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ABSTRACT. The fuzzy ideal (based on the fuzzy point) in unital cycloids is defined and various related properties are investigated. Characterization of fuzzy ideal is discussed, and conditions under which fuzzy sets can be fuzzy ideals are explored. The relationship between the fuzzy ideal and $(\in, \in\vee q)$-fuzzy ideal is established. A set $\tilde{X}$ is constructed, and the conditions under which it is an ideal are explored. Conditions in which the $\in_t$-set and Q_t-set become ideals are provided.

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1. Introduction

The quantum Yang-Baxter equation is named after C. N. Yang and R. J. Baxter, introduced independently in the context of statistical mechanics and precisely solvable models, and it plays an important role in the study of integrable systems, topological quantum field theories, and quantum groups as fundamental equations in mathematical physics and quantum algebra. The study of quantum Yang-Baxter equations and their solutions leads to deep connections between algebra, topology, and physics, making them a central topic in the field of mathematical physics. W. Rump studied the algebraic solution of the quantum Yang-Baxter equation, and introduced the notion of unital cycloids and L-algebras (See [1, 2]). In [2], W. Rump introduced the concept of ideals in unital cycloids. Several things, including ideal in unital cycloids and L-algebas, were covered in [2, 3, 4, 5].

In this paper, we use the fuzzy set theory (based on the fuzzy point) to study ideals of unital cycloids. We introduce the notion of fuzzy ideal in the unital cycloid, and ivestigate several properties. We discuss characterizations of fuzzy ideals, and explore the conditions under which a fuzzy set can be a fuzzy ideal. We consider the relationship between a fuzzy ideal and an $(\in, \in\vee q)$-fuzzy ideal. We build a set $\tilde{X}$ and explore the conditions under which it becomes an ideal. We provide conditions in which the $\in_t$-set and Q_t-set of a fuzzy set ξ in X become ideals.

2. Preliminaries

2.1. **Basic results on L-algebras.** What is already known about L-algebras is mentioned here. For more information, please refer to [2, 4, 5], etc.

Definition 2.1 ([2, 4])**.** Let X be a set with a binary operation "$\to$". An element $1 \in X$ is called a *logical unit*, if it satisfies:

$$\boxed{(\forall \mathfrak{a} \in X)\ (1 \to \mathfrak{a} = \mathfrak{a},\ \ \mathfrak{a} \to \mathfrak{a} = \mathfrak{a} \to 1 = 1).} \tag{2.1}$$

A couple $(X, \to)$ is called a *cycloid*, if it satisfies:

$$\boxed{(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\ ((\mathfrak{a} \to \mathfrak{b}) \to (\mathfrak{a} \to \mathfrak{c}) = (\mathfrak{b} \to \mathfrak{a}) \to (\mathfrak{b} \to \mathfrak{c})).} \tag{2.2}$$

If a cycloid $(X, \to)$ has a logical unit "1", then it is called a *unital cycloid*, and is denoted by $(X, \to, 1)$.

Definition 2.2 ([2, 4])**.** An *L-algebra* is defined to be a unital cycloid $(X, \to, 1)$ that satisfies:

$$\boxed{(\forall \mathfrak{a}, \mathfrak{b} \in X)\ (\mathfrak{a} \to \mathfrak{b} = \mathfrak{b} \to \mathfrak{a} = 1\ \Rightarrow\ \mathfrak{a} = \mathfrak{b}).} \tag{2.3}$$

We define a binary relation "$\leq_X$" in an L-algebra $(X, \to, 1)$ as follows:

$$\boxed{(\forall \mathfrak{a}, \mathfrak{b} \in X)\ (\mathfrak{a} \leq_X \mathfrak{b}\ \Leftrightarrow\ \mathfrak{a} \to \mathfrak{b} = 1).} \tag{2.4}$$

If $(X, \to, 1)$ is an L-algebra, then $(X, \leq_X)$ is a poset.

Definition 2.3 ([2, 4])**.** An L-algebra $(X, \to, 1)$ is called

- a *KL-algebra*, if it satisfies:

$$\boxed{(\forall \mathfrak{a}, \mathfrak{b} \in X)\ (\mathfrak{a} \to (\mathfrak{b} \to \mathfrak{a}) = 1),} \tag{2.5}$$

- a *CL-algebra*, if it satisfies:

$$\boxed{(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\ ((\mathfrak{a} \to (\mathfrak{b} \to \mathfrak{c})) \to (\mathfrak{b} \to (\mathfrak{a} \to \mathfrak{c})) = 1).} \tag{2.6}$$

Definition 2.4 ([2])**.** Let $(X, \to, 1)$ be a unital cycloid. A subset I of X is called an *ideal* of $(X, \to, 1)$, if it satisfies:

$$1 \in I, \tag{2.7}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in I,\ \mathfrak{a} \to \mathfrak{b} \in I\ \Rightarrow\ \mathfrak{b} \in I), \tag{2.8}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in I\ \Rightarrow\ (\mathfrak{a} \to \mathfrak{b}) \to \mathfrak{b} \in I), \tag{2.9}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in I\ \Rightarrow\ \mathfrak{b} \to \mathfrak{a} \in I,\ \mathfrak{b} \to (\mathfrak{a} \to \mathfrak{b}) \in I). \tag{2.10}$$

Remark 2.5 (JoA320-2328,[4])**.** (1) In a KL-algebra $(X, \to, 1)$, a subset I of X is an ideal of $(X, \to, 1)$ if and only if it satisfies (2.7), (2.8) and (2.9).

2ii) In a CL-algebra $(X, \to, 1)$, a subset I of X is an ideal of $(X, \to, 1)$ if and only if it satisfies (2.7) and (2.8).

2.2. **Basic results on the fuzzy set theory.** A fuzzy set ξ in a set X of the form

$$\xi(a) := \begin{cases} t \in (0,1] & \text{if } a = c, \\ 0 & \text{if } a \neq c, \end{cases}$$

is said to be a *fuzzy point* with support c and value t and is denoted by $\langle c/t \rangle$.

For a fuzzy set ξ in a set X, we say that a fuzzy point $\langle c/t \rangle$ is

(i) *contained* in ξ, denoted by $\langle c/t \rangle \in \xi$ (See [6]), if $\xi(c) \geq t$.
(ii) *quasi-coincident* with ξ, denoted by $\langle c/t \rangle \, q \, \xi$ (See [6]), if $\xi(c) + t > 1$.

If a fuzzy point $\langle c/t \rangle$ is contained in ξ or is quasi-coincident with ξ, we denote it $\langle c/t \rangle \in\!\vee q \, \xi$. If $\langle c/t \rangle \, \alpha \, \xi$ is not established for $\alpha \in \{\in, q, \in\!\vee q\}$, it is denoted by $\langle c/t \rangle \, \overline{\alpha} \, \xi$.

Given $t \in (0,1]$ and a fuzzy set ξ in a set X, consider the following sets

$$(\xi, t)_{\in} := \{x \in X \mid \langle x/t \rangle \in \xi\} \text{ and } (\xi, t)_{q} := \{x \in X \mid \langle x/t \rangle \, q \, \xi\}$$

which are called an $\in_t$*-set* and a Q_t*-set* of ξ, respectively, in X.

It is clear that $(\xi, t)_q \subseteq (\xi, s)_q$ for all $t, s \in (0,1]$ with $t \leq s$.

3. Fuzzy ideals based on fuzzy points

Definition 3.1. Let $(X, \to, 1)$ be a unital cycloid. A fuzzy set ξ in X is called a *fuzzy ideal* of $(X, \to, 1)$ if it satisfies:

$$(\forall t \in (0,1]) \left((\xi, t)_{\in} \neq \varnothing \;\Rightarrow\; 1 \in (\xi, t)_{\in} \right), \tag{3.1}$$

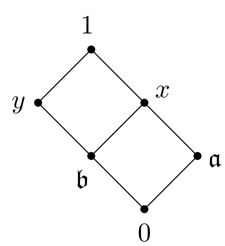
$$(\forall x, b \in X)(\forall t, s \in (0,1]) \begin{pmatrix} x \in (\xi, t)_{\in},\, x \to b \in (\xi, s)_{\in} \\ \Rightarrow\; b \in (\xi, \min\{t, s\})_{\in} \end{pmatrix}, \tag{3.2}$$

$$(\forall x, b \in X)(\forall t \in (0,1])(x \in (\xi, t)_{\in} \;\Rightarrow\; (x \to b) \to b \in (\xi, t)_{\in}), \tag{3.3}$$

$$(\forall x, b \in X)(\forall t \in (0,1])(x \in (\xi, t)_{\in} \;\Rightarrow\; b \to x \in (\xi, t)_{\in}), \tag{3.4}$$

$$(\forall x, b \in X)(\forall t \in (0,1])(x \in (\xi, t)_{\in} \;\Rightarrow\; b \to (x \to b) \in (\xi, t)_{\in}). \tag{3.5}$$

Example 3.2. Let $X = \{1, \mathfrak{a}, \mathfrak{b}, x, y, 0\}$ be a set with the Hasse diagram and Cayley table as follows:



$\to$	1	x	$\mathfrak{a}$	y	$\mathfrak{b}$	0
1	1	x	$\mathfrak{a}$	y	$\mathfrak{b}$	0
x	1	1	x	y	y	$\mathfrak{b}$
$\mathfrak{a}$	1	1	1	y	y	y
y	1	x	$\mathfrak{a}$	1	x	$\mathfrak{a}$
$\mathfrak{b}$	1	1	x	1	1	x
0	1	1	1	1	1	1

Hassee diagram of $(X, \leq_X)$ Cayley table for "$\to$"

Then $(X, \to, 1)$ is a unital cycloid. Let ξ be a fuzzy set in X given as follows:

$$\xi = \begin{pmatrix} 1 & x & \mathfrak{a} & y & \mathfrak{b} & 0 \\ 0.83 & 0.83 & 0.83 & 0.57 & 0.57 & 0.57 \end{pmatrix}.$$

It is routine to verify that ξ is a fuzzy ideal of $(X, \to, 1)$.

Theorem 3.3. *Let $(X, \rightarrow, 1)$ be a unital cycloid. A fuzzy set ξ in X is a fuzzy ideal of $(X, \rightarrow, 1)$ if and only if ξ satisfies:*

(3.6) $$(\forall x \in X)(\xi(1) \geq \xi(x)),$$

(3.7) $$(\forall x, b \in X)(\xi(b) \geq \min\{\xi(x), \xi(x \rightarrow b)\}),$$

(3.8) $$(\forall x, b \in X)(\xi((x \rightarrow b) \rightarrow b) \geq \xi(x)),$$

(3.9) $$(\forall x, b \in X)(\xi(b \rightarrow x) \geq \xi(x)),$$

(3.10) $$(\forall x, b \in X)(\xi(b \rightarrow (x \rightarrow b)) \geq \xi(x)).$$

Proof. Assume that ξ is a fuzzy ideal of $(X, \rightarrow, 1)$ and let $x, b \in X$. Suppose that $\xi(1) < \xi(\mathfrak{b})$ for some $\mathfrak{b} \in X$. If we take $t_0 := \frac{1}{2}(\xi(1) + \xi(\mathfrak{b}))$, then $\xi(1) < t_0$ and $0 < t_0 < \xi(\mathfrak{b}) \leq 1$. Thus $\mathfrak{b} \in (\xi, t_0)_\in$, i.e., $(\xi, t_0)_\in \neq \varnothing$. Using (3.1), we have $1 \in (\xi, t_0)_\in$ and so $\langle 1/t_0 \rangle \in \xi$, i.e., $\xi(1) \geq t_0$ which is a contradiction. Hence $\xi(1) \geq \xi(x)$ for all $x \in X$. If (3.7) is not valid, then

$$\xi(\mathfrak{b}) < t \leq \min\{\xi(\mathfrak{a}), \xi(\mathfrak{a} \rightarrow \mathfrak{b})\}$$

for some $\mathfrak{a}, \mathfrak{b} \in X$ and $t \in (0, 1]$. Then $\xi(\mathfrak{a}) \geq t$ and $\xi(\mathfrak{a} \rightarrow \mathfrak{b}) \geq t$, that is, $\langle \mathfrak{a}/t \rangle \in \xi$ and $\langle (\mathfrak{a} \rightarrow \mathfrak{b})/t \rangle \in \xi$. Thus $\mathfrak{a} \in (\xi, t)_\in$ and $\mathfrak{a} \rightarrow \mathfrak{b} \in (\xi, t)_\in$, and so $\mathfrak{b} \in (\xi, \min\{t, t\})_\in = (\xi, t)_\in$ by (3.2). Hence $\xi(\mathfrak{b}) \geq t$ which is a contradiction. Therefore $\xi(b) \geq \min\{\xi(x), \xi(x \rightarrow b)\}$ for all $x, b \in X$. Let $x, b \in X$ be such that $\xi(x) = t$. Then $x \in (\xi, t)_\in$ and so $(x \rightarrow b) \rightarrow b \in (\xi, t)_\in$, $b \rightarrow x \in (\xi, t)_\in$ and $b \rightarrow (x \rightarrow b) \in (\xi, t)_\in$ by (3.3), (3.4) and (3.5), respectively. Hence

$$\xi((x \rightarrow b) \rightarrow b) \geq t = \xi(x),\ \xi(b \rightarrow x) \geq t = \xi(x)$$

and $\xi(b \rightarrow (x \rightarrow b)) \geq t = \xi(x)$.

Conversely, suppose that ξ satisfies (3.6), (3.7), (3.8), (3.9), and (3.10). Suppose that $(\xi, t)_\in \neq \varnothing$ for all $t \in (0, 1]$. Then there exists $\mathfrak{a} \in (\xi, t)_\in$, and thus $\xi(1) \geq \xi(\mathfrak{a}) \geq t$ by (3.6), and so $1 \in (\xi, t)_\in$. Let $x, b \in X$ and $t, s \in (0, 1]$ be such that $x \in (\xi, t)_\in$ and $x \rightarrow b \in (\xi, s)_\in$. Then $\xi(x) \geq t$ and $\xi(x \rightarrow b) \geq s$, which imply from (3.7) that

$$\xi(b) \geq \min\{\xi(x), \xi(x \rightarrow b)\} \geq \min\{t, s\}.$$

Hence $b \in (\xi, \min\{t, s\})_\in$. Let $x, b \in X$ and $t \in (0, 1]$ be such that $x \in (\xi, t)_\in$. Then $\xi(x) \geq t$, and so $\xi((x \rightarrow b) \rightarrow b) \geq \xi(x) \geq t$, $\xi(b \rightarrow x) \geq \xi(x) \geq t$, and $\xi(b \rightarrow (x \rightarrow b)) \geq \xi(x) \geq t$ by (3.8), (3.9), and (3.10), respectively. It follows that $(x \rightarrow b) \rightarrow b \in (\xi, t)_\in$, $b \rightarrow x \in (\xi, t)_\in$, and $b \rightarrow (x \rightarrow b) \in (\xi, t)_\in$. Therefore ξ is a fuzzy ideal of $(X, \rightarrow, 1)$. □

Theorem 3.4. *In Definition 3.1, if $(X, \rightarrow, 1)$ is a KL-algebra, then (3.4) and (3.5) can be deleted.*

Proof. Assume that $(X, \rightarrow, 1)$ is a KL-algebra. Let $x \in (\xi, t)_\in$ for all $t \in (0, 1]$. Then $1 \in (\xi, t)_\in$ by (3.1), and so $x \rightarrow (b \rightarrow x) = 1 \in (\xi, t)_\in$ by (2.5). It follows from (3.2) that $b \rightarrow x \in (\xi, \min\{t, t\})_\in = (\xi, t)_\in$. Also, using (2.5) induces $b \rightarrow (x \rightarrow b) = 1 \in (\xi, t)_\in$. □

Remark 3.5. In the light of Theorem 3.4, a fuzzy set ξ is a fuzzy ideal of a KL-algebra $(X, \rightarrow, 1)$ if and only if ξ satisfies (3.1), (3.2) and (3.3).

Remark 3.6. A fuzzy set ξ is a fuzzy ideal of a KL-algebra $(X, \to, 1)$ if and only if ξ satisfies (3.6), (3.7) and (3.8).

Corollary 3.7. *In Definition 3.1, if $(X, \to, 1)$ is a CL-algebra, then (3.4) and (3.5) can be deleted.*

Proof. Let $(X, \to, 1)$ be a CL-algebra. Then

$$\begin{aligned} 1 &\overset{(2.6)}{=} (b \to (x \to x)) \to (x \to (b \to x)) \\ &\overset{(2.1)}{=} (b \to 1) \to (x \to (b \to x)) \\ &\overset{(2.1)}{=} 1 \to (x \to (b \to x)) \\ &\overset{(2.1)}{=} x \to (b \to x) \end{aligned}$$

for all $x, b \in X$. Hence $(X, \to, 1)$ is a KL-algebra, and therefore (3.4) and (3.5) can be deleted by Theorem 3.4. □

Theorem 3.8. *In Definition 3.1, if $(X, \to, 1)$ is a CL-algebra, then (3.3) can be deleted.*

Proof. Assume that $(X, \to, 1)$ is a CL-algebra. Let $x \in (\xi, t)_\in$ for all $t \in (0, 1]$. Then $1 \in (\xi, t)_\in$ by (3.1). Using (2.6) leads to

$$x \to (b \to z) = b \to (x \to z)$$

for all $x, b, z \in X$. It follows from (2.1) that

$$x \to ((x \to b) \to b) = (x \to b) \to (x \to b) = 1 \in (\xi, t)_\in.$$

Using (3.2), we have $(x \to b) \to b \in (\xi, \min\{t, t\})_\in = (\xi, t)_\in$. □

Corollary 3.9. *In Theorem 3.3, if $(X, \to, 1)$ is a CL-algebra, then (3.8), (3.9), and (3.10) can be deleted. Hence a fuzzy set ξ is a fuzzy ideal of a CL-algebra $(X, \to, 1)$ if and only if ξ satisfies (3.6) and (3.7).*

Remark 3.10. In the light of Corollary 3.7 and Theorem 3.8, a fuzzy set ξ is a fuzzy ideal of a CL-algebra $(X, \to, 1)$ if and only if ξ satisfies (3.1) and (3.2).

We explore the conditions under which a fuzzy set can be a fuzzy ideal.

Lemma 3.11. *Let ξ be a fuzzy set in a unital cycloid $(X, \to, 1)$. If ξ satisfies:*

$$\langle x/t \rangle \in \xi \;\Rightarrow \begin{cases} \langle 1/t \rangle \in \xi \text{ or } \langle 1/t \rangle \, q \, \xi, \\ \langle ((x \to b) \to b)/t \rangle \in \xi \text{ or } \langle ((x \to b) \to b)/t \rangle \, q \, \xi, \\ \langle (b \to x)/t \rangle \in \xi \text{ or } \langle (b \to x)/t \rangle \, q \, \xi, \\ \langle (b \to (x \to b))/t \rangle \in \xi \text{ or } \langle (b \to (x \to b))/t \rangle \, q \, \xi \end{cases} \tag{3.11}$$

for all $x, b \in X$ and $t \in (0, 1]$, then we have

$$\xi(1) \geq \min\{\xi(x), 0.5\}, \tag{3.12}$$

$$\xi((x \to b) \to b) \geq \min\{\xi(x), 0.5\}, \tag{3.13}$$

$$\xi(b \to x) \geq \min\{\xi(x), 0.5\}, \tag{3.14}$$

$$\xi(b \to (x \to b)) \geq \min\{\xi(x), 0.5\}, \tag{3.15}$$

for all $x, b \in X$. Also, if ξ satisfies:

$$(3.16) \qquad (\forall x, b \in X)(\forall t, s \in (0,1]) \left(\begin{array}{l} \langle x/t \rangle \in \xi,\ \langle (x \to b)/s \rangle \in \xi \ \Rightarrow \\ \langle b/\min\{t,s\} \rangle \in \xi \text{ or } \langle b/\min\{t,s\} \rangle \, q \, \xi \end{array} \right),$$

then $\xi(b) \geq \min\{\xi(x), \xi(x \to b), 0.5\}$ for all $x, b \in X$.

Proof. Assume that ξ satisfies (3.11), and let $x, b \in X$. If the inequality (3.12) is not valid, then $\xi(1) < t \leq \min\{\xi(\mathfrak{a}), 0.5\}$ for some $t \in (0,1]$ and $\mathfrak{a} \in X$. Then $t \in (0, 0.5]$, $\langle \mathfrak{a}/t \rangle \in \xi$ and $\langle 1/t \rangle \overline{\in} \xi$. Also we get $\langle 1/t \rangle \, \overline{q} \, \xi$ because of $\xi(1) + t < 1$. This is a contradiction, and thus (3.12) is valid. Suppose that the inequality (3.13) is not valid. Then

$$\xi((\mathfrak{a} \to \mathfrak{b}) \to \mathfrak{b}) < s \leq \min\{\xi(\mathfrak{a}), 0.5\}$$

for some $\mathfrak{a}, \mathfrak{b} \in X$ and $s \in (0,1]$. Then $\langle ((\mathfrak{a} \to \mathfrak{b}) \to \mathfrak{b})/s \rangle \overline{\in} \xi$, $s \in (0, 0.5]$, and $\langle \mathfrak{a}/t \rangle \in \xi$. Since $\xi((\mathfrak{a} \to \mathfrak{b}) \to \mathfrak{b}) + s < 1$, we have $\langle ((\mathfrak{a} \to \mathfrak{b}) \to \mathfrak{b})/s \rangle \, \overline{q} \, \xi$. This is a contradiction, and so

$$\xi((x \to b) \to b) \geq \min\{\xi(x), 0.5\}$$

for all $x, b \in X$. The inequalities (3.14) and (3.15) are derived in the same way. Suppose that ξ satisfies (3.16). Let $x, b \in X$ and suppose that

$$\min\{\xi(x), \xi(x \to b)\} < 0.5.$$

Then $\xi(b) \geq \min\{\xi(x), \xi(x \to b)\}$. In fact, if it is not true then

$$\xi(b) < s < \min\{\xi(x), \xi(x \to b)\}$$

for some $s \in (0, 0.5)$. It follows that $\langle x/s \rangle \in \xi$ and $\langle (x \to b)/s \rangle \in \xi$, but

$$\langle b/\min\{s,s\} \rangle = \langle b/s \rangle \overline{\in} \xi \text{ and } \langle b/\min\{s,s\} \rangle = \langle b/s \rangle \, \overline{q} \, \xi.$$

This is a contradiction, and so $\xi(b) \geq \min\{\xi(x), \xi(x \to b)\}$ whenever

$$\min\{\xi(x), \xi(x \to b)\} < 0.5.$$

If $\min\{\xi(x), \xi(x \to b)\} \geq 0.5$, then $\langle x/0.5 \rangle \in \xi$ and $\langle (x \to b)/0.5 \rangle \in \xi$ which implies from (3.16) that $\langle b/0.5 \rangle = \langle b/\min\{0.5, 0.5\} \rangle \in \xi$ or $\langle b/0.5 \rangle = \langle b/\min\{0.5, 0.5\} \rangle \, q \, \xi$. Hence $\xi(b) \geq 0.5$ because if $\xi(b) < 0.5$ then $\xi(b) + 0.5 < 0.5 + 0.5 = 1$, and so $\langle b/0.5 \rangle \, \overline{q} \, \xi$, a contradiction. Hence $\xi(b) \geq \min\{\xi(x), \xi(x \to b), 0.5\}$ for all $x, b \in X$. ◻

In a unital cycloid $(X, \to, 1)$, if a fuzzy set ξ satisfies (3.11) and (3.16), we say that ξ is an $(\in, \in\vee q)$-*fuzzy ideal* of $(X, \to, 1)$.

We provide a condition for the $(\in, \in\vee q)$-fuzzy ideal to be a fuzzy ideal.

Theorem 3.12. *Let ξ be an $(\in, \in\vee q)$-fuzzy ideal of a unital cycloid $(X, \to, 1)$ that satisfies $\xi(x) < 0.5$ for all $x \in X$. Then it is a fuzzy ideal of $(X, \to, 1)$.*

Proof. Let $t \in (0,1]$ be such that $(\xi, t)_\in \neq \varnothing$. Then there exists $\mathfrak{a} \in (\xi, t)_\in$, and so $\xi(1) \geq \min\{\xi(\mathfrak{a}), 0.5\} = \xi(\mathfrak{a}) \geq t$ by (3.12). Hence $1 \in (\xi, t)_\in$. Let $x, b \in X$ and $t, s \in (0,1]$ be such that $x \in (\xi, t)_\in$ and $x \to b \in (\xi, s)_\in$. Then $\xi(x) \geq t$ and $\xi(x \to b) \geq s$, which imply from Lemma 3.11 that

$$\xi(b) \geq \min\{\xi(x), \xi(x \to b), 0.5\} = \min\{\xi(x), \xi(x \to b)\} \geq \min\{t, s\}.$$

Thus $b \in (\xi, \min\{t, s\})_{\in}$. Let $x, b \in X$ and $t \in (0, 1]$ be such that $x \in (\xi, t)_{\in}$. Then $\xi(x) \geq t$, and so $\xi((x \to b) \to b) \geq \min\{\xi(x), 0.5\} \geq t$ by (3.13). Hence $\langle((x \to b) \to b)/t\rangle \in \xi$, that is, $(x \to b) \to b \in (\xi, t)_{\in}$. In the same way, (3.14) and (3.15) are used to obtain $b \to x \in (\xi, t)_{\in}$ and $b \to (x \to b) \in (\xi, t)_{\in}$, respectively. Therefore ξ is a fuzzy ideal of $(X, \to, 1)$. □

Given a fuzzy set ξ in X, consider the set:

$$\widetilde{X} := \{x \in X \mid \xi(x) > 0\}. \tag{3.17}$$

We explore the conditions for the set $\widetilde{X}$ to be an ideal.

Theorem 3.13. *Let $(X, \to, 1)$ be a KL-algebra and let ξ be a nonzero fuzzy set in X, i.e., there exists $\mathfrak{a} \in X$ such that $\xi(\mathfrak{a}) \neq 0$. If ξ is a fuzzy ideal of $(X, \to, 1)$, then the set $\widetilde{X}$ is an ideal of $(X, \to, 1)$.*

Proof. Assume that ξ is a nonzero fuzzy ideal of a KL-algebra $(X, \to, 1)$. Then $\widetilde{X}$ is nonempty. If $1 \notin \widetilde{X}$, then $\xi(1) = 0$ and so $\xi(x) = 0$ for all $x \in X$. Thus $\widetilde{X} = \varnothing$, a contradiction. So $1 \in \widetilde{X}$. Let $x, b \in X$ be such that $x \in \widetilde{X}$ and $x \to b \in \widetilde{X}$. Then $\xi(b) \geq \min\{\xi(x), \xi(x \to b)\} > 0$ and $\xi((x \to b) \to b) \geq \xi(x) > 0$ by (3.7) and (3.8), respectively, and so $b \in \widetilde{X}$ and $(x \to b) \to b \in \widetilde{X}$. Therefore $\widetilde{X}$ is an ideal of $(X, \to, 1)$. □

Theorem 3.14. *Let $(X, \to, 1)$ be a KL-algebra and let ξ be a nonzero fuzzy set in X. If ξ satisfies:*

$$(\forall t \in (0, 1])\,((\xi, t)_{\in} \neq \varnothing \;\Rightarrow\; 1 \in (\xi, t)_q)\,, \tag{3.18}$$

$$(\forall x, b \in X)(\forall t, s \in (0, 1])\left(\begin{array}{c} x \in (\xi, t)_{\in},\, x \to b \in (\xi, s)_{\in} \\ \Rightarrow\; b \in (\xi, \min\{t, s\})_q \end{array}\right), \tag{3.19}$$

$$(\forall x, b \in X)(\forall t \in (0, 1])(x \in (\xi, t)_{\in} \;\Rightarrow\; (x \to b) \to b \in (\xi, t)_q), \tag{3.20}$$

then $\widetilde{X}$ is an ideal of $(X, \to, 1)$.

Proof. Suppose that ξ satisfies (3.18), (3.19) and (3.20). Since ξ is nonzero, there exists $\mathfrak{a} \in X$ such that $\xi(\mathfrak{a}) > 0$. Thus $\mathfrak{a} \in (\xi, \xi(\mathfrak{a}))_{\in}$ and so $1 \in (\xi, \xi(\mathfrak{a}))_q$ by (3.18). It follows that $\langle 1/\xi(\mathfrak{a})\rangle q\xi$, that is, $\xi(1) + \xi(\mathfrak{a}) > 1$. Hence $1 \in \widetilde{X}$. Let $x, b \in X$ be such that $x \in \widetilde{X}$ and $x \to b \in \widetilde{X}$. Then $\xi(x) > 0$ and $\xi(x \to b) > 0$. Thus $x \in (\xi, \xi(x))_{\in}$ and $x \to b \in (\xi, \xi(x \to b))_{\in}$, which implies from (3.19) and (3.20) that $b \in (\xi, \min\{\xi(x), \xi(x \to b)\})_q$ and $(x \to b) \to b \in (\xi, \xi(x))_q$. It follows that $\langle b/\min\{\xi(x), \xi(x \to b)\}\rangle\, q\, \xi$ and $\langle((x \to b) \to b)/\xi(x)\rangle\, q\, \xi$. If $\xi(b) = 0$, then $\xi(b) + \min\{\xi(x), \xi(x \to b)\} = \min\{\xi(x), \xi(x \to b)\} \leq 1$. Hence $\langle b/\min\{\xi(x), \xi(x \to b)\}\rangle\, \overline{q}\, \xi$, that is, $b \notin (\xi, \min\{\xi(x), \xi(x \to b)\})_q$. This is a contradiction, and so $\xi(b) \neq 0$. Hence $b \in \widetilde{X}$. If $\xi((x \to b) \to b) = 0$, then $\xi((x \to b) \to b) + \xi(x) = \xi(x) \leq 1$. Thus $\langle((x \to b) \to b)/\xi(x)\rangle\, \overline{q}\, \xi$, that is, $(x \to b) \to b \notin (\xi, \xi(x))_q$. This is a contradiction, and so $\xi((x \to b) \to b) \neq 0$. Hence $(x \to b) \to b \in \widetilde{X}$. Consequently, $\widetilde{X}$ is an ideal of $(X, \to, 1)$. □

Theorem 3.15. *Let $(X, \to, 1)$ be a KL-algebra and let ξ be a nonzero fuzzy set in X. If ξ satisfies:*

$$(\forall t \in (0,1])\left((\xi,t)_q \neq \varnothing \;\Rightarrow\; 1 \in (\xi,t)_\in\right), \tag{3.21}$$

$$(\forall x, b \in X)(\forall t, s \in (0,1])\left(\begin{array}{c} x \in (\xi,t)_q,\ x \to b \in (\xi,s)_q \\ \Rightarrow\; b \in (\xi, \min\{t,s\})_\in \end{array}\right), \tag{3.22}$$

$$(\forall x, b \in X)(\forall t \in (0,1])(x \in (\xi,t)_q \;\Rightarrow\; (x \to b) \to b \in (\xi,t)_\in), \tag{3.23}$$

then $\widetilde{X}$ is an ideal of $(X, \to, 1)$.

Proof. Assume that ξ satisfies (3.21), (3.22) and (3.23). Since ξ is nonzero, $\widetilde{X}$ is nonempty and thus there exists $\mathfrak{b} \in X$ such that $\xi(\mathfrak{b}) > 0$. If we take $t \in (1-\xi(\mathfrak{b}), 1]$, then $\mathfrak{b} \in (\xi,t)_q$, and so $1 \in (\xi,t)_\in$ by (3.21). This implies $\xi(1) \geq t > 0$, and thus $1 \in \widetilde{X}$. Let $x, b \in X$ be such that $x \in \widetilde{X}$ and $x \to b \in \widetilde{X}$. Then $\xi(x) > 0$ and $\xi(x \to b) > 0$ If we take $t_\mathfrak{a} \in (1-\xi(x), 1]$ and $t_\mathfrak{b} \in (1-\xi(x \to b), 1]$, then $x \in (\xi, t_\mathfrak{a})_q$ and $x \to b \in (\xi, t_\mathfrak{b})_q$. It follows from (3.22) that $b \in (\xi, \min\{t_\mathfrak{a}, t_\mathfrak{b}\})_\in$. Hence $\xi(b) \geq \min\{t_\mathfrak{a}, t_\mathfrak{b}\} > 0$, and so $b \in \widetilde{X}$. Also, using (3.23) induces $(x \to b) \to b \in (\xi, t_\mathfrak{a})_\in$. Thus $\xi((x \to b) \to b) \geq t_\mathfrak{a} > 0$ and so $(x \to b) \to b \in \widetilde{X}$. Therefore $\widetilde{X}$ is an ideal of $(X, \to, 1)$. □

Theorem 3.16. *Let $(X, \to, 1)$ be a KL-algebra and let ξ be a nonzero fuzzy set in X. If ξ satisfies:*

$$(\forall t \in (0,1])\left((\xi,t)_q \neq \varnothing \;\Rightarrow\; 1 \in (\xi,t)_q\right), \tag{3.24}$$

$$(\forall x, b \in X)(\forall t, s \in (0,1])\left(\begin{array}{c} x \in (\xi,t)_q,\ x \to b \in (\xi,s)_q \\ \Rightarrow\; b \in (\xi, \min\{t,s\})_q \end{array}\right), \tag{3.25}$$

$$(\forall x, b \in X)(\forall t \in (0,1])(x \in (\xi,t)_q \;\Rightarrow\; (x \to b) \to b \in (\xi,t)_q), \tag{3.26}$$

then $\widetilde{X}$ is an ideal of $(X, \to, 1)$.

Proof. Assume that ξ satisfies (3.24), (3.25) and (3.26). Since $\xi(a) > 0$ for some $a \in X$, we get $a \in (\xi,t)_q$ by taking $t \in (1-\xi(a), 1]$, and so $1 \in (\xi,t)_q$ by (3.24). Thus $\xi(1) > 1 - t \geq 0$, and so $1 \in \widetilde{X}$. Let $x, b \in X$ be such that $x \in \widetilde{X}$ and $x \to b \in \widetilde{X}$. Then $\xi(x) > 0$ and $\xi(x \to b) > 0$. It follows that $x \in (\xi,t)_q$ and $x \to b \in (\xi,s)_q$ where $t \in (1-\xi(x), 1]$ and $s \in (1-\xi(x \to b), 1]$. Hence $b \in (\xi, \min\{t,s\})_q$ by (3.25), and thus

$$\xi(b) > 1 - \min\{t,s\} = \max\{1-t, 1-s\} \geq 0$$

which shows that $b \in \widetilde{X}$. Also, we have $(x \to b) \to b \in (\xi,t)_q$ by (3.26) and so $\xi((x \to b) \to b) > 1 - t \geq 0$, i.e., $(x \to b) \to b \in \widetilde{X}$. Therefore $\widetilde{X}$ is an ideal of $(X, \to, 1)$. □

We provide conditions for the $\in_t$-set and Q_t-set of a fuzzy set ξ in X to be ideals of a KL-algebra $(X, \to, 1)$.

Theorem 3.17. *Let $(X, \to, 1)$ be a KL-algebra. Given a noconstant fuzzy set ξ in X, its $\in_t$-set $(\xi,t)_\in$ is an ideal of $(X, \to, 1)$ for all $t \in (0.5, 1]$ if and only if ξ*

satisfies:

(3.27) $(\forall x \in X)(\xi(x) \leq \max\{\xi(1), 0.5\}),$

(3.28) $(\forall x, b \in X)(\max\{\xi(b), 0.5\} \geq \min\{\xi(x), \xi(x \to b)\}),$

(3.29) $(\forall x, b \in X)(\max\{\xi((x \to b) \to b), 0.5\} \geq \xi(x)).$

Proof. Assume that the $\in_t$-set $(\xi, t)_\in$ of ξ is an ideal of $(X, \to, 1)$ for all $t \in (0.5, 1]$. If the condition (3.27) is not met, then $\xi(c) > \max\{\xi(1), 0.5\}$ for some $c \in X$. Thus $t := \xi(c) \in (0.5, 1]$ and $c \in (\xi, t)_\in$. But $1 \notin (\xi, t)_\in$, a contradiction. Hence $\xi(x) \leq \max\{\xi(1), 0.5\}$ for all $x \in X$. Suppose that there exist $c, a \in X$ such that $\max\{\xi(a), 0.5\} < \min\{\xi(c), \xi(c \to a)\}$ and

$$\max\{\xi((c \to a) \to a), 0.5\} < \xi(c).$$

Taking $t := \min\{\xi(c), \xi(c \to a)\}$ induces $t \in (0.5, 1]$, $c \in (\xi, t)_\in$ and $c \to a \in (\xi, t)_\in$. But $a \notin (\xi, t)_\in$ which is a contradiction. Also, if we put $s := \xi(c)$, then $s \in (0.5, 1]$, and $c \in (\xi, s)_\in$ but $(c \to a) \to a) \notin (\xi, s)_\in$, a contradiction. Therefore

$$\max\{\xi(b), 0.5\} \geq \min\{\xi(x), \xi(x \to b)\}$$

and $\max\{\xi((x \to b) \to b), 0.5\} \geq \xi(x)$ for all $x, b \in X$.

Conversely, suppose that ξ satisfies (3.27), (3.28) and (3.29). Let $t \in (0.5, 1]$. For every $x \in (\xi, t)_\in$, we have $\max\{\xi(1), 0.5\} \geq \xi(x) \geq t > 0.5$ and so $\xi(1) \geq t$, i.e., $\langle 1/t \rangle \in \xi$. Hence $1 \in (\xi, t)_\in$. Let $x, b \in X$ be such that $x \in (\xi, t)_\in$ and $x \to b \in (\xi, t)_\in$. Then $\langle x/t \rangle \in \xi$ and $\langle x \to b/t \rangle \in \xi$, which implies from (3.28) and (3.29) that $\max\{\xi(b), 0.5\} \geq \min\{\xi(x), \xi(x \to b)\} \geq t > 0.5$ and $\max\{\xi((x \to b) \to b), 0.5\} \geq \xi(x) \geq t$. Thus $\xi(b) \geq t$ and $\xi((x \to b) \to b) \geq t$, i.e., $\langle b/t \rangle \in \xi$ and $\langle ((x \to b) \to b)/t \rangle \in \xi$. Hence $b \in (\xi, t)_\in$ and $(x \to b) \to b \in (\xi, t)_\in$, and therefore $(\xi, t)_\in$ is an ideal of $(X, \to, 1)$. □

Theorem 3.18. *Let $(X, \to, 1)$ be a KL-algebra. A fuzzy set ξ in X is a fuzzy ideal of $(X, \to, 1)$ if and only if the nonempty $\in_t$-set $(\xi, t)_\in$ of ξ is an ideal of $(X, \to, 1)$ for all $t \in (0, 1]$.*

Proof. Let ξ be a fuzzy ideal of $(X, \to, 1)$ and assume that $(\xi, t)_\in$ is nonempty for all $t \in (0, 1]$. Then $1 \in (\xi, t)_\in$ by (3.1). Let $x, b \in X$ be such that $x \in (\xi, t)_\in$ and $x \to b \in (\xi, t)_\in$. Then $b \in (\xi, t)_\in$ and $(x \to b) \to b \in (\xi, t)_\in$ by (3.2) and (3.3). Hence $(\xi, t)_\in$ is an ideal of $(X, \to, 1)$ for all $t \in (0, 1]$.

Conversely, suppose that the nonempty $\in_t$-set $(\xi, t)_\in$ of ξ is an ideal of $(X, \to, 1)$ for all $t \in (0, 1]$. If $\xi(1) < \xi(a)$ for some $a \in X$, then $a \in (\xi, \xi(a))_\in$ and $1 \notin (\xi, \xi(a))_\in$. This is a contradiction, and so $\xi(1) \geq \xi(x)$ for all $x \in X$. If there exists $c, a \in X$ such that $\xi(a) < \min\{\xi(c), \xi(c \to a)\}$, then $c \in (\xi, t)_\in$ and $c \to a \in (\xi, t)_\in$ for $t := \min\{\xi(c), \xi(c \to a)\}$. Hence $a \in (\xi, t)_\in$ since $(\xi, t)_\in$ is an ideal of $(X, \to, 1)$. Thus $\xi(a) \geq t = \min\{\xi(c), \xi(c \to a)\}$, a contrsdiction. Hence $\xi(b) \geq \min\{\xi(x), \xi(x \to b)\}$ for all $x, b \in X$. Also, if $\xi((c \to a) \to a) < \xi(c)$ for some $c, a \in X$, then $c \in (\xi, t)_\in$ and $(c \to a) \to a \notin (\xi, t)_\in$ where $t = \xi(c)$. This is a contradiction, and thus $\xi((x \to b) \to b) \geq \xi(x)$ for all $x, b \in X$. Therefore ξ is an ideal of $(X, \to, 1)$ by Remark 3.6. □

Theorem 3.19. *Let* $(X, \to, 1)$ *be a* KL*-algebra. Given a fuzzy set* ξ *in* X*, the nonempty* $\in_t$*-set* $(\xi, t)_\in$ *of* ξ *is an ideal of* $(X, \to, 1)$ *for all* $t \in (0, 0.5]$ *if and only if* ξ *satisfies* (3.16) *and*

$$\text{(3.30)} \qquad \langle x/t\rangle \in \xi \;\Rightarrow \begin{cases} \langle 1/t\rangle \in \xi \text{ or } \langle 1/t\rangle \, q\, \xi, \\ \langle ((x \to b) \to b)/t\rangle \in \xi \text{ or } \langle ((x \to b) \to b)/t\rangle \, q\, \xi, \end{cases}$$

for all $x, b \in X$ *and* $t \in (0, 1]$.

Proof. Let ξ be a fuzzy set in X such that $(\xi, t)_\in \neq \varnothing$ is an ideal of $(X, \to, 1)$ for all $t \in (0, 0.5]$. Let $x \in X$ and $t \in (0, 1]$ be such that $\langle x/t\rangle \in \xi$. Then $\xi(x) \geq t$. If $\xi(1) < \min\{\xi(c), 0.5\}$ for some $c \in X$, then there exists $t \in (0, 1]$ such that $\xi(1) < t \leq \min\{\xi(c), 0.5\}$. Hence $t \in (0, 0.5]$ and $1 \notin (\xi, t)_\in$, a contadiction. Thus $\xi(1) \geq \min\{\xi(x), 0.5\}$ for all $x \in X$. Suppose that $\langle 1/t\rangle \,\overline{\in}\, \xi$. Then $\xi(1) < t$. If $\xi(x) < 0.5$, then $\xi(1) \geq \min\{\xi(x), 0.5\} = \xi(x) \geq t$, a contradiction. Hence $\xi(x) \geq 0.5$, and so $\xi(1) + t > \xi(1) + \xi(1) \geq 2 \min\{\xi(x), 0.5\} = 1$. Hence $\langle 1/t\rangle \, q\, \xi$. If $\xi((c \to a) \to a) < \min\{\xi(c), 0.5\}$ for some $c \in X$, then $\xi((c \to a) \to a) < t \leq \min\{\xi(c), 0.5\}$ for some $t \in (0, 1]$. Then $t \in (0, 0.5]$, $c \in (\xi, t)_\in$ and $(c \to a) \to a \notin (\xi, t)_\in$. This is a contadiction. So

$$\xi((x \to b) \to b) \geq \min\{\xi(x), 0.5\}$$

for all $x, b \in X$. Suppose that $\langle ((x \to b) \to b)/t\rangle \,\overline{\in}\, \xi$. Then $\xi((x \to b) \to b) < t$. If $\xi(x) < 0.5$, then $\xi((x \to b) \to b) \geq \min\{\xi(x), 0.5\} = \xi(x) \geq t$, a contradiction. Hence $\xi(x) \geq 0.5$, and so

$$\begin{aligned} \xi((x \to b) \to b) + t &> \xi((x \to b) \to b) + \xi((x \to b) \to b) \\ &\geq 2 \min\{\xi(x), 0.5\} = 1. \end{aligned}$$

Hence $\langle ((x \to b) \to b)/t\rangle \, q\, \xi$. Therefore ξ satisfies (3.30). Let $x, b \in X$ and $t, s \in (0, 1]$ be such that $\langle x/t\rangle \in \xi$ and $\langle (x \to b)/s\rangle \in \xi$. We now prove that $\xi(b) \geq \min\{\xi(x), \xi(x \to b), 0.5\}$ for all $x, b \in X$. If not, then there exists $c, a \in X$ and $t_c \in (0, 1]$ such that

$$\xi(a) < t_c \leq \min\{\xi(c), \xi(c \to a), 0.5\}.$$

Hene $t_c \in (0, 0.5]$, $c \in (\xi, t_c)_\in$ and $c \to a \in (\xi, t_c)_\in$, but $a \notin (\xi, t_c)_\in$. This is a contradiction, and thus $\xi(b) \geq \min\{\xi(x), \xi(x \to b), 0.5\}$ for all $x, b \in X$. Assume that $\langle b/\min\{t, s\}\rangle \,\overline{\in}\, \xi$. Then $\xi(b) < \min\{t, s\}$. If $\min\{\xi(x), \xi(x \to b)\} < 0.5$, then

$$\xi(b) \geq \min\{\xi(x), \xi(x \to b), 0.5\} = \min\{\xi(x), \xi(x \to b)\} \geq \min\{t, s\}.$$

This is impossible, and so $\min\{\xi(x), \xi(x \to b)\} \geq 0.5$. It follows that

$$\xi(b) + \min\{t, s\} \geq 2\xi(b) \geq 2 \min\{\xi(x), \xi(x \to b), 0.5\} = 1.$$

Hence $\langle b/\min\{t, s\}\rangle \, q\, \xi$, and thus (3.16) is valid.

Conversely, assume that ξ satisfies (3.16) and (3.30) and let $t \in (0, 0.5]$. Then $\xi(1) \geq \min\{\xi(x), 0.5\}$ and $\xi((x \to b) \to b) \geq \min\{\xi(x), 0.5\}$ for every $x \in (\xi, t)_\in$ by Lemma 3.11, which implies that $\xi(1) \geq \min\{t, 0.5\} = t$ and $\xi((x \to b) \to b) \geq \min\{t, 0.5\} = t$. Thus $1 \in (\xi, t)_\in$ and $(x \to b) \to b \in (\xi, t)_\in$. Let $x, b \in X$ be such that $x \in (\xi, t)_\in$ and $x \to b \in (\xi, t)_\in$ for $t \in (0, 0.5]$. Then $\xi(x) \geq t$ and $\xi(x \to b) \geq t$. Using Lemma 3.11, we have

$$\xi(b) \geq \min\{\xi(x), \xi(x \to b), 0.5\} \geq \min\{t, 0.5\} = t,$$

and so $b \in (\xi, t)_\in$. Therefore $(\xi, t)_\in$ is an ideal of $(X, \to, 1)$ for all $t \in (0, 0.5]$. □

Theorem 3.20. *Let $(X, \to, 1)$ be a KL-algebra. If ξ is a fuzzy ideal of $(X, \to, 1)$, then the nonempty Q_t-set $(\xi, t)_q$ of ξ is an ideal of $(X, \to, 1)$ for all $t \in (0, 1]$.*

Proof. Let ξ be a fuzzy ideal of $(X, \to, 1)$ and assume that $(\xi, t)_q \neq \varnothing$ for all $t \in (0, 1]$. Then there exists $c \in (\xi, t)_q$, and so $\xi(1) + t \geq \xi(c) + t > 1$ by (3.6). Thus $1 \in (\xi, t)_q$. Let $x, b \in X$ be such that $x \in (\xi, t)_q$ and $x \to b \in (\xi, t)_q$. Then $\xi(x) + t > 1$ and $\xi(x \to b) + t > 1$. It follows from (3.7) and (3.8) that

$$\begin{aligned} \xi(b) + t &\geq \min\{\xi(x), \xi(x \to b)\} + t \\ &= \min\{\xi(x) + t, \xi(x \to b) + t\} > 1 \end{aligned}$$

and $\xi((x \to b) \to b) + t \geq \xi(x) + t > 1$. Hence $b \in (\xi, t)_q$ and $(x \to b) \to b \in (\xi, t)_q$. Therefore $(\xi, t)_q$ is an ideal of $(X, \to, 1)$ for all $t \in (0, 1]$. □

Proposition 3.21. *Let $(X, \to, 1)$ be a KL-algebra. Given a fuzzy set ξ in X, if its Q_t-set $(\xi, t)_q$ is an ideal of $(X, \to, 1)$ for all $t \leq 0.5$, then the following assertions are established:*

(3.31) $\qquad (\forall t \in (0, 0.5])(1 \in (\xi, t)_\in),$

(3.32) $\qquad (\forall x, b \in X)(\forall t, s \in (0, 0.5]) \left(\begin{array}{c} \langle x/t \rangle \, q \, \xi, \ \langle (x \to b)/s \rangle \, q \, \xi \\ \Rightarrow \ b \in (\xi, \max\{t, s\})_\in \end{array} \right),$

(3.33) $\qquad (\forall x, b \in X)(\forall t \in (0, 0.5]) \left(\langle x/t \rangle \, q \, \xi \ \Rightarrow \ (x \to b) \to b \in (\xi, t)_\in \right).$

Proof. Assume that $(\xi, t)_q$ is an ideal of $(X, \to, 1)$ for all $t \leq 0.5$. If $1 \notin (\xi, t)_\in$ for some $t \in (0, 0.5]$, then $\langle 1/t \rangle \, \overline{\in} \, \xi$ and so $\xi(1) < t \leq 1 - t$ since $t \leq 0.5$. Thus $\langle 1/t \rangle \, \overline{q} \, \xi$, and so $1 \notin (\xi, t)_q$. This is a contradiction, and thus $1 \in (\xi, t)_\in$ for all $t \in (0, 0.5]$. Let $x, b \in X$ and $t, s \in (0, 0.5]$ be such that $\langle x/t \rangle \, q \, \xi$ and $\langle (x \to b)/s \rangle \, q \, \xi$. Then $x \in (\xi, t)_q \subseteq (\xi, \max\{t, s\})_q$ and $x \to b \in (\xi, s)_q \subseteq (\xi, \max\{t, s\})_q$. Since $(\xi, \max\{t, s\})_q$ and $(\xi, t)_q$ are ideals of $(X, \to, 1)$, it follows from (2.8) and (2.9) that $b \in (\xi, \max\{t, s\})_q$ and $(x \to b) \to b \in (\xi, t)_q$. Hence $\xi(b) > 1 - \max\{t, s\} \geq \max\{t, s\}$ and

$$\xi((x \to b) \to b) > 1 - t \geq t,$$

that is, $\langle b/\max\{t, s\} \rangle \in \xi$ and $\langle ((x \to b) \to b)/t \rangle \in \xi$. Thus $b \in (\xi, \max\{t, s\})_\in$ and $(x \to b) \to b \in (\xi, t)_\in$. □

Proposition 3.22. *Let $(X, \to, 1)$ be a KL-algebra. Given a fuzzy set ξ in X, if its Q_t-set $(\xi, t)_q$ is an ideal of $(X, \to, 1)$ for all $t \in (0.5, 1]$, then ξ satisfies:*

(3.34) $\qquad (\forall x, b \in X)(\forall t, s \in (0.5, 1]) \left(\begin{array}{c} \langle x/t \rangle \in \xi, \ \langle (x \to b)/s \rangle \in \xi \\ \Rightarrow \ b \in (\xi, \max\{t, s\})_q, \end{array} \right),$

(3.35) $\qquad (\forall x, b \in X)(\forall t \in (0.5, 1]) \left(\langle x/t \rangle \in \xi \ \Rightarrow \ (x \to b) \to b \in (\xi, t)_q \right).$

Proof. Assume that $(\xi, t)_q$ is an ideal of $(X, \to, 1)$ for all $t \in (0.5, 1]$. Let $x, b \in X$ and $t, s \in (0.5, 1]$ be such that $\langle x/t \rangle \in \xi$ and $\langle (x \to b)/s \rangle \in \xi$. Then $\xi(x) \geq t > 1 - t$ and $\xi(x \to b) \geq s > 1 - s$, that is, $\langle x/t \rangle \, q \, \xi$ and $\langle x \to b/s \rangle \, q \, \xi$. So $x \in (\xi, t)_q \subseteq (\xi, \max\{t, s\})_q$ and $x \to b \in (\xi, s)_q \subseteq (\xi, \max\{t, s\})_q$. It follows from (2.8) and (2.9) that $b \in (\xi, \max\{t, s\})_q$ and $(x \to b) \to b \in (\xi, t)_q$. □

Theorem 3.23. *Let $(X, \to, 1)$ be a KL-algebra. If a fuzzy set ξ in X satisfies:*

$$(\forall x \in X)(\forall t \in (0.5, 1])(\langle x/t\rangle \, q\, \xi \;\Rightarrow\; \langle 1/t\rangle \in \xi \text{ or } \langle 1/t\rangle \, q\, \xi), \tag{3.36}$$

$$(\forall x, b \in X)(\forall t, s \in (0.5, 1])\left(\begin{array}{l} \langle x/t\rangle \, q\, \xi,\ \langle (x \to b)/s\rangle \, q\, \xi \;\Rightarrow \\ \langle b/\min\{t, s\}\rangle \in \xi \text{ or} \\ \qquad\qquad \langle b/\min\{t, s\}\rangle \, q\, \xi \end{array}\right), \tag{3.37}$$

$$(\forall x \in X)(\forall t \in (0.5, 1])\left(\begin{array}{l} \langle x/t\rangle \, q\, \xi \;\Rightarrow\; \langle ((x \to b) \to b)/t\rangle \in \xi \text{ or} \\ \langle ((x \to b) \to b)/t\rangle \, q\, \xi \end{array}\right), \tag{3.38}$$

then the nonempty Q_t-set $(\xi, t)_q$ of ξ is an ideal of $(X, \to, 1)$ for all $t \in (0.5, 1]$.

Proof. Assume that ξ satisfies (3.36), (3.37) and (3.38). Let $t \in (0.5, 1]$ be such that $(\xi, t)_q \neq \varnothing$. Then there exists $c \in (\xi, t)_q$ and so $\langle c/t\rangle \, q\, \xi$. Hence $\langle 1/t\rangle \in \xi$ or $\langle 1/t\rangle \, q\, \xi$ by (3.36). If $\langle 1/t\rangle \, q\, \xi$, then $1 \in (\xi, t)_q$. If $\langle 1/t\rangle \in \xi$, then $\xi(1) \geq t > 1 - t$ since $t > 0.5$. Hence $\langle 1/t\rangle \, q\, \xi$, i.e., $1 \in (\xi, t)_q$. Let $x, b \in X$ be such that $x \in (\xi, t)_q$ and $x \to b \in (\xi, t)_q$. Then $\langle x/t\rangle \, q\, \xi$ and $\langle (x \to b)/t\rangle \, q\, \xi$. It follows from (3.37) and (3.38) that $\langle b/t\rangle \in \xi$ or $\langle b/t\rangle \, q\, \xi$, and $\langle ((x \to b) \to b)/t\rangle \in \xi$ or $\langle ((x \to b) \to b)/t\rangle \, q\, \xi$. If $\langle b/t\rangle \, q\, \xi$, then $b \in (\xi, t)_q$. If $\langle b/t\rangle \in \xi$, then $\xi(b) \geq t > 1 - t$ since $t > 0.5$. Hence $\langle b/t\rangle \, q\, \xi$, i.e., $b \in (\xi, t)_q$. If $\langle ((x \to b) \to b)/t\rangle \, q\, \xi$, then $(x \to b) \to b \in (\xi, t)_q$. If $\langle ((x \to b) \to b)/t\rangle \in \xi$, then $\xi((x \to b) \to b) \geq t > 1 - t$ since $t > 0.5$. Hence $\langle ((x \to b) \to b)/t\rangle \, q\, \xi$, i.e., $(x \to b) \to b \in (\xi, t)_q$. Therefore $(\xi, t)_q$ is an ideal of $(X, \to, 1)$ for all $t \in (0.5, 1]$. $\square$

Proposition 3.24. *Let I be an ideal of a KL-algebra $(X, \to, 1)$ and let ξ be a fuzzy set in X such that*

$$(\forall x \in X \setminus I)(\xi(x) = 0), \tag{3.39}$$

$$(\forall x \in I)(\xi(x) \geq 0.5). \tag{3.40}$$

Then ξ satisfies:

$$(\forall x \in X)(\forall t \in (0, 1])(\langle x/t\rangle \, q\, \xi \;\Rightarrow\; \langle 1/t\rangle \in \xi \text{ or } \langle 1/t\rangle \, q\, \xi), \tag{3.41}$$

$$(\forall x, b \in X)(\forall t, s \in (0, 1])\left(\begin{array}{l} \langle x/t\rangle \, q\, \xi,\ \langle (x \to b)/s\rangle \, q\, \xi \;\Rightarrow \\ \langle b/\min\{t, s\}\rangle \in \xi \text{ or} \\ \qquad\qquad \langle b/\min\{t, s\}\rangle \, q\, \xi \end{array}\right), \tag{3.42}$$

$$(\forall x \in X)(\forall t \in (0, 1])\left(\begin{array}{l} \langle x/t\rangle \, q\, \xi \;\Rightarrow\; \langle ((x \to b) \to b)/t\rangle \in \xi \text{ or} \\ \langle ((x \to b) \to b)/t\rangle \, q\, \xi, \end{array}\right). \tag{3.43}$$

Proof. Let $x \in X$ and $t \in (0, 1]$ be such that $\langle x/t\rangle \, q\, \xi$. Then $\xi(x) + t > 1$. If $x \in X \setminus I$, then $\xi(x) = 0$ and thus $t > 1$. This is impossible, and so $x \in I$. Hence $\xi(x) \geq 0.5$. If $\langle 1/t\rangle \,\overline{\in}\, \xi$, then $t > \xi(1) \geq 0.5$ and so $\xi(1) + t > 1$, i.e., $\langle 1/t\rangle \, q\, \xi$. Also, if $\langle ((x \to b) \to b)/t\rangle \,\overline{\in}\, \xi$, then $t > \xi((x \to b) \to b) \geq 0.5$ and so $\xi((x \to b) \to b) + t > 1$, i.e., $\langle ((x \to b) \to b)/t\rangle \, q\, \xi$. Thus (3.41) and (3.43) are valid. Let $x, b \in X$ and $t, s \in (0, 1]$ be such that $\langle x/t\rangle \, q\, \xi$ and $\langle (x \to b)/s\rangle \, q\, \xi$. Then $\xi(x) + t > 1$ and $\xi(x \to b) + s > 1$. If $x \notin I$ (resp., $x \to b \notin I$), then $\xi(x) = 0$ (resp., $\xi(x \to b) = 0$) and so $t > 1$ (resp., $s > 1$). This is a contradiction, and hence $x \in I$ and $x \to b \in I$. Since I is an ideal of $(X, \to, 1)$, we get $b \in I$ and so $\xi(b) \geq 0.5$. If $t \leq 0.5$ or $s \leq 0.5$, then $\xi(b) \geq 0.5 \geq \min\{t, s\}$. So $\langle b/\min\{t, s\}\rangle \in \xi$. If $t > 0.5$ and $s > 0.5$, then

$\xi(b) + \min\{t, s\} > 0.5 + 0.5 = 1$, and thus $\langle b/\min\{t, s\}\rangle \, q \, \xi$. Consequently, (3.42) is valid. □

Using the combination of Theorem 3.23 and Proposition 3.24, we have the following corollary.

Corollary 3.25. *Let I be an ideal of a KL-algebra $(X, \to, 1)$. If a fuzzy set ξ in X satisfies* (3.39) *and* (3.40)*, then the nonempty Q_t-set $(\xi, t)_q$ of ξ is an ideal of $(X, \to, 1)$ for all $t \in (0.5, 1]$.*

4. Conclusion and future work

The cycloid is a magma $(X, \to)$, that is, a set X with a binary operation " $\to$ ", that satisfies the following condition

$$(\mathfrak{a} \to \mathfrak{b}) \to (\mathfrak{a} \to \mathfrak{c}) = (\mathfrak{b} \to \mathfrak{a}) \to (\mathfrak{b} \to \mathfrak{c})$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$. If a cycloid $(X, \to)$ has a logical unit "1", then it is called a unital cycloid, and is denoted by $(X, \to, 1)$. In 2008, Rump introduced the notion of ideal in the unital cycloid. The purpose of this work is to examine the fuzzy version of the ideal in unital cycloids by Rump. We used the fuzzy point as a tool for this work, and we reviewed various results, as shown in the text. Using the results of this paper, we can study various types of fuzzy versions in the future. In addition, by introducing various forms of ideals in unital cycloids, we will be able to identify related properties and interrelationships and enable various fuzzy studies.

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