Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 27, No. 3, (June 2024) pp. 223–232
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2024.27.3.223

@FMI
© Research Institute for Basic Science, Wonkwang University
http://ribs.wonkwang.ac.kr

On L-almost separation axioms in L-fuzzifying bitopological spaces

Hana M. Binshahnah, A. K. Mousa

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 27, No. 3, June 2024

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 27, No. 3, (June 2024) pp. 223–232
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2024.27.3.223

@FMI

© Research Institute for Basic
Science, Wonkwang University
http://ribs.wonkwang.ac.kr

On L-almost separation axioms in L-fuzzifying bitopological spaces

Hana M. Binshahnah, A. K. Mousa

Received 14 December 2023; Revised 26 January 2024; Accepted 2 March 2024

Abstract. Khalaf and El-Latif [1] introduced and investigated the concept of L-almost separation axioms in L-fuzzifying bitopological spaces, where L is a complete residuated lattice, but we note that some of their results are incorrect (See Theorems 6, 7, 8, 18 in [1]). Firstly in this paper we give some counterexamples to show that these results generally need not be true. Secondly, we introduced the notions of L-almost continuity, L-almost open function and L-completely continuous function in L-fuzzifying bitopological spaces with studying some important results. Finally, under these types of L-fuzzy mappings we study the image of these kinds of L-fuzzifying bitopological spaces.

2020 AMS Classification: 54A40, 54D10, 54E55

Keywords: Fuzzifying topology, Fuzzifying bitopological spaces, Almost separation axioms.

Corresponding Author: A. K. Mousa (akmousa@azhar.edu.eg)

1. Introduction

In 1963, Kelley [2] defined the concept of bitopological spaces by spaces equipped with its two (arbitrary) topologies. In 1993, Shen [3] is considered the first to study the separation axioms in fuzzifying topologies.

In 2000, Zahran [4] introduced the notion of regular open sets in I-fuzzifying topological spaces but some of his results were incorrect so, in 2004, Sayed and Zahran [5] gave corrections of them. Also, Sayed et al. [6, 7, 8, 9, 10, 11, 12, 13] studied many deferent separation axioms in fuzzifying topology and (L,M)-fuzzy convexity spaces. Allam et al. [14, 15] studied separation axioms and semi-separation axioms in fuzzifying bitopological spaces. In 2023, Binshahnah et al. [16] studied strongly separation axioms in fuzzifying bitopological spaces

In 2019, Khalaf and El-Latif [1] deal with regular open sets similar to open sets in their study of L-almost separation axioms. So, they introduced their results as the same as the results on open sets but this is a big mistake because regular open sets and open sets are independent in I-fuzzifying topology (See [5]) and thus in L-fuzzifying bitopology as we illustrate in this paper.

The contents of this paper are arranged as follows: In Section 3, we prove that some results obtained in [1] need not be true, by giving some counterexamples. In Section 4, we introduced the notions of L-almost continuity, L-almost open function and L-completely continuous function in L-fuzzifying bitopological spaces with studying some important results. Finally, in Section 5, under these types of L-fuzzy mappings we study the image of these kinds of L-fuzzifying bitopological spaces

2. Preliminaries

Definition 2.1 ([17, 18])**.** A structure $(L, \vee, \wedge, *, \longrightarrow, \bot, \top)$ is called a *complete residuated lattice*, if the following conditions are satisfied:

(i) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,

(ii) $(L, *, \top)$ is a commutative monoid, i.e.,

(a) $*$ is a commutative and associative binary operation on L and

(b) for every $a \in L, a * \top = a$,

(iii) $\longrightarrow$ is related with $*$ as: $a * b \leq c$ if and only if $a \leq b \longrightarrow c \ \forall a, b, c \in L$, where " $\longrightarrow$ " is defined by: $\alpha \longrightarrow \beta = \bigvee\{\lambda \in L : \alpha * \lambda \leq \beta\} \ \forall \alpha, \beta \in L$.

In each statement in the rest of this paper, L is assumed to be a complete residuated lattice. Sometimes we need to add more conditions on L such as the completely distributive law (briefly, CDL) or the double negation law (briefly, DNL).

Definition 2.2 ([19])**.** *We say that L satisfies CDL*, if the following law is satisfied:

$$\bigwedge_{j\in J} \bigvee A_j = \bigvee_{f \in \prod\limits_{j \in J} A_j} \Big(\bigwedge_{j\in J} f(j)\Big) \ \forall \{A_j | j \in J\} \subseteq 2^L,$$

where 2^L is the power subset of L.

Definition 2.3 ([20])**.** We say that L *satisfies DNL*, if the following law is satisfied:

$$(a \longrightarrow \bot) \longrightarrow \bot = a \ \forall \ a \in L.$$

Definition 2.4 ([20])**.** Let $f, \ g \in L^X$. Then the *L-equality* between f and g, denoted by $[[f, g]]$, is defined as follows: $[[f, g]] = \bigwedge\limits_{x \in X} \big((f(x) \longrightarrow g(x)) \wedge (g(x) \longrightarrow f(x))\big)$.

Definition 2.5 ([20, 21, 22, 23])**.** The *L-fuzzifying topology* is a mapping $\varrho : 2^X \longrightarrow L$ satisfying the following conditions:

(i) $\varrho(X) = \varrho(\varnothing) = \top$,

(ii) $\varrho(\bigcup\limits_{\gamma \in \Upsilon} \mathcal{O}_\gamma) \geq \bigwedge\limits_{\gamma \in \Upsilon} \varrho(\mathcal{O}_\gamma) \ \forall \{\mathcal{O}_\gamma | \gamma \in \Upsilon\} \subseteq 2^X$,

(iii) $\varrho(\mathcal{O} \cap \mathcal{G}) \geq \varrho(\mathcal{O}) \wedge \varrho(\mathcal{G}) \ \forall \mathcal{O}, \ \mathcal{G} \in 2^X$.

A pair (X, ϱ) is called an *L-fuzzifying topological space*.

Definition 2.6 ([24]). Let (X, ϱ_1) and (X, ϱ_2) be two L-fuzzifying topological spaces. Then a system $(X, \varrho_1, \varrho_2)$ is called an *L-fuzzifying bitopological space* (briefly, L-fbs).

Definition 2.7 ([1]). Let $(X, \varrho_1, \varrho_2)$ be an L-fbs.

(i) The set of all L-fuzzifying (s,k)-regular open sets is denoted by $R\varrho_{(s,k)} \in L^{2^X}$ and defined as follows:

$$R\varrho_{(s,k)}(\mathcal{O}) = \min\Big(\bigwedge_{x\in\mathcal{O}} I_k(C_s(\mathcal{O}))(x), \bigwedge_{x\in X-\mathcal{O}} (I_k(C_s(\mathcal{O}))(x) \longrightarrow \perp)\Big),$$

where $s,k = 1,2$ and $s \neq k$, $I_k(A)$ is means the interior of a set A with respect to ϱ_k and $C_s(A)$ is means the closure of a set A with respect to ϱ_s, $\forall A \in 2^X$.

(ii) The set of all L-fuzzifying (s,k)-regular closed sets is denoted by $R\mathcal{F}_{(s,k)} \in L^{2^X}$ and defined as follows: $R\mathcal{F}_{(s,k)}(\mathcal{O}) = R\varrho_{(s,k)}(X-\mathcal{O})$.

Definition 2.8 ([1]). Let $(X, \varrho_1, \varrho_2)$ be an L-fbs and $x \in \mathcal{O}$. Then

(i) an *(s,k)-regular neighborhood system of x*, denoted by $RN_x^{(s,k)} \in L^{2^X}$, is defined as follows: $RN_x^{(s,k)}(\mathcal{O}) = \bigvee_{x\in B\subseteq\mathcal{O}} R\varrho_{(s,k)}(B)$,

(ii) an *(s,k)-regular closure operator*, denoted by $RC_{(s,k)} \in (L^X)^{2^X}$, is defined as follows:

$$RC_{(s,k)}(\mathcal{O})(x) = RN_x^{(s,k)}(\mathcal{O}) \longrightarrow \perp.$$

For simplicity, we take:

$K_R^{(s,k)}(x,y) = (\bigvee_{y\notin A} RN_x^{(s,k)}(A)) \vee (\bigvee_{x\notin A} RN_y^{(s,k)}(A))$,

$H_R^{(s,k)}(x,y) = (\bigvee_{y\notin B} RN_x^{(s,k)}(B)) \wedge (\bigvee_{x\notin C} RN_y^{(s,k)}(C))$,

$M_R^{(s,k)}(x,y) = \bigvee_{C\cap B=\phi} \big(RN_x^{(s,k)}(B) \wedge RN_y^{(s,k)}(A)\big)$,

$V_R^{(s,k)}(x,D) = \bigvee_{A\cap B=\phi, D\subseteq B} (RN_x^{(s,k)}(A) \wedge R\varrho_{(s,k)}(B))$,

where $x,y \in X$ and $A,B,D \in 2^X$.

Definition 2.9 ([1]). Let Ω be the class of all L-fbss. Then the *unary L-predicates L-almost–$T_n^{(s,k)}$* $\in L^\Omega$, denoted by $RT_n^{(s,k)}$, $n = 0,1,2,3$ are defined as follows:

(i) $RT_0^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\neq y} K_R^{(s,k)}(x,y)$,

(ii) $RT_1^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\neq y} H_R^{(s,k)}(x,y)$,

(iii) $RT_2^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\neq y} M_R^{(s,k)}(x,y)$,

(iv) $RT_3^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\notin D} (R\mathcal{F}_{(s,k)}(D) \longrightarrow V_R^{(s,k)}(x,D))$.

Definition 2.10 ([1]). Let $(X, \varrho_1, \varrho_2)$ be an L-fbs. The *L-fuzzifying derived set* of $\mathcal{O} \subseteq X$, denoted by $Rd_{(s,k)} \in (L^X)^{2^X}$, is defined by: for each $x \in X$,

$$Rd_{(s,k)}(\mathcal{O})(x) = RN_x^{(s,k)}\big((X-\mathcal{O}) \cup \{x\}\big) \longrightarrow \perp.$$

Definition 2.11 ([1])**.** Let Ω be the class of all L-fbss. Then the *unary L-predicate* ${}^1RT_3^{(s,k)} \in L^{\Omega}$ are defined as follows:

$$
\begin{aligned}
&{}^1RT_3^{(s,k)}(X,\varrho_1,\varrho_2)\\
&= \bigwedge_{x\notin D}\Big(R\mathcal{F}_s(D) \longrightarrow \bigvee_{A\in 2^X}\big(RN_x^{(s,k)}(A) \wedge (\bigwedge_{y\in D} RC_{(s,k)}(A)(y) \longrightarrow \perp)\big)\Big).
\end{aligned}
$$

3. L-almost separation axioms in L-fbss

First, the following example shows that

(1) For any $\{\mathcal{O}_\gamma : \gamma\in\Upsilon\}$, $R\varrho_{(s,k)}(\bigcup_{\gamma\in\Upsilon}\mathcal{O}_\gamma) \geq \bigwedge_{\gamma\in\Upsilon} R\varrho_{(s,k)}(\mathcal{O}_\gamma)$ and

(2) For any $\mathcal{O},\ \mathcal{G}\subseteq X$, $R\varrho_{(s,k)}(\mathcal{O}\cap\mathcal{G}) \leq R\varrho_{(s,k)}(\mathcal{O}) \wedge R\varrho_{(s,k)}(\mathcal{G})$, generally need not be true.

Example 3.1. Let $S=\{l,t,m\}$ and $L=[0,1]$, ϱ_1, ϱ_2 be two fuzzifying topologies defined on S as follows:

$$
\varrho_1(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S, \{l\}, \{l,m\}\},\\ 1/8 & \text{if} \quad B\in\{\{m\},\{t,m\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,t\}\},\end{cases}
$$

$$
\varrho_2(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S, \{l\}, \{l,m\}\},\\ 1/4 & \text{if} \quad B\in\{\{m\},\{t,m\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,t\}\}.\end{cases}
$$

Note that

$$
R\varrho_{(1,2)}(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S\},\\ 1/4 & \text{if} \quad B\in\{\{m\},\{t,m\}\},\\ 1/8 & \text{if} \quad B\in\{\{l\},\{l,t\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,m\}\},\end{cases}
$$

$$
R\mathcal{F}_{(1,2)}(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S\},\\ 1/4 & \text{if} \quad B\in\{\{l\},\{l,t\}\},\\ 1/8 & \text{if} \quad B\in\{\{m\},\{t,m\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,m\}\}.\end{cases}
$$

We note that $R\varrho_{(1,2)}(\{l,t\}) \wedge R\varrho_{(1,2)}(\{t,m\}) = 1/4\wedge 1/8 = 1/8$ and $R\varrho_{(1,2)}(\{l,t\}\cap \{t,m\}) = R\varrho_{(1,2)}(\{t\}) = 0$. Then $R\varrho_{(1,2)}(\{l,t\}\cap\{t,m\}) \ngeq R\varrho_{(1,2)}(\{l,t\})\wedge\varrho_{(1,2)}(\{t,m\})$. Also, $R\varrho_{(1,2)}(\{l\}\cup\{m\}) = R\varrho_{(1,2)}(\{l,m\}) = 0 \ngeq 1/8 = R\varrho_{(1,2)}(\{l\}) \wedge R\varrho_{(1,2)}(\{m\})$.

In Definition 6 (1) [1], Khalaf and Abd El-Latif said $R\varrho_{(s,k)}(\mathcal{O}) = \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O})$, but generally, this is not true as shown by the following theorem.

Theorem 3.2. *Let (X,ϱ_1,ϱ_2) be an L-fbs. Then $R\varrho_{(s,k)}(\mathcal{O}) \leq \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O})$.*

Proof. Note that $\bigwedge_{y\in\mathcal{O}}\bigvee_{y\in B\subseteq\mathcal{O}} R\varrho_{(s,k)}(B) \geq R\varrho_{(s,k)}(\mathcal{O})$. Then we have

$$
R\varrho_{(s,k)}(\mathcal{O}) \leq \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O}).
$$

□

The following example illustrates that $R\varrho_{(s,k)}(\mathcal{O}) \neq \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O})$ in general.

Example 3.3. From Example 3.1, we have

$$R\varrho_{(1,2)}(\{l,m\}) = 0 \neq 1/8 = \bigwedge_{x\in\{l,m\}} RN_x^{(1,2)}(\{l,m\}).$$

Theorem 3.4 (Theorem 6, [1])**.** *Let (X,ϱ_1,ϱ_2) be an L-fbs. If L satisfies CDL, then $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) = \bigwedge_{x\in X} R\mathcal{F}_{(s,k)}(\{x\})$.*

The following example shows that $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) > \bigwedge_{x\in X} R\mathcal{F}_{(s,k)}(\{x\})$. Then the above theorem is not true in general.

Example 3.5. From Example 3.1, we have

$$RT_1^{(1,2)}(S,\varrho_1,\varrho_2) = 1/8 > 0 = \bigwedge_{x\in S} R\mathcal{F}_{(1,2)}(\{x\}).$$

Theorem 3.6 (Theorem 7, [1])**.** *Let (X,ϱ_1,ϱ_2) be an L-fbs and $\mathcal{O}\subseteq X$. If L satisfies CDL, then $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) \leq \bigwedge_{x\in X} RN_x^{(s,k)}\big((X-\mathcal{O})\cup\{x\}\big)$.*

Theorem 3.7 (Theorem 8, [1])**.** *Let (X,ϱ_1,ϱ_2) be an L-fbs and $\mathcal{O}\subseteq X$. If L satisfies CDL, then $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) \leq [[Rd_{(s,k)}(\mathcal{O}), 1_\varnothing]]$.*

The following example shows that Theorem 3.6 and Theorem 3.7 are incorrect in general.

Example 3.8. From Example 3.1, take $\mathcal{O}=\{l,m\}$, we have
(1) $\bigwedge_{x\in S} RN_t^{(1,2)}(\{t\}\cup\{x\}) = 0 \ngeq 1/8 = RT_1^{(1,2)}(S,\varrho_1,\varrho_2)$,
(2) $[[Rd_{(1,2)}(\mathcal{O}), 1_\varnothing]] = 0 \ngeq 1/8 = RT_1^{(1,2)}(S,\varrho_1,\varrho_2)$.

Theorem 3.9 (Theorem 18, [1])**.** *Let (X,ϱ_1,ϱ_2) be an L-fbs. If L satisfies CDL and DNL, then $RT_3^{(s,k)}(X,\varrho_1,\varrho_2) = {}^1RT_3^{(s,k)}(X,\varrho_1,\varrho_2)$.*

The following example shows that Theorem 3.9 need not be true in general.

Example 3.10. From Example 3.1, we have

$$R\varrho_1(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S\},\\ 1/8 & \text{if} \quad B\in\{\{l\},\{l,t\},\{m\},\{t,m\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,m\}\},\end{cases}$$

$$R\mathcal{F}_1(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S\},\\ 1/8 & \text{if} \quad B\in\{\{l\},\{l,t\},\{m\},\{t,m\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,m\}\}.\end{cases}$$

Then $RT_3^{(1,2)}(S,\varrho_1,\varrho_2) = 7/8 \neq 1 = {}^1RT_3^{(1,2)}(S,\varrho_1,\varrho_2)$.

4. L-almost continuity, L-almost opennes And L-completely continuity in L-fbss

Definition 4.1. Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. A unary L-predicate $AC_{(s,k)}\in L^{(Y^X)}$ is called *L-almost continuous*, provide that

$$AC_{(s,k)}(g)=\bigwedge_{W\in 2^Y}\big(R\xi_{(s,k)}(W)\longrightarrow\varrho_s(g^{-1}(W))\big).$$

Theorem 4.2. *Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. Then $AC_{(s,k)}(g)\leq\bigwedge_{x\in X}\bigwedge_{W\in 2^Y}\big(RN^{(s,k)}_{g(x)}(W)\longrightarrow N^s_x(g^{-1}(W))\big)$.*

Proof. We have $AC_{(s,k)}(g)=\bigwedge_{W\in 2^Y}\big(R\xi_{(s,k)}(W)\longrightarrow\varrho_s(g^{-1}(W))\big)$. Then

$$AC_{(s,k)}(g)\leq R\xi_{(s,k)}(W)\longrightarrow\varrho_s(g^{-1}(W))\ \forall x\in X,\ \forall W\in 2^Y.$$

Thus $AC_{(s,k)}(g)*R\xi_{(s,k)}(W)\leq\varrho_s(g^{-1}(W))$ implies

$$AC_{(s,k)}(g)*R\xi_{(s,k)}(W)\leq\bigvee_{x\in H\subseteq g^{-1}(W)}\varrho_s(H)=N^s_x(g^{-1}(W)).$$

So $\bigvee_{g(x)\in H\subseteq W}\big(AC_{(s,k)}(g)*R\xi_{(s,k)}(H)\big)\leq\bigvee_{g(x)\in H\subseteq W}N^s_x(g^{-1}(H))$ implies

$$AC_{(s,k)}(g)*\bigvee_{g(x)\in H\subseteq W}R\xi_{(s,k)}(H)\leq\bigvee_{x\in g^{-1}(H)\subseteq f^{-1}(W)}N^s_x(g^{-1}(H)).$$

Hence $AC_{(s,k)}(g)*RN^{(s,k)}_{g(x)}(W)\leq N^s_x(g^{-1}(W))$ implies

$$AC_{(s,k)}(g)\leq RN^{(s,k)}_{g(x)}(W)\longrightarrow N^s_x(g^{-1}(W)).$$

Therefore $AC_{(s,k)}(g)\leq\bigwedge_{x\in X}\bigwedge_{W\in\ 2^Y}\big(RN^{(s,k)}_{g(x)}(W)\longrightarrow N^s_x(g^{-1}(W))\big)$. □

Definition 4.3. Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. A unary L-predicate $AO_{(s,k)}\in L^{(Y^X)}$ is called *L-almost open*, provide that

$$AO_{(s,k)}(g)=\bigwedge_{U\in 2^X}\big(R\varrho_{(s,k)}(U)\longrightarrow\xi_s(g(U))\big).$$

Theorem 4.4. *Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. Then $AO_{(s,k)}(g)\leq\bigwedge_{x\in X}\bigwedge_{W\in 2^Y}\big(RN^{(s,k)}_x(g^{-1}(W))\longrightarrow N^s_{g(x)}(W)\big)$.*

Proof. We can prove this theorem in the same way as the proof of Theorem 4.2. □

Definition 4.5. Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. A unary L-predicate $RC_{(s,k)}\in L^{(Y^X)}$ is called *L-completely continuous*, provide that

$$RC_{(s,k)}(g)=\bigwedge_{W\in 2^Y}\big(\xi_s(W)\longrightarrow R\varrho_{(s,k)}(g^{-1}(W))\big).$$

Theorem 4.6. *Let $(X, \varrho_1, \varrho_2), (Y, \xi_1, \xi_2)$ be two L-fbss and $g : (X, \varrho_1, \varrho_2) \to (Y, \xi_1, \xi_2)$ be a mapping. Then* $RC_{(s,k)}(g) \leq \bigwedge_{x \in X} \bigwedge_{W \in 2^Y} \big(N^s_{g(x)}(W) \longrightarrow RN^{(s,k)}_x(g^{-1}(W))\big)$.

Proof. We can prove this theorem in the same way as the proof of Theorem 4.2. □

5. L-almost separation axioms and L-fuzzy mappings In L-fbss

Theorem 5.1. *Let $(X, \varrho_1, \varrho_2), (Y, \xi_1, \xi_2)$ be two L-fbss. If $g : (X, \varrho_1, \varrho_2) \longrightarrow (Y, \xi_1, \xi_2)$ is a bijective mapping and $[AC_{(s,k)}(g)] = \top$, then*

(1) $RT_0^{(s,k)}(Y, \xi_1, \xi_2) \leq T_0(X, \varrho_s)$,
(2) $RT_1^{(s,k)}(Y, \xi_1, \xi_2) \leq T_1(X, \varrho_s)$,
(3) $RT_2^{(s,k)}(Y, \xi_1, \xi_2) \leq T_2(X, \varrho_s)$.

Proof. (1) From Theorem 4.2 and $[AC_{(s,k)}(g)] = \top$, we have

$$\begin{aligned} RT_0^{(s,k)}(Y, \xi_1, \xi_2) &= \bigwedge_{z \neq w} \Big(\bigvee_{w \notin W} RN_z^{(s,k)}(W) \vee \bigvee_{z \notin W} RN_w^{(s,k)}(W)\Big) \\ &= \bigwedge_{gg^{-1}(z) \neq gg^{-1}(w)} \Big(\bigvee_{gg^{-1}(w) \notin W} RN^{(s,k)}_{gg^{-1}(z)}(W) \\ &\qquad \vee \bigvee_{gg^{-1}(z) \notin W} RN^{(s,k)}_{gg^{-1}(w)}(W)\Big) \\ &\leq \bigwedge_{g^{-1}(z) \neq g^{-1}(w)} \Big(\bigvee_{g^{-1}(w) \notin g^{-1}(W)} N^s_{g^{-1}(z)}(g^{-1}(W)) \\ &\qquad \vee \bigvee_{g^{-1}(z) \notin g^{-1}(W)} N^s_{g^{-1}(w)}(g^{-1}(W))\Big) \\ &= \bigwedge_{x \neq y} \Big(\bigvee_{y \notin U} N^s_x(U) \vee \bigvee_{x \notin U} N^s_y(U)\Big) = T_0(X, \varrho_s). \end{aligned}$$

We can prove (2) and (3) in the same way as the proof of (1) above. □

Theorem 5.2. *Let $(X, \varrho_1, \varrho_2), (Y, \xi_1, \xi_2)$ be two L-fbss. If $g : (X, \varrho_1, \varrho_2) \longrightarrow (Y, \xi_1, \xi_2)$ is a bijective mapping and $[AO_{(s,k)}(g)] = \top$, then*

(1) $RT_0^{(s,k)}(X, \varrho_1, \varrho_2) \leq T_0(Y, \xi_s)$,
(2) $RT_1^{(s,k)}(X, \varrho_1, \varrho_2) \leq T_1(Y, \xi_s)$,
(3) $RT_2^{(s,k)}(X, \varrho_1, \varrho_2) \leq T_2(Y, \xi_s)$.

Proof. (1) From Theorem 4.4 and $[AO_{(s,k)}(g)] = \top$, we have

$$\begin{aligned} RT_0^{(s,k)}(X, \varrho_1, \varrho_2) &= \bigwedge_{x\neq y} \Big(\bigvee_{y\notin \mathcal{O}} RN_x^{(s,k)}(\mathcal{O}) \vee \bigvee_{x\notin \mathcal{O}} RN_y^{(s,k)}(\mathcal{O})\Big) \\ &= \bigwedge_{x\neq y} \Big(\bigvee_{y\notin \mathcal{O}} RN_x^{(s,k)}(g^{-1}g(\mathcal{O})) \vee \bigvee_{x\notin \mathcal{O}} RN_y^{(s,k)}(g^{-1}g(\mathcal{O}))\Big) \\ &\leq \bigwedge_{x\neq y} \Big(\bigvee_{y\notin \mathcal{O}} N^s_{g(x)}(g(\mathcal{O})) \vee \bigvee_{x\notin \mathcal{O}} N^s_{g(y)}(g(\mathcal{O}))\Big) \\ &= \bigwedge_{g(x)\neq g(y)} \Big(\bigvee_{g(y)\notin g(\mathcal{O})} N^s_{g(x)}(g(\mathcal{O})) \vee \bigvee_{g(x)\notin g(\mathcal{O})} N^s_{g(y)}(g(\mathcal{O}))\Big) \\ &= \bigwedge_{z\neq w} \Big(\bigvee_{w\notin H} N^s_z(H) \vee \bigvee_{z\notin H} N^s_w(H)\Big) \\ &= T_0(Y, \xi_s). \end{aligned}$$

We can prove (2) and (3) in the same way as the proof of (1) above. □

Theorem 5.3. *Let $(X, \varrho_1, \varrho_2), (Y, \xi_1, \xi_2)$ be two L-fbss. If $g : (X, \varrho_1, \varrho_2) \longrightarrow (Y, \xi_1, \xi_2)$ is an injective mapping and $[RC_{(s,k)}(g)] = \top$, then*
(1) $T_0(Y, \xi_s) \leq RT_0^{(s,k)}(X, \varrho_1, \varrho_2)$,
(2) $T_1(Y, \xi_s) \leq RT_1^{(s,k)}(X, \varrho_1, \varrho_2)$,
(3) $T_2(Y, \xi_s) \leq RT_2^{(s,k)}(X, \varrho_1, \varrho_2)$.

Proof. (1) From Theorem 4.2 and $[RC_{(s,k)}(g)] = \top$, we have for every $W \in 2^Y$ and $x \in X$, $N^s_{g(x)}(W) \leq RN_x^{(s,k)}(g^{-1}(W))$. Therefore

$$\begin{aligned} RT_0^{(s,k)}(X, \varrho_1, \varrho_2) &= \bigwedge_{x\neq y} \Big(\bigvee_{y\notin \mathcal{O}} RN_x^{(s,k)}(\mathcal{O}) \vee \bigvee_{x\notin \mathcal{O}} RN_y^{(s,k)}(\mathcal{O})\Big) \\ &= \bigwedge_{x\neq y} \Big(\bigvee_{y\notin \mathcal{O}} RN_x^{(s,k)}(g^{-1}g(\mathcal{O})) \vee \bigvee_{x\notin \mathcal{O}} RN_y^{(s,k)}(g^{-1}g(\mathcal{O}))\Big) \\ &\geq \bigwedge_{x\neq y} \Big(\bigvee_{y\notin \mathcal{O}} N^s_{g(x)}(g(\mathcal{O})) \vee \bigvee_{x\notin \mathcal{O}} N^s_{g(y)}(g(\mathcal{O}))\Big) \\ &= \bigwedge_{g(x)\neq g(y)} \Big(\bigvee_{g(y)\notin g(\mathcal{O})} N^s_{g(x)}(g(\mathcal{O})) \vee \bigvee_{g(x)\notin g(\mathcal{O})} N^s_{g(y)}(g(\mathcal{O}))\Big) \\ &= \bigwedge_{z\neq w} \Big(\bigvee_{w\notin H} N^s_z(H) \vee \bigvee_{z\notin H} N^s_w(H)\Big) = T_0(Y, \xi_s). \end{aligned}$$

The proof of (2) and (3) is similar to (1) above. □

6. Conclusions

In the present paper, we prove that some results obtained in [1] need not be true, by giving some counterexamples. Also, we study some types of L-fuzzy mappings and the image of these kinds of L-fuzzifying bitopological spaces. In the future, we can take these properties in applications of L-fuzzifying bitopological spaces.

References

[1] M. M. Khalaf and A. A. Abd El-Latif, L-almost separation axioms in L-fuzzifying-bitopologies via complete residuated lattice-valued logic, Int. J. Appl. Comput. Math. 5 (2) (2019) https://doi.org/10.1007/s40819-018-0575-x.

[2] J. L. Kelley, Bitopological spaces, Proc. London Math. Soc. 13 (3) (1963) 71–89.

[3] J. Shen, Separation axiom in fuzzifying topology, Fuzzy Sets and Systems 57 (1993) 111–123.

[4] A. M. Zahran, Almost continuity and δ-continuity in fuzzifying topology, Int. J. Appl. Comput. Math. 116 (2000) 339–352.

[5] O. R. Sayed, and A. M. Zahran, Almost continuity and δ-continuity in fuzzifying topology, Fuzzy Sets and Systems 146 (2004) 153–154.

[6] K. M. Abd El-Hakeim, F. M Zeyada and O. R. Sayed, Pre-separation axioms in fuzzifying topology, Fuzzy Systems and Mathematics 17 (1) (2003) 29–37.

[7] F. H. Khedr, F. M. Zeyada and O. R. Sayed, On separation axioms in fuzzifying topology, Fuzzy Sets and Systems 119 (2001) 439–458.

[8] O. R. Sayed, Fuzzy γ-separation axioms in fuzzifying topology, The Journal of Fuzzy Mathematics 14 (4) (2006) 767–787.

[9] O. R. Sayed, β-Separation axioms based on based on based on based on based Lukasiewicz logic, Engineering Science Letters 1 (1) (2012) 1–24.

[10] O. R. Sayed and A. K. Mousa, Almost separation axioms in fuzzifying topology, Journal of Advanced Studies in Topology 4 (4) (2013) 46–58.

[11] O. R. Sayed, α- Separation axioms based on Lukasiewicz logic logic, Hacettepe Journal of Mathematics and Statistics 43 (2) (2014) 269–287.

[12] O. R. Sayed, E. El-Sanousy and Y. H. Raghp, Separation axioms in (L, M)-fuzzy topology (L, M)-fuzzy convexity spaces, Ann. Fuzzy Math. Inform. 21 (1) (2021) 93–103.

[13] Zhao Hu, O. R. Sayed, E. El-Sanousy, Y. H. Raghp and Chen Guixiu, On separation axioms in (L, M)-fuzzy convex structures, Journal of Intelligent and Fuzzy Systems 40 (2021) 8765–8773.

[14] A. A. Allam, A. M. Zahran, A. K. Mousa and H. M. Binshahnah, On semi separation axioms in fuzzifying bitopological spaces, Journal of Advanced Studies in Topology 6 (4) (2015) 152–163.

[15] A. A. Allam, A. M. Zahran, A. M. Mousa and H. M. Binshahnah, On separation axioms in fuzzifying bitopological spaces, International Journal of Fuzzy Logic and Intelligent Systems 20 (1) (2020) 77–86.

[16] H. M. Binshahnah, A. K. Mousa and A. A. Allam, Strongly separation axioms in fuzzifying bitopological spaces, J. Math. Computer Sci. 31 (2023) 240–246.

[17] J. Pavelka, On fuzzy logic II, Z. Math. Logic Grundlagen Math. 25 (1979) 119–134.

[18] M. S. Ying, Fuzzifying topology based on complete residuated lattice-valued logic (I), Fuzzy Sets and Systems 56 (1993) 337–373.

[19] G. Birkhoff, Lattice theory, 3rd ed, AMS Providence, RI, 2nd 1967.

[20] U. Höhle and A. P. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, The Handbooks of Fuzzy Sets Series 3 (1999) 123–272.

[21] U. Höhle, Upper semicontinuous fuzzy sets and applications, J. Math. Anal. Appl. 78 (1980) 659–673.

[22] T. Kubiak, On fuzzy topologies, Adam Mickiewicz University, Poznan, Poland 1985.

[23] A. P. Šostak, On fuzzy topological structure, Topology and its App. 11 (2) (1985) 89–103.

[24] G. J. Zhang and M. J. Liu, On properties of $\theta_{(i,j)}$-open sets in fuzzifying bitopological spaces, The J. Fuzzy Math. 11 (1) (2003) 165–178.

Hana M. Binshahnah (h.binshahnah@hu.edu.ye)
Department of mathematics, faculty of science, Hadhramout University, Yemen.

A. K. Mousa (akmousa@azhar.edu.eg)
Department of mathematics, faculty of science, Al-Azhar University, Assiut 71524, Egypt.

Department of mathematics, college of Umluj, University of Tabuk, Tabuk, K.S.A.