Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 27, No. 2, (April 2024) pp. 159–168
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2024.27.2.159

@FMI

More on neutrosophic Lie subalgebra

S. Jafari, S. R. Vidhya, N. Rajesh, A. Pigazzini

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 27, No. 1, February 2024

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 27, No. 2, (April 2024) pp. 159–168
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2024.27.2.159

@FMI

More on neutrosophic Lie subalgebra

S. Jafari, S. R. Vidhya, N. Rajesh, A. Pigazzini

Received 24 November 2023; Revised 18 December 2023; Accepted 23 January 2024

Abstract. In this paper, we present some more fundamental properties of the notion of neutrosophic Lie subalgebra of a Lie algebra.

2020 AMS Classification: 03E72, 03F55, 13F20

Keywords: Lie algebras, Neutrosophic set, Neutrosophic Lie subalgebra, Cartesian product, Lie homomorphism.

Corresponding Author: S. Jafari (saeidjafari@topositus.comm)

1. Introduction and preliminaries

Lie (1842-1899)introduced Lie algebras in the field of mathematics and was motivated by his attempt to classify certain "smooth" subgroups of general linear groups. These groups are now called Lie groups. By definition the tangent space at identity element of a Lie group gives us its Lie algebra. Sometimes it is easier and manageable to consider a problem on Lie groups and reduce it to a problem on Lie algebra. The application of Lie algebra is vast, among others, in different branches of physics and mathematics, such as spectroscopy of molecules, atoms, hyperbolic and stochastic differential equations. After the advent of the notion of fuzzy set introduced by Zadeh [1], some useful and important notions have been introduced and investigated. One of them is called a neutrosophic set, introduced by Smarandache [2], which is now this set and its application in pure and applied mathematics are active research fields for many researchers worldwide. Neutrosophic theory and its applications have influenced almost all parts of pure and applied sciences and also our outlook towards the real world and the way we analyse things and our argumentaion theory (See [3]). Moreover, the interested reader can see the influence of neutrosophic theory in Decision making problems, graph theory, image analysis, information theory, algebra, topology etc. in [4].

Recently, Das et al. [5] presented not only the properties of single-valued pentapartitioned neutrosophic Lie algebra by focusing on single-valued pentapartitioned neutrosophic set but also introduced and studied their related Lie ideals. It is worth mentioning that Abdullayev and Nesibova [6] and Akram et al. [7] studied neutrosophic Lie algebras and single-valued neutrosophic Lie algebras respectively and obtained several fundamental properties. Also Parimala et al. [8] the notion of complex neutrosophic Lie algebra and obtained several basic and interesting properties. In the present paper, we further investigate some basic properties of the notion of neutrosophic Lie subalgebras of a Lie algebra. We establish the Cartesian product of neutrosophic Lie subalgebras and in particular, we obtain some results dealing with the homomorphisms between the neutrosophic Lie subalgebras of a Lie algebra, and also obtaining some other properties under the presence of these homomorphisms. Now, we mention some notions which will be used in the sequal.

It is well-know that a *Lie algebra* is a vector space $\mathcal{L}$ over a field F (it can be $\mathbb{R}$ or $\mathbb{C}$) on which $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$, denoted by $(\zeta, \xi) \mapsto [\zeta, \xi]$, for $\zeta, \xi \in \mathcal{L}$ and $[\zeta, \xi]$ is called *Lie bracket* satisfying the following conditions:

- $[\zeta, \xi]$ is bilinear,
- $[\zeta, \zeta] = 0$ for all $\zeta \in \mathcal{L}$,
- $[[\zeta, \xi], \nu] + [[\xi, \nu], \zeta] + [[\nu, \zeta], \xi] = 0$ for all $\zeta, \xi, \nu \in \mathcal{L}$ (Jacobi identity).

It is worth noticing that the multiplication in a Lie algebra is not associative, i.e., $[[\zeta, \xi], \nu] \neq [\zeta, [\xi, \nu]]$. But it is true that $[\zeta, \xi] = -[\xi, \zeta]$, which means it is anti-commutative. We call a subspace $\mathcal{H}$ of $\mathcal{L}$ a *Lie subalgebra*, if it is closed under $[\cdot, \cdot]$. A subspace I of $\mathcal{L}$ with the property $[I, L] \subset I$ is called a *Lie ideal* of $\mathcal{L}$. Observe that any Lie ideal is a Lie subalgebra.
A complex mapping $C = (\mu_C, \gamma_C, \psi_C) : \mathcal{L} \to [0,1] \times [0,1] \times [0,1]$ is called a *neutrosophic set* in $\mathcal{L}$ if $\mu_C(\zeta) + \gamma_C(\zeta) + \psi_C(\zeta) \leq 1$ for all $\zeta \in \mathcal{L}$, where the mappings $\mu_C : \mathcal{L} \to [0,1]$ and $\psi_C : \mathcal{L} \to [0,1]$ denote the degree of truth-membership (namely $\mu_C(\zeta)$), the degree of indeterminancy-membership (namely $\gamma_C(\zeta)$) and the degree of non-membership (namely $\psi_C(\zeta)$) of each element $\zeta \in \mathcal{L}$ to C, respectively.

Definition 1.1 ([6])**.** A neutrosophic set $C = (\mu_C, \gamma_C, \psi_C)$ on $\mathcal{L}$ is called a *neutrosophic Lie subalgebra*, if the following conditions are satisfied:

$$(\forall \zeta, \xi \in \mathcal{L}) \begin{pmatrix} \mu_C(\zeta + \xi) \geq \min\{\mu_C(\zeta), \mu_C(\xi)\} \\ \gamma_C(\zeta + \xi) \geq \min\{\gamma_C(\zeta), \gamma_C(\xi)\} \\ \psi_C(\zeta + \xi) \leq \max\{\psi_C(\zeta), \psi_C(\xi)\} \end{pmatrix}, \tag{1.1}$$

$$(\forall \zeta \in \mathcal{L}, \alpha \in F) \begin{pmatrix} \mu_C(\alpha\zeta) \geq \mu_C(\zeta) \\ \gamma_C(\alpha\zeta) \geq \gamma_C(\zeta) \\ \psi_C(\alpha\zeta) \leq \psi_C(\zeta) \end{pmatrix}, \tag{1.2}$$

$$(\forall \zeta, \xi \in \mathcal{L}) \begin{pmatrix} \mu_C([\zeta, \xi]) \geq \min\{\mu_C(\zeta), \mu_C(\xi)\} \\ \gamma_C([\zeta, \xi]) \geq \min\{\gamma_C(\zeta), \gamma_C(\xi)\} \\ \psi_C([\zeta, \xi]) \leq \max\{\psi_C(\zeta), \psi_C(\xi)\} \end{pmatrix}. \tag{1.3}$$

Definition 1.2 ([6])**.** A neutrosophic set $C = (\mu_C, \gamma_C, \psi_C)$ on $\mathcal{L}$ is called a neutrosophic Lie ideal if it satisfies (1.1) and (1.2) and the following relations

$$(\forall \zeta, \xi \in \mathcal{L}) \begin{pmatrix} \mu_C([\zeta, \xi]) \geq \mu_C(\zeta) \\ \gamma_C([\zeta, \xi]) \geq \gamma_C(\zeta) \\ \psi_C([\zeta, \xi]) \leq \psi_C(\zeta) \end{pmatrix}. \tag{1.4}$$

From (1.2), we have:

(1.5) $$\mu_C(0) \geq \mu_C(\zeta), \gamma_C(0) \geq \gamma_C(\zeta), \psi_C(0) \leq \psi_C(\zeta),$$

(1.6) $$\mu_C(-\zeta) \geq \mu_C(\zeta), \gamma_C(-\zeta) \geq \gamma_C(\zeta), \psi_C(-\zeta) \leq \psi_C(\zeta).$$

2. Neutrosophic Lie ideals

Proposition 2.1 ([6]). *Every neutrosophic Lie ideal is a neutrosophic Lie subalgebra.*

The converse of Proposition 2.1 does not hold in general.

Example 2.2. Consider $F = \mathbb{R}$. Let $\mathcal{L} = \{(\zeta, \xi, \nu) : \zeta, \xi, \nu \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors which forms a Lie algebra and define $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ by $[\zeta, \xi] \to \zeta \times \xi$, where $\times$ is the usual cross product. We define a neutrosophic set $C = (\mu_C, \gamma_C, \psi_C) : \mathcal{L} \to [0,1] \times [0,1]$ by

$$\mu_C(\zeta,\xi,\nu) = \begin{cases} 0.7 & \text{if } \zeta = \xi = \nu = 0 \\ 0.5 & \text{if } \zeta \neq 0, \xi = \nu = 0 \\ 0 & \quad \textit{otherwise}, \end{cases}$$

$$\gamma_C(\zeta,\xi,\nu) = \begin{cases} 0.2 & \text{if } \zeta = \xi = \nu = 0 \\ 0.1 & \text{if } \zeta \neq 0, \xi = \nu = 0 \\ 0 & \quad \textit{otherwise}, \end{cases}$$

$$\psi_C(\zeta,\xi,\nu) = \begin{cases} 0 & \text{if } \zeta = \xi = \nu = 0 \\ 0.3 & \text{if } \zeta \neq 0, \xi = \nu = 0 \\ 1 & \quad \textit{otherwise}. \end{cases}$$

Then $C = (\mu_C, \gamma_C, \psi_C)$ is a neutrosophic Lie subalgebra of $\mathcal{L}$ but $C = (\mu_C, \gamma_C, \psi_C)$ is not a neutrosophic Lie ideal of $\mathcal{L}$ since $\mu_C([(1,0,0)(1,1,1)]) = \mu_C(0,-1,1) = 0 \ngeq 0.3 = \mu_C(1,0,0)$.

Proposition 2.3. *If $C = (\mathcal{L}, \mu_C, \gamma_C, \psi_C)$ is a neutrosophic Lie ideal of $\mathcal{L}$, then, $\mu_C(0) = \sup_{\zeta \in \mathcal{L}} \mu_C(\zeta)$, $\gamma_C(0) = \sup_{\zeta \in \mathcal{L}} \gamma_C(\zeta)$ and $\psi_C(0) = \inf_{\zeta \in \mathcal{L}} \psi_C(\zeta)$.*

Proof. It is straightforward. □

Theorem 2.4. *Let $C = (\mathcal{L}, \mu_C, \gamma_C, \psi_C)$ be a neutrosophic Lie ideal of $\mathcal{L}$. Then for each $\alpha, \beta, \delta \in [0,1]$ with $\alpha \leq \mu_C(0)$, $\beta \leq \gamma_C(0)$ and $\delta \geq \psi_C(0)$ and $\alpha + \beta + \delta \leq 1$, the (α, β, δ)-level subset $\mathcal{L}_C^{(\alpha,\beta,\delta)}$ is a Lie ideal of $\mathcal{L}$.*

Proof. Let $\zeta, \xi \in \mathcal{L}_C^{(\alpha,\beta,\delta)}$ and $r \in F$. Then

$$\mu_C(\zeta + \xi) \geq \min\{\mu_C(\zeta), \mu_C(\xi)\} \geq \alpha,$$
$$\gamma_C(\zeta + \xi) \geq \min\{\gamma_C(\zeta), \gamma_C(\xi)\} \geq \beta,$$
$$\psi_C(\zeta + \xi) \leq \max\{\psi_C(\zeta), \psi_C(\xi)\} \leq \delta,$$

$$\mu_C(r\zeta) \geq \mu_C(\zeta) \geq \alpha, \gamma_C(r\zeta) \geq \gamma_C(\zeta) \geq \beta, \psi_C(r\zeta) \leq \psi_C(\zeta) \leq \delta.$$

Thus $\zeta + \xi \in \mathcal{L}_C^{(\alpha,\beta,\delta)}$ and $r\zeta \in \mathcal{L}_C^{(\alpha,\beta,\delta)}$. So $\mathcal{L}_C^{(\alpha,\beta,\delta)}$ is a Lie subalgebra of $\mathcal{L}$. Now let $\zeta \in \mathcal{L}$ and $\xi \in \mathcal{L}_C^{(\alpha,\beta,\delta)}$. Then we have

$$\mu_C([\zeta,\xi]) \geq \mu_C(\xi) \geq \alpha, \ \gamma_C([\zeta,\xi]) \geq \gamma_C(\xi) \geq \beta, \ \psi_C([\zeta,\xi]) \leq \psi_C(\xi) \leq \delta.$$

Thus $[\zeta,\xi] \in \mathcal{L}_C^{(\alpha,\beta,\delta)}$. So $\mathcal{L}_C^{(\alpha,\beta,\delta)}$ is a Lie ideal of $\mathcal{L}$. □

Theorem 2.5. *Let ω be a fixed element of $\mathcal{L}$. If $C = (\mathcal{L}, \mu_C, \gamma_C, \psi_C)$ is a neutrosophic Lie ideal of $\mathcal{L}$, then the set*

$$C^\omega = \{\zeta \in \mathcal{L} : \mu_C(\zeta) \geq \mu_C(\omega), \gamma_C(\zeta) \geq \gamma_C(\omega), \psi_C(\zeta) \leq \psi_C(\omega)\}$$

is a Lie ideal of $\mathcal{L}$.

Proof. Let $\zeta, \xi \in C^\omega$ and $r \in F$. Then

$$\mu_C(\zeta+\xi) \geq \min\{\mu_C(\zeta), \mu_A(\xi)\} \geq \mu_C(\omega),$$
$$\gamma_C(\zeta+\xi) \geq \min\{\gamma_C(\zeta), \gamma_C(\xi)\} \geq \gamma_C(\omega),$$
$$\psi_C(\zeta+\xi) \leq \max\{\psi_C(\zeta), \psi_A(\xi)\} \leq \psi_C(\omega),$$
$$\mu_C(r\zeta) \geq \mu_C(\zeta) \geq \mu_C(\omega), \gamma_C(r\zeta) \geq \gamma_C(\zeta) \geq \gamma_C(\omega), \psi_C(r\zeta) \leq \psi_C(\zeta) \leq \psi_C(\omega).$$

Thus $\zeta, \xi, r\zeta \in C^\omega$. For every $\zeta \in \mathcal{L}$ and $\xi \in C^\omega$, we have

$$\mu_C([\zeta\xi]) \geq \mu_C(\xi) \geq \mu_C(\omega),\ \gamma_C([\zeta\xi]) \geq \gamma_C(\xi) \geq \gamma_C(\omega),\ \psi_C([\zeta\xi]) \leq \psi_C(\xi) \leq \psi_C(\omega).$$

It follows that $[\zeta\xi] \in C^\omega$. So C^ω is a Lie ideal of $\mathcal{L}$. □

Corollary 2.6. *If $C = (\mathcal{L}, \mu_C, \gamma_C, \psi_C)$ is a neutrosophic Lie ideal of $\mathcal{L}$, then the set $C^0 = \{\zeta \in \mathcal{L} : \mu_C(\zeta) \geq \mu_C(0), \gamma_C(\zeta) \geq \gamma_C(0), \psi_C(\zeta) \leq \psi_C(0)\}$ is a Lie ideal of $\mathcal{L}$.*

Proof. Straightforward. □

Theorem 2.7. *Let $C = (\mu_C, \gamma_C, \psi_C)$ be a neutrosophic Lie subalgebra of Lie algebra $\mathcal{L}$. Define a binary relation $\sim$ on $\mathcal{L}$ by $\zeta \sim \xi$ if and only if $\mu_C(\zeta - \xi) = \mu_C(0)$, $\gamma_C(\zeta - \xi) = \gamma_C(0)$, $\psi_C(\zeta - \xi) = \psi_C(0)$ for all $\zeta, \xi \in \mathcal{L}$. Then $\sim$ is a congruence relation on $\mathcal{L}$.*

Proof. We first prove that $\sim$ is an equivalence relation. Let $\zeta \in \mathcal{L}$. Then $\mu_C(\zeta-\zeta) = \mu_C(0)$, $\gamma_C(\zeta - \zeta) = \gamma_C(0)$ and $\psi_C(\zeta - \zeta) = \psi_C(0)$. Thus $\zeta \sim \zeta$. Let $\zeta, \xi \in \mathcal{L}$. If $\zeta \sim \xi$, then $\mu_C(\zeta - \xi) = \mu_C(0)$, $\gamma_C(\zeta - \xi) = \gamma_C(0)$, $\psi_C(\zeta - \xi) = \psi_C(0)$. Thus

$$\mu_C(\xi - \zeta) = \mu_C(-(\zeta - \xi)) \geq \mu_C(\zeta - \xi) = \mu_C(0),$$
$$\gamma_C(\xi - \zeta) = \gamma_C(-(\zeta - \xi)) \geq \gamma_C(\zeta - \xi) = \gamma_C(0),$$
$$\psi_C(\xi - \zeta) = \psi_C(-(\zeta - \xi)) \leq \psi_C(\zeta - \xi) = \psi_C(0).$$

So $\xi \sim \zeta$. Let $\zeta, \xi, \nu \in \mathcal{L}$. If $\zeta \sim \xi$ and $\xi \sim \nu$, then $\mu_C(\zeta - \xi) = \mu_C(0)$, $\mu_C(\xi - \nu) = \mu_C(0)$, $\mu_C(\zeta - \xi) = \mu_C(0)$, $\mu_C(\xi - \nu) = \mu_C(0)$ and $\psi_C(\zeta - \xi) = \psi_C(0)$, $\psi_C(\xi - \nu) = \psi_C(0)$. Thus it follows that

$$\mu_C(\zeta - \nu) = \mu_C(\zeta - \xi + \xi - \nu) \geq \min\{\mu_C(\zeta - \xi), \mu_C(\xi - \nu)\} = \mu_C(0),$$
$$\gamma_C(\zeta - \nu) = \gamma_C(\zeta - \xi + \xi - \nu) \geq \min\{\gamma_C(\zeta - \xi), \gamma_C(\xi - \nu)\} = \gamma_C(0),$$
$$\psi_C(\zeta - \nu) = \psi_C(\zeta - \xi + \xi - \nu) \leq \max\{\psi_C(\zeta - \xi), \psi_C(\xi - \nu)\} = \psi_C(0).$$

So $\zeta \sim \nu$ for all $\zeta, \xi, \nu \in \mathcal{L}$. Hence $\sim$ is an equivalence relation on $\mathcal{L}$.
We now verify that $\sim$ is a congruence relation on $\mathcal{L}$. For this, we let $\zeta \sim \xi$ and $\xi \sim \nu$. Then

$$\mu_C(\zeta - \xi) = \mu_C(0), \mu_C(\xi - \nu) = \mu_C(0),$$
$$\gamma_C(\zeta - \xi) = \mu_C(0), \gamma_C(\xi - \nu) = \mu_C(0),$$
$$\psi_C(\zeta - \xi) = \psi_C(0), \psi_C(\xi - \nu) = \psi_C(0).$$

Now, for $\zeta_1, \zeta_2, \xi_1, \zeta_2 \in \mathcal{L}$, we have

$$\begin{array}{rcl}
\mu_C((\zeta_1+\zeta_2)-(\xi_1+\xi_2)) & = & \mu_C((\zeta_1-\xi_1)+(\zeta_2-\xi_2)) \\
& \geq & \min\{\mu_C(\zeta_1-\xi_1), \mu_C(\zeta_2-\xi_2)\} \\
& = & \mu_C(0), \\
\gamma_C((\zeta_1+\zeta_2)-(\xi_1+\xi_2)) & = & \gamma_C((\zeta_1-\xi_1)+(\zeta_2-\xi_2)) \\
& \geq & \min\{\gamma_C(\zeta_1-\xi_1), \gamma_C(\zeta_2-\xi_2)\} \\
& = & \gamma_C(0), \\
\psi_C((\zeta_1+\zeta_2)-(\xi_1+\xi_2)) & = & \psi_C((\zeta_1-\xi_1)+(\zeta_2-\xi_2)) \\
& \leq & \max\{\psi_C(\zeta_1-\xi_1), \psi_C(\zeta_2-\xi_2)\} \\
& = & \psi_C(0), \\
\mu_C(\alpha\zeta_1-\alpha\xi_1) & = & \mu_C(\alpha(\zeta_1-\xi_1)) \\
& \geq & \mu_C(\zeta_1-\xi_1) \\
& = & \mu_C(0), \\
\gamma_C(\alpha\zeta_1-\alpha\xi_1) & = & \gamma_C(\alpha(\zeta_1-\xi_1)) \\
& \geq & \gamma_C(\zeta_1-\xi_1) \\
& = & \gamma_C(0), \\
\psi_C(\alpha\zeta_1-\alpha\xi_1) & = & \psi_C(\alpha(\zeta_1-\xi_1)) \\
& \leq & \psi_C(\zeta_1-\xi_1) \\
& = & \psi_C(0), \\
\mu_C([\zeta_1,\zeta_2]-[\xi_1,\xi_2]) & = & \mu_C([\zeta_1-\xi_1],[\zeta_2-\xi_2]) \\
& \geq & \min\{\mu_C(\zeta_1-\xi_1), \mu_C(\zeta_2-\xi_2)\} \\
& = & \mu_C(0), \\
\gamma_C([\zeta_1,\zeta_2]-[\xi_1,\xi_2]) & = & \gamma_C([\zeta_1-\xi_1],[\zeta_2-\xi_2]) \\
& \geq & \min\{\gamma_C(\zeta_1-\xi_1), \gamma_C(\zeta_2-\xi_2)\} \\
& = & \gamma_C(0), \\
\psi_C([\zeta_1,\zeta_2]-[\xi_1,\xi_2]) & = & \psi_C([\zeta_1-\xi_1],[\zeta_2-\xi_2]) \\
& \leq & \max\{\psi_C(\zeta_1-\xi_1), \psi_C(\zeta_2-\xi_2)\} \\
& = & \psi_C(0).
\end{array}$$

That is, $\zeta_1+\zeta_2 \sim \xi_1+\xi_2$, $\alpha\zeta_1 \sim \alpha\xi_1$ and $[\zeta_1,\zeta_2] \sim [\xi_1,\xi_2]$. Thus $\sim$ is indeed a congruence relation on $\mathcal{L}$. □

Definition 2.8. Let $\mathcal{L}$ be a nonempty set. Then we call a complex mapping $C = (\mu_C, \gamma_C, \psi_C) : \mathcal{L}\times\mathcal{L} \to [0,1]\times[0,1]\times[0,1]$ a *neutrosophic relation* on $\mathcal{L}$, if $\mu_C(\zeta,\xi)+\gamma_C(\zeta,\xi)+\psi_C(\zeta,\xi) \leq 1$ for all $(\zeta,\xi) \in \mathcal{L}\times\mathcal{L}$.

Definition 2.9. Let $C = (\mu_C, \gamma_C, \psi_C)$ and $D = (\mu_D, \gamma_D, \psi_D)$ be neutrosophic sets on a set $\mathcal{L}$. If $C = (\mu_C, \gamma_C, \psi_C)$ is a neutrosophic relation on a set $\mathcal{L}$, then $C = (\mu_C, \gamma_C, \psi_C)$ is said to be a *neutrosophic relation* on $D = (\mu_D, \gamma_D, \psi_D)$, if it satisfies the following conditions: for all $\zeta, \xi \in \mathcal{L}$,

$$\mu_C(\zeta,\xi) \leq \min\{(\mu_D(\zeta), \mu_D(\xi)\},$$
$$\gamma_C(\zeta,\xi) \leq \min\{(\gamma_D(\zeta), \gamma_D(\xi)\},$$
$$\psi_C(\zeta,\xi) \geq \max\{\psi_D(\zeta), \psi_D(\xi)\}.$$

Definition 2.10. Let $C = (\mu_C, \gamma_C, \psi_C)$ and $D = (\mu_D, \gamma_D, \psi_D)$ be two neutrosophic sets on a set $\mathcal{L}$. Then the *generalized Cartesian product* $C \times D$ is defined as

$$C \times D = (\mu_C, \gamma_C, \psi_C) \times (\mu_D, \gamma_D, \psi_D) = (\mu_C \times \mu_D, \gamma_C \times \gamma_D, \psi_C \times \psi_D),$$

where $(\mu_C \times \mu_D)(\zeta,\xi) = \min\{\mu_C(\zeta),\mu_D(\xi)\}$, $(\gamma_C \times \gamma_D)(\zeta,\xi) = \min\{\gamma_C(\zeta),\gamma_D(\xi)\}$ and $(\psi_C \times \psi_D)(\zeta,\xi) = \max\{\psi_C(\zeta),\psi_D(\xi)\}$.

Note that the generalized Cartesian product $C \times D$ is a neutrosophic set in $\mathcal{L} \times \mathcal{L}$, if $\min\{\mu_C(\zeta),\mu_D(\xi)\} + \min\{\gamma_C(\zeta),\gamma_D(\xi)\} + \max\{\psi_C(\zeta),\psi_D(\xi)\} \leq 1$.

Proposition 2.11. *Let $C = (\mu_C,\gamma_C,\psi_C)$ and $D = (\mu_D,\gamma_D,\psi_D)$ be neutrosophic sets on a set $\mathcal{L}$. Then*

(1) *$C \times D$ is a neutrosophic relation on $\mathcal{L}$,*
(2) *$U(\mu_C \times \mu_D, t) = U(\mu_C,t) \times U(\mu_D,t)$, $U(\gamma_C \times \gamma_D,t) = U(\gamma_C,t) \times U(\gamma_D,t)$ and $L(\psi_C \times \psi_D,t) = L(\psi_C,t) \times L(\psi_D,t)$ for all $t \in [0,1]$.*

Theorem 2.12. *Let $C = (\mu_C,\gamma_C,\psi_C)$ and $D = (\mu_D,\gamma_D,\psi_D)$ be two neutrosophic Lie subalgebras of a Lie algebras $\mathcal{L}$. Then $C \times D$ is a neutrosophic Lie subalgebra of $\mathcal{L} \times \mathcal{L}$.*

Proof. Let $\zeta = (\zeta_1,\zeta_2)$ and $\xi = (\xi_1,\xi_2) \in \mathcal{L} \times \mathcal{L}$ and $r \in F$. Then

$$\begin{aligned}
(\mu_C \times \mu_D)(\zeta+\xi) &= (\mu_C \times \mu_D)((\zeta_1,\zeta_2)+(\xi_1,\xi_2))\\
&= (\mu_C \times \mu_D)((\zeta_1+\xi_1,\zeta_2+\xi_2))\\
&= \min(\mu_C(\zeta_1+\xi_1),\mu_D(\zeta_2+\xi_2))\\
&\geq \min(\min(\mu_C(\zeta_1),\mu_C(\xi_1)),\min(\mu_D(\zeta_2),\mu_D(\xi_2)))\\
&= \min(\min(\mu_C(\zeta_1),\mu_D(\zeta_2)),\min(\mu_C(\xi_1),\mu_D(\xi_2)))\\
&= \min((\mu_C \times \mu_D)(\zeta_1,\zeta_2)),(\mu_C \times \mu_D)(\xi_1,\xi_2))\\
&= \min((\mu_C \times \mu_D)(\zeta),(\mu_C \times \mu_D)(\xi)),
\end{aligned}$$

$$\begin{aligned}
(\gamma_C \times \gamma_D)(\zeta+\xi) &= (\gamma_C \times \gamma_D)((\zeta_1,\zeta_2)+(\xi_1,\xi_2))\\
&= (\gamma_C \times \gamma_D)((\zeta_1+\xi_1,\zeta_2+\xi_2))\\
&= \min(\gamma_C(\zeta_1+\xi_1),\gamma_D(\zeta_2+\xi_2))\\
&\geq \min(\min(\gamma_C(\zeta_1),\gamma_C(\xi_1)),\min(\gamma_D(\zeta_2),\gamma_D(\xi_2)))\\
&= \min(\min(\gamma_C(\zeta_1),\gamma_D(\zeta_2)),\min(\gamma_C(\xi_1),\gamma_D(\xi_2)))\\
&= \min((\gamma_C \times \gamma_D)(\zeta_1,\zeta_2)),(\gamma_C \times \gamma_D)(\xi_1,\xi_2))\\
&= \min((\gamma_C \times \gamma_D)(\zeta),(\gamma_C \times \gamma_D)(\xi)),
\end{aligned}$$

$$\begin{aligned}
(\psi_C \times \psi_D)(\zeta+\xi) &= (\psi_C \times \psi_D)((\zeta_1,\zeta_2)+(\xi_1,\xi_2))\\
&= (\psi_C \times \psi_D)((\zeta_1+\xi_1,\zeta_2+\xi_2))\\
&= \max(\psi_C(\zeta_1+\xi_1),\psi_D(\zeta_2+\xi_2))\\
&\leq \max(\max(\psi_C(\zeta_1),\psi_C(\xi_1)),\max(\psi_D(\zeta_2),\psi_D(\xi_2)))\\
&= \max(\max(\psi_C(\zeta_1),\psi_D(\zeta_2)),\max(\psi_C(\xi_1),\psi_D(\xi_2)))\\
&= \max((\psi_C \times \psi_D)(\zeta_1,\zeta_2)),(\psi_C \times \psi_D)(\xi_1,\xi_2))\\
&= \max((\psi_C \times \psi_D)(\zeta),(\psi_C \times \psi_D)(\xi)),
\end{aligned}$$

$$\begin{aligned}
(\mu_C \times \mu_D)(\alpha\zeta) &= (\mu_C \times \mu_D)(\alpha(\zeta_1,\zeta_2))\\
&= (\mu_C \times \mu_D)((\alpha\zeta_1,\alpha\zeta_2))\\
&= \min(\mu_C(\alpha\zeta_1),\mu_D(\alpha\zeta_2))\\
&\geq \min(\mu_C(\zeta_1),\mu_D(\zeta_2))\\
&= (\mu_C \times \mu_D)(\zeta_1,\zeta_2)\\
&= (\mu_C \times \mu_D)(\zeta),
\end{aligned}$$

$$\begin{aligned}(\gamma_C \times \gamma_D)(\alpha\zeta) &= (\gamma_C \times \gamma_D)(\alpha(\zeta_1, \zeta_2))\\ &= (\gamma_C \times \gamma_D)((\alpha\zeta_1, \alpha\zeta_2))\\ &= \min(\gamma_C(\alpha\zeta_1), \gamma_D(\alpha\zeta_2))\\ &\geq \min(\gamma_C(\zeta_1), \gamma_D(\zeta_2))\\ &= (\gamma_C \times \gamma_D)(\zeta_1, \zeta_2)\\ &= (\gamma_C \times \gamma_D)(\zeta),\end{aligned}$$

$$\begin{aligned}(\psi_C \times \psi_D)(\alpha\zeta) &= (\psi_C \times \psi_D)(\alpha(\zeta_1, \zeta_2))\\ &= (\psi_C \times \psi_D)((\alpha\zeta_1, \alpha\zeta_2))\\ &= \max(\psi_C(\alpha\zeta_1), \psi_D(\alpha\zeta_2))\\ &\leq \max(\psi_C(\zeta_1), \psi_D(\zeta_2))\\ &= (\psi_C \times \psi_D)(\zeta_1, \zeta_2)\\ &= (\psi_C \times \psi_D)(\zeta),\end{aligned}$$

$$\begin{aligned}(\mu_C \times \mu_D)([\zeta, \xi]) &= (\mu_C \times \mu_D)([(\zeta_1, \zeta_2), (\xi_1, \xi_2)])\\ &\geq \min(\min(\mu_C(\zeta_1), \mu_D(\zeta_2)), \min(\mu_C(\xi_1), \mu_D(\xi_2)))\\ &= \min((\mu_C \times \mu_D)(\zeta_1, \zeta_2)), (\mu_C \times \mu_D)(\xi_1, \xi_2))\\ &= \min((\mu_C \times \mu_D)(\zeta), (\mu_C \times \mu_D)(\xi)),\end{aligned}$$

$$\begin{aligned}(\gamma_C \times \gamma_D)([\zeta, \xi]) &= (\gamma_C \times \gamma_D)([(\zeta_1, \zeta_2), (\xi_1, \xi_2)])\\ &\geq \min(\min(\gamma_C(\zeta_1), \gamma_D(\zeta_2)), \min(\gamma_C(\xi_1), \gamma_D(\xi_2)))\\ &= \min((\gamma_C \times \gamma_D)(\zeta_1, \zeta_2)), (\gamma_C \times \gamma_D)(\xi_1, \xi_2))\\ &= \min((\gamma_C \times \gamma_D)(\zeta), (\gamma_C \times \gamma_D)(\xi)),\end{aligned}$$

$$\begin{aligned}(\psi_C \times \psi_D)([\zeta, \xi]) &= (\psi_C \times \psi_D)([(\zeta_1, \zeta_2), (\xi_1, \xi_2)])\\ &\leq \max(\max(\psi_C(\zeta_1), \psi_D(\zeta_2)), \max(\psi_C(\xi_1), \psi_D(\xi_2)))\\ &= \max((\psi_C \times \psi_D)(\zeta_1, \zeta_2)), (\psi_C \times \psi_D)(\xi_1, \xi_2))\\ &= \max((\psi_C \times \psi_D)(\zeta), (\psi_C \times \psi_D)(\xi)).\end{aligned}$$

This shows that $C \times D$ is a neutrosophic Lie subalgebra of $\mathcal{L} \times \mathcal{L}$. □

Definition 2.13. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lie algebras over a field F. Then a linear transformation $f : \mathcal{L}_1 \to \mathcal{L}_2$ is called a *Lie homomorphism*, if $f([\zeta, \xi]) = [f(\zeta), f(\xi)]$ holds for all $\zeta, \xi \in \mathcal{L}_1$.

For the Lie algebras $\mathcal{L}_1$ and $\mathcal{L}_2$, it can be easily observed that if $f : \mathcal{L}_1 \to \mathcal{L}_2$ is a Lie homomorphism and C is a neutrosophic Lie subalgebra of $\mathcal{L}_2$, then the neutrosophic set $f^{-1}(C)$ of $\mathcal{L}_1$ is also a neutrosophic Lie subalgebra.

Definition 2.14. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lie algebras. Then a Lie homomorphism $f : \mathcal{L}_1 \to \mathcal{L}_2$ is said to *have a natural extension* $f : I^{\mathcal{L}_1} \to I^{\mathcal{L}_2}$ defined by for all $C = (\mu_C, \gamma_C, \psi_C) \in I^{\mathcal{L}_1}, \xi \in \mathcal{L}_2$. $f(\mu_C)(\xi) = \sup\{\mu_C(\zeta) : \zeta \in f^{-1}(\xi)\}$ $f(\psi_C)(\xi) = \inf\{\psi_C(\zeta) : \zeta \in f^{-1}(\xi)\}$.

Theorem 2.15. *The homomorphic image of a neutrosophic Lie subalgebra is also a neutrosophic Lie subalgebra of its co-domain.*

Proof. Let $\xi_1, \xi_2 \in \mathcal{L}_2$. Then $\{\zeta : \zeta \in f^{-1}(\xi_1+\xi_2)\} \supseteq \{\zeta_1+\zeta_2 : \zeta_1 \in f^{-1}(\xi_1)\ and\ \zeta_2 \in f^{-1}(\xi_2)\}$. Now, we have

$$\begin{array}{rcl} f(\mu_C)(\xi_1+\xi_2) & = & \sup\{\mu_C(\zeta) : \zeta \in f^{-1}(\xi_1+\xi_2)\} \\ & \geq & \{\mu_C(\zeta_1+\zeta_2) : \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2)\} \\ & \geq & \sup\{\min\{\mu_C(\zeta_1), \mu_C(\zeta_2)\} : \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2)\} \\ & = & \min\{\sup\{\mu_C(\zeta_1) : \zeta_1 \in f^{-1}(\xi_1)\}, \sup\{\mu_C(\zeta_2) : \zeta_2 \in f^{-1}(\xi_2)\}\} \\ & = & \min\{f(\mu_C)(\xi_1), f(\mu_C)(\xi_2)\}. \end{array}$$

For $\xi \in \mathcal{L}_2$ and $\alpha \in F$, we have

$$\{\zeta : \zeta \in f^{-1}(\alpha\zeta)\} \supseteq \{\alpha\zeta : \zeta \in f^{-1}(\xi)\}.$$

$$\begin{array}{rcl} f(\mu_C)(\alpha\xi) & = & \sup\{\mu_C(\alpha\zeta) : \zeta \in f^{-1}(\xi)\} \\ & \geq & \{\mu_C(\alpha\zeta) : \zeta \in f^{-1}(\alpha\xi)\} \\ & \geq & \sup\{\mu_C(\zeta) : \zeta \in f^{-1}(\xi)\} \\ & = & f(\mu_C)(\xi). \end{array}$$

If $\xi_1, \xi_2 \in \mathcal{L}_2$, then $\{\zeta : \zeta \in f^{-1}(\xi_1+\xi_2)\} \supseteq \{\zeta_1+\zeta_2 : \zeta_1 \in f^{-1}(\xi_1)\ and\ \zeta_2 \in f^{-1}(\xi_2)\}$. Now, we have

$$\begin{array}{rcl} f(\mu_C)(\zeta i_1+\zeta i_2) & = & \sup\{\mu_C(\zeta) : x \in f^{-1}(\xi_1+\xi_2)\} \\ & \geq & \{\mu_C(\zeta_1+\zeta_2) : \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2)\} \\ & \geq & \sup\{\min\{\mu_C(\zeta_1), \mu_C(\zeta_2)\} : \zeta_1 \in f^{-1}(\xi_1), \zeta_2 \in f^{-1}(\xi_2)\} \\ & = & \min\{\sup\{\mu_C(\zeta_1) : \zeta_1 \in f^{-1}(\xi_1)\}, \sup\{\mu_C(\zeta_2) : \zeta_2 \in f^{-1}(\xi_2)\}\} \\ & = & \min\{f(\mu_C)(\xi_1), f(\mu_C)(\xi_2)\}. \end{array}$$

Thus $f(\mu_C)$ is a fuzzy Lie algebra of $\mathcal{L}_2$. In the same manner, we can prove that $f(\psi_C)$ is a fuzzy Lie subalgebra of $\mathcal{L}_2$. So $f(C) = (f(\mu_C), f(\psi_C))$ is a neutrosophic Lie subalgebra of $\mathcal{L}_2$. □

Definition 2.16. Let $C = (\mu_C, \gamma_C, \psi_C)$ and $D = (\mu_D, \gamma_D, \psi_D)$ be neutrosophic subalgebras of $\mathcal{L}$. Then C is said to *be of the same type of* D, if there exists $f \in Aut(L)$ such that $C = D \circ f$, that is, $\mu_C(\zeta) = \mu_D(f(\zeta))$, $\gamma_C(\zeta) = \gamma_D(f(\zeta))$, $\psi_C(\zeta) = \psi_D(f(\zeta))$ for all $\zeta \in \mathcal{L}$.

Theorem 2.17. *Let $C = (\mu_C, \gamma_C, \psi_C)$ and $D = (\mu_D, \gamma_D, \psi_D)$ be two neutrosophic subalgebras of $\mathcal{L}$. Then C is a neutrosophic subalgebra having the same type of D if and only if C is isomorphic to D.*

Proof. We only need to prove the necessity part because the sufficiency part is trivial. Let $C = (\mu_C, \gamma_C, \psi_C)$ be a neutrosophic subalgebra having the same type of D. Then there exists $C \in Aut(L)$ such that $\mu_C(\zeta) = \mu_D(\varphi(\zeta))$, $\gamma_C(\zeta) = \gamma_D(\varphi(\zeta))$, $\psi_C(\zeta) = \psi_D(\varphi(\zeta))\ \forall\zeta \in \mathcal{L}$. Let $f : C(L) \to D(L)$ be a mapping defined by $f(\varphi(\zeta)) = B(\varphi(\zeta))$ for all $\zeta \in \mathcal{L}$, that is, $f(\mu_C(\zeta)) = \mu_D(\varphi(\zeta))$, $f(\gamma_C(\zeta)) = \gamma_D(\varphi(\zeta))$, $f(\psi_C(\zeta)) = \psi_D(\varphi(\zeta))\ \forall\zeta \in \mathcal{L}$. Then it is clear that f is surjective. Also, f is injective because if $f(\mu_C(\zeta)) = f(\mu_C(\xi))$ for all $\zeta, \xi \in \mathcal{L}$, then $\mu_D(\varphi(\zeta)) = \mu_D(\varphi(\xi))$ and thus $\mu_C(\zeta) = \mu_D(\xi)$. By the same token, we have $f(\psi_C(\zeta)) = f(\psi_C(\xi)) \Rightarrow \psi_C(\zeta) = \psi_D(\xi)$ for all $\zeta \in \mathcal{L}$. Finally, f is a homomorphism because for $\zeta, \xi \in \mathcal{L}$,

$$f(\mu_C(\zeta+\xi)) = \mu_D(\varphi(\zeta+\xi)) = \mu_D(\varphi(\zeta)+\varphi(\xi)),$$

$$f(\gamma_C(\zeta+\xi)) = \gamma_D(\varphi(\zeta+\xi)) = \gamma_D(\varphi(\zeta)+\varphi(\xi)),$$

$$f(\psi_C(\zeta+\xi)) = \psi_D(\varphi(\zeta+\xi)) = \psi_D(\varphi(\zeta)+\varphi(\xi)),$$
$$f(\mu_C(\alpha\zeta)) = \mu_D(\varphi(\alpha\zeta)) = \alpha\mu_D(\varphi(\zeta)),$$
$$f(\gamma_C(\alpha\zeta)) = \gamma_D(\varphi(\alpha\zeta)) = \alpha\gamma_D(\varphi(\zeta)),$$
$$f(\psi_C(\alpha\zeta)) = \psi_D(\varphi(\alpha\zeta)) = \alpha\psi_D(\varphi(\zeta)),$$
$$f(\mu_C([\zeta,\xi])) = \mu_D(\varphi([\zeta,\xi])) = \mu_D([\varphi(\zeta),\varphi(\xi)]),$$
$$f(\gamma_C([\zeta,\xi])) = \gamma_D(\varphi([\zeta,\xi])) = \gamma_D([\varphi(\zeta),\varphi(\xi)]),$$
$$f(\psi_C([\zeta,\xi])) = \psi_D(\varphi([\zeta,\xi])) = \psi_D([\varphi(\zeta),\varphi(\xi)]).$$

Hence $C = (\mu_C, \gamma_C, \psi_C)$ is isomorphic to $D = (\mu_D, \gamma_D, \psi_D)$. □

3. Conclusion

Presently, science and technology are featured with complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models are based on an extension of the ordinary set theory such as bifuzzy sets and soft sets. In the present paper, we have presented the basic properties on neutrosophic Lie subalgebra of a Lie algebra. The obtained results probably can be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks , genetic algorithms, neural networks, expert systems, decision making, automata theory and medical diagnosis. In our opinion the future study of Lie algebras can be extended with the study of (i) neutrosophic roughness in Lie algebras and (ii) neutroosphic rough Lie algebras.

Acknowledgements. The authors thank the referees for suggesting to add some more relevant references [6], [7] and [8] and their remarks which improved the quality of the paper.

References

[1] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[2] F. Smarandache, A Unifying Field in Logics: Neutrosophic logic, Neutrosophy, Neutrosophic set, Neutrosophic probability, American Reserch Press, Rehoboth, NM. 1999.
[3] S. Pramanik, Single-valued neutrosophic set: An overview. In: N. Rezaei (Eds) Transdisciplinarity, Integrated Science 5 (2022) 563–608. Springer, Cham. https://doi.org/10.1007/978-3-030-94651-7-26.
[4] F. Smarandache and S. Pramanik, New trends in neutrosophic theory and applications, Vol. II, Ponns Editions, Brussels, Belgium, EU. 2018.
[5] S. Das, R. Das, B. Shil and B. C. Tripathy, Lie-algebra of single-valued pentapartitioned neutrosophic set, Neutrosophic Sets and Systems 51 (2022) 157–171.
[6] S. Abdullayev and L. Nesibova, Neutrosophic Lie Algebra, Inter. Math. Forum 14 (2) (2019) 95–106.
[7] M. Akram, H. Gulzar, K. P. Shum, Single-valued neutrosophic Lie algebras, Journal of Mathematical Research with Applications 39 (2) (2019) 141–152.
[8] M. Parimala, F. Smarandache, M. Al Tahan and C. Ozel, On complex neutrosophic Lie algebras, Palestine Journal of Mathematics, Palestine Polytechnic University-PPU, 11 (1) (2022) 235–242.

S. Jafari (jafaripersia@gmail.com)
Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark

S. R. Vidhya (tmrvani@gmail.com)
Department of Mathematics, Rajah Serfoji Government College (affliated to Bharathdasan University), Thanjavur-613005, Tamilnadu, India

N. Rajesh (nrajesh_topology@yahoo.co.in)
Department of Mathematics, Rajah Serfoji Government College (affliated to Bharathdasan University), Thanjavur-613005, Tamilnadu, India

A. Pigazzini (pigazzini@topositus.com)
Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark