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Abstract. In this paper, we introduce the concepts of positive implicative [resp. implicative and commutative] Γ-KU-algebras, and obtain their some properties (including characterizations) respectively and some relationships among them. Next, we propose the notions of positive implicative [resp. implicative and commutative] Γ-ideals of a Γ-KU-algebra, and deal with their some properties (including characterizations) respectively and some relationships among them. Finally, we define a topological Γ-KU-algebra and discuss its various topological structures.

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Keywords: Γ-KU-algebra, Γ-KU-subalgebra, Γ-KU-ideal, Topological Γ-KU-algebra.

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1. Introduction

In 1978, Iséki and Tanaka [1] introduced the notion of BCK-algebras as a generalization of I-algebras proposed by Imai and Iséki [2] in 1966. Iséki [3] defined BCI-algebras which is a proper subclass of BCK-algebras. Some researchers [4, 5, 6, 7, 8] studied properties of ideals which important role in BCK-algebras and BCI-algebras respectively. Furthermore, Dudek and Zhang [9] introduced a concept of BCC-algebras and discussed relationships between ideals and congruences in BCC-algebras. Also, some researchers [10, 11, 12, 13] investigated topological structures on BCK-algebras and BCI-algebras respectively.

In 2009, Prabpayak and Leerawat [14] defined a KU-algebra as new logical algebras and studied properties ideals and congruences in KU-algebras. Also, They [15] dealt with isomorphisms in KU-algebras. After then, many researchers [16, 17, 18, 19, 20, 21, 22] investigated various properties in KU-algebras. Recently, Hur et al. [23] introduced the notion of square root fuzzy sets and obtained some properties of square root fuzzy ideals of a KU-algebra. KU-algebras were studied by many mathematicians and applied to group theory, functional analysis, probability theory, topology, graph theory and computer science etc.

In 1981, Sen [24] proposed the notion of Γ-semigroups as a generalization of semigroups. Rao [25] introduced the concept of Γ-groups as generalization of groups and studied it various properties. Also, Rao [26] proposed the notion of Γ-semirings of a generalization of semirings. After then, Kaushik and Moin [27] investigated bi-Γ-ideals in a Γ-semiring, Rao and Venkateswarlu [28] studied some properties related to regular Γ-incline and field Γ-semiring.

In 2022, Saeid et al. [29] introduced the concept of Γ-BCK-algebras as a generalization of BCK-algebras and dealt with some properties of subalgebras, ideals, closed ideals, normal subalgebras and normal ideals in Γ-BCK-algebras and quotient Γ-BCK-algebras. After that time, Shi et al. [30] defined positive implicative [resp. implicative and commutative] Γ-BCK-algebras and positive implicative [resp. implicative and commutative] Γ-ideals of a Γ-BCK-algebra, and studied their some properties respectively and some relationships among them. Also, Shi et al. [31] defined a topological Γ-BCK-algebra and studied some of its topological structures.

It is the aim of our study to introduce the notion of Γ-KU-algebras as a generalization of KU-algebras, and define positive implicative [resp. implicative and commutative] Γ-KU-algebras and discuss their some properties (including characterizations) respectively and some relationships among them. Also, we define positive implicative [resp. implicative and commutative] Γ-ideals of a Γ-KU-algebra, and obtain their some properties (including characterizations) respectively and some relationships among them. Furthermore, we introduce the concept of topological Γ-KU-algebras and study its various topological structures.

2. Preliminaries

We recall some definitions needed in next sections. An *algebra* $X = (X, *, 0)$ means a nonempty set X together with a binary operation $*$ and a special element 0.

Definition 2.1 ([14]). An algebra X is called a *KU-algebra*, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
(KU_1) $(x * y) * [(y * z)] * (x * z)] = 0$,
(KU_2) $x * 0 = 0$,
(KU_3) $0 * x = x$,
(KU_4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In KU-algebra X, we define a binary operation $\leq$ on X as follows: for any $x,\ y \in X$,

$$x \leq y \text{ if and only if } y * x = 0.$$

Definition 2.2. Let X be a KU-algebra. Then X is said to be:
(i) *KU-positive implicative* [19], if $(z*x)*(z*y) = z*(x*y)$ for any $x,\ y,\ z \in X$,
(ii) *KU-commutative* [20], if $(y * x) * x = (x * y) * x$ for any $x,\ y \in X$,
(ii) *KU-implicative* [19], if $x = (x * y) * x$ for any $x,\ y \in X$.

Definition 2.3 (See [14]). Let A be a nonempty set of a KU-algebra X. Then A is called a *subalgebra* of X, if $x * y \in A$ for any $x,\ y \in A$.

Definition 2.4. Let I be a nonempty set of a KU-algebra X. Then

(a) I is called an *ideal* of X [19], if it satisfies the following conditions: for any $x,\ y \in X$,

(I_1) $0 \in I$,

(I_2) $x * y \in I$ and $x \in I$ imply $y \in I$.

(a) I is called an *ideal* (briefly, KUI) of X [14], if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(KUI_1) $0 \in I$,

(KUI_2) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$.

Definition 2.5 ([18]). Let I be a nonempty set of a KU-algebra X. Then I is called a *KU-positive implicative ideal* (briefly, KUPII) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(KUI_1) $0 \in I$,

(KUPII_2) $z * (x * y) \in I$ and $z * x \in I$ imply $z * y \in I$.

Definition 2.6 ([18]). Let I be a nonempty set of a KU-algebra X. Then I is called an *KU-implicative ideal* (briefly, KUII) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(KUI_1) $0 \in I$,

(KUII_2) $(x * y) * (z * x) \in I$ and $z \in I$ imply $x \in I$.

Definition 2.7 ([18]). Let I be a nonempty set of a KU-algebra X. Then I is called a *KU-commutative ideal* (briefly, KUCI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(KUI_1) $0 \in I$,

(KUCI_2) $y * (z * x) \in I$ and $z \in I$ imply $[(x * y) * y] * x \in I$.

Definition 2.8 ([26]). Let X and Γ be two nonempty sets. Then X is called a Γ-*semigroup*, if there is a mapping $f : X \times \Gamma \times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X \times \Gamma \times X$, such that it satisfies the following condition: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$x\alpha(y\beta z) = (x\alpha y)\beta z. \tag{2.1}$$

3. Some properties of Γ-KU-algebras

In this section, we introduce the concept of Γ-KU-algebras and study some of its properties.

Definition 3.1. Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is called a Γ-*KU-algebra*, if there is a mapping $f : X \times \Gamma \times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X \times \Gamma \times X$, satisfying the following axioms: for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,

($\Gamma\mathrm{KU}_1$) $(x\alpha y)\beta[(y\alpha z)\beta(x\alpha z)] = 0$,

($\Gamma\mathrm{KU}_2$) $0\alpha x = x$,

($\Gamma\mathrm{KU}_3$) $x\alpha 0 = 0$,

($\Gamma\mathrm{KU}_4$) $x\alpha y = 0 = y\alpha x$ imply $x = y$.

Remark 3.2. From (ΓKU$_1$), (ΓKU$_3$) and (ΓKU$_1$), (ΓKU$_2$), we have

$$x\alpha x = 0,\ z\beta(x\alpha z) = 0 \ \ \textit{for any } x,\ z \in X \textit{ and any } \alpha,\ \beta \in \Gamma. \tag{3.1}$$

We define a binary relation $\leq$ on Γ-KU-algebra X as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$x \leq y \Leftrightarrow y\alpha x = 0. \tag{3.2}$$

Then we obtain the following properties.

Proposition 3.3. *Let X be a Γ-KU-algebra. Then the following inequalities hold: for any $x,\ y,\ z \in X$ and each $\alpha,\ \beta \in \Gamma$,*
(1) $(y\alpha z)\beta(x\alpha z) \leq x\alpha y$,
(2) $0 \leq x$,
(3) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
(4) $x \leq x$,
(5) $x\alpha y \leq y$.

It is clear that for a Γ-KU-algebra X and a fixed $\alpha \in \Gamma$, if we define the operation $* : X \times X \to X$ as follows: for any $x,\ y \in X$,

$$x * y = x\alpha y,$$

then $(X, *, 0)$ is a KU-algebra and is denoted by X_α.

Example 3.4. (1) Let $X = \{0, 1, 2, 3\}$, let $\Gamma = \{\alpha, \beta, \gamma\}$ and let the ternary operation be defined by the table:

α	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	0	0	0
3	0	0	0	0

β	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	0
3	0	0	0	0

γ	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	0	0	0
3	0	1	2	0

Table 3.1

Then we can easily check that X is Γ-KU-algebra. Also, X_α, X_β and X_γ can confirm KU-algebras.

(2) Let $X = \{0, 1, 2, 3, 4, 5\}$, let $\Gamma = \{\alpha, \beta, \gamma, \delta, \psi\}$ and let the ternary operation be defined by the table:
Then we can easily check that X is Γ-KU-algebra.

Proposition 3.5. *Let X be a Γ-KI-algebra. Then the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,*
(1) $x \leq y$ *implies* $y\alpha z \leq x\alpha z$,
(2) $x \leq y$ *and* $y \leq z$, *then* $x \leq z$.

Proof. (1) Suppose $x \leq y$. Then clearly, $y\alpha x = 0$ for each $\alpha \in \Gamma$. Thus by the axiom (ΓKU$_1$), we have

$$(y\alpha x)\beta[(x\alpha z)\beta(y\alpha z)] = 0, \text{ i.e., } 0\beta[(x\alpha z)\beta(y\alpha z)] = 0 \text{ for any } \alpha,\ \beta \in \Gamma.$$

So by the axiom (ΓKU$_2$), $(x\alpha z)\beta(y\alpha z) = 0$. Hence by (3.2), $y\alpha z \leq x\alpha z$.

α	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	4	5
2	0	1	0	3	4	5
3	0	1	2	0	4	5
4	0	1	2	3	0	5
5	0	1	2	3	4	0

α	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	3	4	5	1
2	0	2	0	4	5	1
3	0	2	3	0	4	1
4	0	2	3	4	0	1
5	0	2	3	4	5	0

α	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	4	5	1	2
2	0	3	0	5	1	2
3	0	3	4	0	1	2
4	0	3	4	5	0	2
5	0	3	4	5	1	0

δ	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	5	1	2	3
2	0	4	0	1	2	3
3	0	4	5	0	2	3
4	0	4	5	1	0	3
5	0	4	5	1	2	0

ψ	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	1	2	3	4
2	0	5	0	2	3	4
3	0	5	1	0	3	4
4	0	5	1	2	0	4
5	0	5	1	2	3	0

Table 3.2

(2) Suppose $x \leq y$ and $y \leq z$. Then by (1), $z\alpha x \leq y\alpha x$. Since $x \leq y$, $y\alpha x = 0$. Thus $z\alpha x \leq 0$. By Proposition 3.3 (2), $0 \leq z\alpha x$. So by Proposition 3.3 (3), $z\alpha x = 0$. Hence $x \leq z$. □

From Proposition 3.3 (3), (3.1) and Proposition 3.5 (2), it is obvious that $(X, \leq)$ is a poset with the least element 0.

Proposition 3.6. *Let X be a Γ-KU-algebra. Then the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha,\ \beta \in \Gamma$,*

$$z\alpha(y\beta x) = y\alpha(z\beta x). \tag{3.3}$$

Proof. From the axiom (ΓKU$_1$), $(0\alpha z)\beta[(z\alpha x)\beta(0\alpha x)] = 0$. Then by the axioms ($\Gamma$KU$_2$), $z\beta[(z\alpha x)\beta x] = 0$, i.e.,

$$(z\alpha x)\beta x \leq z. \tag{3.4}$$

Thus by (3.4), Proposition 3.5 (1) and Proposition 3.3 (1), we have

$$z\alpha(y\beta x) \leq [(z\alpha x)\beta x]\alpha(y\beta x) \leq y\alpha(z\beta x). \tag{3.5}$$

Since $x,\ y,\ z$ are arbitrary, by interchanging y and z in the equality (3.5), we get

$$y\alpha(z\beta x) \leq z\alpha(y\beta x). \tag{3.6}$$

So the axiom (ΓKU$_4$), the identity (3.3) holds. □

The followings are immediate consequences of Proposition 3.6.

Corollary 3.7. *Let X be a Γ-KU-algebra. Then the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha,\ \beta \in \Gamma$,*
(1) *$x\alpha y \leq z$ if and only if $x\alpha z \leq y$,*
(2) *$(y\alpha x)\beta x \leq y$.*

The following is an immediate consequence of Theorem 3.3 (1) and Corollary 3.7 (1).

Corollary 3.8. *In a Γ-KU-algebra X, the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,*

$$(x\alpha y)\beta(x\alpha z) \leq y\alpha z,\ \textit{i.e.},\ (y\alpha z)\beta[(x\alpha y)\beta(x\alpha z)] = 0.$$

The following is an immediate consequence of Corollary 3.8.

Corollary 3.9. *In a Γ-KU-algebra X, the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,*

$$x \leq y \textit{ implies } z\alpha x \leq z\alpha y.$$

We define a binary operation $\wedge$ on a Γ-KU-algebra X as follows: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$x \wedge y = (y\alpha x)\beta x.$$

Then it is obvious that $x \wedge y$ is a lower bound of $\{x, y\}$ and $x \wedge x = 0$, $x \wedge 0 = 0 = 0\alpha x$. However, $x \wedge y \neq y \wedge x$ in general.

Proposition 3.10. *In a Γ-KU-algebra X, the followings hold: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,*

$$(y \wedge x)\alpha x = y\alpha x.$$

Proof. Since $y \wedge x \leq y$, by Proposition 3.5 (1), we have

$$y\alpha x \leq (y \wedge x)\alpha x. \tag{3.7}$$

On the other hand, by Corollary 3.7 (2), we get

$$(y \wedge x)\alpha x = [(x\alpha y)\beta y]\alpha x \leq y\alpha x. \tag{3.8}$$

Thus $(y \wedge x)\alpha x = y\alpha x$. □

We obtain a characterization of a Γ-KU-algebra.

Theorem 3.11. *Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is a Γ-KU-algebra if and only if it satisfies axioms (ΓKU$_1$), (ΓKU$_4$) and the following condition: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

$$(y\alpha 0)\beta x = x. \tag{3.9}$$

Proof. ($\Rightarrow$): The proof is straightforward from the axioms (ΓKU$_2$) and (ΓKU$_3$).

($\Leftarrow$): Suppose the necessary conditions hold, let $x \in X$ and let $\alpha,\ \beta \in \Gamma$. Then from the axiom ($\Gamma$KU$_3$), we get $(x\alpha 0)\beta[(0\alpha 0)\beta(x\alpha 0)] = 0$. On the other hand, by (3.9), $(x\alpha 0)\beta(x\alpha 0) = 0$. Again by (3.9), $x\alpha 0 = 0$. Thus the axiom (ΓKU$_3$) holds. By combining (3.9) and the axiom (ΓKU$_3$), $0\alpha x = x$. So the axiom (ΓKU$_2$) holds. Hence X is a Γ-KU-algebra. □

4. Special Γ-KU-algebras

In this section, we define some special Γ-KU-algebras, for example, positive implicative [resp. implicative and commutative] Γ-KU-algebras and obtain some of their properties (including characterizations) respectively and a relationship among them.

Definition 4.1. A Γ-KU-algebra X is said to be *positive implicative*, if it satisfies the following axiom: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$(z\alpha x)\beta(z\alpha y) = z\beta(x\alpha y). \tag{4.1}$$

It is obvious that if X is a positive implicative Γ-KU-algebra, then X_α is a positive implicative KU-algebra for each $\alpha \in \Gamma$.

Example 4.2. Let $X = \{0, 1, 2, 3\}$, let $\Gamma = \{\alpha, \beta\}$ and let the ternary operation be defined by the table:

α	0	1	2	3	β	0	1	2	3
0	0	1	2	3	0	0	1	2	3
1	0	0	2	3	1	0	0	2	3
2	0	1	0	3	2	0	0	0	3
3	0	0	2	0	3	0	0	2	0

Table 4.1

Then we can easily check that X is a positive implicative Γ-KU-algebra. Moreover, we confirm that X_α and X_β are positive implicative KU-algebras.

Proposition 4.3. *In a Γ-KU-algebra X, the following holds: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

$$(x\alpha y)\beta((y\alpha x)\beta x) \leq (((x\alpha y)\beta y)\alpha x)\beta x. \tag{4.2}$$

Proof.
$$\begin{aligned}
&[(((x\alpha y)\beta y)\alpha x)\beta x]\alpha[(x\alpha y)\beta((y\alpha x)\beta x)]\\
&= (x\alpha y)\beta[((((x\alpha y)\beta y)\alpha x)\beta x)\alpha((y\alpha x)\beta x)] \text{ [By the identity (3.3)]}\\
&\leq (x\alpha y)\beta[((x\alpha y)\beta y)\alpha((y\alpha x)\beta x)] \text{ [By Corollary 3.7 (2)]}\\
&\leq (x\alpha y)\beta(x\alpha y) \text{ [By Corollary 3.7 (2)]}\\
&= 0. \text{ [By (3.1)]}
\end{aligned}$$
Then by Proposition 3.3 (2) and (3), we have

$$[(((x\alpha y)\beta y)\alpha x)\beta x]\alpha[(x\alpha y)\beta((y\alpha x)\beta x)] = 0.$$

Thus the inequality (4.2) holds. □

We have a characterization of a positive implicative Γ-KU-algebra.

Theorem 4.4. *Let X be a Γ-KU-algebra. Then the followings are equivalent:*
(1) *X is positive implicative,*
(2) *$x\alpha y = x\alpha(x\beta y)$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$.*

Proof. (1) $\Rightarrow$ (2): The proof follows from the axiom (ΓKU_2), (3.1) and the identity (4.1).

(2) $\Rightarrow$ (1): Suppose the condition (2) holds, let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have
$$\begin{aligned}
&[z\alpha(x\beta y)]\alpha[(z\beta x)\alpha(z\beta y)]\\
&= [z\alpha(x\beta y)]\alpha[(z\beta x)\alpha(z\beta(z\alpha y))] \text{ [By (2)]}\\
&\leq [z\alpha(x\beta y)]\alpha[x\alpha(z\beta y)] \text{ [By Corollary 3.8]}\\
&= [z\alpha(x\beta y)]\alpha[z\alpha(x\beta y)] \text{ [By Proposition 3.6]}
\end{aligned}$$

$= 0.$ [By (3.1)]
Thus $(z\beta x)\alpha(z\beta y) \leq z\alpha(x\beta y)$. The proof of the converse inequality is easy. So $(z\beta x)\alpha(z\beta y) = z\alpha(x\beta y)$. Hence X is positive implicative. □

We give another characterization of a Γ-KU-algebra.

Theorem 4.5. *Let X be a Γ-KU-algebra. Then the followings are equivalent: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,*
(1) *X is positive implicative,*
(2) *$z\alpha(x\beta y) = 0$ implies $(z\alpha x)\beta(z\alpha y) = 0$,*
(3) *$y\alpha(y\beta x) = 0$ implies $y\alpha x = 0$.*

Proof. (1) ⇒ (2): The proof is straightforward.
(2) ⇒ (3): Suppose (2) holds and $y\alpha(y\beta x) = 0$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned} y\alpha x &= 0\beta(y\alpha x) \text{ [By the axiom } (\Gamma\mathrm{KU}_2)] \\ &= (y\alpha y)\beta(y\alpha x) \text{ [By (3.1)]} \\ &= 0. \text{ [By (2)]} \end{aligned}$$

Thus (3) holds.
(3) ⇒ (1): Suppose (3) holds. For any $x,\ y \in X$ and any $\alpha,\ \beta \in \tau$, let $u = y\beta(y\alpha x)$. Then we have

$$\begin{aligned} y\alpha(y\beta(u\alpha x)) &= y\alpha(u\beta(y\alpha x)) \text{ [By Proposition 3.6]} \\ &= u\alpha(y\beta(y\alpha x)) \text{ [By Proposition 3.6]} \\ &= (y\beta(y\alpha x))\alpha(y\beta(y\alpha x)) \\ &= 0. \text{ [By (3.1)]} \end{aligned}$$

Thus by the hypothesis and (3.1), we get

$$0 = y\beta(u\alpha x) = y\beta(y\beta(y\alpha x)\alpha x)) = (y\beta(y\alpha x)\beta(y\alpha x).$$

So $y\alpha x \leq y\beta(y\alpha x)$. On the other hand, from Proposition 3.6 and (3.1), it is obvious that $y\beta(y\alpha x) \leq y\alpha x$. Hence $y\beta(y\alpha x) = y\alpha x$. Therefore by Theorem 4.4, X is positive implicative. □

Definition 4.6. A Γ-KU-algebra X is said to be *commutative*, if it satisfies the following axiom:

(4.3) $(y\alpha x)\beta x = (x\alpha y)\beta y$, *i.e.*, $x \wedge y = y \wedge x$ *for any* $x,\ y \in X$ *and any* $\alpha,\ \beta \in \Gamma$.

We can easily see that if X is a commutative Γ-KU-algebra, then X_α is a commutative kU-algebra for each $\alpha \in \Gamma$.

Example 4.7. (1) Let $X = \{0, 1, 2, 3, 4\}$, let $\Gamma = \{\alpha, \beta\}$ and let the ternary operation be defined by the table:

α	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	1	3
2	0	1	0	3	4
3	0	0	0	0	1
4	0	0	0	0	0

β	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	1	3
2	0	1	0	3	4
3	0	0	0	0	1
4	0	0	0	0	0

Table 4.2

Then clearly, X is a Γ-KU-algebra but $(2\alpha 3)\beta 3 = 3 \neq 2 = (3\alpha 2)\beta 2$. Thus X is not commutative.

(2) Let $X = \{0, 1, 2, 3\}$, let $\Gamma = \{\alpha, \beta\}$ and let the the ternary operation be defined as the following table:

α	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	2
3	0	1	0	0

β	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Table 4.3

Then we can easily check that X is commutative Γ-KU-algebra.

The following is an immediate consequence of the axiom (ΓKU$_4$) and Definition 4.6.

Theorem 4.8. *Let X be a Γ-KU-algebra. Then the followings are equivalent: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*
(1) *X is commutative,*
(2) $(y\alpha x)\beta x \leq (x\alpha y)\beta y$,
(3) $((x\alpha y)\beta y)\alpha((y\alpha x)\beta x) = 0$.

We obtain a characterization of commutative Γ-BCK-algebras.

Theorem 4.9. *Let X be a Γ-KU-algebra. Then the followings are equivalent: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,*
(1) *$x \leq z$ and $y\alpha z \leq x\alpha z$ imply $x \leq y$,*
(2) *$x,\ y \leq z$ and $y\alpha z \leq x\alpha z$ imply $x \leq y$,*
(3) *$x \leq y$ implies $x = (x\alpha y)\beta y$,*
(4) *X is commutative,*
(5) *$x \leq y$ implies $((x\alpha y)\beta y)\alpha x = 0$.*

Proof. (1) $\Rightarrow$ (2): The proof is clear.

(2) $\Rightarrow$ (3): Suppose $x \leq y$ and let $\alpha,\ \beta \in \Gamma$. Then by Corollary 3.7 (2), $(x\alpha y)\beta y \leq x$. Moreover, $((x\alpha y)\beta y)\alpha y \leq x\alpha y$. Thus by the hypothesis, $x \leq (x\alpha y)\beta y$. So $x = (x\alpha y)\beta y$.

(3) $\Rightarrow$ (4): Suppose the condition (3) holds, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. From Corollary 3.7 (2), it is clear that $(x\alpha y)\beta y \leq x$. Then by the hypothesis, we have

$$(x\alpha y)\beta y = (((x\alpha y)\beta y)\alpha x)\beta x. \tag{4.4}$$

On the other hand, we get

$$\begin{aligned}
&[(y\alpha x)\beta x]\alpha[(x\alpha y)\beta y] \\
&= [(y\alpha x)\beta x]\alpha[(((x\alpha y)\beta y)\alpha x)\beta x] \text{ [By (4.4)]} \\
&= [((x\alpha y)\beta y)\alpha x]\alpha[((y\alpha x)\beta x)\beta x] \text{ [By Proposition 3.6]} \\
&\leq [((x\alpha y)\beta y)\alpha x]\alpha(y\beta x) \text{ [Corollary 3.7 (2)]} \\
&\leq y\alpha[(x\alpha y)\beta y] \text{ [By Proposition 3.3 (1)]} \\
&= (x\alpha y)\alpha(y\beta y) \text{ [By Proposition 3.6]} \\
&= (x\alpha y)\alpha 0 \text{ [By (3.1)]}
\end{aligned}$$

$= 0$ [By (ΓKU$_3$)]
Thus $(x\alpha y)\beta y \leq (y\alpha x)\beta x$. So by Theorem 4.8, X is commutative.

(4) $\Rightarrow$ (1): Suppose the condition (4) holds, and suppose $x \leq z$ and $y\alpha z \leq x\alpha z$ for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$. Then clearly, $z\alpha x = 0$ and $(x\alpha z)\beta(y\alpha z) = 0$ for any $\beta \in \Gamma$. Thus we have

$y\alpha x = y\alpha(0\beta x)$ [By (ΓKU$_2$)]
$= y\alpha[(z\alpha x)\beta x]$ [Since $z\alpha x = 0$]
$= y\alpha[(x\alpha z)\beta z]$ [Since X is commutative]
$= (x\alpha z)\beta(y\alpha z)$ [By Proposition 3.6]
$= 0.$

So $x \leq y$. Hence (1) holds.

(3) $\Leftrightarrow$ (5): The proof is obvious. □

The following is an immediate consequence of Theorem 4.9.

Theorem 4.10. *Let X be a Γ-KU-algebra. Then followings are equivalent: ,*
(1) *X is commutative,*
(2) *$x\alpha(x\beta y) = y\alpha(y\beta(x\alpha(x\beta y)))$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$*

For a Γ-KU-algebra X and each $x \in X$, the set

$$A(x) = \{y \in X : y \leq x\}$$

is called an *initial section* of x.

Theorem 4.11. *a Γ-KU-algebra X is commutative if and only if for any $x,\ y \in X$,*

$$A(x) \cap A(y) = A(x \wedge y).$$

Proof. The proof follows from the property of $\wedge$ and Theorem 4.8 (2). □

Definition 4.12. Let X be a Γ-KU-algebra. Then X is said to be *implicative*, if it satisfies the following condition:

$$x = (x\alpha y)\beta x \text{ for any } x,\ y \in X \text{ and any } \alpha,\ \beta \in \Gamma. \tag{4.5}$$

It is clear that if X is an implicative Γ-BCK-algebra, then X_α is an implicative BCK-algebra fore each $\alpha \in \Gamma$.

Example 4.13. (1) Let $X = \{0, 1, 2, 3\}$, let $\Gamma = \{\alpha, \beta\}$ and let the the ternary operation be defined as the following table:

α	0	1	2	3	β	0	1	2	3
0	0	1	2	3	0	0	1	2	3
1	0	0	2	2	1	0	0	2	2
2	0	1	0	3	2	0	1	0	3
3	0	0	2	0	3	0	2	2	0

Table 4.4

Then clearly, X is an implicative Γ-KU-algebra.

(2) Let $X = \{0, 1, 2, 3, 4\}$, let $\Gamma = \{\alpha, \beta\}$ and let the the ternary operation be defined as the following table:
Then X is a Γ-KU-algebra. But it is neither implicative nor commutative.

α	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	1	4
2	0	1	0	0	4
3	0	0	0	0	4
4	0	0	0	0	0

β	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	1	4
2	0	1	0	3	4
3	0	0	1	0	4
4	0	0	0	0	0

Table 4.5

We obtain a relationship among implicativeness, commutativity and positive implicativeness.

Theorem 4.14. *Let X be a Γ-KU-algebra. Then X is implicative if and only if it is commutative and positive implicative.*

Proof. Suppose X is implicative, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by Proposition 3.6, we have

$$x\alpha y = [(x\alpha y)\beta x]\beta(x\alpha y) = x\alpha(x\beta y).$$

Thus by Theorem 4.4, X is positive implicative. On the other hand, we get

$$\begin{aligned}(y\alpha x)\beta x &= (y\alpha x)\beta((x\alpha y)\beta x) \text{ [By the hypothesis]}\\ &\leq (x\alpha y)\beta y. \text{ [By Proposition 3.3 (1)]}\end{aligned}$$

So by Theorem 4.8 (1), X is commutative.

Conversely, suppose the necessary conditions hold and let $x,\ y \in X$ and $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned}[(x\alpha y)\beta x]\alpha x &= [x\beta(x\alpha y)]\beta(x\alpha y) \text{ [Since } X \text{ is commutative]}\\ &= [x\beta(x\alpha y)]\beta[x\beta(x\alpha y)] \text{ [By Theorem 4.4 (2)]}\\ &= 0. \text{ [By (3.1)]}\end{aligned}$$

Thus $x \leq (x\alpha y)\beta x$. On the other hand, by Proposition 3.3 (5), $(x\alpha y)\beta x \leq x$. So by Proposition 3.3 (3), $x = (x\alpha y)\beta x$. Hence X is implicative. This completes the proof. □

Now we provide a sufficient condition of implicative Γ-KU-algebras.

Proposition 4.15. *Let X be a Γ-KU-algebra. Suppose the following holds: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

$$(y\alpha x)\beta[(y\alpha x)\beta x] = (x\alpha y)\beta[(x\alpha y)\beta y]. \tag{4.6}$$

Then X is implicative

Proof. Suppose the condition (4.6) holds, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned}y\alpha x &= 0\alpha[0\beta(y\alpha x)] \text{ [By the axiom } (\Gamma\mathrm{KU}_2)]\\ &= [y\beta(x\alpha x)]\alpha[(y\beta(x\alpha x))\beta(y\alpha x)] \text{ [By the axiom } (\Gamma\mathrm{KU}_3) \text{ and } (3.1)]\\ &= [x\beta(y\alpha x)]\alpha[(x\beta(y\alpha x))\beta(y\alpha x)] \text{ [By Proposition 3.6]}\\ &= (x\beta y)\alpha[(x\beta y)\beta(y\alpha x)] \text{ [Putting } y\alpha y = y]\\ &= [x\alpha(x\beta y)]\alpha(x\beta y) \text{ [By the condition (2)]}\\ &= [x\alpha(x\beta(x\alpha y))]\alpha[x\beta(x\alpha y)] \text{ [Since } y = x\alpha y]\\ &= (x\alpha y)\alpha[x\beta(x\alpha y)] \text{ [By Proposition 3.6 (6)]}\\ &= [x\alpha(x\beta(x\alpha y))]\alpha y \text{ [By the identity (3.1)]}\end{aligned}$$

$= (x\alpha y)\alpha y$. [By Proposition 3.6 (6)]
Thus X is positive implicative. On the other hand, we have

$$\begin{aligned} x\alpha(x\beta y) &= [x\alpha(x\beta y)]\alpha(x\beta y) \text{ [By Proposition 3.6 (6)]} \\ &= [y\alpha(y\beta x)]\alpha(y\beta x) \text{ [By the condition (2)]} \\ &= y\alpha(y\beta x). \text{ [By Proposition 3.6 (6)]} \end{aligned}$$

So X is commutative. Hence by Theorem 4.14, X is implicative. □

5. Some Γ-ideals of Γ-KU-algebras

In this section, we introduce the concepts of Γ-KU-ideals, positive implicative Γ-KU-ideals, implicative Γ-KU-ideals and commutative Γ-KU-ideals in Γ-KU-algebras and discuss some of their properties respectively and a relationship among them.

Definition 5.1. Let X be a Γ-KU-algebra and let A be a nonempty subset of X. Then A is called a Γ- *subalgebra* of X, if it satisfies the following condition:

(5.1) $\qquad x\alpha y \in A$ for any $x,\ y \in A$ and for each $\alpha \in \Gamma$.

Example 5.2. Let X be the Γ-KU-algebra given in Example 3.2 (3), $\{0, 1, 2\}$ is a Γ-subalgebra of X.

Definition 5.3. Let X be Γ-KU-algebra and let I be a nonempty set of X. Then I is called a Γ- *ideal* (briefly, ΓI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(ΓI_1) $0 \in I$,

(ΓI_2) if $x\alpha y \in I$ and $x \in I$, then $y \in I$.

An ideal I is said to be *proper*, if $I \neq X$. It is obvious that X and $\{0\}$ are ideals of X. In particular, X is called a *trivial* Γ*-ideal* of X.

Example 5.4. (1) Consider the Γ-KU-algebra given in Example 3.4. Then $\{0, 2\}$ is a Γ-ideal but $\{0, 1\}$ not a Γ-ideal of X.

(2) Let X be the commutative Γ-KU-algebra given in Example 4.7 (2). Then we can easily see that X has only two ΓIs $\{0\}$ and X.

The following is an immediate consequence of Definition 5.3.

Proposition 5.5. *Let I be a ΓI of a Γ-KU-algebra X and let $x \in I$. If $y \leq x$, then $y \in I$.*

Proposition 5.6. *Every ΓI of a Γ-KU-algebra X is a Γ-subalgebra of X.*

Proof. Let I be a ΓI of X and let $x,\ y \in I$, $\alpha \in \Gamma$. Then by Proposition 3.6 and the axiom (ΓKU_3), $y\beta(x\alpha y) = 0$. Thus $x\alpha y \leq y$. So by Proposition 5.5, $x\alpha y \in I$. Hence I is a Γ-subalgebra of X. □

Definition 5.7. Let X be Γ-KU-algebra and let $a,\ b \in X$ and $\alpha \in \Gamma$. Then the subset $A_\alpha(a, b)$ of X is defined as follows:

$$A_\alpha(a, b) = \{x \in X : b\alpha x \leq a\}.$$

It is obvious that $0,\ a,\ b \in A_\alpha(a, b)$.

Example 5.8. Let X be the Γ-BCK-algebra in Example 3.4. Then clearly,

$$A_\alpha(1, 2) = X,\ A_\beta(1, 2) = \{0, 1, 2\} = A_\gamma(1, 2).$$

We have a characterization of ΓIs of a Γ-KU-algebra.

Theorem 5.9. *Let I be a nonempty subset of a Γ-KU-algebra X. Then I is a ΓI of X if and only if $A_\alpha(a,b) \subset I$ for any $a,\ b \in I$ and each $\alpha \in \Gamma$.*

Proof. ($\Rightarrow$): Suppose I is a ΓI of X and let $x \in A_\alpha(a,b)$. Then clearly, $b\alpha x \leq a$. Thus by Proposition 5.5, $b\alpha x \in I$. Since $b \in I$ and I is a ΓI of X, $x \in I$. So $A_\alpha(a,b) \subset I$.

($\Leftarrow$): Suppose the necessary condition holds. Since $I \neq \varnothing$, there is $a \in I$. Then by (3.1), $a\alpha 0 \leq a$. Thus $0 \in A_\alpha(a,a)$. Since $A_\alpha(a,a) \subset I$, $0 \in I$. So the condition (ΓI_1) holds. Now let $b\beta a \in I$ and $b \in I$. Then by Corollary 3.7 (2), $(b\beta a)\alpha x \leq b$. Thus $x \in A_\alpha(b\beta a, b) \subset I$. So the condition ($\Gamma I_2$) holds. Hence I is a ΓI of X. □

The following is an immediate consequence of Theorem 5.9.

Corollary 5.10. *I is a ΓI of a Γ-KU-algebra X if and only if for any $a,\ b \in I$ and any $\alpha,\ \beta \in \Gamma$, $(b\beta a)\alpha x = 0$ implies $x \in I$.*

Definition 5.11. Let X be Γ-KU-algebra and let I be a nonempty set of X. Then I is called a *positive implicative* Γ-*KU-ideal* (briefly, PIΓKUI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(ΓI_1) $0 \in I$,

($PI\Gamma KUI_2$) if $z\alpha(x\beta y) \in I$ and $z\alpha x \in I$, then $z\alpha y \in I$.

It is obvious that X is a PIΓKUI of X.

Example 5.12. Let X be the Γ-BCK-algebra given in Example 4.13 (2). Then we can easily check that $\{0,1,3\}$ and $\{0,1,2,3\}$ are PIΓKUIs of X. Furthermore, $\{0\}$, $\{0,2\}$ and $\{0,2,4\}$ are ΓIs but not PIΓKUIs of X.

Proposition 5.13. *Every PIΓKUI of Γ-KU-algebra X is a ΓI of X but the converse is not true.*

Proof. Let I be a PIΓKUI of X. Suppose $x\alpha y \in I$ and $x \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then clearly, $0\beta(x\alpha y) \in I$ and $0\alpha y \in I$. Thus by ($PI\Gamma KUI_2$), $x = 0\alpha x \in I$. So I is a ΓI of X. See Example 5.12 for the converse. □

We have a characterization of positive implicative Γ-KU-ideals.

Theorem 5.14. *Let I be a ΓKUI of a Γ-KU-algebra X. Then I is positive implicative if and only if the set $A_a = \{x \in X : a\alpha x \in I \text{ for each } \alpha \in \Gamma\ \}$ is a ΓI of X for each each $a \in X$.*

Proof. Suppose I is positive implicative and $x\alpha y \in A_a$, $x \in A_a$ for each $a \in X$ and each $\alpha \in \Gamma$. Then clearly, $a\beta(x\alpha y) \in I$ and $a\alpha y \in I$. Thus by the condition ($PI\Gamma KUI_2$), $a\alpha x \in I$. So $x \in A_a$. Hence A_a is a ΓI of X.

Now suppose the necessary condition holds, and $z\alpha(x\beta y) \in I$ and $z\alpha y \in I$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Then clearly, $y\alpha x \in A_z$ and $x \in A_z$. Thus by the hypothesis, $y \in A_z$. So $z\alpha y \in I$. Hence I is positive implicative. □

The following is an immediate consequence of Theorem 5.14.

Corollary 5.15. *If I is a PIΓKUI of a Γ-KU-algebra X, then for each each $a \in X$, A_a is the least ΓI of X such that $I \cup \{a\} \subset A_a$.*

We obtain a characterization of PIΓKUIs.

Theorem 5.16. *Let I be a nonempty subset of a Γ-KU-algebra X. Then the followings are equivalent:*
(1) *I is a PIΓKUI of X,*
(2) *I is a ΓI of X and $y\alpha(y\beta x) \in I$ implies $y\alpha x \in I$ for any $x,\ y \in X$ and $\alpha,\ \beta \in \Gamma$,*
(3) *I is a ΓI of X and $z\alpha(y\beta x) \in I$ implies $(z\alpha y)\beta(z\alpha x) \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,*
(4) *$0 \in I$, and $z\alpha[y\beta(y\alpha x)] \in I$ and $z \in I$ imply $y\alpha x \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$.*

Proof. (1)⇒(2): Suppose I is a PIΓKUI of X. Then by Proposition 5.13, I is a ΓI of X. Now suppose $y\alpha(y\beta x) \in I$ for any $x,\ y \in X$ and $\alpha,\ \beta \in \Gamma$. From (3.1), $y\alpha y = 0 \in I$. Then by (PIΓKUI$_2$), $x\alpha y \in I$. Thus the condition (2) holds.

(2)⇒(3): Suppose the condition (2) holds and suppose $z\alpha(y\beta x) \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned} z\alpha[z\alpha((z\beta y)\alpha x)] &= z\beta[(z\beta y)\alpha(z\beta x)] \text{ [By (3.3)]} \\ &\leq z\alpha(y\beta x). \text{ [By Corollaries 3.8 and Corollaries 3.7 (2)]} \end{aligned}$$

Since I is a ΓI of X, by Proposition 5.5, $z\alpha[z\alpha((z\beta y)\alpha x)] \in I$. By the condition (2), $z\alpha[(z\beta y)\alpha x] \in I$. On the other hand, by Proposition 3.6, $z\alpha[(z\beta y)\alpha x] = (z\beta y)\alpha(z\beta x)$. Thus $(z\beta y)\alpha(z\beta x) \in I$. So the condition (3) holds.

(3)⇒(4): Suppose the condition (3) holds. Then clearly, $0 \in I$. Suppose $z\alpha[y\beta(y\alpha x)] \in I$ and $z \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$. Then by Proposition 3.6, we get

$$z\alpha[y\beta(y\alpha x)] = y\alpha[y\beta(z\alpha x)].$$

Thus $y\alpha[y\beta(z\alpha x)] \in I$. On the other hand, from Proposition 3.6, (3.1) and the condition (3), we have

$$z\beta(y\alpha x) = y\beta(z\alpha x) = (y\alpha y)\beta(z\alpha x) \in I.$$

Since I is a ΓI of X and $z \in I$, $y\alpha x \in I$. So the condition (4) holds.

(4)⇒(1): Suppose the condition (4) holds. Suppose $x\alpha y \in I$ and $x \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then by the axiom (ΓKU$_2$), we get

$$x\alpha y = x\alpha[0\alpha(0\beta y)].$$

Thus $x\alpha[(y\alpha 0)\beta 0] \in I$ and $x \in I$. By the condition (4), $0\beta y \in I$. By the axiom (ΓKU$_2$), $0\beta y = y$. So $y \in I$. Hence I is a ΓI of X.

Now suppose $z\alpha(x\beta y) \in I$ and $z\alpha x \in I$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Then from Corollary 3.9 and Proposition 3.6, we have

$$(z\alpha y)\beta[z\alpha(z\alpha x) \leq y\beta(z\alpha x) = z\beta(y\alpha x).$$

Since $z\alpha(x\beta y) \in I$, $(z\alpha y)\beta[z\alpha(z\alpha x) \in I$. Since $z\alpha x \in I$, by the condition (4), $y\alpha x \in I$. Thus I is a PIΓKUI of X. This completes the proof. □

Proposition 5.17. *Let I and J be ΓIs of a Γ-KU-algebra X such that $I \subset J$. If I is positive implicative, then so is J.*

Proof. Suppose $z\beta(x\alpha y) \in J$ and $z\alpha x \in J$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Let $u = z\beta(x\alpha y)$. Then from Proposition 3.6, (3.1) and the hypothesis, we have

$$z\alpha[x\beta(u\alpha y)] = u\alpha[z\beta(x\alpha y)] = 0 \in I.$$

Since I is positive implicative, by Theorem 5.16 (3), we get

$$(z\alpha x)\beta[z\alpha(u\alpha y)] \in I.$$

On the other hand, by Proposition 3.6, we have

$$(z\alpha x)\beta[z\alpha(u\alpha y)] = u\beta[(z\alpha x)\beta(z\alpha y) = [z\beta(x\alpha y)]\beta[(z\alpha x)\beta(z\alpha y)].$$

Thus $[z\beta(x\alpha y)]\beta[(z\alpha x)\beta(z\alpha y)] \in I$. Since $I \subset J$, $[z\beta(x\alpha y)]\beta[(z\alpha x)\beta(z\alpha y)] \in J$. Since $z\beta(x\alpha y) \in J$ and J is a ΓI of X, $(z\alpha x)\beta(z\alpha y) \in J$. So by Theorem 5.16 (3), J is positive implicative. □

From the following Theorem, we can see that in Γ-KU-algebras, the zero ΓIs play important roles.

Theorem 5.18. *Let X be a Γ-KU-algebra. Then the followings are equivalent:*
(1) *X is positive implicative,*
(2) *$\{0\}$ is a PIΓKUI of X,*
(3) *every ΓI of X is positive implicative,*
(4) *the set $A(a) = \{x \in X : x \leq a\}$ is a ΓI of X for each $a \in X$.*

Proof. (1)⇒(2): Suppose X is positive implicative. It is obvious that $\{0\}$ is a ΓI of X. Suppose $y\beta(y\alpha x) \in \{0\}$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. Since X is positive implicative, by Theorem 4.4, $y\alpha x = y\alpha(y\beta x)$. Then by the hypothesis, $y\alpha x \in \{0\}$. Thus by Theorem 5.16 (2), $\{0\}$ is a PIΓKUI of X.

(2)⇒(3): The proof follows from Proposition 5.17.

(3)⇒(4): Suppose the condition (3) holds and $x\alpha y,\ y \in A(a)$ for each $a \in X$ and each $\alpha \in \Gamma$. Then clearly, $y\alpha x \leq a$ and $y \leq a$. Thus $a\beta(y\alpha x) = 0 \in \{0\}$ and $a\alpha y = 0 \in \{0\}$ for any $\beta \in \Gamma$. By the hypothesis, $\{0\}$ is positive implicative. So $a\alpha x \in \{0\}$, i.e., $a\alpha x = 0$, i.e., $x \leq a$. Hence $x \in A(a)$. Therefore $A(a)$ is a ΓI of X.

(4)⇒(1): Suppose the condition (4) holds and $y\beta(y\alpha x) = 0$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. Then clearly, $y\alpha x \leq y$, i.e., $y\alpha x \in A(y)$. By the condition (4), $A(y)$ is a ΓI of X. It is obvious that $y \in A(y)$. Thus $x \in A(y)$. So $y\alpha x = 0$. Hence by Theorem 4.5, X id positive implicative. □

We have a characterization of a positive implicative Γ-KU-algebra by ΓIs.

Theorem 5.19. *Let X be a Γ-KU-algebra. Then X is positive implicative if and only if A_a is a ΓI of X for each ΓI I of X and each $a \in X$.*

Proof. Suppose X is positive implicative, let I be any ΓI of X and let $a \in X$. Then by Theorem 5.18, I is a PIΓKUI of X. Thus by Theorem 5.14, A_a is a ΓI of X.

Conversely, suppose the necessary condition holds and let J be any ΓI of X. Suppose $z\alpha(x\beta y) \in J$ and $z\alpha x \in J$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Consider the set $A_z = \{u \in X : z\alpha u \in J\}$. Then clearly, $x\beta y \in A_z$ and $x \in A_z$. Since A_z is a ΓI of X, $y \in A_z$. Thus $z\alpha y \in J$. So J is positive implicative. Hence by Theorem 5.18, X is positive implicative. □

Definition 5.20. Let X be Γ-KU-algebra and let I be a nonempty subset of X. Then I is called an *implicative* Γ-KU-*ideal* (briefly, IΓKUI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(ΓI$_1$) $0 \in I$,

(IΓKUI$_2$) if $z\alpha[(x\beta y)\alpha x] \in I$ and $z \in I$, then $x \in I$.

For any Γ-KU-algebra X, it is obvious that X is always an IΓKUI of X which is called the *trivial implicative* Γ-KU-*ideal*.

We can easily show that every ΓI of an implicative Γ-KU-algebra X is implicative.

Example 5.21. Let X be the Γ-KU-algebra given in Example 4.13 (2). Then we can easily check that $\{0, 1, 2, 3\}$ is an IΓKUI of X. Furthermore, $\{0\}$ is a ΓI of X but not implicative, since $0\alpha[(1\beta 2)\alpha 1)] \in \{0\}$ and $0 \in \{0\}$ but $1 \notin \{0\}$.

Proposition 5.22. *Every IΓKUI is a ΓI but the converse is not true.*

Proof. The proof is straightforward from Definitions 5.3 and 5.20. See Example 5.21) for the converse. □

Proposition 5.23. *Every IΓKUI is positive implicative but the converse is not true.*

Proof. Let I be an IΓKUI of a Γ-KU-algebra X and $z\alpha(y\beta x),\ z\beta y \in I$ for any $x,\ y\ zinX$ and any $\alpha,\ \beta \in \Gamma$. Then we get

$$\begin{aligned}(z\beta y)\alpha[z\alpha(z\beta x)] &\leq y\alpha(z\beta x) \text{ [By Corollary 3.7]}\\ &= z\alpha(y\beta x). \text{ [Proposition 3.6]}\end{aligned}$$

Since $z\alpha(y\beta x) \in I$, by Proposition 5.5, $(z\beta y)\alpha[z\alpha(z\beta x)] \in I$. Since $z\beta y \in I$ and I is a ΓI of X by Proposition 5.22, $z\alpha(z\beta x) \in I$. On the other hand, we have

$$\begin{aligned}[z\alpha(z\beta x)]\alpha(z\beta x) &= z\alpha[(z\alpha(z\beta x))\alpha x] \text{ [By Proposition 3.6]}\\ &= z\alpha(z\beta x) \in I. \text{ [By Proposition 3.10]}\end{aligned}$$

Thus $0\beta[(z\alpha(z\beta x))\alpha(z\beta x)] \in I$. Since $0 \in I$ and I is implicative, $z\beta x \in I$. So I is positive implicative.

In Example 5.12, $\{0, 1, 3\}$ is positive implicative but not implicative. □

We obtain a condition for a ΓI to become a IΓKUI.

Theorem 5.24. *Let I be a ΓI of a Γ-KU-algebra X. Then I is implicative if and only if the following holds:*

$$(x\alpha y)\beta x \in I \text{ implies } x \in I \text{ for any } x,\ y \in X \text{ and any } \alpha,\ \beta \in \Gamma. \tag{5.2}$$

Proof. Suppose I is implicative and $(x\alpha y)\beta x \in I$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. It is obvious that $0\beta[(x\alpha y)\beta x] \in I$ and $0 \in I$. Then by the hypothesis, $x \in I$. Thus (5.2) holds.

The proof of the converse is easy. □

Now we obtain a condition for a PIΓKUI to become a IΓKUI.

Theorem 5.25. *Let I be a PIΓKUI of a Γ-KU-algebra X. Then I is implicative if and only if the following holds:*

$$(x\alpha y)\beta y \in I \text{ implies } (y\alpha x)\beta x \in I \text{ for any } x,\ y \in X \text{ and any } \alpha,\ \beta \in \Gamma. \tag{5.3}$$

Proof. Suppose I is implicative and $(x\alpha y)\beta y \in I$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. Then by Corollary 3.7 (2), $(y\alpha x)\beta x \leq y$. Thus by Proposition 3.5 (1), $x\beta y \leq x\beta[(y\alpha x)\beta x]$. Furthermore, we get

$$\begin{aligned}[x\beta((y\alpha x)\beta x)]\alpha[(y\alpha x)\beta x] &\leq (x\beta y)\alpha[(y\alpha x)\beta x](\text{ [By Proposition 3.5 (1)]}\\ &= (y\alpha x)\beta[(x\alpha y)\beta x]. \text{ [By Proposition 3.6]}\\ &\leq (x\alpha y)\beta y. \text{ [By Proposition 3.3] (1)]}\end{aligned}$$

Since I is a ΓI of X by the hypothesis and Proposition 5.22, we get

$$0\beta([x\beta((y\alpha x)\beta x)]\alpha[(y\alpha x)\beta x]) \in I.$$

Since $0 \in I$, by the condition ($I\Gamma KUI_2$), $y\alpha(x\beta x) \in I$. So (5.3) holds.

Conversely, suppose necessary condition (5.3) holds, and $z\beta[(x\alpha y)\beta x] \in I$ and $z \in I$ Since I is positive implicative, by Proposition 5.13, I is a ΓI of X. Then $(x\alpha y)\beta x \in I$. By Proposition 3.6 (3), we have

$$(x\beta y)\beta[(x\alpha y)\beta y] \leq (x\alpha y)\beta x) \in I.$$

Thus $(x\beta y)\beta[(x\alpha y)\beta y] \in I$. Since I is positive implicative, by Theorem 5.16 (2), $(x\alpha y)\beta y \in I$. By the condition (5.3), we get

$$(y\alpha x)\beta x \in I. \tag{5.4}$$

Furthermore, from (3.1) and the axiom (ΓKU_3), we have

$$z\beta(y\alpha x) \leq y\alpha x \leq (y\alpha x)\beta x \in I.$$

So $z\beta(y\alpha x) \in I$. Since $z \in I$ and I is a ΓI of X, $y\alpha x \in I$. By the condition (5.4), $x \in I$. Hence I is implicative. □

We obtain a similar consequence to Proposition 5.17.

Proposition 5.26. *If I is an $I\Gamma KUI$ of a Γ-KU-algebra X, then every ΓI containing I is implicative.*

Proof. Suppose I is implicative and let J be any ΓI of X such that $I \subset J$. From Proposition 5.23, it is obvious that I is positive implicative. By Proposition 5.17, J is positive implicative. To prove that I is implicative, it is sufficient to prove that J satisfies the condition (5.3). Suppose $(x\beta y)\alpha y \in J$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$ and let $u = (x\beta y)\alpha y$. Then clearly, $u\alpha[(x\beta y)\alpha y] = 0 \in I$. Since I is positive implicative, by Theorem 5.16 (3) and Proposition 3.6, we have

$$[u\alpha(y\beta x)]\alpha(u\beta x) = [y\alpha(u\beta x)]\alpha(u\beta x) \in I.$$

Since I is implicative, by the condition (5.3), $[(u\beta x)\alpha y]\beta y \in I$. Since $I \subset J$, $[(u\beta x)\alpha y]\beta y \in J$. On the other hand, by Corollary 3.7 (2), $[(u\beta x)\alpha y]\beta y \leq u\beta x$ and $(x\beta y)\alpha y \leq x$. Thus we get

$$\begin{aligned}[((u\beta x)\alpha y)\beta y]\alpha[(x\beta y)\alpha y] &\leq (x\beta y)\alpha[(u\beta x)\alpha y] \text{ [By Proposition 3.3 (1)]}\\ &\leq (u\beta x)\alpha x \text{ [By Proposition 3.5 (1)]}\\ &= [((x\beta y)\alpha y)\beta x]\alpha x \text{ [Since } u = x\alpha(x\beta y)]\\ &\leq (x\beta y)\alpha y \in J.\end{aligned}$$

So $[((u\beta x)\alpha y)\beta y]\alpha[(x\beta y)\alpha y] \in J$. Since $[(u\beta x)\alpha y]\beta y \in J$, $(x\beta y)\alpha y] \in J$. Hence by Theorem 5.25, J is implicative. □

Now we obtain a similar consequence of Theorem 5.18.

Theorem 5.27. *Let X be a Γ-KU-algebra. The the followings are equivalent:*
(1) $\{0\}$ *is implicative,*
(2) *every* ΓI *of* X *is implicative,*
(3) $A(a)$ *is implicative for each* $a \in X$,
(4) X *is implicative.*

Proof. (1)⇔(2): The proof follows from Proposition 5.26.
(2)⇔(3): The prof is straightforward from Proposition 5.23 and Theorem 5.18.
(4)⇒(1): The proof is obvious.
(1)⇒(4): Suppose $\{0\}$ is implicative. Then by Proposition 5.23, $\{0\}$ is positive implicative. By Theorem 5.18, $A((x\beta y)\alpha x))$ is a ΓI of X for any $x,\ y,\ z \in X$. By the hypothesis, $A((x\beta y)\alpha x))$ is implicative. It is clear that $(x\beta y)\alpha x \in A((x\beta y)\alpha x))$. Thus $x \in A((x\beta y)\alpha x))$. So $(x\beta y)\alpha x \leq x$. Note that $x \leq (x\beta y)\alpha x$. Hence $x = (x\beta y)\alpha x$. Therefore X is implicative. □

Definition 5.28. Let X be Γ-KU-algebra and let I be a nonempty subset of X. Then I is called a *commutative* Γ-KU*-ideal* (briefly, CΓKUI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,
($\Gamma \mathrm{I}_1$) $0 \in I$,
($\mathrm{C}\Gamma\mathrm{KUI}_2$) if $z\alpha(y\beta x),\ z \in I$, then $[(x\alpha y)\beta y]\alpha x \in I$.
It is obvious that X is always a CΓKUI of a Γ-KU-algebra X which is called the *trivial commutative* Γ-KU*-ideal*.

Example 5.29. Let X be the Γ-KU-algebra given in Example 4.13 (2). Then we can easily see that $\{0, 4\}$ is commutative but not positive implicative, $\{0, 1, 3\}$ is positive implicative but not commutative and $\{0, 1, 2, 3\}$ is implicative.

Proposition 5.30. *Every CΓKUI of a* Γ-KU*-algebra* X *is a* ΓI *of* X *but the converse is not true.*

Proof. Let I be any CΓKUI of X and $y\alpha x \in I$ and $y \in I$ for any $x,\ y \in X$ and each $\beta \in \Gamma$. Then clearly, $y\alpha(0\beta x) \in I$ for each $\alpha \in \Gamma$. Since I is commutative, $x = [(x\alpha 0)\beta 0]\alpha x \in I$. Then I is a ΓI of X. See Example 5.29 for the converse. □

We have an equivalent condition of CΓKUIs.

Theorem 5.31. *Let* X *be a* Γ-KU*-algebra and let* I *be a* ΓI *of* X. *Then* I *is commutative if and only if it satisfies the following condition:*

$$y\alpha x \in I \text{ implies } [(x\alpha y)\beta y]\alpha x \in I \text{ for any } x,\ y \in X \text{ any } \alpha,\ \beta \in \Gamma. \tag{5.5}$$

Proof. Suppose I is commutative and $y\alpha x \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then clearly, $0\beta(y\alpha x) \in I$ for any $\beta \in \Gamma$ and $0 \in I$. Thus by the condition ($C\Gamma\mathrm{KUI}_2$), $[(x\alpha y)\beta y]\alpha x \in I$. So the condition (5.5) holds.
Conversely, suppose the condition (5.5) holds and $z\beta(y\alpha x),\ z \in I$ for any $x,\ y, z \in X$ and any $\alpha,\ \beta \in \Gamma$. Since I is a ΓI of X, $y\alpha x \in I$. Then by the condition (5.5), $[(x\alpha y)\beta y]\alpha x \in I$. Thus I is commutative. □

We obtain a similar consequence of Theorem 4.14 for ΓIs.

Theorem 5.32. *Let* X *be a* Γ-KU*-algebra and let* I *be a nonempty subset of* X. *Then* I *is implicative if and only if it is both commutative and positive implicative.*

Proof. Suppose I is implicative. Then by Proposition 5.23, I is positive implicative. It is sufficient to prove that I is commutative.

Suppose $y\alpha x \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. From (3.1) and the axiom (ΓKU$_3$), $[(x\alpha y)\beta y]\alpha x \leq x$ for each $\beta \in \Gamma$. Then $y\alpha x \leq y\beta[((x\alpha y)\beta y)\alpha x]$. Let $u = [(x\alpha y)\beta y]\alpha x$. Then we have

$$\begin{aligned}(u\beta y)\alpha u &= [(((x\alpha y)\beta y)\alpha x)\beta y]\alpha[((x\alpha y)\beta y)\alpha x]\\ &\leq (x\alpha y)\alpha[((x\alpha y)\beta y)\alpha x]\\ &= [(x\alpha y)\beta y]\alpha[(x\alpha y)]\alpha x]\\ &\leq y\alpha x \in I.\end{aligned}$$

Thus $u\beta(y\alpha u) \in I$. Since I is implicative, by Theorem 5.24, $u \in I$, i.e., $[(x\alpha y)\beta y]\alpha x \in I$. So by Theorem 5.31, I is commutative.

Conversely, suppose the necessary condition holds and $(x\alpha y)\beta x \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. It is obvious that

$$(x\alpha y)\beta[(x\alpha y)\beta y] \leq (x\alpha y)\beta x \in I.$$

Then $(x\alpha y)\beta[(x\alpha y)\beta y] \in I$. Since I is positive implicative, by Theorem 5.16 (2), we have

$$(x\alpha y)\beta y \in I. \tag{5.6}$$

Furthermore, by Propositions 3.3 (1) and 3.6, we have

$$y\beta x \leq (x\alpha y)\beta x.$$

Since $(x\alpha y)\beta x \in I$, $y\beta x \in I$, i.e., $y\alpha x \in I$. Since I is commutative, by Theorem 5.31,

$$[(x\alpha y)\beta y]\alpha x \in I. \tag{5.7}$$

Thus by (5.6) and (5.7), $x \in I$. So I is implicative. □

We obtain a similar consequence of Proposition 5.17 for IΓKUIs.

Proposition 5.33. *Let I and J be ΓKUIs of a Γ-KU-algebra X such that $I \subset J$. If I is commutative, then so is J.*

Proof. Suppose I is commutative and $y\alpha x \in J$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. In order to show that J is commutative, it is sufficient to show that $[(x\alpha y)\beta y]\alpha x \in J$ by using Theorem 5.31. Let $u = y\alpha x$. Then we get

$$y\beta(u\alpha x) = u\beta(y\alpha x) = 0 \in I.$$

Since I is commutative, by Theorem 5.31, we have

$$[((u\alpha x)\beta y)\alpha y]\beta(u\alpha x) \in I.$$

By Proposition 3.6, we have

$$[((u\alpha x)\beta y)\alpha y]\beta(u\alpha x) = u\beta[(((u\alpha x)\beta y)\alpha y)\alpha x] \in I.$$

Since $I \subset J$, $u\beta[(((u\alpha x)\beta y)\alpha y)\alpha x] \in J$. Since J is a ΓI of X and $u \in J$, $[((u\alpha x)\beta y)\alpha y]\alpha x \in J$. On the other hand, from Proposition 3.3 (1), (1) and (ΓKU$_3$), we get

$$\begin{aligned}[(((u\alpha x)\beta y)\alpha y)\alpha x]\beta[((x\alpha y)\beta y)\alpha x] &\leq [(x\alpha y)\beta y]\alpha[((u\alpha x)\beta y)\alpha y]\\ &\leq [(u\alpha x)\beta y]\alpha(x\beta y)\\ &\leq x\beta(u\alpha x)\\ &= u\beta(x\alpha x).\end{aligned}$$

$$= 0 \in J.$$

Thus $[(x\alpha y)\beta y]\alpha x \in J$. So by Theorem 5.31, J is implicative. □

Finally, we obtain a characterization of commutative Γ-KU-algebras.

Theorem 5.34. *Let X be a Γ-KU-algebra. The the followings are equivalent:*
(1) $\{0\}$ *is commutative,*
(2) *every* ΓI *of* X *is commutative,*
(3) X *is commutative.*

Proof. (1)$\Leftrightarrow$(2): The proof is clear from Proposition 5.33.
(1)$\Leftrightarrow$(3): The proof follows from Theorem 4.9. □

6. Topological structures on Γ-KU-algebras

We recall some terms and notations related for a general topology (See [32, 33]). For a subset A of a topological space (X, τ), the closure and the interior of A are denoted by $cl_\tau(A)$, $cl(A)$ or $\bar{A}$ and $int_\tau(A)$, $int(A)$ or A°. A subfamily $\mathcal{B}$ of τ is called a *base* for τ, if for each $U \in \tau$ either $U = \varnothing$ or there is $\mathcal{B}' \subset \mathcal{B}$ such that $U = \bigcup \mathcal{B}'$. A subset A of X is called a *neighborhood* of $x \in X$, if there is $U \in \tau$ such that $x \in U \subset A$. The set of all neighborhoods of x write as $N_\tau(x)$ or $N(x)$ and $N(x)$ is called the *neighborhood filter* of $x \in X$. A subfamily $\mathcal{N}(x)$ of $N(x)$ is called a *fundamental system of neighborhoods* of x, if for each $U \in N(x)$ there is $V \in \mathcal{N}(x)$ such that $V \subset U$. In fact, $\mathcal{N}(x)$ is a filter base of $N(x)$. Moreover, it is well-known ([32]) that $N_\tau(x)$ satisfies the following properties:
(N_1) $x \in U$ for each $U \in N_\tau(x)$,
(N_2) if $U \in N_\tau(x)$ and $U \subset V \subset X$, then $V \in N_\tau(x)$,
(N_3) if $U_1,\ U_2 \in N_\tau(x)$, then $U_1 \cap U_2 \in N_\tau(x)$,
(N_4) if $V \in N_\tau(x)$, there is $W \in N_\tau(x)$ such that $V \in N_\tau(x)$ for each $y \in W$.
Furthermore, it is well-known (Proposition 1.1.2, [32]) that for each $x \in X$ if $\mathcal{B}(x)$ be a set of subsets of X satisfying the properties (N_1)–(N_4), then a unique topology on X such that $\mathcal{B}(x) = N_\tau(x)$, where

$$\tau = \{V \subset X : \forall x \in V,\ \exists U \in \mathcal{B}(x) \text{ such that } U \subset V\}.$$

Definition 6.1. Let X be a KU-algebra and let τ be a topology on X. Then X is called a *topological KU-algebra* (briefly, TKU-algebra), if $* : (X \times X, \tau \times \tau) \to (X, \tau)$ is continuous, i.e., for any $x,\ y \in X$ and each $W \in N(x * y)$ there are $U \in N(x)$ and $V \in N(y)$ such that $U * V \subset W$, where $U * V = \{x * y \in X : x \in U,\ y \in V\}$.

Definition 6.2. Let X be a Γ-KU-algebra and let τ be a topology on X. Then X is called a *topological Γ-KU-algebra* (briefly, TΓ-KU-algebra), if a mapping $f : (X, \tau) \times \Gamma \times (X, \tau) \to (X, \tau)$ is continuous at each $(x, \alpha, y) \in X \times \Gamma \times X$, i.e., for each $\alpha \in \Gamma$, any $x,\ y \in X$ and each $W \in N(x\alpha y)$ there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset W$, where $U\alpha V \subset W = \{x\alpha y : x \in U,\ y \in V\}$.

It is clear that if X is a TΓ-KU-algebra, then X_α is a TKU-algebra for each $\alpha \in \Gamma$.

Example 6.3. (1) Let $X = \{0, 1, 2, 3, 4\}$ be the Γ-KU-algebra given in Example 4.13 (2). Consider the topology τ on X given by:

$$\tau = \{\varnothing, \{4\}, \{0, 1, 2, 3\}, X\}.$$

Then we can easily check that (X, τ) is a TΓ-KU-algebra. Moreover, X_α and X_β are TKU-algebras.

(2) Let $X = \{0, 1, 2, 3\}$ be the Γ-KU-algebra given in Example 3.4 (1). Consider a topology τ on X given by:

$$\tau = \{\varnothing, \{0\}, \{0, 1\}, \{0, 2, 3\}, X\}.$$

Then we can easily see that (X, τ) is a TΓ-KU-algebra.

Proposition 6.4. *Let X be a TΓ-KU-algebra. If $\{0\}$ is open in X, then X is discrete.*

Proof. Let $x \in X$ and let $\alpha \in \Gamma$. Then clearly, $x\alpha x = 0 \in \{0\} \in N(0)$. Thus there are $U,\ V \in N(x)$ such that $U\alpha V = \{0\}$. Let $W = U \cap V$. Then $W\alpha W \subset U\alpha V = \{0\}$. Thus $W\alpha W = \{0\}$. Since $x \in U \cap V$, $x \in W$. So $W = \{x\}$ and W is open in X. Hence X is discrete. □

The following is an immediate consequence of Proposition 6.4.

Corollary 6.5. *Let X be a TΓ-KU-algebra. If $\{0\}$ is open in X_α for each $\alpha \in \Gamma$, then X_α is discrete.*

Theorem 6.6. *Let X be a TΓ-KU-algebra. Then $\{0\}$ is closed in X if and only if X is Hausdorff.*

Proof. Suppose $\{0\}$ is closed in X, let $x,\ y \in X$ such that $x \neq y$ and let $\alpha \in \Gamma$. Then $x\alpha y \neq 0$ or $y\alpha x \neq 0$, say $x\alpha y \neq 0$. Since $\{0\}$ is closed in X and $x\alpha y \neq 0$, $\{0\}^c$ is open in X and $x\alpha y \in \{0\}^c$. Thus $\{0\}^c \in N(x\alpha y)$. Since X is a TΓ-KU-algebra, by Definition 6.2, there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset \{0\}^c$. So $U \cap V = \varnothing$. Hence X is Hausdorff.

Conversely, suppose X is Haousdorff and let $x \in \{0\}^c$. Then $x \neq 0$. By the hypothesis, there are $U \in N(x)$ and $V \in N(0)$ such that $U \cap V = \varnothing$. Thus $0 \notin U$. So $U \subset \{0\}^c$. Hence $\{0\}^c$ is open in X. Therefore $\{0\}$ is closed in X. □

The following is an immediate consequence of Theorem 6.6.

Corollary 6.7. *Let X be a TΓ-KU-algebra. Then $\{0\}$ is closed in X_α if and only if X_α is Hausdorff for each $\alpha \in \Gamma$.*

Proposition 6.8. *Let X be a TΓ-KU-algebra and let A be open in X. If A is a Γ-subalgebra of X, then A is a TΓ-KU-algebra.*

Proof. Let τ be the topology on X and let τ_A be the subspace topology on A with respect to τ. Let $x,\ y \in A$ and let $\alpha \in \Gamma$. Since A is a Γ-subalgebra of X, $x\alpha y \in A$. Let $W_A \in N_{\tau_A}(x\alpha y)$, where $N_{\tau_A}(x\alpha y)$ denotes the neighborhood of $x\alpha y$ in the subspace (A, τ_A) of (X, τ). Then there is $W \in N(x\alpha y)$ such that $W_A = A \cap W$. Since X is a TΓ-KU-algebra, there are $U \in N(x)$ and $V \in N(y)$ such that $U\alpha V \subset W$. Thus $U_A = A \cap U \in N_{\tau_A}(x)$ and $V_A = A \cap V \in N_{\tau_A}(x)$. It is clear that

$$U_A\alpha V_A = (A \cap U)\alpha(A \cap V) \subset W \text{ and } U_A\alpha V_A \subset A.$$

So $U_A \alpha V_A \subset A \cap W = W_A$. Hence A is a TΓ-KU-algebra. $\square$

Corollary 6.9. *Let X be a TΓ-KU-algebra and let A be open in X_α for each $\alpha \in \Gamma$. If A is a Γ-subalgebra of X_α, then A is a TKU-algebra.*

Proposition 6.10. *Let X be a TΓ-KU-algebra and let I be open in X. If I is a ΓI of X, then I is closed in X.*

Proof. Let $x \in I^c$ and let $\alpha \in \Gamma$. Since $x\alpha x = 0 \in I$ and I is open, $I \in N(0)$. Since X is a TΓ-KU-algebra, there is $U \in N(x)$ such that $U\alpha U \subset I$. Assume that $U \not\subset I^c$. Then there is $y \in X$ such that $y \in U \cap I$. It is obvious that $y\alpha z \in U\alpha U \subset I$ for each $z \in U$. Since I is a ΓI of X and $y \in I$, $z \in I$. Thus $U \subset I$. This is a contradiction. So $U \subset I^c$, i.e., I^c is open in X. Hence I is closed in X. $\square$

Corollary 6.11. *Let X be a TΓ-KU-algebra and let I be open in X_α for each $\alpha \in \Gamma$. If I is a ΓI of X_α, then I is closed in X_α.*

Proposition 6.12. *Let X be a TΓ-KU-algebra and let I be a ΓI of X. If $0 \in int(I)$, then I is open in X.*

Proof. Let $x \in I$ and let $\alpha \in \Gamma$. Since $0 \in int(I)$ and $x\alpha x = 0 \in I$, there is $W \in N(0) = N(x\alpha x)$ such that $W \subset I$. Since X is a TΓ-KU-algebra, there are U, $V \in N(x)$ such that $U\alpha V \subset W \subset I$. It is obvious that $x\alpha y \in U\alpha V \subset I$ for each $y \in U$. Since I is a ΓI of X and $x \in I$, $y \in I$. Then $y \in I$. Thus $U \subset I$. So I is open in X. $\square$

Corollary 6.13. *Let X be a TΓ-KU-algebra and let I be a ΓI of X_α for each $\alpha \in \Gamma$. If $0 \in int(I)$, then I is open in X_α.*

In Proposition 6.12, when $0 \neq x \in int(I)$, I need not open in X (See Example 6.14).

Example 6.14. For a set $\Gamma = \{\alpha, \beta\}$, let $X = \{0, 1, 2, 3\}$ be a Γ-KU-algebra with the ternary operation be defined by the table:

α	0	1	2	3	β	0	1	2	3
0	0	1	2	3	0	0	1	2	3
1	0	0	2	2	1	0	0	1	3
2	0	0	0	3	2	0	0	0	3
3	0	1	2	0	3	0	2	1	0

Table 6.1

Consider a topology τ on X given by:

$$\tau = \{\varnothing, \{2\}, \{3\}, \{0,1\}, \{2,3\}, \{0,1,3\}, X\}.$$

Let $I = \{0, 3\}$. Then clearly, $3 \in int(I)$. But $I \notin \tau$.

Proposition 6.15. *Let X be TΓ-KU-algebra. Then $\bigcap N(0) = \{0\}$ and thus $\bigcap \mathcal{N}(0) = \{0\}$.*

Proof. Assume that $0 \neq x \notin \bigcap N(0)$. Then clearly, there is $U \in N(0)$ such that $0 \in U$ but $x \notin U$. Thus $x \notin \bigcap N(0)$. This is a contradiction. So $\bigcap N(0) = \{0\}$. $\square$

Proposition 6.16. *Let* (X,τ) *be a* $T\Gamma$-KU*-algebra and let* $\mathcal{B}_1,\ \mathcal{B}_2$ *be the families of subsets of* X *given by:*

$$\mathcal{B}_1=\{U\alpha x : x\in X,\ \alpha\in\Gamma,\ U\in\mathcal{N}(0)\},\ \mathcal{B}_2=\{x\alpha U : x\in X,\ \alpha\in\Gamma,\ U\in\mathcal{N}(0)\},$$

where $U\alpha x=\{u\alpha x : u\in U\}$ *and* $x\alpha U=\{x\alpha u : u\in U\}$. *Then* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are bases for* τ.

Proof. Let $x\in X$. Since $0\in U\in\mathcal{N}(0)$, $0\alpha x=x$. Then $\bigcup\mathcal{B}_1=X$. Suppose $B_1,\ B_2\in\mathcal{B}_1$ and $z\in B_1\cap B_2$. Then there are $U_1,\ U_2\in\mathcal{N}(0)$ such that $B_1=U_1\alpha x$, $B_2=U_2\alpha x$ and $B_1\cap B_2=(U_1\cap U_2)\alpha x$. Since $z\in B_1\cap B_2$, there is $y\in U_1\cap U_2$. Since $U_1,\ U_2\in N(0)$, $U_1\cap U_2\in N(0)$. So there is $V\in\mathcal{N}(0)$ such that $y\in V\subset U_1\cap U_2$. Hence $z=y\alpha x\in V\alpha x\in\mathcal{B}_1$. Therefore $\mathcal{B}_1$ is a base for τ. Similarly, we can prove that $\mathcal{B}_2$ is a base for τ. □

Now in order to give a filter base on X generating a topology on a Γ-KU-algebra, let us define the subset $U(a)$ of X generated by each $a\in X$ and each subset U of X as follows:

$$U(a)=\{x\in X : x\alpha a\in U,\ a\alpha x\in U,\ \alpha\in\Gamma\}.$$

Proposition 6.17. *Let* X *be a* Γ-KU*-algebra. Suppose* $\mathcal{B}$ *is a filter base on* X *satisfying the following condition:*

(1) *for each* $u\in U\in\mathcal{B}$ *there is* $B\in\mathcal{B}$ *such that* $B(u)\subset U$,

(2) *for each* $u\in U\in\mathcal{B}$ *and each* $\alpha\in\Gamma$ *if* $u\alpha x=0$, *then* $x\in U$,

(3) *for each* $U\in\mathcal{B}$ *there is* $B\in\mathcal{B}$ *such that* $B(b)\subset U$ *for each* $b\in B$, *i.e.,* $B(B)\subset U$.

Then there is a unique topology τ *on* X *such that* $\mathcal{B}=\mathcal{N}_\tau(0)$ *and* (X,τ) *is a* $T\Gamma$-KU*-algebra.*

Proof. Let $\tau=\{O\in P(X) :$ for each $a\in O$ there is $B\in\mathcal{B}$ such that $B(a)\subset O\}$. Then we can easily prove that τ is a topology on X. To accomplish to the proof, consider the following Claims.

Claim 1: $B(a)\in\tau$. Let $x\in B(a)$. Then $x\alpha a,\ a\alpha x\in B$ for each $\alpha\in\Gamma$. Thus by the condition (1), there are $B_1,\ B_2\in\mathcal{B}$ such that $B_1(x\alpha a)\subset B$ and $B_2(a\alpha x)\subset B$. Since $\mathcal{B}$ is a filter base on X, there is $U\in\mathcal{B}$ such that $U\in B_1\cap B_2$. Let $x\alpha y,\ y\alpha x\in U$, i.e., $y\in U(x)$. By Proposition 3.3 (1), we have

$$(x\alpha a)\beta(y\alpha a)\le y\alpha x,\ (y\alpha a)\beta(x\alpha a)\le x\alpha y.$$

Then $(y\alpha x)\beta[(x\alpha a)\beta(y\alpha a)]=0,\ (x\alpha y)\beta[(y\alpha a)\beta(x\alpha y)]=0$. By the condition (2), $(x\alpha a)\beta(y\alpha a),\ (y\alpha a)\beta(x\alpha y)\in U$. Thus we get

$$y\alpha a\in U(x\alpha a)\subset B_1(x\alpha a)\subset B.$$

So $y\alpha a\in B$. Similarly, we can show that $a\alpha y\in U$. Hence $y\in U(a)$, i.e., $U(x)\subset B(a)$. Therefore $B(a)\in\tau$.

Claim 2: $\mathcal{B}=\mathcal{N}_\tau(0)$. Let $A\in\mathcal{B}$ and let $x\in A$. Since X is a Γ-KU-algebra, by the axiom (ΓKU_3), $x\alpha 0=0$. By the condition (2), $0\in A$. By the condition (1), there is $B\in\mathcal{B}$ such that $B(0)\subset A$. Then by Claim 1, $B(0)\in\tau$. Thus $A\in N_\tau(0)$. So $\mathcal{B}\subset N_\tau(0)$. Hence by the condition (3), $\mathcal{B}\subset\mathcal{N}_\tau(0)$. It can be easily proved that $\mathcal{N}_\tau(0)\subset\mathcal{B}$. Therefore $\mathcal{B}=\mathcal{N}_\tau(0)$.

Claim 3: A mapping $f : (X,\tau) \times \Gamma \times (X,\tau) \to (X,\tau)$ is continuous at each $(x,\alpha,y) \in X \times \Gamma \times X$. Let $x,\ y \in X$, let $\alpha \in \Gamma$ and let $W \in N_\tau(x\alpha y)$. Since $x\alpha y \in W$, by the condition (1), there is $W^{'} \in \mathcal{B}$ such that $W^{'}(x\alpha y) \subset W$. Since $W^{'} \in \mathcal{B}$, by the condition (3), there is $B \in \mathcal{B}$ such that $B(b) \subset W^{'}$ for each $b \in W^{'}$. Let $U = B(x),\ V = B(y)$ and let $u \in U,\ v \in V$. Then we have

$$\begin{aligned}(x\alpha u)\beta[(u\alpha v)\beta(x\alpha y)] &= (u\alpha v)\beta[(x\alpha u)\beta(x\alpha y)] \text{ [By Proposition 3.6]}\\ &\leq (u\alpha v)\beta(u\alpha y) \text{ [By Corollary 3.8]}\\ &\leq v\alpha y. \text{ [By Corollary 3.8]}\end{aligned}$$

Thus $(\alpha y)\beta[(x\alpha u)\beta((u\alpha v)\beta(x\alpha y))] = 0$. Since $v\alpha y \in B$, by the condition (2), $(x\alpha u)\beta[(u\alpha v)\beta(x\alpha y)] \in B$. Similarly, we have $[(u\alpha v)\beta(x\alpha y)]\beta(x\alpha u) \in B$. So we get

$$(u\alpha v)\beta(x\alpha y) \in B(x\alpha u) \subset W^{'}, \text{i.e., } (u\alpha v)\beta(x\alpha y) \in W^{'}.$$

Similarly, $(x\alpha y)\beta(u\alpha v) \in W^{'}$. Hence we have

$$u\alpha v \in W^{'}(x\alpha y), \text{ i.e., } U\alpha V = B(x)\alpha B(y) \subset W^{'}(x\alpha y) \subset W.$$

Therefore f is continuous. The proof of uniqueness for τ is easy. This completes the proof. □

Example 6.18. (1) Let X be the Γ-KU-algebra and let $\mathcal{I}$ be the collection of all ΓIs of X. Let $x \in I \in \mathcal{I}$. Then clearly, $I(x) \subset I$. Thus $\mathcal{I}$ satisfies the conditions (1) and (3) in Proposition 6.17. Let $y \in I \in \mathcal{I}$ and suppose $y\alpha x = 0$. Then $y\alpha x = 0 \in I$. Thus $x \in I$. So $\mathcal{I}$ satisfies the condition (2) in Proposition 6.17. So $\mathcal{I}$ forms a filter base of X satisfying all the conditions in Proposition 6.17. Hence (X,τ) is a (X,τ) is a $T\Gamma$-KU-algebra, where τ is the topology on X generated by $\mathcal{I}$.

(2) Let $X = \{0,1,2,3\}$ be the Γ-KU-algebra given in Example 4.7 (2). Consider the family $\mathcal{B}$ of subsets of X given by:

$$\mathcal{B} = \{\{0,1\},\{0,2\},\{0,3\},\{0,1,2\},\{0,1,3\},\{0,2,3\}\}.$$

Then we can easily check that $\mathcal{B}$ is a filter base on X. Moreover, we have

$$\{0,1\}(0) = \{0,1\}(1) = \{0,1\},\ \{0,1\}(2) = \{2\},\ \{0,1\}(3) = \{3\},$$

$$\{0,2\}(0) = \{0,2\}(2) = \{0,2\},\ \{0,2\}(1) = \{1\},\ \{0,2\}(3) = \{3\},$$

$$\{0,3\}(0) = \{0,3\}(3) = \{0,3\},\ \{0,3\}(1) = \{1\},\ \{0,3\}(2) = \{2\},$$

$$\{0,1,2\}(0) = \{0,1,2\}(1) = \{0,1,2\}(2) = \{0,1,2\},\ \{0,1,2\} = \{3\},$$

$$\{0,1,3\}(0) = \{0,1,3\}(1) = \{0,1,3\}(3) = \{0,1,3\},\ \{0,1,3\}(2) = \{2\},$$

$$\{0,2,3\}(0) = \{0,2,3\}(2) = \{0,2,3\}(3) = \{0,2,3\},\ \{0,2,3\}(1) = \{1\}.$$

Thus $\mathcal{B}$ is a filter base on X satisfying all the conditions in Proposition 6.17. So the topology τ on X generated by $\mathcal{B}$ is given as follows:

$$\tau = \{\varnothing,\{0,1\},\{0,2\},\{0,3\},\{0,1,2\},\{0,1,3\},\{0,2,3\},X\}.$$

Hence (X,τ) is a TΓ-KU-algebra.

Lemma 6.19. *Let X be a Γ-KU-algebra and let τ be the topology on X generated by $\mathcal{B}$, where $\mathcal{B}$ is a filter base on satisfying all the conditions in Proposition 6.17. Then for each $B \in \mathcal{B}$ and each $a \in X$,*
(1) $B(a) \in N_\tau(a)$,
(2) $B(A) = \bigcup_{a \in A} B(a) \in N_\tau(A)$ *for each* $A \in P(X)$.

Proof. The proof is straightforward. □

Proposition 6.20. *Let X be a Γ-KU-algebra and let τ be the topology on X generated by $\mathcal{B}$, where $\mathcal{B}$ is a filter base on X satisfying all the conditions in Proposition 6.17. Then for each $B \in \mathcal{B}$, $cl_\tau(A) = \bigcap_{B \in \mathcal{B}} B(A)$.*

Proof. Let $x \in cl_\tau(A)$ and let $B \in \mathcal{B}$. By Lemma 6.19 (1), $B(x) \in N_\tau(x)$. Then $B(x) \cap A \neq \varnothing$. Thus there is $a \in A$ such that $a\alpha x,\ x\alpha a \in B$ for each $\alpha \in \Gamma$. So $x \in B(a) \subset B(A)$, i.e., $x \in \bigcap_{B \in \mathcal{B}} B(A)$. Hence $cl_\tau(A) \subset \bigcap_{B \in \mathcal{B}} B(A)$. Conversely, let $x \in \bigcap_{B \in \mathcal{B}} B(A)$. Then $x \in U(A)$ for each $U \in \mathcal{B}$. Thus there is $a \in A$ such that $x \in B(a)$, i.e., $x\alpha a,\ a\alpha x \in B$ for each $\alpha \in \Gamma$. So $a \in B(x)$, i.e., $B(x) \cap A \neq \varnothing$. Hence $x \in cl_\tau(A)$, i.e., $\bigcap_{B \in \mathcal{B}} B(A) \subset cl_\tau(A)$. Therefore $cl_\tau(A) = \bigcap_{B \in \mathcal{B}} B(A)$. □

Corollary 6.21. *Let (X, τ) be a $T\Gamma$-KU-algebra, where $\mathcal{B}$ is a filter base on X satisfying all the conditions in Proposition 6.17 and τ is the topology on X generated by $\mathcal{B}$. Then every $B \in \mathcal{B}$ is closed in X, i.e., $\mathcal{B}$ is a collection of clopen subsets of X.*

Proof. Let $B \in \mathcal{B}$. It is obvious that $B(B) \subset B$. Then by Proposition 6.20, $B \subset cl_\tau(B) = \bigcap_{U \in \mathcal{B}} U(B) \subset B(B) \subset B$. Thus $cl_\tau(B) = B$. So B is closed in X. From Proposition 6.17, it is clear that B is open in X. So B is clopen in X. □

The following shows that every neighborhood of a compact set contains a neighborhood $B(A)$ for some $B \in \mathcal{B}$

Proposition 6.22. *Let A be a compact subset of a $T\Gamma$-KU-algebra. If U is a neighborhood of A, then there is $B \in \mathcal{B}$ such that $A \subset B(A) \subset U$.*

Proof. Suppose U is a neighborhood of A and let $a \in A$. Then there is $B_a \in \mathcal{B}$ such that $B_a \subset U$. Thus by the condition (3), there is $W_a \in \mathcal{B}$ such that $W_a(W_a) \subset B_a$. Since A is a compact subset of X and $A \subset \bigcup_{a \in A} W_a(a)$, there are $a_1,\ a_2,\ \cdots,\ a_n \in A$ such that

$$A \subset W_{a_1}(a_1) \cup W_{a_2}(a_2) \cup \cdots W_{a_n}(a_n). \tag{6.1}$$

Now let $W = \bigcap_{i=1}^{n} W_{a_i}$ and let $a \in A$. Then by (6.1), there is $i \in \{1, 2, \cdots, n\}$ such that $a \in W_{a_i}(a_i)$ Thus $a\alpha a_i,\ a_i\alpha a \in W_{a_i}$ for each $\alpha \in \Gamma$. Suppose $a\alpha y,\ y\alpha a \in W$ for each $y \in X$. By Proposition 3.3 (1). we have

$$(y\alpha a_i)\beta(a\alpha a_i) \leq a\alpha y \in W \text{ for each } \beta \in \Gamma. \tag{6.2}$$

Then $(y\alpha a_i)\beta(a\alpha a_i) \in W$. Thus we get

$$y\alpha a_i \in W_{a_i}(a\alpha a_i) \subset W_{a_i}(W_{a_i}) \subset B_{a_i}.$$

Similarly, $a_i\alpha y \in B_{a_i}$. So $y \in B_{a_i}(a_i) \subset U$ and $W(a) \subset U$. Hence $W(A) \subset U$. □

The following is an immediate consequence of Proposition 6.22.

Corollary 6.23. *Let A be a compact subset of a $T\Gamma$-KU-algebra and let F is closed in X. If $A \cap F = \varnothing$, then there is $B \in \mathcal{B}$ such that $B(A) \cap B(F) = \varnothing$.*

7. Conclusions

By proposing positive implicative [resp. implicative and commutative] Γ-KU-algebras, we obtained some of their properties respectively and a relationship among them (See Theorem 4.14). Also, by defining positive implicative [resp. implicative and commutative] Γ-KU-ideals of a Γ-KU-algebra, we studied their various properties respectively and a relationship among them (See Theorem 5.32). Moreover, we discussed some topological structures on a Γ-KU-algebra.

In the future, we will use our proposed Γ-KU-algebras to address quotient Γ-KU-algebras, homorphism problems, graph theory and Zariski topological structures. Furthermore, we want to study some ideals of a Γ-KU-algebra in the sense of the fuzzy set theory.

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