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On the new type of hybrid structures in ordered semigroups

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Abstract. The hybrid structures consist of the soft sets and the fuzzy sets. In this paper, we apply a hybrid structure theory to an ordered semigroup theory. The new many types of hybrid structures are introduced and some algebraic properties of such new hybrid structures are studied.

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1. Introduction

The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty and was introduced by Zadeh [1] in 1965. After the introduction of the concept of fuzzy sets by Zadeh, several researchers researched the generalizations of the notions of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine, graph theory, logic, operations research and many branches of pure and applied mathematics. For example, Xie et al. [2] applied fuzzy set theory to the switching method.

Molodtsov [3] introduced the concept of the soft set as a new mathematical tool for dealing with uncertainties that are free from the difficulties which have troubled the usual theoretical approaches. The soft sets have many applications in several branches of both pure and applied sciences (See [4, 5, 6]).

As a parallel circuit of fuzzy sets and soft sets, Jun, Song, and Muhiuddin [7] introduced the notion of hybrid structures in a set of parameters over an initial universe set. The hybrid structures can be applied in many areas including mathematics, statistics, computer science, electrical instruments, industrial operations, business, engineering, social decisions, etc.

Ani et al. [8] introduced the notions of hybrid subsemigroups and hybrid left (resp., right) ideals in semigroups and discussed characterizations of subsemigroups and left (resp., right) ideals. They also characterizations of hybrid subsemigroups and hybrid left (resp., right) ideals are considered by using the notion of hybrid product. Relations between the hybrid intersection and hybrid product are displayed. Mekwian and Lekkoksung [9] applied hybrid structure to ordered semigroups. They characterized regular, intra-regular, and weakly regular ordered semigroups in terms of hybrid left and right ideals. In 2022, Porselvi et al. [10] introduced the new concepts of hybrid interior ideals and hybrid simple in an ordered semigroup. They discussed characteristic hybrid structures using ideals and interior ideals, and characterize ordered semigroup in terms of different hybrid ideal structures.

In this present paper, we apply a hybrid structure theory to an ordered semigroup theory. New many types of hybrid structures are introduced and some algebraic properties of such new hybrid structures are studied.

2. Preliminaries

In this section, we will recall the basic terms and definitions from the ordered semigroup theory and the hybrid structure theory that we will use in this paper.

Let S be a nonempty set and a binary operation $\cdot$ on the set S. The structure $(S;\cdot)$ is called a *groupoid*. If the binary operation $\cdot$ satisfied associative property, that is

$$x\cdot(y\cdot z)=(x\cdot y)\cdot z$$

for all $x,\ y,\ z\in S$, the groupoid $(S;\cdot)$ is called a *semigroup*.

A binary relation $\leq$ on S, that is, $\leq\subseteq S\times S$, is called *partial ordered relation on* S, if it satisfies the following three properties:

(i) reflexive, i.e., $a\leq a$ for all $a\in S$,

(ii) antisymmetric, i.e., if $a\leq b$ and $b\leq a$, then $a=b$ for all $a,\ b\in S$,

(iii) transitive, i.e., if $a\leq b$ and $b\leq c$, then $a\leq c$ for all $a,\ b,\ c\in S$.

The structure $(S;\leq)$ is called a *partial ordered set*, if the relation $\leq$ is a partial ordered relation on S.

Definition 2.1 ([11])**.** The structure $(S;\cdot,\leq)$ is called an *ordered semigroup*, if the following conditions are satisfied:

(i) $(S;\cdot)$ is a semigroup,
(ii) $(S;\leq)$ is a partially ordered set,
(iii) for every $a,\ b,\ c\in S$ if $a\leq b$, then $a\cdot c\leq b\cdot c$ and $c\cdot a\leq c\cdot b$.

For simplicity, we denoted an ordered semigroup $(S;\cdot,\leq)$ by its carrier set as a bold letter $\mathbf{S}$ and if $a,\ b\in S$, we will instead of $a\cdot b$ by ab.

Let A and B be two nonempty subsets of S. Then we define

$$AB:=\{ab\ :\ a\in A\text{ and }b\in B\}.$$

Definition 2.2 ([11])**.** Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is called a *left* (resp. *right*) *ideal* of $\mathbf{S}$, if it satisfies

(i) $SA\subseteq A$ (resp., $AS\subseteq A$),
(ii) for $x,\ y\in S$, if $x\leq y$ and $y\in A$, then $x\in A$.

A nonempty subset I of S is called *two-sided ideal* or simply *ideal*, if I is both a left and a right ideal of $\mathbf{S}$.

Let $I = [0,1]$ be the unit interval, X a set of parameters and $\mathcal{P}(U)$ the power set of an initial universe set U.

Definition 2.3 ([8]). A *hybrid structure* in X over U is defined to be a mapping

$$f := (f^*, f^+) : X \to \mathcal{P}(U) \times I, x \mapsto (f^*(x), f^+(x)),$$

where

$$f^* : X \to \mathcal{P}(U) \text{ and } f^+ : X \to I$$

are mappings.

Example 2.4 ([8]). Let $U := \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the set of six wooden houses under consideration and let X be the set of parameters that consist of "cheap", "expensive", "beautiful" and "in good location". Consider a hybrid structure in X over U which is given in the following table:

X	cheap	expensive	beautiful	in good location
$f^*(x)$	$\{h_1, h_3\}$	$\{h_2, h_5\}$	$\{h_2, h_3, h_5\}$	$\{h_2, h_4\}$
$f^+(x)$	0.3	0.7	0.6	0.5

Definition 2.5. Let $\mathbf{S}$ be an ordered semigroup. A hybrid structure $f = (f^*, f^+)$ in S over U is called a *hybrid left ideal* in $\mathbf{S}$ over U, if satisfies the following conditions: for $x,\ y \in S$,

(i) $f^*(xy) \supseteq f^*(y)$ and $f^+(xy) \leq f^+(y)$,
(ii) If $x \leq y$, then $f^*(x) \supseteq f^*(y)$ and $f^+(x) \leq f^+(y)$.

Example 2.6. Let $S = \{a, b, c\}$. We define a binary operation $\circ$ and a binary relation $\leq$ on S as follows:

$\circ$	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

and $\leq := \{(a,b)\} \cup \Delta_S$, where Δ_S is an identity relation on S. Then $\mathbf{S} := (S; \circ, \leq)$ is an ordered semigroup. Let $U = \mathbb{N}$. Define a hybrid structure $f = (f^*, f^+)$ in $\mathbf{S}$ over U as follows:

S	a	b	c
$f^*(x)$	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$
$f^+(x)$	0.2	0.7	0.8

Then f is a hybrid left ideal in $\mathbf{S}$ over U.

Definition 2.7. Let $\mathbf{S}$ be an ordered semigroup. A hybrid structure $f = (f^*, f^+)$ in S over U is called a *hybrid right ideal* in $\mathbf{S}$ over U, if satisfies the following conditions: for $x,\ y \in S$,

(i) $f^*(xy) \supseteq f^*(x)$ and $f^+(xy) \leq f^+(x)$,
(ii) If $x \leq y$, then $f^*(x) \supseteq f^*(y)$ and $f^+(x) \leq f^+(y)$.

Example 2.8. Let $S = \{a, b, c\}$. We define a binary operation $\circ$ and a binary relation $\leq$ on S as follows:

$\circ$	a	b	c
a	a	a	a
b	a	a	a
c	c	c	c

and $\leq := \{(a, c), (b, c)\} \cup \Delta_S$, where Δ_S is an identity relation on S. Then $\mathbf{S} := (S; \circ, \leq)$ is an ordered semigroup. Let $U = \mathbb{N}$. Define a hybrid structure $f = (f^*, f^+)$ in S over U as follows:

S	a	b	c
$f^*(x)$	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$
$f^+(x)$	0.2	0.7	0.8

Then f is a hybrid right ideal in $\mathbf{S}$ over U.

A hybrid structure f in S over U is called a *hybrid two-sided ideal* or *hybrid ideal* in $\mathbf{S}$ over U, if f is both a hybrid left and a hybrid right ideal in $\mathbf{S}$ over U.

Example 2.9. Let $S = \{a, b, c, d\}$. Then we define an associative binary operation $\circ$ and a partial order relation $\leq$ on S as follows:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and

$$\leq := \{(a, b)\} \cup \Delta_S,$$

where Δ_S is an identity relation on S. It is easy to verify that $\mathbf{S} := (S; \circ, \leq)$ is an ordered semigroup. Now, we let $U = \{u_1, u_2, u_3, u_4, u_5\}$. Then we define the hybrid structure $f = (f^*, f^+)$ in S over U as follows:

S	a	b	c	d
$f^*(x)$	U	$\{u_2, u_3, u_4\}$	$\{u_3\}$	$\{u_2, u_3\}$
$f^+(x)$	0.2	0.5	0.9	0.7

Then f is a hybrid ideal in $\mathbf{S}$ over U.

3. Main Results

In the main section, we construct the new hybrid structures from given the hybrid structure and study some algebraic properties of such new hybrid structures.

Let $f = (f^*, f^+)$ be a hybrid structure in S over U. We define a new set as follows:

$$f[\alpha, t] := \{x \in S \mid f^*(x) \supseteq \alpha \text{ and } f^+(x) \leq t\},$$

where $\alpha \in \mathcal{P}(U)$, $t \in I$. We see that $f[\alpha, t]$ is not empty set, since $f^*(x) \supseteq \varnothing$ and $f^+(x) \leq 1$ for all $x \in S$.

We now consider the connection between the hybrid ideal $f = (f^*, f^+)$ in $\mathbf{S}$ over U and the such new set $f[\alpha, t]$.

Theorem 3.1. *Let* $\mathbf{S}$ *be an ordered semigroup and* $f = (f^*, f^+)$ *a hybrid structure in* S *over* U*. Then the following conditions are equivalent:*

(1) f *is a hybrid left ideal in* $\mathbf{S}$ *over* U.
(2) $f[\alpha, t]$ *is a left ideal in* $\mathbf{S}$ *for all* $\alpha \in \mathcal{P}(U),\ t \in [0, 1]$.

Proof. (1)⇒(2). Suppose $f = (f^*, f^+)$ is a hybrid left ideal in $\mathbf{S}$ over U. Let $\alpha \in \mathcal{P}(U)$ and $t \in [0, 1]$ and $xy \in Sf[\alpha, t]$. By the hypothesis, we have $f^*(xy) \supseteq f^*(y) \supseteq \alpha$ and $f^+(xy) \leq f^+(y) \leq t$. This means that $xy \in f[\alpha, t]$, and we obtain $Sf[\alpha, t] \subseteq f[\alpha, t]$. Let $x \in S$ and $y \in f[\alpha, t]$ be such that $x \leq y$. Then by the hypothesis, we obtain $f^*(x) \supseteq f^*(y) \supseteq \alpha$ and $f^+(x) \leq f^+(y) \leq t$. It follows that $x \in f[\alpha, t]$. Thus $f[\alpha, t]$ is a left ideal of $\mathbf{S}$.

(2)⇒(1). Suppose $f[\alpha, t]$ is a left ideal in $\mathbf{S}$ for all $\alpha \in \mathcal{P}(U),\ t \in [0, 1]$. Let $x,\ y \in S$. We set $\alpha = f^*(y)$ and $t = f^+(y)$. Then $y \in f[\alpha, t]$. By the hypothesis, we have $xy \in f[\alpha, t]$. It follows that $f^*(xy) \supseteq \alpha = f^*(y)$ and $f^+(xy) \leq t = f^+(y)$. Let $x,\ y \in S$ be such that $x \leq y$. If $y \in f[\alpha, t]$ such that $\alpha = f^*(y),\ t = f^+(y)$. Then by the hypothesis, we have $x \in f[\alpha, t]$. Thus $f^*(x) \supseteq \alpha = f^*(y)$ and $f^+(x) \leq t = f^+(y)$. So f is a hybrid left ideal in $\mathbf{S}$ over U. □

Similar to Theorem 3.1, we have the following theorem.

Theorem 3.2. *Let* $\mathbf{S}$ *be an ordered semigroup and* $f = (f^*, f^+)$ *a hybrid structure in* S *over* U*. Then the following conditions are equivalent:*

(1) f *is a hybrid right ideal in* $\mathbf{S}$ *over* U,
(2) $f[\alpha, t]$ *is a right ideal in* $\mathbf{S}$ *for all* $\alpha \in \mathcal{P}(U),\ t \in I$.

Combining Theorem 3.1 and 3.2, we obtain the following corollary.

Corollary 3.3. *Let* $\mathbf{S}$ *be an ordered semigroup and* $f = (f^*, f^+)$ *a hybrid structure in* S *over* U*. Then the following conditions are equivalent:*

(1) f *is a hybrid ideal in* $\mathbf{S}$ *over* U,
(2) $f[\alpha, t]$ *is an ideal in* $\mathbf{S}$ *for all* $\alpha \in \mathcal{P}(U),\ t \in I$.

Let $f = (f^*, f^+)$ be a hybrid structure in S over U, and let $\widetilde{f} = (\widetilde{f}^*, \widetilde{f}^+)$ be a hybrid structure in S over U defined by

$$\widetilde{f}^*(x) := \begin{cases} f^*(x) & \text{if } x \in f[\alpha, t] \\ \beta & \text{otherwise,} \end{cases}$$

and

$$\widetilde{f}^+(x) := \begin{cases} f^+(x) & \text{if } x \in f[\alpha, t] \\ s & \text{otherwise,} \end{cases}$$

where $\alpha,\ \beta \in \mathcal{P}(U)$ and $s,\ t \in I$ with $f^*(x) \supset \beta$ and $f^+(x) < s$.

Theorem 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup and* $f = (f^*, f^+)$ *a hybrid structure in* S *over* U*. If* f *is a hybrid left ideal in* $\mathbf{S}$ *over* U*, then* $\widetilde{f}$ *is a hybrid left ideal in* $\mathbf{S}$ *over* U.

Proof. Suppose that f is a hybrid left ideal in $\mathbf{S}$ over U. Let $x, y \in S$ and $\alpha,\ \beta \in \mathcal{P}(U),\ s,\ t \in I$. Then we consider 2 cases as follows.

Case 1. If $y \notin f[\alpha, t]$, then by the hypothesis, we have $xy \notin f[\alpha, t]$. Thus

$$\widetilde{f}^*(xy) = \beta = \widetilde{f}^*(y) \text{ and } \widetilde{f}^+(xy) = s = \widetilde{f}^+(y).$$

Case 2. If $y \in f[\alpha, t]$, then we obtain

$$\widetilde{f}^*(xy) = f^*(xy) \supseteq f^*(y) = \widetilde{f}^*(y)$$

and

$$\widetilde{f}^+(xy) = f^+(xy) \leq f^+(y) = \widetilde{f}^+(y).$$

Let $x, y \in S$ be such that $x \leq y$. If $y \notin f[\alpha, t]$, then

$$\widetilde{f}^*(x) \supseteq \beta = \widetilde{f}^*(y) \text{ and } \widetilde{f}^+(x) \leq s = \widetilde{f}^+(y).$$

If $y \in f[\alpha, t]$, then we obtain

$$\widetilde{f}^*(x) = f^*(x) \supseteq f^*(y) = \widetilde{f}^*(y)$$

and

$$\widetilde{f}^+(x) = f^+(x) \leq f^+(y) = \widetilde{f}^+(y).$$

Thus $\widetilde{f}$ is a hybrid left ideal in $\mathbf{S}$ over U. □

Similar to Theorem 3.4, we obtain the following theorem.

Theorem 3.5. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$ a hybrid structure in S over U. If f is a hybrid right ideal in $\mathbf{S}$ over U, then we have $\widetilde{f}$ is a hybrid right ideal in $\mathbf{S}$ over U.*

Combining Theorem 3.4 and 3.5, we have the following corollary.

Corollary 3.6. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$ a hybrid structure in S over U. If f is a hybrid ideal in $\mathbf{S}$ over U, then we have $\widetilde{f}$ is a hybrid ideal in $\mathbf{S}$ over U.*

Let $\mathbf{L} := (L; \circ_L, \leq_L)$ and $\mathbf{M} := (M; \circ_M, \leq_M)$ be ordered semigroups. Let $\theta : L \to M$ be a mapping from a set L to a set M. For a hybrid structure $g = (g^*, g^+)$ in M over U, consider a hybrid structure $\theta^{-1}(g) = (\theta^{-1}(g^*), \theta^{-1}(g^+))$ in L over U where

$$\theta^{-1}(g^*)(x) := g^*(\theta(x)) \text{ and } \theta^{-1}(g^+)(x) := g^+(\theta(x))$$

for all $x \in L$. We say that $\theta^{-1}(g)$ is the *hybrid preimage* of g under θ. The mapping $\theta : L \to M$ is called a *homomorphism* of ordered semigroups, if its satisfied the following conditions: for $a,\ b \in S$,

(i) $\theta(a \circ_T b) = \theta(a) \circ_M \theta(b)$,
(ii) if $a \leq_T b$, then $\theta(a) \leq_M \theta(b)$.

Theorem 3.7. *Let $\mathbf{L} := (L; \circ_L, \leq_L)$ and $\mathbf{M} := (M; \circ_M, \leq_M)$ be ordered semigroups. Let $\theta : L \to M$ be a homomorphism of ordered semigroups and $g = (g^*, g^+)$ a hybrid structure in M over U. If $g = (g^*, g^+)$ is a hybrid left ideal in $\mathbf{M}$ over U, then $\theta^{-1}(g)$ is a hybrid left ideal in $\mathbf{L}$ over U.*

Proof. Let $\theta : L \to M$ be a homomorphism of ordered semigroups. Let $g = (g^*, g^+)$ be a hybrid left ideal in $\mathbf{M}$ over U and $x,\ y \in L$. Then

$$\begin{aligned} \theta^{-1}(g^*)(x \circ_L y) &= g^*(\theta(x \circ_L y)) \\ &= g^*(\theta(x) \circ_M \theta(y)) \\ &\supseteq g^*(\theta(y)) \\ &= \theta^{-1}(g^*)(y) \end{aligned}$$

and

$$\begin{aligned}\theta^{-1}(g^+)(x \circ_L y) &= g^+(\theta(x \circ_L y)) \\ &= g^+(\theta(x) \circ_M \theta(y)) \\ &\leq g^+(\theta(y)) \\ &= \theta^{-1}(g^+)(y).\end{aligned}$$

Let $x,\ y \in L$ be such that $x \leq y$. Then

$$\begin{aligned}\theta^{-1}(g^*)(x) &= g^*(\theta(x)) \\ &\supseteq g^*(\theta(y)) \\ &= \theta^{-1}(g^*)(y)\end{aligned}$$

and

$$\begin{aligned}\theta^{-1}(g^+)(x) &= g^+(\theta(x)) \\ &\leq g^+(\theta(y)) \\ &= \theta^{-1}(g^+)(y).\end{aligned}$$

Thus $\theta^{-1}(g)$ is a hybrid left ideal in $\mathbf{L}$ over U. □

Similar to Theorem 3.7, we obtain the following theorem.

Theorem 3.8. *Let* $\mathbf{L} := (L; \circ_L, \leq_L)$ *and* $\mathbf{M} := (M; \circ_M, \leq_M)$ *be ordered semigroups. Let* $\theta : L \to M$ *be a homomorphism of ordered semigroups and* $g = (g^*, g^+)$ *a hybrid structure in* M *over* U*. If* $g = (g^*, g^+)$ *is a hybrid right ideal in* $\mathbf{M}$ *over* U*, then* $\theta^{-1}(g)$ *is a hybrid right ideal in* $\mathbf{L}$ *over* U*.*

Combining Theorem 3.7 and 3.8, we have the following corollary.

Corollary 3.9. *Let* $\mathbf{L} := (L; \circ_L, \leq_L)$ *and* $\mathbf{M} := (M; \circ_M, \leq_M)$ *be ordered semigroups. Let* $\theta : L \to M$ *be a homomorphism of ordered semigroups and* $g = (g^*, g^+)$ *a hybrid structure in* M *over* U*. If* $g = (g^*, g^+)$ *is a hybrid ideal in* $\mathbf{M}$ *over* U*, then* $\theta^{-1}(g)$ *is a hybrid ideal in* $\mathbf{L}$ *over* U*.*

A mapping $\theta : L \to M$ is called *onto homomorphism* of ordered semigroups, if
(i) θ is an onto mapping,
(ii) θ is a homomorphism of ordered semigroups.
Let $f = (f^*, f^+)$ be a hybrid structure in L over U, the *hybrid image* of f under θ is defined to be a hybrid structure $\theta(f) = (\theta(f^*), \theta(f^+))$ in M over U, where

$$\theta(f^*)(y) := \begin{cases} \bigcup\limits_{x \in \theta^{-1}(y)} f^*(x) & \text{if } \theta^{-1}(y) \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$\theta(f^+)(y) := \begin{cases} \bigwedge\limits_{x \in \theta^{-1}(y)} f^+(x) & \text{if } \theta^{-1}(y) \neq \varnothing \\ \varnothing & \text{otherwise,} \end{cases}$$

for every $y \in M$.

Theorem 3.10. *Let* $\mathbf{L} := (L; \circ_L, \leq_L)$ *and* $\mathbf{M} := (M; \circ_M, \leq_M)$ *be an ordered semigroups. Let* $\theta : L \to M$ *be an onto homomorphism of ordered semigroups and* $g = (g^*, g^+)$ *a hybrid structure in* M *over* U*. If the preimage* $\theta^{-1}(g)$ *of* g *under* θ *is a hybrid left ideal in* $\mathbf{L}$ *over* U*, then* g *is a hybrid left ideal in* $\mathbf{M}$ *over* U*.*

Proof. Let $\theta : L \to M$ be an onto homomorphism of ordered semigroups and $a,\ b \in M$. There exist $x,\ y \in L$ such that $\theta(x) = a$ and $\theta(y) = b$. We obtain

$$\begin{aligned} g^*(a \circ_M b) &= g^*(\theta(x) \circ_M \theta(y)) \\ &= g^*(\theta(x \circ_L y)) \\ &= \theta^{-1}(g^*)(x \circ_L y) \\ &\supseteq \theta^{-1}(g^*)(y) \\ &= g^*(\theta(y)) \\ &= g^*(b) \end{aligned}$$

and

$$\begin{aligned} g^+(a \circ_M b) &= g^+(\theta(x) \circ_M \theta(y)) \\ &= g^+(\theta(x \circ_L y)) \\ &= \theta^{-1}(g^+)(x \circ_L y) \\ &\leq \theta^{-1}(g^+)(y) \\ &= g^+(\theta(y)) \\ &= g^+(b). \end{aligned}$$

Let $a,\ b \in M$ be such that $a \leq_M b$. Since θ is onto, there exist $x,\ y \in L$ such that $\theta(x) = a$ and $\theta(y) = b$ and then $\theta(x) \leq_M \theta(y)$. Then we obtain

$$\begin{aligned} g^*(a) &= g^*(\theta(x)) \\ &= \theta^{-1}(g^*)(x) \\ &\supseteq \theta^{-1}(g^*)(y) \\ &= g^*(\theta(y)) \\ &= g^*(b) \end{aligned}$$

and

$$\begin{aligned} g^+(a) &= g^+(\theta(x)) \\ &= \theta^{-1}(g^+)(x) \\ &\leq \theta^{-1}(g^+)(y) \\ &= g^+(\theta(y)) \\ &= g^+(b). \end{aligned}$$

Thus g is a hybrid left ideal in $\mathbf{M}$ over U. □

Similar to Theorem 3.10, we obtain the following theorem.

Theorem 3.11. *Let* $\mathbf{L} := (L; \circ_L, \leq_L)$ *and* $\mathbf{M} := (M; \circ_M, \leq_M)$ *be an ordered semigroups. Let* $\theta : L \to M$ *be an onto homomorphism of ordered semigroups and*

$g = (g^*, g^+)$ a hybrid structure in M over U. If the preimage $\theta^{-1}(g)$ of g under θ is a hybrid right ideal in $\mathbf{L}$ over U, then g is a hybrid right ideal in $\mathbf{M}$ over U.

Combining Theorem 3.10 and 3.11, we have the following corollary.

Corollary 3.12. *Let $\mathbf{L} := (L; \circ_L, \leq_L)$ and $\mathbf{M} := (M; \circ_M, \leq_M)$ be an ordered semigroups. Let $\theta : L \to M$ be an onto homomorphism of ordered semigroups and $g = (g^*, g^+)$ a hybrid structure in M over U. If the preimage $\theta^{-1}(g)$ of g under θ is a hybrid ideal in $\mathbf{L}$ over U, then g is a hybrid ideal in $\mathbf{M}$ over U.*

Let $\mathbf{S}$ be an ordered semigroup. We define a binary operation $\star : S \times S \to S$ on S as follows:

$$(x, a) \star (y, b) := (xy, ab)$$

and we define a binary relation $\preceq$ on $S \times S$ as follows:

$$(x, a) \preceq (y, b) \text{ if, } x \leq y \text{ and } a \leq b$$

for all $(x, a), (y, b) \in S \times S$ and we see that

$$\begin{aligned} [(x, a) \star (y, b)] \star (z, c) &= (xy, ab) \star (z, c) \\ &= ((xy)z, (ab)c) \\ &= (x(yz), a(bc)) \\ &= (x, a) \star (yz, bc) \\ &= (x, a) \star [(y, b) \star (z, c)] \end{aligned}$$

for all $(x, a), (y, b), (z, c) \in S \times S$. Thus $(S \times S; \star)$ is a semigroup and it is easy to verify that $(S \times S; \star, \preceq)$ is an ordered semigroup.

Let $f = (f^*, f^+)$ and $g = (g^*, g^+)$ be hybrid structures in S over U. The cartesian product of f and g, denoted by $f \times g$, is a hybrid structure $f \times g : S \times S \to \mathcal{P}(U) \times I$ in $S \times S$ over U and is defined by: for all $(x, y) \in S \times S$,

$$(f^* \times g^*)(x, y) := f^*(x) \cap g^*(y)$$

and

$$(f^+ \times g^+)(x, y) := \max\{f^+(x), g^+(y)\}.$$

Theorem 3.13. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$, $g = (g^*, g^+)$ hybrid structures in S over U. If $f = (f^*, f^+)$ and $g = (g^*, g^+)$ are hybrid left ideals in $\mathbf{S}$ over U, then $f \times g$ is a hybrid left ideal in $\mathbf{S} \times \mathbf{S}$ over U.*

Proof. Let $f = (f^*, f^+)$ and $g = (g^*, g^+)$ be hybrid left ideals in $\mathbf{S}$ over U and let $(x, a), (y, b) \in S \times S$. Then we obtain

$$\begin{aligned} (f^* \times g^*)((x, a) \star (y, b)) &= (f^* \times g^*)(xy, ab) \\ &= f^*(xy) \cap g^*(ab) \\ &\supseteq f^*(y) \cap g^*(b) \\ &= (f^* \times g^*)(y, b) \end{aligned}$$

and

$$\begin{array}{rcl}(f^+ \times g^+)((x,a) \star (y,b)) & = & (f^+ \times g^+)(xy, ab) \\ & = & \max\{f^+(xy)g^+(ab)\} \\ & \leq & \max\{f^+(y), g^+(b)\} \\ & = & (f^+ \times g^+)(y,b).\end{array}$$

Let $(x,a),\ (y,b) \in S \times S$ be such that $(x,a) \preceq (y,b)$, which means that $x \leq y$ and $a \leq b$. Then we obtain

$$\begin{array}{rcl}(f^* \times g^*)(x,a) & = & f^*(x) \cap g^*(a) \\ & \supseteq & f^*(y) \cap g^*(b) \\ & = & (f^* \times g^*)(y,b)\end{array}$$

and

$$\begin{array}{rcl}(f^+ \times g^+)(x,a) & = & \max\{f^+(x),\ g^+(a)\} \\ & \leq & \max\{f^+(y),\ g^+(b)\} \\ & = & (f^+ \times g^+)(y,b).\end{array}$$

Thus $f \times g$ is a hybrid left ideal in $\mathbf{S} \times \mathbf{S}$ over U. □

Similar to Theorem 3.13, we obtain the following result.

Theorem 3.14. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$, $g = (g^*, g^+)$ hybrid structures in S over U. If $f = (f^*, f^+)$ and $g = (g^*, g^+)$ are hybrid right ideals in $\mathbf{S}$ over U, then $f \times g$ is a hybrid right ideal in $\mathbf{S} \times \mathbf{S}$ over U.*

Combining Theorem 3.13 and 3.14, we obtain the following Corollary.

Corollary 3.15. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$, $g = (g^*, g^+)$ hybrid structures in S over U. If $f = (f^*, f^+)$ and $g = (g^*, g^+)$ are hybrid right ideals in $\mathbf{S}$ over U, then $f \times g$ is a hybrid right ideal in $\mathbf{S} \times \mathbf{S}$ over U.*

By Theorem 3.13, Theorem 3.14 and Corollary 3.15, we obtain the following Corollaries.

Corollary 3.16. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$ a hybrid structure of S over U. If $f = (f^*, f^+)$ is hybrid left ideal in $\mathbf{S}$ over U, then $f \times f$ is a hybrid left ideal in $\mathbf{S} \times \mathbf{S}$ over U.*

Corollary 3.17. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$ a hybrid structure of S over U. If $f = (f^*, f^+)$ is hybrid right ideal in $\mathbf{S}$ over U, then $f \times f$ is a hybrid right ideal in $\mathbf{S} \times \mathbf{S}$ over U.*

Corollary 3.18. *Let $\mathbf{S}$ be an ordered semigroup and $f = (f^*, f^+)$ a hybrid structure of S over U. If $f = (f^*, f^+)$ is hybrid ideal in $\mathbf{S}$ over U, then $f \times f$ is a hybrid ideal in $\mathbf{S} \times \mathbf{S}$ over U.*

4. Conclusions

In this present paper, we applied a hybrid structure to an ordered semigroup. The new many types of hybrid structures are introduced and some algebraic properties of such new hybrid structures are studied. In the future woke, we will use this ideal for hyperstructures as hypersemigroups, ordered hypersemigroups, etc.

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