Rep}(f \cap g)(r) = \text{Rep}(f)(r) \cap \text{Rep}(g)(r)$ $\forall$ $r \in J$. By Theorem 3.8 and 3.1, both $\mathcal{C}_{qi}$ and $\mathcal{C}_i$ are closed under projection. Then $\text{Rep}(f)(r) \in \mathcal{C}_{qi}$ and $Rep(g)(r) \in \mathcal{C}_i$. Thus $\text{Rep}(f \cap g)(r) \in \mathcal{C}_{qi,i}$ $\forall$ $r \in J$. So $\mathcal{C}_{qi,i}$ is closed under projection. Also, $\mathcal{C}_{qi}$ is closed under projection by Theorem 3.8. By Corollary 3.11 of [4], the intersection of a quasi-interior ideal and a two sided ideals of a semiring S is a quasi-interior ideal of S and Chi is an isomorphism from $P(S)$ to $C(S)$. Hence $C_{qi,i} \subseteq C_{qi}$. Therefore by Proposition 2.7, $\mathcal{C}_{qi,i} \subseteq \mathcal{C}_{qi}$. □

Theorem 3.24. *The following statements hold in a semiring S.*

(1) *If $f \in \mathcal{C}_{qir}$ and $g \in \mathcal{C}_{qil}$, then $f \cap g \in \mathcal{C}_{qi}$.*

(2) *If* $f \in \mathcal{C}_r$ *and* $g \in \mathcal{C}_l$*, then* $f \cap g \in \mathcal{C}_{qi}$.
(3) *If* $f \in \mathcal{C}_{qir}$ *and* $g \in \mathcal{C}_{qil}$*, then* $f \circ g \in \mathcal{C}_{qi}$.
(4) *If* $f \in \mathcal{C}_{qil}(\mathcal{C}_{qir})$ *and* $g \in \mathcal{C}_r(\mathcal{C}_l)$*, then* $f \cap g \in \mathcal{C}_{qil}(\mathcal{C}_{qir})$.
(5)*If* $f \in \mathcal{C}_{qil}$ *and* $g \in \mathcal{C}_{in}$*, then* $f \cap g \in \mathcal{C}_{qil}$.
(6) *If* $f \in \mathcal{C}_{qi}$ *and* $g \in \mathcal{C}_{ss}$*, then* $f \cap g \in \mathcal{C}_{qi}$.

Proof. We establish (3). We define the classes $C_{qir,qil} = \{f_1 \circ f_2 : f_1 \in C_{qir}, f_2 \in C_{qil}\}$ and $\mathcal{C}_{qir,qil} = \{f \circ g : f \in \mathcal{C}_{qir}, g \in \mathcal{C}_{qil}\}$, where C_{qir} and C_{qil} are classes of all crisp right quasi-interior ideals and crisp left quasi-interior ideals of S respectively. First we show that $\mathcal{C}_{qir,qil}$ is closed under projection. Let $f \circ g \in \mathcal{C}_{qir,qil}$. Since Rep commutes with operation '$\circ$' (Proposition 2.5), we have for all $f \in \mathcal{C}_{qir}$ and $g \in \mathcal{C}_{qil}$, $\mathrm{Rep}(f \circ g)(r) = \mathrm{Rep}(f)(r) \circ \mathrm{Rep}(g)(r)\ \forall\ r \in J$. By Theorem 3.7 and 3.6, $\mathcal{C}_{qir}$ and $\mathcal{C}_{qil}$ are closed under projection. Then we have $\mathrm{Rep}(f)(r) \in \mathcal{C}_{qir}$, $Rep(g)(r) \in \mathcal{C}_{qil}$. Thus $\mathrm{Rep}(f \circ g)(r) \in \mathcal{C}_{qir,qil}\ \ \forall\ r \in J$. So $\mathcal{C}_{qir,qil}$ is closed under projection. Also, $\mathcal{C}_{qi}$ is closed under projection by Theorem 3.8. By Theorem 3.1 of [4], the product of a right quasi-interior ideal and a left quasi-interior ideals of a semiring S is a quasi-interior ideal of S and Chi is an isomorphism from $P(S)$ to $C(S)$. Hence $C_{qir,qil} \subseteq C_{qi}$. Therefore by Proposition 2.7, $\mathcal{C}_{qir,qil} \subseteq \mathcal{C}_{qi}$.

Other results are extension of Theorem 3.1, 3.8, 3.14 and Corollary 3.10 of [4] in fuzzy setting and can be proved similarly. □

Theorem 3.25. *Let* h *ba a fuzzy subsemiring of a semiring* S*. Then* $h \in \mathcal{C}_{qil}$ *if and only if there exist* $f, g \in \mathcal{C}_l$ *such that* $f \circ g \subseteq h \subseteq f \cap g$.

Proof. We define the classes $C = \{\chi_K \in C(S)$, where K is a subsemiring of S : $L_1L_2 \subseteq K \subseteq L_1 \cap L_2$ for some left ideals L_1, L_2 of $S\}$ and $\mathcal{C} = \{h \in \mathcal{C}_{ss} : f \circ g \subseteq h \subseteq f \cap g$ for some $f, g \in \mathcal{C}_l\}$. First we show that $\mathcal{C}$ is closed under projection. Let $h \in \mathcal{C}$. Then $h \in \mathcal{C}_{ss}$ such that $f \circ g \subseteq h \subseteq f \cap g$ for some $f, g \in \mathcal{C}_l$. By Proposition 2.3, $\mathrm{Rep}(f \circ g)(r) \leq \mathrm{Rep}(h)(r) \leq \mathrm{Rep}(f \cap g)(r)$ for some $f, g \in \mathcal{C}_l$ and $\forall\ r \in J$. Since Rep commutes with infs of finite sets of fuzzy subset sets and with operation '$\circ$' (Proposition 2.3 and 2.5), we have for all $r \in J$, $\mathrm{Rep}(f)(r) \circ \mathrm{Rep}(g)(r) \leq \mathrm{Rep}(h)(r) \leq Rep(f)(r) \cap Rep(g)(r)$. By Theorem 3.1, both the classes $\mathcal{C}_{ss}$ and $\mathcal{C}_l$ of fuzzy subsemirings and fuzzy left ideals are closed under projection. Thus we have $\mathrm{Rep}(f)(r), \mathrm{Rep}(g)(r) \in \mathcal{C}_l$ and $\mathrm{Rep}(h)(r) \in \mathcal{C}_{ss}$. So $\mathrm{Rep}(h)(r) \in \mathcal{C}\ \ \forall\ r \in J$, i.e., $\mathcal{C}$ is closed under projection. By Theorem 3.6, $\mathcal{C}_{qil}$, the class of all fuzzy left quasi-interior ideals of S is closed under projection. By Theorem 3.7 of [4], a subsemiring B of a semiring S is a left quasi-interior ideal of S if and only if there exist left ideals L_1 and L_2 such that $L_1L_2 \subseteq B \subseteq L_1 \cap L_2$ and Chi is an isomorphism from $P(S)$ to $C(S)$. Hence we have $C = C_{qil}$. Therefore by Proposition 2.7, $\mathcal{C} = \mathcal{C}_{qil}$. □

The following results are extension of Corollary 3.9 of [4] to the fuzzy setting and can be proved similarly.

Theorem 3.26. *Let* h *ba a fuzzy subsemiring of a semiring* S*. Then* $h \in \mathcal{C}_{qir}$ *if and only if there exist* $f, g \in \mathcal{C}_r$ *such that* $f \circ g \subseteq h \subseteq f \cap g$.

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