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$(3,3,\sqrt[3]{\cdot})$-picture fuzzy structures of a BCC-ideal of BCC-algebras and their correlation coefficient

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Abstract. A picture fuzzy set is one of the generalizations of Atanassove's (IFSs) fuzzy set. Under this environment, in this manuscript, we familiarize a new type of extensions of fuzzy sets called cubic root fuzzy sets (briefly, $\sqrt[3]{\cdot}$-Fuzzy sets) and Fermatean fuzzy sets to contrast $(3,3,\sqrt[3]{\cdot})$-picture sets. We introduce the notion of $(3,3,\sqrt[3]{\cdot})$- picture fuzzy BCC-ideals of BCC-algebras. After then, we study the homomorphic image and inverse image of $(3,3,\sqrt[3]{\cdot})$-picture fuzzy BCC-ideals under homomorphism of BCC-algebras. Moreover, the Cartesian product of $(3,3,\sqrt[3]{\cdot})$-picture fuzzy BCC- ideals of BCC-algebras is given. Finally, we introduce the concept of correlation for $(3,3,\sqrt[3]{\cdot})$-picture fuzzy sets, which is a new extension of the correlation of Atanassove's IFSs and investigated several properties.

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Keywords: BCC-algebra, Fuzzy BCC-ideal, $(3,3,\sqrt[3]{\cdot})$-picture fuzzy BCC-ideal, The pre-image of a $(3,3,\sqrt[3]{\cdot})$-picture fuzzy BCC-ideal under a homomorphism of BCC-algebras, The Cartesian product of $(3,3,\sqrt[3]{\cdot})$-picture BCC-ideal, Correlation coefficient for $(3,3,\sqrt[3]{\cdot})$-picture fuzzy sets.

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1. Introduction

In 1978, Iséki and Tanaka [1] introduced the notion of BCK-algebras. Iséki [2] introduced the concept of a BCI-algebra which is a generalization of BCK-algebra. Since then, numerous mathematical papers have been written investigating the algebraic properties of BCK/BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras. In particular, the emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. Iséki posed an interesting problem (solved by Wron$'$ski [3]) whether the class of BCK-algebras is a variety. In connection with this problem, Komori [4] introduced a notion of BCC-algebras, and Dudek [5] redefined the concept of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori. In [6], Dudek and Zhang introduced the notion of BCC-ideals of BCC-algebras and described connections between such ideals and congruences. In 1956, Zadeh [7] introduced the concept of fuzzy sets. At present, this concept has been applied to many mathematical branches. The idea of "intuitionistic fuzzy set" was first published by Atanassov [8] as a generalization of the notion of fuzzy sets. In 1991, Xi [9] applied fuzzy sets to BCI, BCK, MV-algebras. Dudek and Jun [10] considered the fuzzification of BCC-ideals of BCC-algebras. In 2013, including the measure of neutral membership and generalizing the notion of intuitionistic fuzzy set, the concept of picture fuzzy set was initiated by Cuong [11] (See [12, 13] for further research). After the initiation of the picture fuzzy set, different types of research works in the context of picture fuzzy set. Yager [14, 15] put forward the concept of Pythagorean fuzzy sets which is a generalization of intuitionistic fuzzy sets, and it is a more powerful tool to solve uncertain problems. Ibrahim et al. [16] defined a new generalization of Pythagorean fuzzy sets called $(3;2)$-*fuzzy sets*. The main advantage of $(3;2)$-fuzzy sets is that they can characterize more vague cases than Pythagorean fuzzy sets, which can be exploited in many decision-making problems. In 2019, Senapati et al. [17] initiated Fermatean fuzzy sets. Fermatean fuzzy set is characterized by membership and non membership grade restricted that cube sum of its membership grade and non membership grade is less than or equal to one. Correlation coefficient is one of the hot research topics in IFS theory, and it has received a lot of attention from various researchers (See [18, 19, 20, 21, 22, 23, 24]). It is widely used in statistical analysis and engineering sciences. For instance, Gerstenkorn and Manko [18] introduced the correlation coefficients of Atanassove's IFSs. In this paper, we introduce the notion of $(3,3,\sqrt[3]{\cdot})$-picture fuzzy BCC-ideals of BCC-algebras and we study the homomorphic image and inverse image of $(3,3,\sqrt[3]{\cdot})$-picture fuzzy BCC-ideals under a homomorphism of BCC-algebras. Moreover, the Cartesian product of $(3,3,\sqrt[3]{\cdot})$-picture fuzzy BCC-ideal of BCC-algebras is given. Finally, we introduce the concept of correlation for $(3,3,\sqrt[3]{\cdot})$-picture fuzzy sets, which is a new extension of the correlation of Atanassove's IFSs and investigated several properties.

2. Preliminaries

Now we review some definitions and properties that will be useful in our results.

Definition 2.1 (See [5, 25, 26])**.** An algebra system $(X,*,0)$ of type $(2,0)$ is called a *BCC-algebra*, if it satisfies the following axioms: for any $x,\ y,\ z\in X$,

(BCC_1) $[(x*y)*(z*y)]*(x*z)=0$,

(BCC_2) $x*0=x$,

(BCC_3) $x*x=0$,

(BCC_4) $0*x=0$,

(BCC_5) $x*y=0$ and $y*x=0$ imply $x=y$.

On a BCC-algebra X, we can define a partial order $\leq$ on X by: for any $x,\ y \in X$,

$$x \leq y \Leftrightarrow x * y = 0.$$

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (See Dudek [25]). Note that a BCC-algebra is a BCK-algebra if and only if

$$(x * y) * z = (x * z) * y \text{ for any } x,\ y,\ z \in X.$$

In what follows, X will denote a BCC-algebra unless otherwise specified.

Result 2.2 (See [25]). *The followings hold: for any* $x,\ y,\ z \in X$,
(1) $(x * y) * x = 0$,
(2) $x \leq y$ *implies* $x * z \leq y * z$,
(3) $x \leq y$ *implies* $z * y \leq z * x$,
(4) $(x * y) * (z * y) \leq x * z$.

For any $x,\ y \in X$, we denote $x \wedge y = y * (y * x)$.
We will refer to is a KU-algebra unless otherwise indicated.

Definition 2.3 ([25]). Let I be a nonempty subset of X. Then I is called an BCK*-ideal* of X, if it satisfies the following axioms: for any $x,\ y \in X$,
($BCKI_1$) $0 \in I$,
($BCKI_2$) $x * y,\ y \in I$ imply $x \in I$.

Definition 2.4 ([25]). Let I be a nonempty subset of X. Then I is called an BCC*-ideal* of X, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
($BCCI_1$) $0 \in I$,
($BCCI_2$) $(x * y) * z,\ y \in I$ imply $x * z \in I$.

Example 2.5 (See [5]). Let $X = \{0, 1, 2, 3\}$ be a in which the operation $*$ is defined as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Table 2.1

Then we can easily check that $(X, *, 0)$ is a BCC-algebra.

Definition 2.6 (See[8]). Let X be a nonempty set of X. Then a mapping $A = (A^{\in}, A^{\notin})X \to [0,1] \times [0,1]$ is called an *intuitionistic fuzzy set* (briefly, IFS) in X, if $0 \leq A^{\in}(x) + A^{\notin}(x)$ for each $x \in X$. In this case, $A^{\in}(x)$ [resp. $A^{\notin}(x)$] is called the *degree of membership* [resp. *nonmembership*] of A at $x \in X$.

Definition 2.7 ([11, 12]). Let X be a nonempty universe set. Then a mapping $A = \langle A_{_P}, A_{_I}, A_{_N} \rangle : X \to [0,1] \times [0,1] \times [0,1]$ is called a *picture fuzzy set* (briefly, PFS) in X, if $0 \leq A_{_P}(x) + A_{_I}(x) + A_{_N}(x) \leq 1$ for each $x \in X$. In this case, $A_{_P}(x)$ [resp. $A_{_I}(x)$ and $A_{_N}(x)$] is called the *positive-membership* [resp. the *neutral-membership* and the *negativ-emembership*] *degree* of A at $x \in X$.

Definition 2.8. Let X be a nonempty set and let $A = (A^{\in}, A^{\notin}) : X \to [0,1] \times [0,1]$ be a mapping. Then A is called:

(i) a Pythagorean fuzzy set in X (See [14]), if $0 \leq A^{\in 2}(x) + A^{\notin 2}(x) \leq 1$ for each $x \in X$,

(ii) an n-Pythagorean fuzzy set in X (See [27]), if $0 \leq A^{\in n}(x) + A^{\notin n}(x) \leq 1$ for each $x \in X$,

(iii) a Fermatean fuzzy set in X (See [17]), if $0 \leq A^{\in 3}(x) + A^{\notin 3}(x) \leq 1$ for each $x \in X$,

(iv) a $(3;2)$-fuzzy set in X (See [16, 28, 29]), if $0 \leq A^{\in 3}(x) + A^{\notin 2}(x) \leq 1$ for each $x \in X$,

(v) an $(n;m)$-fuzzy set in X (See [29]), if $0 \leq A^{\in n}(x) + A^{\notin m}(x) \leq 1$ for each $x \in X$,

(vi) an SR-fuzzy set in X (See [30]), if $0 \leq A^{\in 2}(x) + \sqrt{A^{\notin}(x)} \leq 1$ for each $x \in X$,

(vii) a CR-fuzzy set in X (See [31]), if $0 \leq A^{\in 3}(x) + \sqrt[3]{A^{\notin}(x)} \leq 1$ for each $x \in X$.

3. $(3, 3, \sqrt[3]{\cdot})$-picture fuzzy BCC-ideals of BCC-algebras

First of all, we introduce a new notion called $(3, 3, \sqrt[3]{\cdot})$-*picture fuzzy set* in a nonempty set X and study some of its properties. Next, we propose the concepts of subalgebras and ideals of BCC-algebras and deal with some of their properties.

Definition 3.1. Let X be a nonempty set. Then a mapping $\overline{A} = \langle A_P, A_I, A_N \rangle : X \to [0,1] \times [0,1] \times [0,1]$ is called a $(3, 3, \sqrt[3]{\cdot})$-*picture fuzzy set* in X (briefly, $(3, 3, \sqrt[3]{\cdot})$-PFS), if $0 \leq A_P^3(x) + A_I^3(x) + \sqrt[3]{A_N(x)} \leq 1$ for each $x \in X$. The *whole* [resp. *empty*] $(3, 3, \sqrt[3]{\cdot})$-*picture fuzzy set* in X, denoted by $\overline{1}$ [resp. $\overline{0}$], is a $(3, 3, \sqrt[3]{\cdot})$-picture fuzzy set in X defined as follows: for each $x \in X$,

$$\overline{1}(x) = \langle 1, 0, 0 \rangle \ \text{[resp. } \overline{0}(x) = \langle 0, 0, 1 \rangle].$$

We will denote the set of all $(3, 3, \sqrt[3]{\cdot})$-PFSs in X as $PFS_{(3,3,\sqrt[3]{\cdot})}(X)$.

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a reference set in which the picture fuzzy set $A = \langle A_P, A_I, A_N \rangle$ is given by the following table:

X	0	a	b	c	d
$A_P(x)$	0.4	0.3	0.3	0.2	0.1
$A_I(x)$	0.1	0.1	0.2	0.2	0.3
$A_N(x)$	0.006	0.004	0.002	0.001	0.001

Table 3.1

Then we obtain the $(3, 3, \sqrt[3]{\cdot})$-PFS $\overline{A}$ in X given by the following table:

X	0	a	b	c	d
$A_P^3(x)$	0.064	0.027	0.027	0.008	0.001
$A_I^3(x)$	0.001	0.001	0.008	0.008	0.027
$\sqrt[3]{A_N(x)}$	0.182	0.159	0.126	0.1	0.1

Table 3.2

Definition 3.3. Let X be a nonempty set and let $\overline{A},\ \overline{B} \in PFS_{(3,3,\sqrt[3]{\cdot})}(X)$. Then the *inclusion* $\overline{A} \subset \overline{B}$ and the *equality* $\overline{A} = \overline{B}$ as follows:
(i) $\overline{A} \subset \overline{B}$ iff $A^3_P(x) \leq B^3_P(x),\ A^3_I(x) \geq B^3_I(x),\ \sqrt[3]{A_N(x)} \leq \sqrt[3]{B_N(x)}$ for each $x \in X$,
(ii) $\overline{A} = \overline{B}$ iff $\overline{A} \subset \overline{B}$, $\overline{B} \subset \overline{A}$.

Definition 3.4. Let X be a nonempty set, let $\overline{A},\ \overline{B} \in PFS_{(3,3,\sqrt[3]{\cdot})}(X)$ and let $(\overline{A}_j)_{j\in J} \subset PFS_{(3,3,\sqrt[3]{\cdot})}(X)$, where J denotes an index set. Then the *complement* $\overline{A}^c = \left\langle A^c_P, A^c_I, A^c_N \right\rangle$ of $\overline{A}$, the *intersection* $\overline{A} \cap \overline{B}$ and the *union* $\overline{A} \cup \overline{B}$ of $\overline{A}$ and $\overline{B}$, the *intersection* $\bigcap_{j\in J} \overline{A}_j$ and the *union* $\bigcup_{j\in J} \overline{A}_j$ of $(\overline{A}_j)_{j\in J}$ are $(3, 3, \sqrt[3]{\cdot})$-PFSs in X defined respectively as follows: for each $x \in X$,
(i) $\overline{A}^c(x) = \langle 1 - A_N(x), 1 - A_I(x), 1 - A_P(x)\rangle$,
(ii) $(\overline{A} \cap \overline{B})(x) = \langle (A_P \cap B_P)(x), (A_I \cup B_I)(x), (A_N \cap B_N)(x)\rangle$, where
$(A_P \cap B_P)^3(x) = A^3_P(x) \wedge B^3_P(x),\ (A_I \cup B_I)^3(x) = A^3_I(x) \vee B^3_I(x),$
$\sqrt[3]{(A_N \cap B_N)(x)} = \sqrt[3]{A_N(x)} \wedge \sqrt[3]{B_N(x)},$
(iii) $(\overline{A} \cup \overline{B})(x) = \langle (A_P \cup B_P)(x), (A_I \cap B_I)(x), (A_N \cup B_N)(x)\rangle$, where
$(A_P \cup B_P)^3(x) = A^3_P(x) \vee B^3_P(x),\ (A_I \cap B_I)^3(x) = A^3_I(x) \wedge B^3_I(x),$
$\sqrt[3]{(A_N \cup B_N)(x)} = \sqrt[3]{A_N(x)} \vee \sqrt[3]{B_N(x)},$
(iv) $\left(\bigcap_{j\in J} \overline{A}_j\right)(x) = \left\langle \left(\bigcap_{j\in J} A_{P,j}\right)(x), \left(\bigcup_{j\in J} A_{I,j}\right)(x), \left(\bigcap_{j\in J} A_{N,j}\right)(x)\right\rangle$, where
$\left(\bigcap_{j\in J} A_{P,j}\right)^3(x) = \bigwedge_{j\in J} A^3_{P,j}(x),\ \left(\bigcup_{j\in J} A_{I,j}\right)^3(x) = \bigvee_{j\in J} A^3_{I,j}(x),$
$\sqrt[3]{\left(\bigcap_{j\in J} A_{N,j}\right)(x)} = \bigwedge_{j\in J} \sqrt[3]{A_{N,j}(x)},$
(v) $\left(\bigcup_{j\in J} \overline{A}_j\right)(x) = \left\langle \left(\bigcup_{j\in J} A_{P,j}\right)(x), \left(\bigcap_{j\in J} A_{I,j}\right)(x), \left(\bigcup_{j\in J} A_{N,j}\right)(x)\right\rangle$, where
$\left(\bigcup_{j\in J} A_{P,j}\right)^3(x) = \bigvee_{j\in J} A^3_{P,j}(x),\ \left(\bigcap_{j\in J} A_{I,j}\right)^3(x) = \bigwedge_{j\in J} A^3_{I,j}(x),$
$\sqrt[3]{\left(\bigcup_{j\in J} A_{N,j}\right)(x)} = \bigvee_{j\in J} \sqrt[3]{A_{N,j}(x)}.$

Proposition 3.5. *Let X be a nonempty set, let $\overline{A} \in PFS_{(3,3,\sqrt[3]{\cdot})}(X)$ and let $C(\overline{A}) = \langle C(A_P), C(A_I), C(A_N)\rangle$ be defined by: for each $x \in X$,*

$$C(A_P)(x) = A_N(x),\ C(A_N)(x) = A_P(x),$$

$$C(A_I)(x) = \sqrt[3]{1 - \left(\sqrt[3]{A_P(x)} + A^3_N(x) + \sqrt[3]{A_N(x)}\right)}.$$

Then $C(\overline{A}) \in PFS_{(3,3,\sqrt[3]{\cdot})}(X)$.

Proof. Let $x \in X$. Then we have

$$\begin{aligned} &C(A_P)^3(x) + C(A_I)^3(x) + \sqrt[3]{C(A_N)(x)} \\ &= A^3_N(x) + \left(1 - \left(\sqrt[3]{A_P(x)} + A^3_N(x) + \sqrt[3]{A_N(x)}\right)\right) + \sqrt[3]{A_P(x)} \\ &= 1 - \sqrt[3]{A_N(x)}. \end{aligned}$$

Since $0 \leq 1 - \sqrt[3]{A_N(x)} \leq 1$, $0 \leq C(A_P)^3(x) + C(A_I)^3(x) + \sqrt[3]{C(A_N)(x)} \leq 1$. Thus $C(\overline{A}) \in PFS_{(3,3,\sqrt[3]{\cdot})}(X)$. □

The following is an immediate consequence of Definition 3.4.

Lemma 3.6. *Let X be a nonempty sets and let $\overline{A}$, $\overline{B}$, $\overline{B} \in PFS_{(3,3,\sqrt[3]{\cdot})}(X)$. Then*
(1) $\overline{A} \cap \overline{A} = \overline{A}$, $\overline{A} \cup \overline{A} = \overline{A}$,
(2) $\overline{A} \cap \overline{B} = \overline{B} \cap \overline{A}$, $\overline{A} \cup \overline{B} = \overline{B} \cup \overline{A}$,
(3) $(\overline{A} \cap \overline{B}) \cap \overline{C} = \overline{A} \cap (\overline{B} \cap \overline{C})$, $(\overline{A} \cup \overline{B}) \cup \overline{C} = \overline{A} \cup (\overline{B} \cup \overline{C})$,
(4) $\overline{A} \cap \overline{B}) \cup \overline{C} = (\overline{A} \cup \overline{C}) \cap (\overline{B} \cup \overline{C})$, $\overline{A} \cup \overline{B}) \cap \overline{C} = (\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C})$,
(5) $\left(\overline{A} \cap \overline{B}\right)^c = \overline{A}^c \cup \overline{B}^c$, $\left(\overline{A} \cup \overline{B}\right)^c = \overline{A}^c \cap \overline{B}^c$,
(6) $\left(\overline{A}^c\right)^c = \overline{A}$,
(7) $\overline{A} \cap \overline{A}^c \neq \overline{0}$, $\overline{A} \cup \overline{A}^c \neq \overline{1}$ *in general* (See Example 3.7).

Example 3.7. Consider $\overline{A} = \langle A_P, A_I, A_N \rangle = \langle 0.04, 0.2, 0.006 \rangle \in PFS_{(3,3,\sqrt[3]{\cdot})}(X)$. Then clearly, $\overline{A}^c = \langle 0.994, 0.8, 0.96 \rangle$. Thus we have: for each $x \in X$,

$$\left(A_P \cup A_P^c\right)^3(x) = 0.04^3 \vee 0.994^3 = 0.997 \neq 1,$$

$$\left(A_P \cap A_P^c\right)^3(x) = 0.04^3 \wedge 0.994^3 = 0.000064 \neq 0.$$

So $A_P \cup A_P^c \neq \overline{1}$, $A_P \cap A_P^c \neq \overline{0}$.

Definition 3.8. Let $\overline{A}$ be a $(3,3,\sqrt[3]{\cdot})$-PFS in a BCC-algebra X. Then $\overline{A}$ is called a $(3,3,\sqrt[3]{\cdot})$-*picture fuzzy subalgebra* (briefly, $(3,3,\sqrt[3]{\cdot})$-PFSA) of X, if it satisfies the following conditions: for any x, $y \in X$,
(PFSA_1) $A_P^3(x * y) \geq A_P^3(x) \wedge A_P^3(y)$,
(PFSA_2) $A_I^3(x * y) \leq A_I^3(x) \vee A_I^3(y)$,
(PFSA_2) $\sqrt[3]{A_N(x * y)} \geq \sqrt[3]{A_N(x)} \wedge \sqrt[3]{A_N(y)}$.

Example 3.9. Let $X = \{0, 1, 2, 3\}$ be a set with the binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Table 3.3

Then we can easily check that $(X, *, 0)$ is a BCC-algebra. Define A_P, A_I, A_N as the following table:

X	0	1	2	3
$A_P(x)$	0.04	0.03	0.02	0.01
$A_I(x)$	0.02	0.02	0.02	0.01
$A_N(x)$	0.006	0.004	0.002	0.001

Table 3.4

Then we obtain the following table:
Thus we can easily see that $\overline{A} = \langle A_P, A_I, A_N \rangle$ is a $(3,3,\sqrt[3]{\cdot})$-PFSA of X.

X	0	1	2	3
$A^3_P(x)$	0.064	0.027	0.008	0.001
$A^3_I(x)$	0.008	0.008	0.027	0.027
$\sqrt[3]{A_N(x)}$	0.1817	0.1587	0.1260	0.1

Table 3.5

Lemma 3.10. *If $\overline{A} = \langle A_P, A_I, A_N\rangle$ is a $(3, 3, \sqrt[3]{\cdot})$-PFSA of a BCC-algebra X, then the following inequalities hold: for each $x \in X$,*

$$A^3_P(0) \geq A^3_P(x),\ A^3_I(0) \leq A^3_I(x),\ \sqrt[3]{A_N(0)} \geq \sqrt[3]{A_N(x)}.$$

Proof. Suppose $\overline{A}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFSA of X and let $x \in X$. Then we have

$$\begin{aligned} A^3_P(0) &= A^3_P(x * x) \text{ [By the axiom (BCC}_3)] \\ &\geq A^3_P(x) \wedge A^3_P(x) \text{ [By the condition (PFSA}_1)] \\ &= A^3_P(x), \\ A^3_I(0) &= A^3_I(x * x) \\ &\leq A^3_I(x) \vee A^3_I(x) \text{ [By the condition (PFSA}_2)] \\ &= A^3_I(x), \\ \sqrt[3]{A_N(0)} &= \sqrt[3]{A_N(x * x)} \\ &\geq \sqrt[3]{A_N(x)} \wedge \sqrt[3]{A_N(x)} \text{ [By the condition (PFSA}_3)] \\ &= \sqrt[3]{A_N(x)}. \end{aligned}$$

Thus the results hold. □

Definition 3.11. Let $\overline{A}$ be a $(3, 3, \sqrt[3]{\cdot})$-PFSA of a BCC-algebra X. Then $\overline{A}$ is called a $(3, 3, \sqrt[3]{\cdot})$-*picture fuzzy BCK-ideal* (briefly, $(3, 3, \sqrt[3]{\cdot})$-PFBCKI) of X, if it satisfies the following conditions: for any $x,\ y \in X$,

(PFBCKI$_1$) $A^3_P(0) \geq A^3_P(x),\ A^3_I(0) \leq A^3_I(x),\ \sqrt[3]{A_N(0)} \geq \sqrt[3]{A_N(x)}$,
(PFBCCI$_2$) $A^3_P(x) \geq A^3_P(x * y) \wedge A^3_P(y)$,
(PFBCCI$_3$) $A^3_I(x) \leq A^3_I(x * y) \vee A^3_I(y)$,
(PFBCCI$_4$) $\sqrt[3]{A_N(x)} \geq \sqrt[3]{A_N(x * y)} \wedge \sqrt[3]{A_N(y)}$.

Definition 3.12. Let $\overline{A}$ be a $(3, 3, \sqrt[3]{\cdot})$-PFSA of a BCC-algebra X. Then $\overline{A}$ is called a $(3, 3, \sqrt[3]{\cdot})$-*picture fuzzy BCC-ideal* (briefly, $(3, 3, \sqrt[3]{\cdot})$-PFBCCI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(PFBCCI$_1$) $A^3_P(0) \geq A^3_P(x),\ A^3_I(0) \leq A^3_I(x),\ \sqrt[3]{A_N(0)} \geq \sqrt[3]{A_N(x)}$,
(PFBCCI$_2$) $A^3_P(x * z) \geq A^3_P((x * y) * z) \wedge A^3_P(y)$,
(PFBCCI$_3$) $A^3_I(x * z) \leq A^3_I((x * y) * z) \vee A^3_I(y)$,
(PFBCCI$_4$) $\sqrt[3]{A_N(x * z)} \geq \sqrt[3]{A_N((x * y) * z))} \wedge \sqrt[3]{A_N(y)}$.

Example 3.13. Let X be the BCC-algebra and $\overline{A}$ be the $(3, 3, \sqrt[3]{\cdot})$-PFS in X given in Example 3.9. Then we can easily check that $\overline{A}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of X.

Lemma 3.14. *Let $\overline{A}$ be a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of a BCC-algebra X. If $x \leq y$ in X, then $A^3_P(x) \geq A^3_P(y),\ A^3_I(x) \leq A^3_I(y),\ \sqrt[3]{A_N(x)} \geq \sqrt[3]{A_N(y)}$.*

Proof. Let $x,\ y \in X$ such that $x \leq y$. Then clearly, $x * y = 0$. Since $\overline{A}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of X, by the condition (PFBCCI$_1$),

$$A^3_P(0) \geq A^3_P(x),\ A^3_I(0) \leq A^3_I(x),\ \sqrt[3]{A_N(0)} \geq \sqrt[3]{A_N(x)}. \tag{3.1}$$

Thus we have

$$\begin{aligned}
A_P^3(x) &= A_P^3(x*0) \text{ [By the axiom (BCC}_2\text{)]}\\
&\geq A_P^3((x*y)*0) \wedge A_P^3(y) \text{ [By the condition (PFBCCI}_2\text{)]}\\
&= A_P^3(0*0) \wedge A_P^3(y) \text{ [Since } x*y=0\text{]}\\
&= A_P^3(0) \wedge A_P^3(y) \text{ [By the axiom (BCC}_3\text{)]}\\
&= A_P^3(y), \text{ [By (3.1)]}\\
A_I^3(x) &= A_I^3(x*0)\\
&\leq A_I^3((x*y)*0) \vee A_I^3(y) \text{ [By the condition (PFBCCI}_3\text{)]}\\
&= A_I^3(0*0) \vee A_I^3(y)\\
&= A_I^3(0) \vee A_I^3(y)\\
&= A_I^3(y),\\
\sqrt[3]{A_N(x)} &= \sqrt[3]{A_N(x*0)}\\
&\geq \sqrt[3]{A_N((x*y)*0)} \wedge \sqrt[3]{A_N(y)} \text{ [By the condition (PFBCCI}_4\text{)]}\\
&= \sqrt[3]{A_N(0*0)} \wedge \sqrt[3]{A_N(y)}\\
&= \sqrt[3]{A_N(0)} \wedge \sqrt[3]{A_N(y)}\\
&= \sqrt[3]{A_N(y)}.
\end{aligned}$$

Thus the results hold. □

Lemma 3.15. *Let $\overline{A}$ be a $(3,3,\sqrt[3]{\cdot})$-PFBCCI of a BCC-algebra X. If $x*y \leq z$ in X, then the following inequalities:*

$$A_P^3(x) \geq A_P^3(z) \wedge A_P^3(y),\ A_I^3(x) \leq A_I^3(z) \vee A_I^3(y),\ \sqrt[3]{A_N(x)} \geq \sqrt[3]{A_N(z)} \wedge \sqrt[3]{A_N(y)}.$$

Proof. Let $x,\ y,\ z \in X$ such that $x*y \leq z$. Then by Lemma 3.14, we get

$$A_P^3(x*y) \geq A_P^3(z),\ A_I^3(x*y) \leq A_I^3(z),\ \sqrt[3]{A_N(x*y)} \geq \sqrt[3]{A_N(z)}. \tag{3.2}$$

Thus we have

$$\begin{aligned}
A_P^3(x) &= A_P^3(x*0) \text{ [By the axiom (BCC}_2\text{)]}\\
&\geq A_P^3((x*y)*0) \wedge A_P^3(y) \text{ [By the condition (PFBCCI}_2\text{)]}\\
&= A_P^3(x*y) \wedge A_P^3(y) \text{ [By the axiom (BCC}_2\text{)]}\\
&\geq A_P^3(z) \wedge A_P^3(y), \text{ [By the axiom (3.2)]}\\
A_I^3(x) &= A_I^3(x*0)\\
&\leq A_I^3((x*y)*0) \vee A_I^3(y) \text{ [By the condition (PFBCCI}_3\text{)]}\\
&= A_I^3(x*y) \vee A_I^3(y)\\
&\geq A_I^3(z) \vee A_I^3(y),\\
\sqrt[3]{A_N(x)} &= \sqrt[3]{A_N(x*0)}\\
&\geq \sqrt[3]{A_N((x*y)*0)} \wedge \sqrt[3]{A_N(y)} \text{ [By the condition (PFBCCI}_4\text{)]}\\
&= \sqrt[3]{A_N(x*y)} \wedge \sqrt[3]{A_N(y)}\\
&\geq \sqrt[3]{A_N(z)} \wedge \sqrt[3]{A_N(y)}.
\end{aligned}$$

Thus the inequalities hold. □

Proposition 3.16. *Every $(3,3,\sqrt[3]{\cdot})$-PFBCKI of a BCK-algebra X is a $(3,3,\sqrt[3]{\cdot})$-PFBCCI of X.*

Proof. Let $\overline{A}$ be a $(3,3,\sqrt[3]{\cdot})$-PFBCKI of a BCK-algebra X. Then clearly, $\overline{A}$ satisfies the condition (PFBCCI$_1$). Let $x,\ y,\ z \in X$. Since X is a BCK-algebra,

$$(x*y)*z = (x*z)*y \text{ for any } x,\ y,\ z \in X.$$

Then we have

$$\begin{aligned} A^3_P(x*z) &\geq A^3_P((x*z)*y) \wedge A^3_P(y) \text{ [By the condition (PFBCKI}_2\text{)]} \\ &= A^3_P((x*y)*z) \wedge A^3_P(y), \\ A^3_I(x*z) &\leq A^3_I((x*z)*y) \vee A^3_I(y) \text{ [By the condition (PFBCKI}_3\text{)]} \\ &= A^3_I((x*y)*z) \vee A^3_I(y), \\ \sqrt[3]{A_N(x*z)} &\leq \sqrt[3]{A_N((x*z)*y)} \wedge \sqrt[3]{A_N(y)} \text{ [By the condition (PFBCKI}_4\text{)]} \\ &= \sqrt[3]{A_N((x*y)*z)} \wedge \sqrt[3]{A_N(y)}. \end{aligned}$$

Thus $\overline{A}$ satisfies the conditions (PFBCCI$_2$), (PFBCCI$_3$) and (PFBCCI$_4$). So $\overline{A}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of X. □

Proposition 3.17. *Every $(3, 3, \sqrt[3]{\cdot})$-PFBCKI of a BCC-algebra X is a $(3, 3, \sqrt[3]{\cdot})$-PFSA of X.*

Proof. Let $\overline{A}$ be a $(3, 3, \sqrt[3]{\cdot})$-PFBCKI of a BCC-algebra X and let $x,\ y \in X$. Then by Result 2.2 (1) and the condition (PFBCKI$_1$), we get

$$A^3_P(((x*y)*x) = A^3_P(0) \geq A^3_P(x), \tag{3.3}$$

$$A^3_I(((x*y)*x) = A^3_I(0) \leq A^3_I(x), \tag{3.4}$$

$$\sqrt[3]{A_N((x*y)*x)} = \sqrt[3]{A_N(0)} \geq \sqrt[3]{A_N(x)}. \tag{3.5}$$

Thus we have

$$\begin{aligned} A^3_P(x*y) &\geq A^3_P((x*y)*x) \wedge A^3_P(x) \text{ [By the condition (PFBCKI}_2\text{)]} \\ &= A^3_P(0) \wedge A^3_P(x) \text{ [By Result 2.2 (1)]} \\ &= A^3_P(x) \text{ [By (3.3)]} \\ &\geq A^3_P(x) \wedge A^3_P(y), \\ A^3_I(x*y) &\leq A^3_I((x*y)*x) \vee A^3_I(x) \text{ [By the condition (PFBCKI}_3\text{)]} \\ &= A^3_P(0) \vee A^3_P(x) \\ &= A^3_P(x) \text{ [By (3.4)]} \\ &\geq A^3_P(x) \vee A^3_P(y), \\ \sqrt[3]{A_N(x*y)} &\geq \sqrt[3]{A_N((x*y)*x)} \wedge \sqrt[3]{A_N(x)} \text{ [By the condition (PFBCKI}_4\text{)]} \\ &= \sqrt[3]{A_N(0)} \wedge \sqrt[3]{A_N(x)} \text{ (1)]} \\ &= \sqrt[3]{A_N(x)} \text{ [By (3.4)]} \\ &\geq \sqrt[3]{A_N(x)} \wedge \sqrt[3]{A_N(y)}. \end{aligned}$$

So $\overline{A}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFSA of X. □

Proposition 3.18. *Let $(\overline{M}_j)_{j\in J}$ be a family of $(3, 3, \sqrt[3]{\cdot})$-PFBCCIs of a BCC-algebra X. Then $\bigcap_{j\in J} \overline{M}_j$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of X.*

Proof. Let $\overline{M} = \bigcap_{j\in J} \overline{M}_j$ and let $x \in X$. Then we have

$$M^3_P(0) = \bigwedge_{j\in J} M^3_{P,j}(0) \geq \bigwedge_{j\in J} M^3_{P,j}(x) = M^3_P(x),$$

$$M^3_I(0) = \bigvee_{j\in J} M^3_{I,j}(0) \leq \bigvee_{j\in J} M^3_{I,j}(x) = M^3_I(x),$$

$$\sqrt[3]{M_{N}(0)} = \bigwedge_{j\in J} \sqrt[3]{A_{N,j}(0)} \geq \bigwedge_{j\in J} \sqrt[3]{A_{N,j}(x)}) = \sqrt[3]{M_{N}(x)}.$$

Thus $\overline{M}$ satisfies the condition (PFBCCI$_1$).

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} M_{P}^3(x*z) &= \textstyle\bigwedge_{j\in J} M_{P,j}^3(x*z) \\ &\geq \textstyle\bigwedge_{j\in J}[M_{P,j}^3((x*y)*z) \wedge M_{P,j}^3(y)] \text{ [By (PFBCCI}_2\text{)]} \\ &= (\textstyle\bigwedge_{j\in J}[M_{P,j}^3((x*y)*z)) \wedge (\textstyle\bigwedge_{j\in J} M_{P,j}^3(y)) \\ &= M_{P}^3((x*y)*z) \wedge M_{P}^3(y), \\ M_{I}^3(x*z) &= \textstyle\bigvee_{j\in J} M_{I,j}^3(x*z) \\ &\leq \textstyle\bigvee_{j\in J}[M_{I,j}^3((x*y)*z) \vee M_{I,j}^3(y)] \text{ [By (PFBCCI}_3\text{)]} \\ &= (\textstyle\bigvee_{j\in J}[M_{I,j}^3((x*y)*z)) \vee (\textstyle\bigvee_{j\in J} M_{I,j}^3(y)) \\ &= M_{I}^3((x*y)*z) \vee M_{I}^3(y), \\ \sqrt[3]{M_{N}(x*z)} &= \textstyle\bigwedge_{j\in J} \sqrt[3]{M_{N,j}(x*z)} \\ &\geq \textstyle\bigwedge_{j\in J}[\sqrt[3]{M_{N,j}((x*y)*z)} \wedge \sqrt[3]{M_{N,j}(y)}] \text{ [By (PFBCCI}_2\text{)]} \\ &= (\textstyle\bigwedge_{j\in J}[\sqrt[3]{M_{N,j}((x*y)*z)}) \wedge (\textstyle\bigwedge_{j\in J} \sqrt[3]{M_{N,j}(y)}) \\ &= \sqrt[3]{M_{N}((x*y)*z)} \wedge \sqrt[3]{M_{N}(y)}. \end{aligned}$$

Thus $\overline{M}$ satisfies the conditions (PFBCCI$_2$), (PFBCCI$_3$) and (PFBCCI$_4$). So $\overline{M}$ is a $(3,3,\sqrt[3]{\cdot})$-PFBCCI of X. □

4. The image (pre-image) of a $(3,3,\sqrt[3]{\cdot})$-PFBCCI under a homomorphism of BCC-algebras

Definition 4.1. Let $(X,*,0)$ and $(Y,*',0')$ be two BCC-algebras. Then a mapping $f: X \to Y$ is called a *homomorphism*, if $f(x*y) = f(x) *' f(y)$ for any $x,\ y \in X$. Note that if $f: X \to Y$ is a homomorphism of BCC-algebras, then $f(0) = 0'$.

Definition 4.2. Let X and Y be two nonempty set, and let $\overline{A}$ be a $(3,3,\sqrt[3]{\cdot})$-PFS in X and let $\overline{B}$ be a $(3,3,\sqrt[3]{\cdot})$-PFS in Y.

(i) The *image* of $\overline{A}$, denoted by $f(\overline{A})$, is a $(3,3,\sqrt[3]{\cdot})$-PFS in Y defined as follows: for each $y \in Y$,

(4.1)
$$f(\overline{A})(y) = \begin{cases} \left\langle \bigwedge_{x\in f^{-1}(y)} A_{P}^3(x), \bigvee_{x\in f^{-1}(y)} A_{I}^3(x), \bigwedge_{x\in f^{-1}(y)} \sqrt[3]{A_{N}(x)} \right\rangle & \text{if } f^{-1}(y) \neq \varnothing \\ \langle 0,1,0\rangle & \text{otherwise.} \end{cases}$$

(ii) The *pre-image* of $\overline{B}$, denoted by $f^{-1}(\overline{B}) = \langle f^{-1}(B_{P}), f^{-1}(B_{I}), f^{-1}(A_{N})\rangle$, is a $(3,3,\sqrt[3]{\cdot})$-PFS in X defined as follows: for each $x \in X$,

$$f^{-1}(\overline{B})(x) = \left\langle B_{P}^3(f(x)), B_{I}^3(f(x)), \sqrt[3]{B_{N}(f(x))} \right\rangle. \tag{4.2}$$

In fact, $(f^{-1}(B_{P}))^3(x) = B_{P}^3(f(x))$, $(f^{-1}(B_{I}))^3(x) = B_{I}^3(f(x))$, $\sqrt[3]{f^{-1}(A_{N})(x)} = \sqrt[3]{B_{N}(f(x))}$.

Proposition 4.3. *Let $f: X \to Y$ be a homomorphism of BCC-algebras. If $\overline{B}$ is a $(3,3,\sqrt[3]{\cdot})$-PFBCCI of Y, then $f^{-1}(\overline{B})$ is a $(3,3,\sqrt[3]{\cdot})$-PFBCCI of X.*

Proof. Suppose $\overline{B}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of Y and let $x \in X$. Then we have

$$\begin{aligned}(f^{-1}(B_P))^3(0) &= B_P^3(f(0)) \\ &= B_P^3(0) \text{ [Since } f \text{ is a homomorphism]} \\ &\geq B_P^3(f(x)) \text{ [By the condition (PFBCCI}_1\text{)]} \\ &= (f^{-1}(B_P))^3(x).\end{aligned}$$

Similarly, we get $(f^{-1}(B_P))^3(0 \leq (f^{-1}(B_P))^3(x)$ and $\sqrt[3]{f^{-1}(A_N)(0)} \geq \sqrt[3]{f^{-1}(A_N)(x)}$. Thus $f^{-1}(\overline{B})$ satisfies the condition (PFBCCI$_1$).

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned}(f^{-1}(B_P))^3(x * z) &= B_P^3(f(x * z)) \\ &= B_P^3(f(x) * f(z)) \text{ [Since } f \text{ is a homomorphism]} \\ &\geq B_P^3((f(x) * f(y)) * f(z)) \wedge B_P^3(f(y)) \\ &\quad \text{[By the condition (PFBCCI}_2\text{)]} \\ &= B_P^3(f((x * y) * z))) \wedge B_P^3(f(y)) \\ &\quad \text{[Since } f \text{ is a homomorphism]} \\ &= (f^{-1}(B_P))^3((x * y) * z) \wedge (f^{-1}(B_P))^3(z), \\ (f^{-1}(B_I))^3(x * z) &= B_I^3(f(x * z)) \\ &= B_I^3(f(x) * f(z)) \\ &\leq B_I^3((f(x) * f(y)) * f(z)) \vee B_I^3(f(y)) \\ &\quad \text{[By the condition (PFBCCI}_3\text{)]} \\ &= B_I^3(f((x * y) * z))) \vee B_I^3(f(y)) \\ &= (f^{-1}(B_I))^3((x * y) * z) \vee (f^{-1}(B_I))^3(z), \\ \sqrt[3]{f^{-1}(B_N)(x * z)} &= \sqrt[3]{B_N(f(x * z))} \\ &= \sqrt[3]{B_N(f(x) * f(z))} \\ &\geq \sqrt[3]{B_N((f(x) * f(y)) * f(z))} \wedge \sqrt[3]{B_N(f(y))} \\ &\quad \text{[By the condition (PFBCCI}_4\text{)]} \\ &= \sqrt[3]{B_N(f((x * y) * z))} \wedge \sqrt[3]{B_N(f(y))} \\ &= \sqrt[3]{f^{-1}(B_N)((x * y) * z)} \wedge \sqrt[3]{f^{-1}(B_N)(z)}.\end{aligned}$$

Thus $f^{-1}(\overline{B})$ satisfies the conditions (PFBCCI$_2$), (PFBCCI$_3$) and (PFBCCI$_4$). So $f^{-1}(\overline{B})$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of X. □

Proposition 4.4. *Let $f : X \to Y$ be an epiorphism of BCC-algebras and let $\overline{B}$ be a $(3, 3, \sqrt[3]{\cdot})$-PFS in Y. If $f^{-1}(\overline{B})$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of X, then $\overline{B}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of Y.*

Proof. Suppose $f^{-1}(\overline{B})$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of X and let $a \in Y$. Since f is surjective, there is $x \in X$ such that $a = f(x)$. Then we have

$$\begin{aligned}B_P^3(0) &= B_P^3(f(0)) \text{ [Since } f \text{ is a homomorphism]} \\ &= (f^{-1}(B_P))^3(0) \\ &\geq (f^{-1}(B_P))^3(x) \text{ [By the condition (PFBCCI}_1\text{)]} \\ &= B_P^3(f(x)) \\ &= B_P^3(a), \\ B_I^3(0) &= B_I^3(f(0)) \\ &= (f^{-1}(B_I))^3(0) \\ &\leq (f^{-1}(B_I))^3(x) \\ &= B_I^3(f(x)) \\ &= B_I^3(a),\end{aligned}$$

$$\begin{aligned}\sqrt[3]{B_N(0)} &= \sqrt[3]{B_N(f(0))}\\ &= \sqrt[3]{f^{-1}(B_N)(0)}\\ &\geq \sqrt[3]{f^{-1}(B_N)(x)}\\ &= \sqrt[3]{B_N(f(x))}\\ &= \sqrt[3]{B_N(a)}.\end{aligned}$$

Thus $\overline{B}$ satisfies the condition (PFBCCI_1).

Now let $a,\ b,\ c \in X$. Then clearly, there are $x,\ y,\ z \in X$ such that $a = f(x),\ b = f(y),\ c = f(z)$. Thus we get

$$\begin{aligned}B_P^3(a*c) &= B_P^3((f(x)*f(z))\\ &= B_P^3(f(x*z)) \text{ [Since } f \text{ is a homomorphism]}\\ &= (f^{-1}(B_P)^3(x*z)\\ &\geq (f^{-1}(B_P)^3((x*y)*z) \wedge (f^{-1}(B_P)^3(y)\\ &\quad \text{[By the condition (PFBCCI}_2)\\ &= B_P^3(f((x*y)*z))) \wedge B_P^3(f(y))\\ &= B_P^3(((f(x)*f(y))*f(z)) \wedge B_P^3(f(y))\\ &\quad \text{[Since } f \text{ is a homomorphism]}\\ &= B_P^3((a*b)*c) \wedge B_P^3(b),\end{aligned}$$

$$\begin{aligned}B_I^3(a*c) &= B_I^3((f(x)*f(z))\\ &= B_I^3(f(x*z))\\ &= (f^{-1}(B_I)^3(x*z)\\ &\leq (f^{-1}(B_I)^3((x*y)*z) \vee (f^{-1}(B_I)^3(y)\\ &\quad \text{[By the condition (PFBCCI}_3)\\ &= B_I^3(f((x*y)*z))) \vee B_I^3(f(y))\\ &= B_I^3(((f(x)*f(y))*f(z)) \vee B_I^3(f(y))\\ &= B_I^3((a*b)*c) \vee B_I^3(b),\end{aligned}$$

$$\begin{aligned}\sqrt[3]{B_N(a*c)} &= \sqrt[3]{B_N((f(x)*f(z))}\\ &= \sqrt[3]{B_N(f(x*z))}\\ &= \sqrt[3]{f^{-1}(A_N)(x*z)}\\ &\geq \sqrt[3]{f^{-1}(A_N)((x*y)*z)} \wedge \sqrt[3]{f^{-1}(A_N)(y)}\\ &\quad \text{[By the condition (PFBCCI}_4)\\ &= \sqrt[3]{B_N(f((x*y)*z)))} \wedge \sqrt[3]{B_N(f(y))}\\ &= \sqrt[3]{B_N(((f(x)*f(y))*f(z))} \wedge \sqrt[3]{B_N(f(y))}\\ &= \sqrt[3]{B_N((a*b)*c)} \wedge \sqrt[3]{B_N(b)}.\end{aligned}$$

Thus $\overline{B}$ satisfies the conditions (PFBCCI_2), (PFBCCI_3) and (PFBCCI_4). So $\overline{B}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of Y. □

5. The product of $(3, 3, \sqrt[3]{\cdot})$-PFBCCIs

Definition 5.1. Let $X,\ Y$ be two nonempty sets, let $\overline{A}$ be a $(3, 3, \sqrt[3]{\cdot})$-PFS in X and let $\overline{B}$ be a $(3, 3, \sqrt[3]{\cdot})$-PFS in Y. Then the *Cartesian product* $\overline{A} \times \overline{B} = \langle A_P \times B_P, A_I \times B_I, A_N \times B_N\rangle$ of $\overline{A}$ and $\overline{B}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFS in $X \times Y$ defined as follows: for each $(x, y) \in X \times Y$,

$$(A_P \times B_P)^3(x, y) = A_P^3(x) \wedge B_P^3(y),\ (A_I \times B_I)^3(x, y) = A_I^3(x) \vee B_I^3(y),$$

$$\sqrt[3]{(A_N \times B_N)(x, y)} = \sqrt[3]{A_N(x)} \wedge \sqrt[3]{B_N(y)}.$$

Remark 5.2. Let X and Y be two BCC-algebras. We define $*$ on $X \times Y$ by: for any $(x, y),\ (u, v) \in X \times Y$,

$$(x, y) * (u, v) = (x * u, y * v).$$

Then clearly, $(X \times Y, *, (0, 0))$ is a BCC-algebra.

Proposition 5.3. *Let X be a BCC-algebra. If $\overline{A}$ and $\overline{B}$ are two $(3, 3, \sqrt[3]{\cdot})$-PFBCCIs of X, then $\overline{A} \times \overline{B}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Then by the condition (PFBCCI$_1$), we have

$$(A_P \times B_P)^3(0, 0) = A_P(0) \wedge B_P(0) \geq A_P(x) \wedge B_P(y) = (A_P \times B_P)^3(x, y),$$
$$(A_I \times B_I)^3(0, 0) = A_I(0) \vee B_I(0) \leq A_I(x) \vee B_I(y) = (A_I \times B_I)^3(x, y),$$
$$\sqrt[3]{(A_N \times B_N)(0, 0)} = \sqrt[3]{A_N(0)} \wedge \sqrt[3]{B_N(0)} \geq \sqrt[3]{A_N(x)} \wedge \sqrt[3]{B_N(y)} = \sqrt[3]{(A_N \times B_N)(x, y)}.$$

Thus $\times \overline{B}$ satisfies the condition (PFBCCI$_1$).

Now let $(x_1, x_2),\ (y_1, y_2),\ (z_1, z_2) \in X \times X$. Then we get

$$\begin{aligned}
&(A_P \times B_P)^3(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)) \wedge (A_P \times B_P)^3(y_1, y_2)\\
&= (A_P \times B_P)^3((x_1 * y_1) * z_1, (x_2 * y_2) * z_2) \wedge (A_P \times B_P)^3(y_1, y_2)\\
&= [A_P((x_1 * y_1) * z_1) \wedge B_P((x_2 * y_2) * z_2)] \wedge [A_P(y_1) \wedge B_P(y_2)]\\
&= [A_P((x_1 * y_1) * z_1) \wedge A_P(y_1)] \wedge [B_P((x_2 * y_2) * z_2) \wedge B_P(y_2)]\\
&\leq A_P(x_1 * z_1) \wedge B_P(x_2 * z_2)\\
&= (A_P \times B_P)^3(x_1 * z_1, x_2 * z_2),
\end{aligned}$$

$$\begin{aligned}
&(A_I \times B_I)^3(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)) \vee (A_I \times B_I)^3(y_1, y_2)\\
&= (A_I \times B_I)^3((x_1 * y_1) * z_1, (x_2 * y_2) * z_2) \vee (A_I \times B_I)^3(y_1, y_2)\\
&= [A_I((x_1 * y_1) * z_1) \vee B_I((x_2 * y_2) * z_2)] \vee [A_I(y_1) \vee B_P(y_2)]\\
&= [A_I((x_1 * y_1) * z_1) \vee A_P(y_1)] \vee [B_I((x_2 * y_2) * z_2) \vee B_P(y_2)]\\
&\geq A_I(x_1 * z_1) \vee B_I(x_2 * z_2)\\
&= (A_I \times B_I)^3(x_1 * z_1, x_2 * z_2),
\end{aligned}$$

$$\begin{aligned}
&\sqrt[3]{(A_N \times B_N)(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2))} \wedge \sqrt[3]{(A_N \times B_N)(y_1, y_2)}\\
&= \sqrt[3]{(A_N \times B_N)((x_1 * y_1) * z_1, (x_2 * y_2) * z_2)} \wedge \sqrt[3]{(A_N \times B_N)(y_1, y_2)}\\
&= [\sqrt[3]{A_N((x_1 * y_1) * z_1)} \wedge \sqrt[3]{B_N((x_2 * y_2) * z_2)}] \wedge [\sqrt[3]{A_N(y_1)} \wedge \sqrt[3]{B_N(y_2)}]\\
&= [\sqrt[3]{A_N((x_1 * y_1) * z_1)} \wedge \sqrt[3]{A_N(y_1)}] \wedge [\sqrt[3]{B_N((x_2 * y_2) * z_2)} \wedge \sqrt[3]{B_N(y_2)}]\\
&\leq \sqrt[3]{A_N(x_1 * z_1)} \wedge \sqrt[3]{B_N(x_2 * z_2)}\\
&= \sqrt[3]{(A_N \times B_N)(x_1 * z_1, x_2 * z_2)}.
\end{aligned}$$

Thus $\overline{A} \times \overline{B}$ satisfies the conditions (PFBCCI$_2$), (PFBCCI$_3$) and (PFBCCI$_4$). So $\overline{A} \times \overline{B}$ is a $(3, 3, \sqrt[3]{\cdot})$-PFBCCI of $X \times X$. □

6. Correlation coefficients for $(3, 3, \sqrt[3]{\cdot})$-picture fuzzy sets

Correlation plays an important role in statistics, engineering sciences and so on (See [2, 27, 32]). In this section, we propose some correlation coefficients for any two $(3, 3, \sqrt[3]{\cdot})$-picture fuzzy sets.

Definition 6.1. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse, and let $\overline{A}$ and $\overline{B}$ be two $(3, 3, \sqrt[3]{\cdot})$-PFSs.

(i) The *informational* $(3,3,\sqrt[3]{\cdot})$*-picture fuzzy energy* of $\overline{A}$, denoted by $E_{(3,3,\sqrt[3]{\cdot})}(A)$, is defined as follows: for each $i \in \{1,2,\cdots,n\}$,

$$E_{(3,3,\sqrt[3]{\cdot})}(A) = \Sigma_{i=1}^{n}\left[\left(A_P^3(x_i)\right)^2 + \left(A_I^3(x_i)\right)^2 + \left(\sqrt[3]{A_N(x_i)}\right)^2\right].$$

(ii) The *correlation* between $\overline{A}$ and $\overline{B}$, denoted by $C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})$, is defined as follows: for each $i \in \{1,2,\cdots,n\}$,

$$C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B}) = \Sigma_{i=1}^{n}(\overline{A}\times\overline{B})(x_i,x_i), \text{ i.e.,}$$

$$\begin{aligned}&C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})\\ &= \Sigma_{i=1}^{n}\left[\left(A_P^3(x_i)\times B_P^3(x_i)\right) + \left(A_P^3(x_i)\times B_P^3(x_i)\right) + \left(\sqrt[3]{A_N(x_i)}\times\sqrt[3]{B_N(x_i)}\right)\right].\end{aligned}$$

(iii) The *correlation coefficient* between $\overline{A}$ and $\overline{B}$, denoted by $\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})$, is defined as follows: for each $i \in \{1,2,\cdots,n\}$,

$$\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B}) = \frac{C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})}{\sqrt{E_{(3,3,\sqrt[3]{\cdot})}(A)\cdot E_{(3,3,\sqrt[3]{\cdot})}(B)}}, \text{ i.e.,}$$

$$\begin{aligned}&\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})\\ &= \frac{\Sigma_{i=1}^{n}\left[\left(A_P^3(x_i)\times B_P^3(x_i)\right)+\left(A_P^3(x_i)\times B_P^3(x_i)\right)+\left(\sqrt[3]{A_N(x_i)}\times\sqrt[3]{B_N(x_i)}\right)\right]}{\sqrt{\Sigma_{i=1}^{n}\left[\left(A_P^3(x_i)\right)^2+\left(A_I^3(x_i)\right)^2+\left(\sqrt[3]{A_N(x_i)}\right)^2\right]\cdot\Sigma_{i=1}^{n}\left[\left(B_P^3(x_i)\right)^2+\left(B_I^3(x_i)\right)^2+\left(\sqrt[3]{B_N(x_i)}\right)^2\right]}}.\end{aligned}$$

On the other hand, by using idea of Xu [33], we can suggest an alternative form of the correlation coefficient $\kappa(\overline{A},\overline{B})$ between two $\overline{A}$ and $\overline{B}$ as follows: for each $i \in \{1,2,\cdots,n\}$,

$$\begin{aligned}&\kappa(\overline{A},\overline{B})\\ &= \frac{\Sigma_{i=1}^{n}\left[\left(A_P^3(x_i)\times B_P^3(x_i)\right)+\left(A_P^3(x_i)\times B_P^3(x_i)\right)+\left(\sqrt[3]{A_N(x_i)}\times\sqrt[3]{B_N(x_i)}\right)\right]}{\sqrt{\Sigma_{i=1}^{n}\left[\left(A_P^3(x_i)\right)^2+\left(A_I^3(x_i)\right)^2+\left(\sqrt[3]{A_N(x_i)}\right)^2\right]\bigvee\Sigma_{i=1}^{n}\left[\left(B_P^3(x_i)\right)^2+\left(B_I^3(x_i)\right)^2+\left(\sqrt[3]{B_N(x_i)}\right)^2\right]}}.\end{aligned}$$

The following is an immediate consequence of Definition 6.1.

Proposition 6.2. *Let* $\overline{A}$ *and* $\overline{B}$ *be two* $(3,3,\sqrt[3]{\cdot})$*-PFSs in a universe of discourse* $X = \{x_1,x_2,\cdots,x_n\}$. *Then*

(1) $C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{A}) = E_{(3,3,\sqrt[3]{\cdot})}(A)$,

(2) $C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B}) = C_{(3,3,\sqrt[3]{\cdot})}(\overline{B},\overline{A})$.

Proposition 6.3. *Let* $\overline{A}$ *and* $\overline{B}$ *be two* $(3,3,\sqrt[3]{\cdot})$*-PFSs in a universe of discourse* $X = \{x_1,x_2,\cdots,x_n\}$. *Then*

(1) $\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B}) = \kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{B},\overline{A})$,

(2) $0 \leq \kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B}) \leq 1$,

(3) *if* $\overline{A} = \overline{B}$, *then* $\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B}) = 1$.

Proof. (1) The proof is straightforward.

(2) It is clear that $0 \leq \kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})$. We prove that $\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B}) \leq 1$. Recall the well-known Cauchy-Schwarz inequality.

$$(\Sigma_{i=1}^{n}a_ib_i)^2 \leq (\Sigma_{i=1}^{n}a_i)\cdot(\Sigma_{i=1}^{n}b_i) \text{ for any } a_i,\ b_i \in \mathbb{R}. \tag{6.1}$$

Then by (5.1), we get

$$\left(C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})\right)^2$$
$$=\left(\Sigma_{i=1}^n\left[\left(A_P^3(x_i)\times B_P^3(x_i)\right)+\left(A_P^3(x_i)\times B_P^3(x_i)\right)+\left(\sqrt[3]{A_N(x_i)}\times\sqrt[3]{B_N(x_i)}\right)\right]\right)^2$$
$$\leq\left(\Sigma_{i=1}^n\left[\left(A_P^3(x_i)\right)^2+\left(A_I^3(x_i)\right)^2+\left(\sqrt[3]{A_N(x_i)}\right)^2\right]\right)$$
$$\cdot\left(\Sigma_{i=1}^n\left[\left(A_P^3(x_i)\right)^2+\left(A_I^3(x_i)\right)^2+\left(\sqrt[3]{A_N(x_i)}\right)^2\right]\right)$$
$$=E_{(3,3,\sqrt[3]{\cdot})}(A)\cdot E_{(3,3,\sqrt[3]{\cdot})}(B).$$

Thus $\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})\leq 1$. So $0\leq\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})\leq 1$.

(3) The proof is straightforward. □

Example 6.4. Let $\overline{A}$ and $\overline{B}$ be two $(3,3,\sqrt[3]{\cdot})$-PFSs in the universe of discourse $X=\{0,1,2,3\}$ defined by the following tables:

X	0	1	2	3
$A_P(x)$	0.04	0.03	0.02	0.01
$A_I(x)$	0.02	0.02	0.03	0.03
$A_N(x)$	0.006	0.004	0.002	0.001
$A_P^3(x)$	0.000064	0.000027	0.000008	0.000001
$A_I^3(x)$	0.000008	0.000008	0.000027	0.000027
$\sqrt[3]{A_N(x)}$	0.181712	0.15874	0.12599	0.1

Table 6.1

X	0	1	2	3
$B_P(x)$	0.04	0.03	0.02	0.01
$B_I(x)$	0.01	0.02	0.03	0.04
$B_N(x)$	0.003	0.002	0.002	0.001
$B_P^3(x)$	0.000064	0.000027	0.000008	0.000001
$B_I^3(x)$	0.000001	0.000008	0.000027	0.000064
$\sqrt[3]{B_N(x)}$	0.144225	0.125992	00.125992	0.1

Table 6.2

Then we have

$$E_{(3,3,\sqrt[3]{\cdot})}(A)=0.496409,\ E_{(3,3,\sqrt[3]{\cdot})}(B)=0.566444.$$

Thus $C_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})=0.0711$. So we have

$$\kappa_{(3,3,\sqrt[3]{\cdot})}(\overline{A},\overline{B})=0.053306, \tag{6.2}$$

$$\kappa(\overline{A},\overline{B})=0.09451. \tag{6.3}$$

Hence from (6.2) and (6.3),we see that $(3,3,\sqrt[3]{\cdot})$-PFSs $\overline{A}$ and $\overline{B}$ are not correlated.

7. Conclusions

We discussed some properties of $(3, 3, \sqrt[3]{\cdot})$-PFBCCIs of BCC-algebras. The image and the pre-image of $(3, 3, \sqrt[3]{\cdot})$-PFBCCIs of BCC-algebras under a homomorphism of BCC-algebras were defined and how the image and the pre-image of $(3, 3, \sqrt[3]{\cdot})$-PFBCCIs of BCC-algebras become a $(3, 3, \sqrt[3]{\cdot}))$-PFBCCI were studied. Moreover, the product of $(3, 3, \sqrt[3]{\cdot}))$-PFBCCIs was established. Finally, we introduced the concept of correlations for $(3, 3, \sqrt[3]{\cdot}))$-PFSs, which is a new extension of the correlation of Atanassov's IFSs and investigated its some properties. The main purpose of our future work is to investigate $(3, 3, \sqrt[3]{\cdot}))$-picture fuzzy topologies and $(3, 3, \sqrt[3]{\cdot}))$- picture fuzzy m- polar of BCC-ideals of BCC-algebras.

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