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Abstract. In this paper, we redefine Γ-BCK-algebras in the sense of Saeid et al. [1] and we introduce the concepts of positive implicative [resp. commutative and implicative] Γ-BCK-algebras and study some of their properties. Also, we define Γ-subalgebras [resp. positive implicative, commutative and implicative] Γ-ideals in a Γ-BCK-algebra and investigate several of their properties, relationships among them and their characterizations respectively.

2020 AMS Classification: 03G25, 06F35, 03E72, 08A35

Keywords: Positive implicative [resp. commutative and implicative] Γ-BCK-algebra, Γ-subalgebra, Positive implicative [resp. commutative and implicative] Γ-ideal.

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1. Introduction

In 1966, Imai and Iséki [2] introduced the concept of I-algebras as a class of abstract algebras and studied its several properties. In 1978, Iséki and Tanaka [3] introduced the notion of BCK-algebras as a generalization of I-algebras. Iséki [4] defined BCI-algebras as a generalization of BCK-algebras. Hu and Li [5, 6] introduced the concept of BCH-algebras which any BCI-algebra is a proper BCH-algebra. Jun et al. [7] defined BH-algebras as a generalization of $BCH/BCI/BCK$-algebras. Ahn and Kim [8], and Neggers et al. [9] introduced the notions of QS-algebras and Q-algebras as another generalizations of $BCH/BCI/BCK$-algebras. Such logical algebras have been applied by many mathematicians to group theory, computer science, topology, decision making problems, etc. Since ideals in logical algebras play an important role, numerous mathematicians have studied ideal problems.

In 1964, Nobusawa [10] introduced the concept of Γ-rings as a generalization of rings. Rao [11] defined Γ-semirings as a generalization of Γ-rings, ternary semiring and semiring. As a generalization of semigroups, Sen [12] introduced the notion of Γ-semigroups. Rao [13] defined Γ-groups as a generalization of groups and study its various properties. Recently, Saeid et al. [1] introduced the new concept of Γ-BCK-algebras and discussed some of its properties, for examples, subalgebras, ideals, closed ideals, normal subalgebras, normal ideals in Γ-BCK-algebras and quotient Γ-BCK-algebras. However, in their paper [1], Theorem 3.15 is not true (See the first statement in Section 3 for the reason). So, it is necessary to redefine the concept of Γ-BCK-algebras they defined.

The purpose of our study is to redefine Γ-BCK-algebras in the sense of Saeid et al. [1] and proceed with our study as follows: First, we define positive implicative [resp. commutative and implicative] Γ-BCK-algebras and investigate some of their properties. Next, we introduce the notions of positive implicative [resp. commutative and implicative] Γ-ideals in a Γ-BCK-algebra and discuss several of their properties, relationships among them and their characterizations respectively.

2. Preliminaries

We recall some definitions needed in next sections.

Definition 2.1 ([3, 4])**.** Let X be a nonempty set with a constant 0 and a binary operation $*$. Consider the following axioms: for any $x,\ y,\ z \in X$,

(A_1) $[(x*y)*(x*z)]*(z*y)=0$,
(A_2) $[x*(x*y)]*y=0$,
(A_3) $x*x=0$,
(A_4) $x*y=0$ and $y*x=0$ imply $x=y$,
(A_5) $0*x=0$..

Then X is called a:

(i) *BCI-algebra*, if it satisfies axioms (A_1)–(A_4),
(ii) *BCK-algebra*, if it satisfies axioms (A_1)–(A_5).

In BCI-algebra or BCK-algebra X, we define a binary operation $\leq$ on X as follows: for any $x\ ,y \in X$,

$$x \leq y \text{ if and only if } x*y=0.$$

Definition 2.2. Let X be a BCK-algebra. Then X is said to be:

(i) *positive implicative* [14], if $(x*z)*(y*z)=(x*y)*z$ for any $x,\ y,\ z \in X$,
(ii) *commutative* [15, 16], if $x*(x*y)=y*(y*x)$ for any $x,\ y \in X$,
(ii) *implicative* [17], if $x=x*(y*x)$ for any $x,\ y \in X$.

Example 2.3. Let $X=\{0,1,2,3\}$ be a set having the binary operation $*$ on X given by the table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	2	3	0

Table 2.1

Then we can easily check that X is a BCK-algebra. Moreover, X is implicative. See Example 1 (p.21) and Example 3 (p.23) in [17] respectively for examples of a positive implicative BCK-algebra and a commutative BCK-algebra.

Definition 2.4 ([18]). Let I be a nonempty set of a BCK-algebra X. Then I is called a *subalgebra* of X, if $x * y \in I$ for any $x,\ y \in X$.

Definition 2.5 ([18]). Let I be a nonempty set of a BCK-algebra X. Then I is called an *ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$,
(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.6 ([19]). Let I be a nonempty set of a BCK-algebra X. Then I is called a *positive implicative ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(i) $0 \in I$,
(ii) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$.

Definition 2.7 ([20]). Let I be a nonempty set of a BCK-algebra X. Then I is called an *implicative ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(i) $0 \in I$,
(ii) $[x * (y * x)] * z \in I$ and $z \in I$ imply $x \in I$.

Definition 2.8 ([21]). Let I be a nonempty set of a BCK-algebra X. Then I is called a *commutative ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(i) $0 \in I$,
(ii) $(x * y) * z \in I$ and $z \in I$ imply $x * [y * (y * x)] \in I$.

Definition 2.9 ([11]). Let X and Γ be two nonempty sets. Then X is called a Γ-*semigroup*, if there is a mapping $f : X \times \Gamma \times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X \times \Gamma \times X$, such that it satisfies the following condition: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$x\alpha(y\beta z) = (x\alpha y)\beta z. \tag{2.1}$$

Definition 2.10 ([11]). Let $(X, +)$ and $(\Gamma, +)$ be commutative semigroups. Then X is called a Γ-*semiring*, if there is a mapping $f : X \times \Gamma \times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X \times \Gamma \times X$, such that it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,
(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$,
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

It is obvious that every semiring X is a Γ-semiring with $\Gamma = X$ and ternary operation as the usual semiring multiplication.

Definition 2.11 ([11]). Let X be a Γ-semiring. Then X is said to:
(i) *have zero element*, if there is $0 \in X$ such that for each $x \in X$ and each $\alpha \in \Gamma$,

$$0 + x = x = x + 0 \text{ and } 0\alpha x = 0 = x\alpha 0, \tag{2.2}$$

(ii) *commutative*, if $x\alpha y = y\alpha x$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$.

3. Some properties of Γ-BCK-algebras

[Theorem 3.15, [1]] is wrong. In order to prove it, Saeid et al. used the first axiom of Γ-BCK-algebra. However, $x\beta 0 = x$ cannot be derived from axioms of a Γ-BCK-algebra and then we redefine Γ-BCK-algebra as follows and study some of its properties. Also we introduce some special Γ-BCK-algebras and obtain some of their properties.

Definition 3.1 (See [1]). Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is called a Γ-*BCK-algebra*, if there is a mapping $f : X \times \Gamma \times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X \times \Gamma \times X$, satisfying the following axioms: for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,

(ΓA_1) $[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) = 0$,

(ΓA_2) $[x\alpha(x\beta y)]\alpha y = 0$,

(ΓA_3) if $x\alpha y = 0 = y\alpha x$, then $x = y$,

(ΓA_4) $x\alpha x = 0$,

(ΓA_5) $0\alpha x = 0$.

It is obvious that for a Γ-BCK-algebra X and a fixed $\alpha \in \Gamma$, if we define the operation $* : X \times X \to X$ as follows: for any $x,\ y \in X$,

$$x * y = x\alpha y, \tag{3.1}$$

then $(X, *, 0)$ is a BCK-algebra and is denoted by X_α.

Example 3.2. (1) Let A be an arbitrary nonempty set, let $X = \{f : A \to \mathbb{R}\}$, let $\Gamma = \{\alpha\}$ and let the ternary operation be defined as follows: for any $f,\ g \in X$ and each $a \in A$,

$$(f\alpha g)(a) = \begin{cases} 0 & \text{if } f(a) \le g(a) \\ f(a) - g(a) & \text{if } g(a) < f(a). \end{cases}$$

Then X is a Γ-BCK-algebra.

(2) Let X be a nonempty set, let $P(X)$ be the power set of X, let let $\Gamma = \{\alpha\}$ and let the ternary operation be defined as follows: for any $A,\ B \in P(X)$,

$$A\alpha B = \begin{cases} \varnothing & \text{if } A \subset B \\ A - B & \text{otherwise .} \end{cases}$$

Then we can easily see that $P(X)$ is a Γ-BCK-algebra.

(3) Let $X = \{0, 1, 2, 3\}$, let $\Gamma = \{\alpha, \beta, \gamma\}$ and let the ternary operation be defined by the table:

α	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	0	0

β	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	0	0

γ	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	2
3	3	3	0	0

Table 3.1

Then we can easily check that X is Γ-BCK-algebra. Moreover, X_α, X_β and X_γ are BCK-algebras.

We define a binary relation $\leq$ on a Γ-BCK-algebra X as follows (See [1]): for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$x \leq y \text{ if and only if } x\alpha y = 0. \tag{3.2}$$

In this case, $\leq$ is called a Γ-BCK *ordering*. Then from (3.2), we obtain a characterization of a Γ-BCK-algebra.

Theorem 3.3. *A Γ-algebra X is a Γ-BCK-algebra if and only if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,*
(1) $(x\alpha y)\beta(x\alpha z) \leq z\alpha y$,
(2) $x\alpha(x\beta y) \leq y$,
(3) $x \leq x$,
(4) *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$,
(5) $0 \leq x$,
(6) $x \leq y$ *if and only if* $x\alpha y = 0$.

The followings are immediate consequences of Definition 3.1 and Theorem 3.3.

Proposition 3.4. *Let X be a Γ-BCK-algebra. Then the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,*
(1) *if* $x \leq y$, *then* $z\alpha y \leq z\alpha x$,
(2) *if* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$.

From Theorem 3.3 (3), (4) and Proposition 3.4 (2), it is clear that $(X, \leq)$ is a poset with the least element 0.

Proposition 3.5. *Let X be a Γ-BCK-algebra. Then the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha,\ \beta \in \Gamma$,*

$$(x\alpha y)\beta z = (x\alpha z)\beta y. \tag{3.3}$$

Proposition 3.6. *Let X be a Γ-BCK-algebra. Then the followings hold: for any $x,\ y,\ z \in X$ and each $\alpha,\ \beta \in \Gamma$,*
(1) $x\alpha y \leq z$ *if and only if* $x\alpha z \leq y$,
(2) $(x\alpha z)\beta(y\alpha z) \leq x\alpha y$,
(3) *if* $x \leq y$, *then* $x\alpha z \leq y\alpha z$,
(4) $x\alpha y \leq x$,
(5) $x\alpha 0 = x$,
(6) $x\alpha[x\beta(x\alpha y)] = x\alpha y$.

We give a characterization of a Γ-BCK-algebra.

Theorem 3.7. *Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is a Γ-BCK-algebra if and only if it satisfies axioms (ΓA_1), (ΓA_3) and the following condition: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,*

$$x\alpha(0\beta y) = x. \tag{3.4}$$

Proof. ($\Rightarrow$): Suppose X is a Γ-BCK-algebra, let $x,\ y \in X$ and let $\alpha,\ \beta \in \gamma$. Then by the axiom ($\Gamma A_5$), $x\alpha(0\beta y) = x\alpha 0$. Thus by Proposition 3.6 (5), $x\alpha 0 = x$. So (3.4) holds.

($\Leftarrow$): Suppose the necessary conditions hold, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_1$), $[(0\alpha y)\beta(0\alpha 0)]\beta(0\alpha y) = 0$. By the condition (3.4),

$(0\alpha y)\beta(0\alpha 0) = 0$. Also by the condition (3.4), $0\alpha y = 0$. Thus the axiom (ΓA_5) holds. So by the condition (3.4) and the axiom (ΓA_5), we get

$$x\alpha 0 = x. \tag{3.5}$$

On the other hand, by the axiom (ΓA_1), $[(x\beta 0)\alpha(x\beta 0)]\alpha(0\beta 0) = 0$. Then by the identity (3.5), $x\alpha x = 0$. Thus the axiom (ΓA_4) holds. Also by the axiom (ΓA_1), $[(x\beta 0)\alpha(x\beta y)]\alpha(y\beta 0) = 0$. By the identity (3.5), $[x\alpha(x\beta y)]\alpha y = 0$. So the axiom ($\Gamma A_2$) holds. Hence X is a Γ-BCK-algebra. □

Now we define some special Γ-BCK-algebras and discuss some of their properties.

Definition 3.8. A Γ-BCK-algebra X is said to be *positive implicative*, if it satisfies the following axiom: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$(x\alpha z)\beta(y\alpha z) = (x\alpha y)\beta z. \tag{3.6}$$

It is obvious that if X is a positive implicative Γ-BCK-algebra, then X_α is a positive implicative BCK-algebra for each $\alpha \in \Gamma$.

Example 3.9. (1) Let X be a poset with the least element 0, let $\Gamma = \{\alpha\}$ and let the ternary operation α be defined as followings: for any $x,\ y \in X$,

$$x\alpha y = \begin{cases} 0 & \text{if } x \leq y \\ x & \text{otherwise.} \end{cases}$$

Then X is a positive implicative Γ-BCK-algebra. Furthermore, the Γ-BCK-algebra $P(X)$ given in Example 3.2 (2) is positive implicative.

(2) Consider the Γ-BCK-algebra X given in Example 3.2 (3). Then we have

$$(3\alpha 2)\gamma(1\alpha 2) = 1 \neq 0 = (3\alpha 1)\gamma 2.$$

Thus X is not positive implicative.

(3) Let $X = \{0, 1, 2, 3\}$, let $\Gamma = \{\alpha, \beta\}$ and let the ternary operation be defined by the table:

α	0	1	2	3	β	0	1	2	3
0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	1	1	0	0	0
2	2	2	0	2	2	2	2	0	2
3	3	3	3	0	3	3	3	3	0

Table 3.2

Then we can easily check that X is a positive implicative Γ-BCK-algebra. Moreover, X_α and X_β are positive implicative BCK-algebras.

Proposition 3.10. *Let X be a Γ-BCK-algebra. Then the following holds: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

$$(x\beta(x\alpha y))\beta(y\alpha x) \leq x\beta(x\alpha(y\beta(y\alpha x))). \tag{3.7}$$

Proof.
$$\begin{aligned}
&[((x\beta(x\alpha y))\beta(y\alpha x))]\alpha(x\beta(x\alpha(y\beta(y\alpha x))))\\
&= [(x\beta(x\beta(x\alpha(y\beta(y\alpha x)))))\alpha(x\alpha y))]\beta(y\alpha x) \text{ [By the identity (3.3)]}\\
&= [((x\alpha(y\beta(y\alpha x))))\alpha(x\alpha y))]\beta(y\alpha x) \text{ [By Proposition 3.6 (5)]}
\end{aligned}$$

$\leq (y\alpha(y\beta(y\alpha x)))\beta(y\alpha x)$ [By Proposition 3.6 (2)]
$= 0$. [By the axiom (ΓA_2)]

Then by Theorem 3.3 (5) and (4), we have

$$[((x\beta(x\alpha y))\beta(y\alpha x))]\alpha(x\beta(x\alpha(y\beta(y\alpha x)))) = 0.$$

Thus the inequality (3.7) holds. □

Theorem 3.11. *Let X be a Γ-BCK-algebra. Then the followings are equivalent: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,*
(1) *X is positive implicative,*
(2) *$x\alpha y = (x\alpha y)\beta y$,*

Proof. (1) $\Rightarrow$ (2): Suppose X is positive implicative, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the identity (3.6), we get

$$x\alpha y = (x\alpha y)\beta(y\alpha y) = (x\alpha y)\beta y.$$

Thus the condition (2) holds.

(2) $\Rightarrow$ (1): Suppose the condition (2) holds, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$((x\alpha z)\beta(y\alpha z))\beta((x\alpha y)\beta z)$
$= (((x\alpha z)\beta z)\beta(y\alpha z))\beta((x\alpha y)\beta z)$ [By the condition (2)]
$\leq ((x\alpha z)\beta y)\beta((x\alpha y)\beta z)$ [By Proposition 3.6 (2)]
$= ((x\alpha y)\beta z)\beta((x\alpha y)\beta z)$ [By the identity (3.3)]
$= 0$.[By the axiom (ΓA_4)]

Thus $((x\alpha z)\beta(y\alpha z))\beta((x\alpha y)\beta z) = 0$. So $(x\alpha z)\beta(y\alpha z) \leq (x\alpha y)\beta z$. The proof of converse inequality is easy. Hence $(x\alpha z)\beta(y\alpha z) = (x\alpha y)\beta z$. Therefore the condition (1) holds. □

Theorem 3.12. *Let X be a Γ-BCK-algebra. Then the followings are equivalent: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,*
(1) *X is positive implicative,*
(2) *if $(x\alpha y)\beta z = 0$, then $(x\alpha z)\beta(y\alpha z) = 0$,*
(3) *if $(x\alpha y)\beta y = 0$, then $x\alpha y = 0$.*

Proof. (1) $\Rightarrow$ (2): The proof follows from the identity (3.6).

(2) $\Rightarrow$ (3): Suppose the condition (2) holds and $(x\alpha y)\beta y = 0$ for any $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$x\alpha y = (x\alpha y)\beta 0$ [By Proposition 3.6 (5)]
$= (x\alpha y)\beta(y\alpha y)$ [By the axiom (ΓA_4)]
$= 0$. [By the condition (2)]

Thus the condition (3) holds.

(3) $\Rightarrow$ (1): Suppose the condition (3) holds. For any $x,\ y \in X$ and any $\alpha,\ \beta \in \tau$, let $u = (x\alpha y)\beta y$. Then we have

$((x\alpha u)\beta y)\alpha y = ((x\alpha y)\beta u)\alpha y$ [By the identity (3.3)]
$= ((x\alpha y)\beta y)\alpha u$ [By the identity (3.3)]
$= ((x\alpha y)\beta y)\alpha((x\alpha y)\beta y)$
$= 0$. [By the axiom (ΓA_4)]

Thus by the hypothesis and the identity (3.3), $0 = (x\alpha u)\beta y = (x\alpha y)\beta u$, i.e.,

$$(x\alpha y)\beta((x\alpha y)\beta y) = 0.$$

So $x\alpha y \leq (x\alpha y)\beta y$. On the other hand, from the identity (3.3) and the axiom (ΓA_5), $((x\alpha y)\beta y)\beta(x\alpha y) = 0$, i.e., $(x\alpha y)\beta y \leq x\alpha y$. Hence by Proposition 3.6 (3), $x\alpha y = (x\alpha y)\beta y$. Therefore by Theorem 3.10, X is positive implicative. □

Definition 3.13 ([1]). A Γ-BCK-algebra X is said to be *commutative*, if it satisfies the following axiom: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,

$$y\alpha(y\beta x) = x\alpha(x\beta y). \tag{3.8}$$

We can easily see that if X is a commutative Γ-BCK-algebra, then X_α is a commutative BCK-algebra for each $\alpha \in \Gamma$.

Example 3.14. (1) Let X be the Γ-BCK-algebra in given Example 3.2 (3). Then

$$3\alpha(3\beta 2) = 0 \neq 2 = 2\alpha(2\beta 3).$$

Thus X is not commutative. But the Γ-BCK-algebra $P(X)$ given in Example 3.2 (2) is commutative.

(2) Let $X = \{0,1,2,3\}$, let $\Gamma = \{\alpha,\beta\}$ and let the the ternary operation be defined as the following table:

α	0	1	2	3	β	0	1	2	3
0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	1	1	0	1	1
2	2	2	0	0	2	2	2	0	2
3	3	3	2	0	3	3	3	3	0

Table 3.3

Then we can easily check that X is commutative Γ-BCK-algebra.

From the identity (3.8), we obtain a characterization of commutative Γ-BCK-algebras.

Theorem 3.15. *Let X be a Γ-BCK-algebra. Then the followings are equivalent: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

(1) *X is commutative,*
(2) $x\alpha(x\beta y) \leq y\alpha(y\beta x)$,
(3) $(x\alpha(x\beta y))\alpha(y\alpha(y\beta x)) = 0$.

Lemma 3.16 (See Theorem 3.16, [1]). *Let X be a Γ-BCK-algebra. Then the followings are equivalent: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

(1) *X is commutative,*
(2) *if $x \leq y$, then $x = y\alpha(y\beta x)$.*

Theorem 3.17. *Let X be a Γ-BCK-algebra. Then the followings are equivalent: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,*

(1) *if $x \leq z$ and $z\alpha y \leq z\alpha x$, then $x\alpha y$,*
(2) *if $x,\ y \leq z$ and $z\alpha y \leq z\alpha x$, then $x\alpha y$,*
(3) *if $x \leq y$, then $x = y\alpha(y\beta x)$,*
(4) *X is commutative,*
(5) *if $x\alpha y = 0$, then $x\alpha(y\beta(y\alpha x)) = 0$.*

Proof. The proofs of (1) $\Rightarrow$ (2) and (3) $\Leftrightarrow$ (5) are straightforward.

(2) $\Rightarrow$ (3): Suppose $x \leq y$. By Proposition 3.6 (4), $y\alpha(y\beta x) \leq y$. Then by Proposition 3.6 (3), $y\beta(y\alpha(y\beta x)) \leq y\beta x$. Thus by the hypothesis, $x \leq y\alpha(y\beta x)$. It is clear that $y\alpha(y\beta x) \leq x$. So $x = y\alpha(y\beta x)$.

(3) $\Rightarrow$ (4): The proof follows from Lemma 3.16.

(4) $\Rightarrow$ (1): Suppose the condition (4) holds, and suppose $x \leq z$ and $z\alpha y \leq z\alpha x$. Then clearly, $x\alpha z = 0$ and $(z\alpha y)\beta(z\alpha x) = 0$. Thus we have

$$\begin{aligned} x\alpha y &= (x\alpha(x\alpha z))\beta y \text{ [By Proposition 3.6 (5)]} \\ &= (z\alpha(z\alpha x))\beta y \text{ [Since } X \text{ is commutative]} \\ &= (z\alpha y)\beta(z\alpha x) \text{ [By the identity (3.3)]} \\ &= 0. \end{aligned}$$

So $x \leq y$. Hence (1) holds. □

Also, we obtain another characterization of a commutative Γ-BCK-algebra.

Theorem 3.18. *Let X be a Γ-BCK-algebra. Then followings are equivalent: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

(1) *X is commutative,*

(2) $x\alpha(x\beta y) = y\alpha(y\beta(x\alpha(x\beta y)))$.

Definition 3.19. Let X be a Γ-BCK-algebra. Then X is said to be *implicative*, if it satisfies the following condition:

$$x = x\alpha(y\beta x) \text{ for any } x,\ y \in X \text{ and any } \alpha,\ \beta \in \Gamma. \tag{3.9}$$

It is clear that if X is an implicative Γ-BCK-algebra, then X_α is an implicative BCK-algebra fore each $\alpha \in \Gamma$.

Example 3.20. (1) Let $X = \{0, 1\}$, let $\Gamma = \{\alpha\}$ and let the ternary operation be defined as follows:

$$1\alpha 0 = 1,\ 0\alpha 0 = 0\alpha 1 = 1\alpha 1 = 0.$$

Then clearly X is an implicative Γ-BCK-algebra. Furthermore, the Γ-BCK-algebra $P(X)$ given in Example 3.2 (2) is implicative.

(2) Let $X = \{0, 1, 2, 3\}$, let $\Gamma = \{\alpha, \beta\}$ and let the the ternary operation be defined as the following table:

α	0	1	2	3	β	0	1	2	3
0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	1	1	0	1	2
2	2	2	0	2	2	2	2	0	2
3	3	2	3	0	3	3	2	3	0

Table 3.4

Then clearly, X is an implicative Γ-BCK-algebra.

(3) Let $X = \{0, 1, 2, 3, 4\}$, let $\Gamma = \{\alpha, \beta\}$ and let the the ternary operation be defined as the following table:

Then X is a Γ-BCK-algebra. But it is neither implicative nor commutative.

α	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	1	0	0	0
4	4	4	4	4	0

β	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	1	0
3	3	1	3	0	0
4	4	4	4	4	0

Table 3.5

We give a relationship among implicativeness, commutativity and positive implicativeness.

Theorem 3.21. *Let X be a Γ-BCK-algebra. Then X is implicative if and only if it is commutative and positive implicative.*

Proof. Suppose X is implicative, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$$x\alpha y = (x\alpha y)\beta(y\beta(x\alpha y)) = (x\alpha y)\beta y.$$

Thus by Theorem 3.11, X is positive implicative. On the other hand, we get

$$\begin{aligned} x\alpha(x\beta y) &= (x\alpha(y\beta x))\alpha(x\beta y) \text{ [By the hypothesis]} \\ &\leq y\alpha(y\beta x). \text{ [By Theorem 3.3 (1)]} \end{aligned}$$

Similarly, we get $y\alpha(y\beta x) \leq x\alpha(x\beta y)$. So $x\alpha(x\beta y) = y\alpha(y\beta x)$. Hence X is commutative.

Conversely, suppose the necessary conditions hold and let $x,\ y \in X$ and $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned} x\beta(x\alpha(y\beta x)) &= (y\beta x)\alpha((y\beta x)\alpha x) \text{ [Since } X \text{ is commutative]} \\ &= ((y\beta x)\alpha x)\alpha((y\beta x)\alpha x) \text{ [By Theorem 3.2 (1)]} \\ &= 0. \text{ [By the axiom } (\Gamma A_4)] \end{aligned}$$

Thus $x \leq x\alpha(y\beta x)$. On the other hand, by Proposition 3.6 (4), $(x\alpha(y\beta x))\alpha x = 0$. So $x\alpha(y\beta x) \leq x$. By Theorem 3.3 (4), $x = x\alpha(y\beta x)$. Hence X is implicative. This completes the proof. □

Finally, we give a sufficient condition of implicative Γ-BCK-algebras.

Proposition 3.22. *Let X be a Γ-BCK-algebra. Suppose the following condition holds: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*

$$[x\alpha(x\beta y)]\alpha(x\beta y) = [y\alpha(y\beta x)]\alpha(y\beta x). \tag{3.10}$$

Then X is implicative.

Proof. Suppose the condition (3.10) holds, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we get

$$\begin{aligned} x\alpha y &= [(x\alpha y)\alpha 0]\alpha 0 \text{ [By Proposition 3.6 (5)]} \\ &= [(x\alpha y)\alpha((x\alpha x)\beta y)]\alpha[(x\alpha x)\beta y] \text{ [By the axioms } (\Gamma A_4) \text{ and } (\Gamma A_5)] \\ &= [(x\alpha y)\alpha((x\alpha y)\beta x)]\alpha[(x\alpha y)\beta x] \text{ [By the identity (3.3)]} \\ &= [y\alpha(y\beta x)]\alpha(y\beta x) \text{ [Putting } x\alpha y = y] \\ &= [x\alpha(x\beta y)]\alpha(x\beta y) \text{ [By the condition (3.10)]} \\ &= [x\alpha(x\beta(x\alpha y))]\alpha[x\beta(x\alpha y)] \text{ [Since } y = x\alpha y] \\ &= (x\alpha y)\alpha[x\beta(x\alpha y)] \text{ [By Proposition 3.6 (6)]} \\ &= [x\alpha(x\beta(x\alpha y))]\alpha y \text{ [By the identity (3.3)]} \end{aligned}$$

$= (x\alpha y)\alpha y$. [By Proposition 3.6 (6)]

Thus X is positive implicative. On the other hand, we have

$$\begin{aligned} x\alpha(x\beta y) &= [x\alpha(x\beta y)]\alpha(x\beta y) \text{ [By Proposition 3.6 (6)]} \\ &= [y\alpha(y\beta x)]\alpha(y\beta x) \text{ [By the condition (3.10)]} \\ &= y\alpha(y\beta x). \text{ [By Proposition 3.6 (6)]} \end{aligned}$$

So X is commutative. Hence by Theorem 3.21, X is implicative. □

Let X be a Γ-BCK-algebra in the sense of Saeid et al. [1]. If X is a Γ-semigroup, then we can easily see that (ΓA_2) holds. Thus we redefine a Γ-BCK-algebra as follows:

Definition 3.23. Let X be a Γ-semigroup. Then X is called a Γ-*BCK-algebra*, if it satisfies the following axioms: for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,

(ΓA_1) $[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) = 0$,

(ΓA_2) if $x\alpha y = 0 = y\alpha x$, then $x = y$,

(ΓA_3) $x\alpha x = 0$,

(ΓA_4) $0\alpha x = 0$.

4. Some Γ-ideals of Γ-BCK-algebras

Iséki [19] introduced the concepts of ideals and positive implicative ideals (called *implicative ideals*) in BCK-algebras, and Meng [18, 21] proposed the notions of implicative ideals and commutative ideals in BCK-algebras respectively. By modifying them, we introduce the concepts of ideals, positive implicative ideals, implicative ideals and commutative ideals in Γ-BCK-algebras and discuss some of their properties.

Definition 4.1 ([1])**.** Let X be a Γ-BCK-algebra and let A be a nonempty subset of X. Then A is called a Γ-*subalgebra* of X, if it satisfies the following condition:

$$x\alpha y \in A \text{ for any } x,\ y \in A \text{ and for each } \alpha \in \Gamma. \tag{4.1}$$

Example 4.2. Let $\Gamma = \{\alpha\}$ and let $X = \{0, 1, 2, 3, 4\}$ be the Γ-BCK-algebra with the ternary operation be defined by the following table:

α	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	1
3	3	3	3	0	0
4	4	4	4	4	0

Table 4.1

Then $\{0, 1, 2\}$ is a Γ-subalgebra of X. Also, in Γ-BCK-algebra X given in Example 3.2 (3), $\{0, 1, 2\}$ is a Γ-subalgebra of X.

(2) Let X be the Γ-BCK-algebra given in Example 3.20 (3). Then clearly, $\{0, 1, 2, 3\}$ is a Γ-subalgebra of X.

Definition 4.3 ([1]). Let X be Γ-BCK-algebra and let I be a nonempty set of X. Then I is called a Γ-*ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

(ΓI_1) $0 \in I$,

(ΓI_2) if $x\alpha y \in I$ and $y \in I$, then $x \in I$.

A Γ-ideal I is said to be *proper*, if $I \neq X$. It is obvious that X and $\{0\}$ are Γ-ideals of X. In particular, X is called a *trivial* Γ-*ideal* of X.

Example 4.4. (1) Let X be the Γ-BCK-algebra given in Example 3.2 (3). Consider two subsets I_1 and I_2 of X given by:

$$I_1 = \{0, 1\},\ I_2 = \{0, 1, 2\}.$$

Then clearly, I_1 is a Γ-ideal of X. On the other hand, $3\alpha 2 = 1 \in I_2$ and $2 \in I_2$ but $3 \notin I_2$. Thus I_2 is not a Γ-ideal of X. In fact, I_2 is a Γ-subalgebra of X.

(2) Let X be the commutative Γ-BCK-algebra given in Example 3.14 (2). Then we can easily see that X has only two Γ-ideals $\{0\}$ and X.

Proposition 4.5 (See Theorem 4.4, [1]). *Let I be a Γ-ideal of a Γ-BCK-algebra X and let $x \in I$. If $y \leq x$, then $y \in I$.*

Definition 4.6. Let X be Γ-BCK-algebra and let $a,\ b \in X$ and $\alpha \in \Gamma$ be fixed. Then the subset $A_\alpha(a, b)$ of X is defined as follows:

$$A_\alpha(a, b) = \{x \in X : x\alpha a \leq b\}.$$

It is obvious that $0,\ a,\ b \in A_\alpha(a, b)$.

Example 4.7. Let X be the Γ-BCK-algebra in Example 3.2 (3). Then clearly,

$$A_\alpha(1, 2) = X,\ A_\beta(1, 2) = \{0, 1, 2\} = A_\gamma(1, 2).$$

We obtain an equivalent condition of Γ-ideals in a Γ-BCK-algebra.

Theorem 4.8. *Let I be a nonempty subset of a Γ-BCK-algebra X. Then I is a Γ-ideal of X if and only if $A_\alpha(x, y) \subset I$ for any $x,\ y \in I$ and each $\alpha \in \Gamma$.*

Proof. Suppose I is a Γ-ideal of X, let $x,\ y \in I$ and let $\alpha \in \Gamma$. Let $z \in A_\alpha(x, y)$. Then clearly, $z\alpha x \leq y$. Thus by Proposition 4.5, $z\alpha x \in I$. Since $x \in I$ and I is a Γ-ideal of X, $z \in I$. So $A_\alpha(x, y) \subset I$.

Suppose the necessary condition holds. Since $I \neq \varnothing$, there is $x \in I$. Then by the axiom (ΓA_5), $0\alpha x \leq x$. Thus $0 \in A_\alpha(x, x)$. Since $A_\alpha(x, x) \subset I$, $0 \in I$. So the condition (ΓI_1) holds. Now let $x\alpha y \in I$ and $y \in I$. Then by Theorem 3.3 (2), $x\alpha(x\beta y) \leq y$. Thus $x \in A_\alpha(x\beta y, y) \subset I$. So the condition ($\Gamma I_2$) holds. Hence I is a Γ-ideal of X. □

The following is an immediate consequence of Theorem 4.8.

Corollary 4.9. *I be a Γ-ideal of a Γ-BCK-algebra X if and only if for any $x,\ y \in I$, each $x \in X$ and any $\alpha,\ \beta \in \Gamma$, $(z\alpha x)\beta y = 0$ implies $z \in I$.*

Proposition 4.10. *Every Γ-ideal of a Γ-BCK-algebra X is a Γ-subalgebra of X.*

Proof. The proof is straightforward. □

Definition 4.11. Let X be Γ-BCK-algebra and let I be a nonempty set of X. Then I is called a *positive implicative* Γ*-ideal* (briefly, PIΓI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(ΓI$_1$) $0 \in I$,

(ΓPII$_2$) if $(x\alpha y)\beta z \in I$ and $y\alpha z \in I$, then $x\alpha z \in I$.

It is obvious that X is a PIΓI of X.

Example 4.12. Let X be the Γ-BCK-algebra given in Example 3.20 (3). Then we can easily check that $\{0, 1, 3\}$ and $\{0, 1, 2, 3\}$ are positive implicative Γ-ideals of X.

Proposition 4.13. *Every positive implicative* Γ*-ideal of* Γ*-*BCK*-algebra is a* Γ*-ideal but the converse is not true.*

Proof. Let X be a Γ-BCK-algebra and let I be a positive implicative Γ-ideal of X. Suppose $x\alpha y \in I$ and $y \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then clearly, $(\alpha y)\beta 0 \in I$ and $y\alpha 0 \in I$. Thus by the condition (ΓPII$_2$), $x = x\alpha 0 \in I$. So I is a Γ-ideal of X. On the other hand, in Example 4.12, it is clear that $\{0\}$ is a Γ-ideal of X. But $(3\alpha 1)\beta 1 = 0 \in \{0\}$ and $1\alpha 1 = 0 \in \{0\}$ but $3\alpha 1 = 1 \notin \{0\}$. So $\{0\}$ is not a positive implicative Γ-ideal of X. □

We provide an equivalent of positive implicative Γ-ideals.

Theorem 4.14. *Let I be a* Γ*-ideal of a* Γ*-*BCK*-algebra X. Then I is positive implicative if and only if the set* $A_a = \{x \in X : x\alpha a \in I \text{ for each } \alpha \in \Gamma\}$ *is a* Γ*-ideal of X for each $a \in X$.*

Proof. Suppose I is positive implicative and let $\alpha \in \Gamma$, $a \in X$. Suppose $x\alpha y \in A_a$ and $y \in A_a$. Then clearly, $(x\alpha y)\beta a \in I$ and $y\alpha a \in I$. Thus by the condition (ΓPII$_2$), $x\alpha a \in I$. So $x \in A_a$. Hence A_a is a Γ-ideal of X.

Now suppose the necessary condition holds. Suppose $(x\alpha y)\beta z \in I$ and $y\alpha z \in I$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Then clearly, $x\alpha y \in A_z$ and $y \in A_z$. Thus by the hypothesis, $x \in A_z$. So $x\alpha z \in I$. Hence I is positive implicative. □

The following is an immediate consequence of Theorem 4.14.

Corollary 4.15. *If I is a positive implicative* Γ*-ideal of a* Γ*-*BCK*-algebra X, then for each $a \in X$, A_a is the least* Γ*-ideal of X such that* $I \cup \{a\} \subset A_a$.

The following is a characterization of positive implicative Γ-ideals.

Theorem 4.16. *Let I be a nonempty subset of a* Γ*-*BCK*-algebra X. Then the followings are equivalent:*

(1) *I is a positive implicative* Γ*-ideal of X,*

(2) *I is an ideal of X and $(x\alpha y)\beta y \in I$ implies $x\alpha y \in I$ for any $x,\ y \in X$ and $\alpha,\ \beta \in \Gamma$,*

(3) *I is a* Γ*-ideal of X and $(x\alpha y)\beta z \in I$ implies $(x\alpha z)\beta(y\alpha z) \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,*

(4) *$0 \in I$, and $[(x\alpha y)\beta y]\alpha z \in I$ and $z \in I$ imply $x\alpha y \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$.*

Proof. (1)⇒(2): Suppose I is a positive implicative Γ-ideal of X. Then by Proposition 4.13, I is a Γ-ideal of X. Now suppose $(x\alpha y)\beta y \in I$ for any $x,\ y \in X$ and $\alpha,\ \beta \in \Gamma$. Since $y\alpha y = 0 \in I$, by (ΓPII$_2$), $x\alpha y \in I$. Then the condition (2) holds.

(2)⇒(3): Suppose the condition (2) holds and suppose $(x\alpha y)\beta z \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned}[(x\beta(y\alpha z))\alpha z]\beta z &= [(x\alpha z)\beta(y\alpha z)]\beta z \text{ [By the identity (3.3)]}\\ &\leq (x\alpha y)\beta z \text{ [By Proposition 3.6 (2) and (3)]}\end{aligned}$$

Thus by Proposition 4.5, $[(x\beta(y\alpha z))\alpha z]\beta z \in I$. By the hypothesis, $[x\beta(y\alpha z)]\alpha z \in I$. By the identity (3.3), $[x\beta(y\alpha z)]\alpha z = (x\alpha z)\beta(y\alpha z)$. So $(x\alpha z)\beta(y\alpha z) \in I$. Hence the condition (3) holds.

(3)⇒(4): Suppose the condition (3) holds. Then it is obvious that $0 \in I$. Suppose $[(x\alpha y)\beta y]\alpha z \in I$ and $z \in I$ for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$. Then by the identity (3.3), we get

$$[(x\alpha y)\beta y]\alpha z = [(x\alpha z)\beta y]\alpha z.$$

Thus $[(x\alpha z)\beta y]\alpha z \in I$. On the other hand, from the identity (3.3), the axiom (ΓA_4) and the condition (3), we have

$$(x\alpha z)\beta y = (x\alpha y)\beta z = [(x\alpha z)\beta y]\alpha(y\alpha y) \in I.$$

Since I is a Γ-ideal of X and $z \in I$, $x\alpha y \in I$. So the condition (4) holds.

(4)⇒(1): Suppose the condition (4) holds. Suppose $x\alpha y \in I$ and $y \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then by Proposition 3.6 (5), we get

$$x\alpha y = [(x\alpha 0)\beta 0]\alpha y.$$

Thus $[(x\alpha 0)\beta 0]\alpha y \in I$ and $y \in I$. By the condition (4), $x\alpha 0 \in I$. By Proposition 3.6 (5), $x\alpha 0 = x$. So $x \in I$. Hence I is a Γ-ideal of X.

Now suppose $(x\alpha y)\beta z \in I$ and $y\alpha z \in I$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Then from the identity (3.3), and Proposition 3.6 (2) and (3), we have

$$[(x\alpha z)\beta z]\beta(y\alpha z) = [(x\alpha z)\beta(y\alpha z)]\beta z \leq (x\alpha y)\beta z.$$

Since $(x\alpha y)\beta z \in I$, $[(x\alpha z)\beta z]\beta(y\alpha z) \in I$. Since $y\alpha z \in I$, by the condition (4), $x\alpha y \in I$. Thus I is a positive implicative Γ-ideal of X. This completes the proof. □

Proposition 4.17. *Let I and J be Γ-ideals of a Γ-BCK-algebra X such that $I \subset J$. If I is positive implicative, then so is J.*

Proof. Suppose $(x\alpha y)\beta z \in J$ and $y\alpha z \in J$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Let $u = (x\alpha y)\beta z$. Then from the identity (3.3), the axiom (ΓA_4) and the hypothesis, we have

$$[(x\alpha u)\beta y]\alpha z = [(x\alpha y)\beta u]\alpha z = [(x\alpha y)\beta z]\alpha u = 0 \in I.$$

Since I is positive implicative, by Theorem 4.16 (3), we get

$$[(x\alpha u)\beta z]\alpha(y\alpha z) \in I.$$

On the other hand, from the identity (3.3), we have

$$[(x\alpha u)\beta z]\alpha(y\alpha z) = [(x\alpha z)\beta(y\alpha z)]\beta u = [(x\alpha z)\beta(y\alpha z)]\beta[(x\alpha y)\beta z].$$

Thus $[(x\alpha z)\beta(y\alpha z)]\beta[(x\alpha y)\beta z] \in I$. Since $I \subset J$, $[(x\alpha z)\beta(y\alpha z)]\beta[(x\alpha y)\beta z] \in J$. Since $(x\alpha y)\beta z \in J$ and J is an ideal of X, $(x\alpha z)\beta(y\alpha z) \in J$. So by Theorem 4.16 (3), J is positive implicative. □

Theorem 4.18. *Let X be a Γ-BCK-algebra. Then the followings are equivalent:*
(1) *X is positive implicative,*
(2) *$\{0\}$ is a positive implicative Γ-ideal of X,*
(3) *every Γ-ideal of X is positive implicative,*
(4) *the set $A(a) = \{x \in X : x \leq a\}$ is a Γ-ideal of X for each $a \in X$.*

Proof. (1)⇒(2): Suppose X is positive implicative. It is clear that $\{0\}$ is a Γ-ideal of X. Suppose $(x\alpha y)\beta y \in \{0\}$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. Since X is positive implicative, by Theorem 3.12, $x\alpha y = (x\alpha y)\beta y$. Then by the hypothesis, $x\alpha y \in \{0\}$. Thus by Theorem 4.16 (2), $\{0\}$ is positive implicative.

(2)⇒(3): The proof is clear from Proposition 4.17.

(3)⇒(4): Suppose the condition (3) holds and suppose $x\alpha y,\ y \in A(a)$ for each $a \in X$ and each $\alpha \in \Gamma$. Then clearly, $x\alpha y \leq a$ and $y \leq a$. Thus $(x\alpha y)\beta a = 0 \in \{0\}$ and $y\alpha a = 0 \in \{0\}$ for any $\beta \in \Gamma$. By the hypothesis, $\{0\}$ is positive implicative. So $x\alpha a \in \{0\}$, i.e., $x\alpha a = 0$, i.e., $x \leq a$. Hence $x \in A(a)$. Therefore $A(a)$ is a Γ-ideal of X.

(4)⇒(1): Suppose the condition (4) holds and suppose $(x\alpha y)\beta y = 0$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. Then clearly, $x\alpha y \leq y$, i.e., $x\alpha y \in A(y)$. By the condition (4), $A(y)$ is a Γ-ideal of X. It is obvious that $y \in A(y)$. Thus $x \in A(y)$. So $x\alpha y = 0$. Hence by Theorem 3.12, X id positive implicative. □

We obtain a characterization of a positive implicative Γ-BCK-algebra by Γ-ideals.

Theorem 4.19. *Let X be a Γ-BCK-algebra. Then X is positive implicative if and only if A_a is a Γ-ideal of X for each ideal I of X and each $a \in X$.*

Proof. Suppose X is positive implicative, let I be any Γ-ideal of X and let $a \in X$. Then by Theorem 4.18, I is a positive implicative Γ-ideal of X. Thus by Theorem 4.14, A_a is a Γ-ideal of X.

Conversely, suppose the necessary condition holds and let J be any Γ-ideal of X. Suppose $(x\alpha y)\beta z \in J$ and $y\alpha z \in J$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$. Consider the set $A_z = \{u \in X : u\alpha z \in J\}$. Then clearly, $x\alpha y \in A_z$ and $y \in A_z$. Since A_z is a Γ-ideal of X, $x \in A_z$. Thus $x\alpha z \in J$. So J is positive implicative. Hence by Theorem 4.18, X is positive implicative. □

Definition 4.20. Let X be Γ-BCK-algebra and let I be a nonempty subset of X. Then I is called an *implicative* Γ*-ideal* (briefly, IΓI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(ΓI$_1$) $0 \in I$,

(ΓII$_2$) if $[x\alpha(y\beta x]\alpha z \in I$ and $z \in I$, then $x \in I$.

For any Γ-BCK-algebra X, it is obvious that X is always an IΓI of X which is called the *trivial implicative* Γ*-ideal*.

It is obvious that every Γ-ideal of an implicative Γ-BCK-algebra X is implicative.

Example 4.21. Let X be the Γ-BCK-algebra given in Example 3.20 (3). Then we can easily check that $\{0, 1, 2, 3\}$ is an implicative Γ-ideal of X. Furthermore, $\{0\}$ is an ideal of X but not implicative, because $[1\alpha(2\beta 1)]\alpha 0 \in \{0\}$ and $0 \in \{0\}$ but $1 \notin \{0\}$.

Proposition 4.22. *Every implicative* Γ*-ideal is an ideal but the converse is not true.*

Proof. The proof is clear. See Example 4.21 for the converse. □

Proposition 4.23. *Every implicative* Γ*-ideal is positive implicative but the converse is not true.*

Proof. Let I be an implicative Γ-ideal of a Γ-BCK-algebra X and suppose $(x\alpha y)\beta z,\ y\alpha z \in I$ for any $x,\ y\ zinX$ and any $\alpha,\ \beta \in \Gamma$. Then we get

$$\begin{aligned}[(x\alpha z)\beta z]\beta(y\alpha z) &= [(x\alpha z)\beta(y\alpha z)]\beta z \text{ [By the identity (3.5)]}\\ &\leq (x\alpha y)\beta z. \text{ [Proposition 3.6 (2)]}\end{aligned}$$

Since $(x\alpha y)\beta z \in I$, by Proposition 4.5, $[(x\alpha z)\beta z]\beta(y\alpha z) \in I$. Since $y\alpha z \in I$ and I is an ideal by Proposition 4.22, $(x\alpha z)\beta z \in I$. On the other hand, we have

$$\begin{aligned}(x\alpha z)\beta(x\beta(x\alpha z)) &= [(x\alpha(x\beta(x\alpha z))]\beta z \text{ [By the identity (3.5)]}\\ &= (x\alpha z)\beta z \in I. \text{ [By Proposition 3.6 (6)]}\end{aligned}$$

Thus $[(x\alpha z)\beta(x\beta(x\alpha z))]\alpha 0 \in I$. Since $0 \in I$ and I is implicative, $x\alpha z \in I$. So I is positive implicative.

In Example, 4.12, $\{0, 1, 3\}$ is positive implicative but not implicative. □

The following is an equivalent condition of implicative Γ-ideals.

Theorem 4.24. *Let* I *be a* Γ*-ideal of a* Γ*-*BCK*-algebra* X*. Then* I *is implicative if and only if the following holds:*

(4.2) $\quad x\alpha(y\beta x) \in I$ *implies* $x \in I$ *for any* $x,\ y \in X$ *and any* $\alpha,\ \beta \in \Gamma$.

Theorem 4.25. *Let* I *be a positive implicative* Γ*-ideal of a* Γ*-*BCK*-algebra* X*. Then* I *is implicative if and only if the following holds:*

(4.3) $\quad y\alpha(y\beta x) \in I$ *implies* $x\alpha(x\beta y) \in I$ *for any* $x,\ y \in X$ *and any* $\alpha,\ \beta \in \Gamma$.

Proof. Suppose I is implicative and suppose $y\alpha(y\beta x) \in I$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$. It is clear that $[x\alpha(x\beta y)]\alpha x = 0$. Then $x\alpha(x\beta y) \leq x$. Thus by Proposition 3.4 (1), $y\beta x \leq y\beta(x\alpha(x\beta y)]$. Furthermore, we get

$$\begin{aligned}[x\alpha(x\beta y)]\alpha[y\beta(x\alpha(x\beta y))] &\leq [x\alpha(x\beta y)]\alpha(y\beta x) \text{ [By Proposition 3.4 (1)]}\\ &= [x\alpha(y\beta x)]\alpha(x\beta y). \text{ [By the identity (3.3)]}\end{aligned}$$

Note that $[x\alpha(y\beta x)]\alpha(x\beta y) \leq y\alpha(y\beta x)$. So we we have

$$[x\alpha(x\beta y)]\alpha[y\beta(x\alpha(x\beta y))] \leq y\alpha(y\beta x).$$

Since I is a Γ-ideal of X by the hypothesis and Proposition 4.22, we get

$$([x\alpha(x\beta y)]\alpha[y\beta(x\alpha(x\beta y))])\alpha 0 \in I.$$

Since $0 \in I$, by the condition (ΓII_2), $x\alpha(x\beta y) \in I$. Hence (4.3) holds.

Conversely, suppose necessary condition (4.3) holds and suppose $[x\alpha(y\beta x)]\alpha z \in I,\ z \in I$. Since I is positive implicative, by Proposition 4.13, I is an ideal of X. Then $x\alpha(y\beta x) \in I$. By Theorem 3.3 (1), we have

$$[y\alpha(y\beta x)]\alpha(y\beta x) \leq x\alpha(y\beta x) \in I.$$

Thus $[y\alpha(y\beta x)]\alpha(y\beta x) \in I$. Since I is positive implicative, by Theorem 4.16 (2), $y\alpha(y\beta x) \in I$. By the condition (4.3), we get

$$x\alpha(x\beta y) \in I. \tag{4.4}$$

Furthermore, we have

$$(x\alpha y)\beta z \leq x\alpha y \leq x\alpha(y\beta x) \in I.$$

So $(x\alpha y)\beta z \in I$. Since $z \in I$ and I is a Γ-ideal of X, $x\alpha y \in I$. By the condition (4.4), $x \in I$. Hence I is implicative. This completes the proof. □

We obtain a similar consequence to Proposition 4.17.

Proposition 4.26. *If I is an implicative Γ-ideal of a Γ-BCK-algebra X, then every Γ-ideal containing I is implicative.*

Proof. Suppose I is implicative and let J be any Γ-ideal of X such that $I \subset J$. From Proposition 4.23, it is obvious that I is positive implicative. By Proposition 4.17, J is positive implicative. To prove that I is implicative, it is sufficient to show that J satisfies the condition (4.3). Suppose $x\alpha(x\beta y) \in J$ for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$ and let $u = x\alpha(x\beta y)$. Then clearly, $[x\alpha(x\beta y)]\alpha u = 0 \in I$. Since I is positive implicative, by Theorem 4.16 (3) and the identity (3.3), we have

$$(x\alpha u)\beta[(x\alpha y)\beta u] = (x\alpha u)\beta[(x\alpha u)\beta y] \in I.$$

Since I is implicative, by the condition (4.3), $y\alpha[y\beta(x\alpha u)] \in I$. Since $I \subset J$, $y\alpha[y\beta(x\alpha u)] \in J$. By Theorem 3.3 (2), $y\beta(y\alpha x) \leq x$ and $y\alpha[y\beta(x\alpha u)] \leq x\alpha u$. Thus we get

$$\begin{aligned}[y\beta(y\alpha x)]\beta[y\alpha(y\beta(x\alpha u))] &\leq [y\beta(y\alpha u)]\beta(y\alpha x) \text{ [By Theorem 3.3 (1)]}\\ &\leq x\alpha(x\beta u) \text{ [By Proposition 3.4 (1)]}\\ &= x\alpha[x\beta(x\alpha(x\beta y))] \text{ [Since } u = x\alpha(x\beta y)]\\ &\leq x\alpha(x\beta y) \in J.\end{aligned}$$

So $[y\beta(y\alpha x)]\beta[y\alpha(y\beta(x\alpha u))] \in J$. Since $y\alpha(y\beta(x\alpha u)) \in J$, $y\beta(y\alpha x) \in J$. Hence by Theorem 4.25, J is implicative. □

Also, we give a similar consequence of Theorem 4.18.

Theorem 4.27. *Let X be a Γ-BCK-algebra. The the followings are equivalent:*

(1) $\{0\}$ *is implicative,*
(2) *every Γ-ideal of X is implicative,*
(3) $A(a)$ *is implicative for each $a \in X$,*
(4) X *is implicative.*

Proof. (1)⇔(2): The proof is obvious from Proposition 4.26.

(2)⇔(3): The prof follows from Proposition 4.23 and Theorem 4.18.

(4)⇒(1): The proof is clear.

(1)⇒(4): Suppose $\{0\}$ is implicative. Then by Proposition 4.23, $\{0\}$ is positive implicative. By Theorem 4.18, $A(x\alpha(y\beta x))$ is a Γ-ideal of X for any $x,\ y,\ z \in X$. By the hypothesis, $A(x\alpha(y\beta x))$ is implicative. It is obvious that $x\alpha(y\beta x) \in A(x\alpha(y\beta x))$. Thus $x \in A(x\alpha(y\beta x))$. So $x \leq x\alpha(y\beta x)$. It is clear that $x\alpha(y\beta x) \leq x$. Hence $x = x\alpha(y\beta x)$. Therefore X is implicative. □

Definition 4.28. Let X be Γ-BCK-algebra and let I be a nonempty subset of X. Then I is called a *commutative Γ-ideal* (briefly, CΓI) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

($\Gamma \mathrm{I}_1$) $0 \in I$,

(ΓCI$_2$) if $[(x\alpha y)\beta z] \in I$ and $z \in I$, then $x\alpha[y\beta(y\alpha x)] \in I$.

For any Γ-BCK-algebra X, it is clear that X is always a CI of X which is called the *trivial commutative* Γ*-ideal*.

Example 4.29. Let X be the Γ-BCK-algebra given in Example 3.20 (3). Then we can easily check that $\{0, 4\}$ is commutative but not positive implicative, $\{0, 1, 3\}$ is positive implicative but not commutative and $\{0, 1, 2, 3\}$ is implicative.

Proposition 4.30. *Every commutative Γ-ideal of a Γ-BCK-algebra X is a Γ-ideal of X but the converse is not true.*

Proof. The proof is obvious. See Example 4.29 for the converse. □

The following is an equivalent condition of commutative Γ-ideals.

Theorem 4.31. *Let X be a Γ-BCK-algebra and let I be a Γ-ideal of X. Then I is commutative if and only if it satisfies the following condition:*

(4.5) $\qquad x\alpha y \in I$ *implies* $x\alpha[y\beta(y\alpha x)] \in I$ *for any* $x,\ y \in X$ *any* $\alpha,\ \beta \in \Gamma$.

Proof. The proof is straightforward. □

We obtain a similar consequence of Theorem 3.21 for Γ-ideals.

Theorem 4.32. *Let X be a Γ-BCK-algebra and let I be anonempty subset of X. Then I is implicative if and only if it is both commutative and positive implicative.*

Proof. Suppose I is implicative. Then by Proposition 4.23, I is positive implicative. It is sufficient to show that I is commutative.

Suppose $x\alpha y \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. It is clear that

$$x\alpha[y\beta(y\alpha x)] \leq x \text{ for each } \beta \in \Gamma.$$

Then we have

$$y\alpha x \leq y\beta[x\alpha(y\beta(y\alpha x))].$$

Let $u = x\alpha[y\beta(y\alpha x)]$. Then we get

$$\begin{aligned} u\beta(y\alpha u) &= [x\alpha(y\beta(y\alpha x))]\beta[y\alpha(x\alpha(y\beta(y\alpha x)))] \\ &\leq [x\alpha(y\beta(y\alpha x))]\beta(y\alpha x) \\ &= [x\alpha(y\alpha x)]\beta[y\beta(y\alpha x)] \\ &\leq x\alpha y \in I. \end{aligned}$$

Thus $u\beta(y\alpha u) \in I$. Since I is implicative, by Theorem 4.24, $u \in I$, i.e., $x\alpha[y\beta(y\alpha x)] \in I$. So by Theorem 4.31, I is commutative.

Conversely, suppose the necessary condition holds and suppose $x\alpha(y\beta x) \in I$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. It is clear that

$$[y\alpha(y\beta x)]\alpha(y\beta x) \leq x\alpha(y\beta x).$$

Then $[y\alpha(y\beta x)]\alpha(y\beta x) \in I$. Since I is positive implicative, by Theorem 4.16 (2), we get

$$y\alpha(y\beta x) \in I. \tag{4.6}$$

Furthermore, by Theorem 3.3 (1) and the identity (3.3), we have

$$x\alpha y \leq x\alpha(y\beta x).$$

Since $x\alpha(y\beta x) \in I$, $x\alpha y \in I$. Since I is commutative, by Theorem 4.31,

$$x\beta[y\alpha(y\beta x)] \in I. \tag{4.7}$$

Thus by (4.6) and (4.7), $x \in I$. So I is implicative. □

We obtain a similar consequence of Proposition 4.17 for implicative Γ-ideals.

Proposition 4.33. *Let I and J be Γ-ideals of a Γ-BCK-algebra X such that $I \subset J$. If I is commutative, then so is J.*

Proof. Suppose I is commutative and suppose $x\alpha y \in J$ for any x, $y \in X$ and each $\alpha \in \Gamma$. In order to prove that J is commutative, it is sufficient to show that $x\alpha[y\beta(y\alpha x)] \in J$ by using Theorem 4.31. Let $u = x\alpha y$. Then we have

$$(x\alpha u)\beta y = (x\alpha y)\beta u = 0 \in I.$$

Since I is commutative, by Theorem 4.31, we get

$$(x\alpha u)\alpha[y\beta(y\alpha(x\alpha u))] \in I.$$

By the identity (3.3), we have

$$(x\alpha u)\alpha[y\beta(y\alpha(x\alpha u))] = [x\alpha(y\beta(y\alpha(x\alpha u)))]\alpha u.$$

Since $I \subset J$, $x\alpha[y\beta(y\alpha(x\alpha u))]\alpha u \in J$. Since J is a Γ-ideal of X and $u \in J$, $x\alpha[y\beta(y\alpha(x\alpha u))] \in J$. On the other hand,

$$\begin{aligned}[x\alpha(y\beta(y\alpha x))]\beta[x\alpha(y\beta(y\alpha(x\alpha u)))] &\leq [y\beta(y\alpha(x\beta u))]\alpha[y\beta(y\alpha x)] \\ &\leq (y\alpha x)\beta[y\alpha(x\beta u)] \\ &\leq (x\beta u)\alpha x \\ &= 0\alpha u \in J.\end{aligned}$$

Thus $x\alpha[y\beta(y\alpha x)] \in J$. So by Theorem 4.31, J is commutative. □

The following is a characterization of commutative Γ-BCK-algebras.

Theorem 4.34. *Let X be a Γ-BCK-algebra. The the followings are equivalent:*
(1) $\{0\}$ *is commutative,*
(2) *every* Γ*-ideal of* X *is commutative,*
(3) X *is commutative.*

Proof. (1)⇔(2): The proof is clear from Proposition 4.33.
(1)⇔(3): The proof follows from Theorem 3.17. □

5. Conclusions

We introduced various concepts of positive implicative [resp. commutative and implicative] Γ-BCK-algebras and Γ-subalgebras [resp. positive implicative, commutative and implicative] Γ-ideals and discussed these notions related to BCK-algebras can be naturally extended to Γ-BCK-algebras.

In the future, we will use our proposed Γ-BCK-algebras to address quotient Γ-BCK-algebras, homorphism problems and topological structures. Also, we will discuss Γ-BCK-algebras in terms of Definition 3.23. Furthermore, by using the concepts proposed in Appendix, we are going to apply Γ-BCK-algebras to fuzzy sets.

6. Appendix

For a nonempty set X, a mapping $A : X \to [0, 1]$ is called a fuzzy set in X (See [22]). Saeid et al. [23] applied the concept of fuzzy sets to ideals of Γ-BCK-algebra.

Definition 6.1 (See [23])**.** Let X be a Γ-BCK-algebra and let A be a fuzzy set in X. Then A is called:

(i) a *fuzzy* Γ*-subalgebra* of X, if $A(x\alpha y) \geq A(x) \wedge A(y)$,

(ii) a (*anti*) *fuzzy* Γ*-ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

(*a*) $A(0) \geq A(x)$ $(A(0) \leq A(x))$,

(*b*) $A(x) \geq A(x\alpha y) \wedge A(y)$ $(A(x) \leq A(x\alpha y) \vee A(y))$.

Definition 6.2. Let X be a Γ-BCK-algebra and let A be a fuzzy set in X. Then A is called a *fuzzy positive implicative* Γ*-ideal* (briefly, FPIΓI) of X, if for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(FΓI$_1$) $A(0) \geq A(x)$,

(FPIΓI$_2$) $A(x\alpha y) \geq A((x\alpha y)\beta z) \wedge A(y\alpha z)$.

Definition 6.3. Let X be a Γ-BCK-algebra and let A be a fuzzy set in X. Then A is called a *fuzzy commutative* Γ*-ideal* (briefly, FCΓI) of X, if for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(FΓI$_1$) $A(0) \geq A(x)$,

(FCΓI$_2$) $A(x\alpha[y\beta(y\alpha x)]) \geq A((x\alpha y)\beta z) \wedge A(z)$.

Definition 6.4. Let X be a Γ-BCK-algebra and let A be a fuzzy set in X. Then A is called a *fuzzy implicative* Γ*-ideal* (briefly, FIΓI) of X, if for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$,

(FΓI$_1$) $A(0) \geq A(x)$,

(FIΓI$_2$) $A(x) \geq A([x\alpha(y\beta x)]\alpha z) \wedge A(z)$.

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