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Abstract. In this paper, the means by which fuzzy Baire sets are obtained from fuzzy simply*open sets in fuzzy hyperconnected spaces are discussed. It is obtained that fuzzy Baire sets in fuzzy fraction dense spaces are not fuzzy dense sets. Also the conditions under which fuzzy Baire sets are generated from fuzzy nowhere dense sets, fuzzy dense and fuzzy open sets in fuzzy fraction dense and fuzzy DG_δ-spaces are obtained. It is established that existence of a fuzzy co-σ-boundary set in fuzzy weakly Baire spaces ensures the existence of a pair of disjoint fuzzy Baire sets and fuzzy open sets in fuzzy hyperconnected and fuzzy nodef spaces are fuzzy Baire sets.

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Keywords: Fuzzy residual set, Fuzzy dense set, Fuzzy simply* open set, Fuzzy perfectly disconnected space, Fuzzy hyperconnected space.

Corresponding Author: G. Thangaraj (g.thangaraj@rediffmail.com)

1. Introduction

Any application of mathematical concepts depends firmly and closely how one introduces basic ideas that may yield various theories in various directions. If the basic idea is suitably introduced, then not only the existing theories stand but also the possibility of emerging new theories increases. In 1965, the concept of fuzzy sets as a new approach for modelling uncertainties was introduced and described using membership functions by Zadeh [1]. The notion of fuzzy topological space was introduced by Chang [2] in 1968. Based on this concept, many studies have been conducted in general theoretical areas and in different application sides.

The concept of Baire sets in classical topology was introduced and studied by Szymanski [3]. The notion of fuzzy Baire sets in fuzzy topological spaces was introduced and studied by Thangaraj and Palani [4] by means of fuzzy open sets and fuzzy residual sets. Bakry and Hosny [5] introduced a new class of sets in topological spaces, namely simply* open sets and this concept in fuzzy setting was introduced and studied by Thangaraj and Dinakaran [6] by means of fuzzy open sets and fuzzy first category sets. The purpose of this paper is to study more the notion of fuzzy Baire sets and several characterizations of fuzzy Baire sets are established by means of fuzzy simply*-open sets. It is obtained that fuzzy Baire sets in fuzzy fraction dense spaces are not fuzzy dense sets. A condition under which fuzzy Baire sets are obtained from fuzzy co-σ-boundary sets is established in fuzzy weakly Baire spaces. It is shown that existence of a fuzzy co-σ -boundary set ensures the existence of a pair of disjoint fuzzy Baire sets in fuzzy weakly Baire spaces and fuzzy open sets in fuzzy hyperconnected and fuzzy nodef spaces are fuzzy Baire sets.

In recent years, the topological space theory has been embedding in the soft set theory to obtain some interesting applications [7, 8, 9, 10]. Many authors redefined the classical topological concepts via soft topological structure. Recently, Senel et al. [11] applied the concept of octahedron sets proposed by Lee et al. [12] to multi-criteria group decision making problems. On these lines, there is a need and scope of investigation considering different types of fuzzy sets such as fuzzy Baire sets, fuzzy simply* open sets for applying some fuzzy topological concepts to information science and decision-making problems.

2. Preliminaries

Some basic notions and results used in the sequel, are given in order to make the exposition self - contained. In this work, by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval [0,1]. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1 ([2]). Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The *interior*, the *closure* and the *complement* of λ are defined respectively as follows:

(i) $int(\lambda) = \bigvee\{\mu/\mu \leq \lambda, \mu \in T\}$,
(ii) $cl(\lambda) = \bigwedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$,
(iii) $\lambda'(x) = 1 - \lambda(x)$ for all $x \in X$.

For a family $(\lambda_i)_{i\in I}$ of fuzzy sets in (X, T), the *union* $\bigvee_{i\in I} \lambda_i$ and the *intersection* $\bigwedge_{i\in I} \lambda_i$ are defined respectively as follows: for each $x \in X$,
(iv) $(\bigvee_{i\in I} \lambda_i)(x) = sup_{i\in I}\lambda_i(x)$,
(v) $(\bigwedge_{i\in I} \lambda_i)(x) = inf_{i\in I}\lambda_i(x)$.

Lemma 2.2 ([13]). *For a fuzzy set λ of a fuzzy topological space X,*

(1) $(int(\lambda))' = cl(\lambda')$,
(2) $(cl(\lambda))' = int(\lambda')$.

Definition 2.3 ([2]). A *fuzzy topology* on a set X is a family T of fuzzy sets in X which satisfies the following conditions:
(i) $0_X,\ 1_X \in T$,
(ii) if $\lambda,\ \mu \in T$, then $\lambda \wedge \mu \in T$,

(iii) if $(\lambda_i)_{i \in I} \subset T$, then $\bigvee_{i \in I} \lambda_i \in T$, where I denotes an index set. The pair (X, T) is a *fuzzy topological space* (briefly, fts). Members of T are called *fuzzy open sets* in X and their complements are called *fuzzy closed sets* in X.

Definition 2.4. A fuzzy set λ in a fuzzy topological space (X, T) is called a:

(i) *fuzzy G_δ-set* in (X, T), if $\lambda = \bigwedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for each $i \in I$ and *fuzzy F_σ-set* in (X, T), if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i' \in T$ for each $i \in I$ [14],

(ii) *fuzzy dense set*, if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$, i.e., $cl(\lambda) = 1$ [15],

(iii) *fuzzy nowhere dense set*, if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$, i.e., $int(cl(\lambda)) = 0$ [15],

(iv) *fuzzy first category set*, if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where each λ_i is a fuzzy nowhere dense set in (X, T). Any other fuzzy set in (X, T) is said to be of *fuzzy second category* [15],

(v) *fuzzy residual set*, if λ' is a fuzzy first category set in (X, T) [16],

(vi) *fuzzy strongly nowhere dense set* in (X, T), if $\lambda \wedge \lambda'$ is a fuzzy nowhere dense set in (X, T) [17],

(vii) *fuzzy Baire set* in (X, T), if $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy residual set in (X, T) [18],

(viii) *fuzzy simply* open set* in (X, T), if $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in (X, T) [6].

Definition 2.5. A fuzzy topological space (X, T) is called a:

(i) *fuzzy perfectly disconnected space*, if for any two non -zero fuzzy sets λ and μ in X such that $\lambda \leq \mu'$, then $cl(\lambda) \leq (cl(\mu))'$ [19],

(ii) *fuzzy Baire space*, if $int(\bigvee_{i=1}^{\infty} \lambda_i) = 0$, where each λ_i is a fuzzy nowhere dense set in (X, T) [16],

(iii) *fuzzy second category space*, if $int(\bigvee_{i=1}^{\infty} \lambda_i) \neq 0$, where each λ_i is a fuzzy nowhere dense set in (X, T) [16],

(iv) *fuzzy hyperconnected space*, if every non null fuzzy open subset of (X, T) is fuzzy dense in (X, T) [20],

(v) *fuzzy DG_δ-space*, if each fuzzy dense (but not fuzzy open) set in (X, T) is a fuzzy G_δ-set in (X, T) [21],

(vi) *fuzzy nodef space*, if each fuzzy nowhere dense set is a fuzzy F_σ-set in (X, T) [21],

(vii) *fuzzy fraction dense space*, if for each fuzzy open set λ in (X, T), $cl(\lambda) = cl(\mu)$, where μ is a fuzzy F_σ-set in (X, T) [22],

(viii) *fuzzy extraresolvable space*, if whenever λ_i and $\lambda_j (i \neq j)$ are fuzzy dense sets in (X, T), then $\lambda_i \wedge \lambda_j$ is a fuzzy nowhere dense set in (X, T) [23],

(ix) *fuzzy strongly hyperconnected space*, if the following conditions hold :

(a) if λ is a fuzzy dense set in (X, T), then λ is a fuzzy open set in (X, T),

(b) if λ is a fuzzy open set in (X, T), then λ is a fuzzy dense set in (X, T) [24],

(x) *fuzzy ∂^* space*, if for each fuzzy G_δ-set λ in (X, T), $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in (X, T) [25],

(xi) *fuzzy strongly Baire space*, if $cl(\bigvee_{i=1}^{\infty} \lambda_i) = 1$, where each λ_i is a fuzzy strongly nowhere dense set in (X, T) [17],

(xii) *fuzzy submaximal space*, if for each fuzzy set λ in (X, T) such that $cl(\lambda) = 1, \lambda \in T$ [27].

Theorem 2.6 ([26]). *If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then λ is a fuzzy residual set in (X, T).*

Theorem 2.7 ([25]). *If λ is a fuzzy residual set in a fuzzy ∂^* space (X, T), then there exists a fuzzy simply* open set η in (X, T) such that $\eta < \lambda$.*

Theorem 2.8 ([16]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) *(X, T) is a fuzzy Baire space,*
(2) *$int(\lambda) = 0$, for every fuzzy first category set λ in (X, T),*
(3) *$cl(\mu) = 1$ for every fuzzy residual set μ in (X, T).*

Theorem 2.9 ([18]). *If λ is a fuzzy Baire set in a fuzzy Baire space (X, T), then λ is not a fuzzy dense set in (X, T).*

Theorem 2.10 ([16]). *If a fuzzy topological space (X, T) is a fuzzy Baire space, then (X, T) is a fuzzy second category space.*

Theorem 2.11 ([4]). *If λ is a fuzzy residual set in a fuzzy topological space (X, T), then there exists a fuzzy G_δ-set μ in (X, T) such that $\mu \leq \lambda$.*

Theorem 2.12 ([18]). *If λ is a fuzzy Baire set in a fuzzy topological space (X, T), then there exists a fuzzy Baire set β in (X, T) such that $\beta \leq \lambda$.*

Theorem 2.13 ([27]). *In any fuzzy topological space (X, τ), the following conditions are equivalent:*

(1) *(X, τ) is fuzzy hyperconnected space,*
(2) *every fuzzy subset of X is either fuzzy dense or fuzzy nowhere dense set in (X, τ).*

Theorem 2.14 ([21]). *If λ is a fuzzy dense and fuzzy open set in a fuzzy nodef space (X, T), then λ is a fuzzy residual set in (X, T).*

Theorem 2.15 ([22]). *If λ is a fuzzy open set in a fuzzy fraction dense space (X, T), then λ is not a fuzzy dense set in (X, T).*

Theorem 2.16 ([22]). *If λ is a fuzzy nowhere dense set in a fuzzy fraction dense and fuzzy DG_δ-space (X, T), then λ is a fuzzy first category set in (X, T).*

Theorem 2.17 ([22]). *If (X, T) is a fuzzy fraction dense and fuzzy DG_δ-space, then (X, T) is a fuzzy nodef space.*

Theorem 2.18 ([23]). *If λ is a fuzzy first category set in a fuzzy extraresolvable space (X, T), then λ is a fuzzy dense set in (X, T).*

Theorem 2.19 ([24]). *If λ is a fuzzy residual set in a fuzzy extraresolvable and fuzzy strongly hyperconnected space (X, T), then λ is a fuzzy nowhere dense set in (X, T).*

Theorem 2.20 ([28]). *If γ is a fuzzy co-σ-boundary set in a fuzzy topological space (X, T), then γ' is a fuzzy σ-boundary set in (X, T).*

Theorem 2.21 ([28]). *If λ is a fuzzy σ-boundary set in a fuzzy weakly Baire space (X,T), then $\lambda^{'}$ is a fuzzy dense and fuzzy G_δ-set in (X,T).*

Theorem 2.22 ([28]). *If (X,T) is a fuzzy weakly Baire space, then $int(\lambda) \wedge int(1-\lambda) = 0$, for any fuzzy set λ defined on X.*

Theorem 2.23 ([17]). *If δ is a fuzzy residual set in a fuzzy strongly Baire space (X,T), then $int(\delta) = 0$.*

3. Fuzzy simply* open sets and fuzzy Baire sets

Proposition 3.1. *If a fuzzy set λ in X is a fuzzy simply* open set in a fuzzy hyperconnected space (X,T), then $\lambda^{'}$ is a fuzzy nowhere dense set in (X,T).*

Proof. Suppose λ is a fuzzy simply* open set in (X,T). Then $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in (X,T). Thus $int(\mu) = \mu$ and $intcl(\delta) = 0$. Now $int(\delta) \leq intcl(\delta)$, implies that $int(\delta) = 0$ and $intcl(1-\lambda) = intcl(1-[\mu \vee \delta]) = 1 - clint[\mu \vee \delta]) \leq 1 - cl[int(\mu) \vee int(\delta)] = 1 - cl[\mu \vee int(\delta)]) = 1-[cl(\mu) \vee cl(0)] = 1-[cl(\mu) \vee 0] = 1 - cl(\mu)$. Since (X,T) is a fuzzy hyperconnected space, μ is a fuzzy dense set. So $cl(\mu) = 1$. Hence $intcl(1-\lambda) \leq 1 - 1 = 0$ and this implies that $intcl(1-\lambda) = 0$. Therefore $1-\lambda$ is a fuzzy nowhere dense set in (X,T). □

Proposition 3.2. *If each λ_i ($i = 1$ to ∞) is a fuzzy simply* open set in a fuzzy hyperconnected space (X,T), then $\bigwedge_{i=1}^{\infty} \lambda_i$ is a fuzzy residual set in (X,T).*

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X,T). Since (X,T) is a fuzzy hyperconnected space, by Proposition 3.1, each $1-\lambda_i$ is a fuzzy nowhere dense set in (X,T). Then $\bigvee_{i=1}^{\infty}(1-\lambda_i)$ is a fuzzy first category set in (X,T). Now $\bigvee_{i=1}^{\infty}(1-\lambda_i) = 1 - \bigwedge_{i=1}^{\infty} \lambda_i$ implies that $1 - \bigwedge_{i=1}^{\infty} \lambda_i$ is a fuzzy first category set. Thus $\bigwedge_{i=1}^{\infty} \lambda_i$ is a fuzzy residual set in (X,T). □

The following proposition shows that fuzzy simply* open sets in fuzzy hyperconnected spaces are generating fuzzy Baire sets.

Proposition 3.3. *If each λ_i ($i = 1$ to ∞) is a fuzzy simply* open set in a fuzzy hyperconnected space (X,T), then for any fuzzy open set μ in (X,T), $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is a fuzzy Baire set in (X,T).*

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X,T). Since (X,T) is a fuzzy hyperconnected space, by Proposition 3.2, $\bigwedge_{i=1}^{\infty} \lambda_i$ is a fuzzy residual set in (X,T). Then for any fuzzy open set μ in (X,T), $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is a fuzzy Baire set in (X,T). □

The following proposition shows that fuzzy Baire sets generated by simply* open sets in fuzzy hyperconnected spaces are not fuzzy dense sets.

Proposition 3.4. *If each λ_i ($i = 1$ to ∞) is a fuzzy simply* open set in a fuzzy hyperconnected space (X,T), then for any fuzzy open set μ in (X,T), $cl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) \neq 1$.*

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X, T). Since (X, T) is a fuzzy hyperconnected space, by Proposition 3.3, for any fuzzy open set μ in (X, T), $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is a fuzzy Baire set in (X, T). Now $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i) \leq \bigwedge_{i=1}^{\infty} \lambda_i$ implies that $cl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) \leq cl\,(\bigwedge_{i=1}^{\infty} \lambda_i)$. Assume that $cl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) = 1$. Then $1 \leq cl\,(\bigwedge_{i=1}^{\infty} \lambda_i)$, i.e., $cl\,(\bigwedge_{i=1}^{\infty} \lambda_i) = 1$ and thus for the fuzzy residual set $\bigwedge_{i=1}^{\infty} \lambda_i$, $cl\,(\bigwedge_{i=1}^{\infty} \lambda_i) = 1$. So by Theorem 2.8, (X, T) is a fuzzy Baire space. This means that the fuzzy Baire set $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ in the fuzzy Baire space (X, T) is a fuzzy dense set in (X, T), a contradiction by Theorem 2.9. Hence it must be that for any fuzzy open set μ in (X, T), $cl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) \neq 1$. □

Proposition 3.5. *If each* λ_i ($i = 1$ *to* ∞) *is a fuzzy simply* open set in a fuzzy hyperconnected space* (X, T), *then there exists a fuzzy closed set* δ *such that* $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i) \leq \delta$, *where* μ *is a fuzzy open set in* (X, T).

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X, T). Since (X, T) is a fuzzy hyperconnected space, by Proposition 3.4, for any fuzzy open set μ in (X, T), $cl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) \neq 1$. Then there exists a fuzzy closed set δ in (X, T) such that $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i) \leq \delta$. □

Proposition 3.6. *If each* λ_i ($i = 1$ *to* ∞) *is a fuzzy simply* open set in a fuzzy hyperconnected space* (X, T), *then for any fuzzy open set* μ *in* (X, T), $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ *is a fuzzy Baire set in* (X, T) *such that* $intcl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) = 0$.

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X, T). Since (X, T) is a fuzzy hyperconnected space, by Proposition 3.3, for any fuzzy open set μ, $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is a fuzzy Baire set in (X, T). Then by Proposition 3.4, $cl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) \neq 1$. Thus $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is not a fuzzy dense set in (X, T). So by Theorem 2.13, $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is a fuzzy nowhere dense set. Hence $intcl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) = 0$. □

Corollary 3.7. *If each* λ_i ($i = 1$ *to* ∞) *is a fuzzy simply* open set in a fuzzy hyperconnected space* (X, T), *then for any fuzzy open set* μ *in* (X, T), $int\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) = 0$.

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X, T). Since (X, T) is a fuzzy hyperconnected space, by Proposition 3.6, for any fuzzy open set μ, $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is a fuzzy nowhere dense set and $intcl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) = 0$. Now $int\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) \leq intcl\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i))$ implies that $int\,(\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)) = 0$. □

Remark 3.8. From the above corollary, one will have the following result :"Fuzzy Baire sets generated from the fuzzy simply* open sets in fuzzy hyperconnected spaces are having zero interior".

Proposition 3.9. *If each* λ_i ($i = 1$ *to* ∞) *is a fuzzy simply* open set in a fuzzy hyperconnected space* (X, T) *such that* $cl\,(\bigwedge_{i=1}^{\infty} \lambda_i) = 1$, *then* (X, T) *is a fuzzy second category space.*

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X, T). Since (X, T) is a fuzzy hyperconnected space, by Proposition 3.2, $\bigwedge_{i=1}^{\infty} \lambda_i$ is a fuzzy residual set in (X, T). Then by the hypothesis, $cl\,(\bigwedge_{i=1}^{\infty} \lambda_i) = 1$. Thus by theorem 2.8,

(X,T) is a fuzzy Baire space. So by Theorem 2.10, (X,T) is a fuzzy second category space. □

Proposition 3.10. *If each λ_i ($i = 1$ to ∞) is a fuzzy simply* open set in a fuzzy hyperconnected space (X,T), then there exists a fuzzy G_δ-set γ in (X,T) such that $\gamma \leq \mu \wedge (\wedge_{i=1}^{\infty}\lambda_i)$, where $\mu \in T$.*

Proof. Suppose each λ_i ($i = 1$ *to* ∞) is a fuzzy simply* open set in (X,T). Since (X,T) is a fuzzy hyperconnected space, by Proposition 3.2, $\bigwedge_{i=1}^{\infty} \lambda_i$ is a fuzzy residual set in (X,T). Then by Theorem 2.11, there exists a fuzzy G_δ-set δ in (X,T) such that $\delta \leq \bigwedge_{i=1}^{\infty} \lambda_i$. Now, for any fuzzy open set μ in (X,T), $\mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$ is a fuzzy Baire set in (X,T). Also $\delta \leq \bigwedge_{i=1}^{\infty} \lambda_i$ implies that $\mu\wedge\delta \leq \mu\wedge(\wedge_{i=1}^{\infty}\lambda_i)$. Let $\mu\wedge\delta = \gamma$. Then γ is a fuzzy G_δ-set in (X,T). Thus for the fuzzy Baire set $\mu\wedge(\bigwedge_{i=1}^{\infty} \lambda_i)$ (where $\mu \in T$), there exists a fuzzy G_δ-set γ in (X,T) such that $\gamma \leq \mu \wedge (\bigwedge_{i=1}^{\infty} \lambda_i)$. □

4. Fuzzy Baire sets in fuzzy perfectly disconnected spaces, fuzzy nodef spaces, fuzzy fraction dense spaces and fuzzy weakly Baire spaces

Proposition 4.1. *If λ is a fuzzy Baire set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy Baire set β in (X,T) such that $cl(\beta) \leq int(\lambda)$.*

Proof. Suppose λ is a fuzzy Baire set in (X,T). Then by Theorem 2.12, there exists a fuzzy Baire set β in (X,T) such that $\beta \leq \lambda$. Since (X,T) is a fuzzy perfectly connected space, for the fuzzy sets β and λ with $\beta \leq 1-(1-\lambda)$, $cl(\beta) \leq 1-cl(1-\lambda)$. Then $cl(\beta) \leq 1 - (1 - int(\lambda))$ and $cl(\beta) \leq int(\lambda)$. Thus $cl(\beta) \leq int(\lambda)$. □

Corollary 4.2. *If λ is a fuzzy Baire set in a fuzzy perfectly disconnected space (X,T), then there exists a fuzzy Baire set β in (X,T) such that $cl(\beta) \leq \lambda$.*

Proposition 4.3. *If λ is a fuzzy dense and fuzzy open set in a fuzzy nodef space (X,T), then for any fuzzy open set μ in (X,T), $\mu \wedge \lambda$ is a fuzzy Baire set in (X,T).*

Proof. Suppose λ is a fuzzy dense and fuzzy open set in (X,T). Since (X,T) is a fuzzy nodef space, by Theorem 2.14, λ is a fuzzy residual set in (X,T). Then for any fuzzy open set μ in (X,T), $\mu \wedge \lambda$ is a fuzzy Baire set in (X,T). □

Proposition 4.4. *If δ is a fuzzy dense set in a fuzzy submaximal and fuzzy nodef space (X,T), then for any fuzzy open set μ in (X,T), $\mu \wedge \delta$ is a fuzzy Baire set in (X,T).*

Proof. Suppose δ is a fuzzy dense set in (X,T). Since (X,T) is a fuzzy submaximal space, δ is a fuzzy open set in (X,T). This implies that δ is a fuzzy dense and fuzzy open set in the fuzzy nodef space (X,T). By Proposition 4.3, for a fuzzy open set μ in (X,T), $\mu \wedge \delta$ is a fuzzy Baire set in (X,T). □

Proposition 4.5. *If δ is a fuzzy open set in a fuzzy hyperconnected and fuzzy nodef space (X,T), then δ is a fuzzy Baire set in (X,T).*

Proof. Suppose δ is a fuzzy open set in (X,T). Since (X,T) is a fuzzy hyperconnected space, δ is a fuzzy dense set in (X,T). This implies that δ is a fuzzy dense and fuzzy open set in the fuzzy nodef space (X,T). By Proposition 4.3, for a fuzzy open set μ in (X,T), $\mu \wedge \delta$ is a fuzzy Baire set in (X,T). Let us take $\mu = \delta$. Then $\delta \wedge \delta$ is a fuzzy Baire set in (X,T), i.e., δ is a fuzzy Baire set in (X,T). □

Remark 4.6. In view of the above proposition, one will have the following result : "Fuzzy open sets in fuzzy hyperconnected and fuzzy nodef spaces are fuzzy Baire sets".

The following proposition shows that fuzzy Baire sets in fuzzy fraction dense spaces are not fuzzy dense sets.

Proposition 4.7. *If λ is a fuzzy Baire set in a fuzzy fraction dense space (X,T), then λ is not a fuzzy dense set in (X,T).*

Proof. Suppose λ is a fuzzy Baire set in (X,T). Then $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy residual set in (X,T). Since (X,T) is a fuzzy fraction dense space, by Theorem 2.15, μ is not a fuzzy dense set in (X,T). Now $cl(\lambda) = cl(\mu \wedge \delta) \leq cl(\mu) \wedge cl(\delta)$. Assume that $cl(\lambda) = 1$. Then $cl(\mu) \wedge cl(\delta) = 1$. Thus $cl(\mu) = 1$. This is a contradiction. So $cl(\lambda) \neq 1$. Hence λ is not a fuzzy dense set in (X,T). □

The following propositions show the means by which fuzzy Baire sets are generated from fuzzy nowhere dense sets, fuzzy dense and fuzzy open sets in fuzzy fraction dense and fuzzy DG_δ-spaces.

Proposition 4.8. *If λ is a fuzzy nowhere dense set in a fuzzy fraction dense and fuzzy DG_δ-space (X,T), then for any fuzzy open set μ, $\mu \wedge (1-\lambda)$ is a fuzzy Baire set in (X,T).*

Proof. Suppose λ is a fuzzy nowhere dense set in (X,T). Since (X,T) is a fuzzy fraction dense and fuzzy DG_δ-space, by Theorem 2.16, λ is a fuzzy first category set in (X,T). Then, $1-\lambda$ is a fuzzy residual set in (X,T). Thus for any fuzzy open set μ in (X,T), $\mu \wedge (1-\lambda)$ is a fuzzy Baire set in (X,T). □

Proposition 4.9. *If λ is a fuzzy dense and fuzzy open set in a fuzzy fraction dense and fuzzy DG_δ-space (X,T), then for any fuzzy open set μ in (X,T), $\mu \wedge \lambda$ is a fuzzy Baire set in (X,T).*

Proof. The proof follows from Proposition 4.3 and Theorem 2.17. □

Proposition 4.10. *If λ is a fuzzy residual set in a fuzzy extraresolvable space (X,T), then $int(\lambda) = 0$.*

Proof. Suppose λ is a fuzzy residual set in (X,T). Then $1-\lambda$ is a fuzzy first category set in (X,T). Since (X,T) is a fuzzy extraresolvable space, by Theorem 2.18, $1-\lambda$ is a fuzzy dense set in (X,T) and $cl(1-\lambda) = 1$. Then by Lemma 2.2, $1 - int(\lambda) = cl(1-\lambda) = 1$. Thus $int(\lambda) = 0$. □

Proposition 4.11. *If λ is a fuzzy Baire set in a fuzzy extraresolvable space (X,T), then $int(\lambda) = 0$.*

Proof. Suppose λ is a fuzzy Baire set in (X,T). The, $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy residual set in (X,T). Now $int(\lambda) = int(\mu \wedge \delta) = int(\mu) \wedge int(\delta) = \mu \wedge int(\delta)$. Since (X,T) is a fuzzy extraresolvable space, by

Proposition 4.10, for the fuzzy residual set δ, $int(\delta) = 0$. Then $int(\lambda) = int(\mu \wedge \delta) = int(\mu) \wedge int(\delta) = \mu \wedge int(\delta) = \mu \wedge 0 = 0$. □

Proposition 4.12. *If λ is a fuzzy Baire set in a fuzzy extraresolvable and fuzzy strongly hyperconnected space (X, T), then $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in (X, T).*

Proof. Suppose λ is a fuzzy Baire set in a fuzzy extraresolvable and fuzzy strongly hyperconnected space (X, T). Then $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy residual set in (X, T). Since (X, T) is a fuzzy extraresolvable space, by Theorem 2.19, δ is a fuzzy nowhere dense set in (X, T). Thus $\lambda = \mu \wedge \delta$, where $\mu \in T$ and δ is a fuzzy nowhere dense set in (X, T). □

Proposition 4.13. *If λ is a fuzzy Baire set in a fuzzy extraresolvable and fuzzy strongly hyperconnected space (X, T), then there exists a fuzzy simply* open set η in (X, T) such that $\lambda \leq \eta$.*

Proof. Suppose λ is a fuzzy Baire set in a fuzzy extraresolvable and fuzzy strongly hyperconnected space (X, T). Then by Proposition 4.12, $\lambda = \mu \wedge \delta$, where $\mu \in T$ and δ is a fuzzy nowhere dense set in (X, T). Now $\mu \wedge \delta \leq \mu \vee \delta$. Let $\eta = \mu \vee \delta$. Then η is a fuzzy simply* open set in (X, T). Thus there exists a fuzzy simply* open set η in (X, T) such that $\lambda \leq \eta$. □

Proposition 4.14. *If λ is a fuzzy Baire set in a fuzzy ∂^* space (X, T), then for any fuzzy open set μ in (X, T), there exists a fuzzy simply* open set η in (X, T) (where $\mu \in T$) such that $\mu \wedge \eta < \lambda$.*

Proof. Suppose λ is a fuzzy Baire set in a fuzzy ∂^* space (X, T). Then $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy residual set in (X, T). Since (X, T) is a fuzzy ∂^* space, by Theorem 2.7, for the fuzzy residual set δ, there exists a fuzzy simply* open set η in (X, T) such that $\eta < \delta$. Now, for the fuzzy open set $\mu, \mu \wedge \eta < \mu \wedge \delta = \lambda$. Thus for the fuzzy Baire set λ, there exists a fuzzy simply* open set η in (X, T) (where $\mu \in T$) such that $\mu \wedge \eta < \lambda$. □

Proposition 4.15. *If γ is a fuzzy co-σ-boundary set in a fuzzy weakly Baire space (X, T), then γ is a fuzzy residual set in (X, T).*

Proof. Suppose γ is a fuzzy co-σ-boundary set in a fuzzy weakly Baire space (X, T). Then by Theorem 2.20, $1 - \gamma$ is a fuzzy σ-boundary set in (X, T). Since (X, T) is a fuzzy weakly Baire space, by Theorem 2.21, $1 - (1 - \gamma)$ is a fuzzy dense and fuzzy G_δ-set in (X, T). Thus γ is a fuzzy dense and fuzzy G_δ-set in (X, T). So By Theorem 2.6, γ is a fuzzy residual set in (X, T). □

The following proposition gives a condition under which fuzzy Baire sets are generated from fuzzy co-σ-boundary sets in fuzzy weakly Baire spaces.

Proposition 4.16. *If γ is a fuzzy co-σ-boundary set in a fuzzy weakly Baire space (X, T), then for any fuzzy open set μ in (X, T), $\mu \wedge \gamma$ is a fuzzy Baire set in (X, T).*

Proof. Suppose γ is a fuzzy co-σ-boundary set in a fuzzy weakly Baire space (X, T). Then by Proposition 4.15, γ is a fuzzy residual set in (X, T). Thus for any fuzzy open set μ in (X, T), $\mu \wedge \gamma$ is a fuzzy Baire set in (X, T). □

In a fuzzy weakly Baire space, it seems that existence of a fuzzy co-σ-boundary set ensures a pair of disjoint fuzzy Baire sets. The following proposition establishes this result.

Proposition 4.17. *If (X, T) is a fuzzy weakly Baire space, then there exist fuzzy Baire sets λ and μ in (X, T) such that $\lambda \wedge \mu = 0$.*

Proof. Let γ be a fuzzy co-σ-boundary set in (X, T). Since (X, T) is a fuzzy weakly Baire space, by Theorem 2.22, for the fuzzy set γ, $int(\gamma) \wedge int(1-\gamma) = 0$. Now $int(\gamma)$ and $int(1-\gamma)$ are fuzzy open sets in (X, T). By Proposition 4.16, $int(\gamma) \wedge \gamma$ and $int(1-\gamma) \wedge \gamma$ are fuzzy Baire sets in (X, T). Let $\lambda = int(\gamma) \wedge \gamma$ and $\mu = int(1-\gamma) \wedge \gamma$. Then we have

$$\lambda \wedge \mu = [int(\gamma) \wedge \gamma] \wedge [int(1-\gamma) \wedge \gamma] = [int(\gamma) \wedge int(1-\gamma)] \wedge \gamma = 0 \wedge \gamma = 0.$$

Thus there exist fuzzy Baire sets λ and μ in a fuzzy weakly Baire space (X, T) such that $\lambda \wedge \mu = 0$. □

Proposition 4.18. *If λ is a fuzzy Baire set in a fuzzy strongly Baire space (X, T), then $int(\lambda) = 0$.*

Proof. Suppose λ is a fuzzy Baire set in a fuzzy strongly Baire space (X, T). Then $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy residual set in (X, T). Thus we get

$$int(\lambda) = int(\mu \wedge \delta) = int(\mu) \wedge int(\delta) = \mu \wedge int(\delta).$$

Since (X, T) is a fuzzy strongly Baire space, by Theorem 2.23, for the fuzzy residual set δ, $int(\delta) = 0$ in (X, T) . So we have

$$int(\lambda) = int(\mu \wedge \delta) = int(\mu) \wedge int(\delta) = \mu \wedge int(\delta) = \mu \wedge 0 = 0.$$

□

5. Conclusion

In this paper, the means by which fuzzy Baire sets are obtained from fuzzy simply*open sets in fuzzy hyperconnected spaces are discussed. It is established that fuzzy Baire sets generated from simply* open sets in fuzzy hyperconnected spaces are not fuzzy dense sets and are having zero interior. It is obtained that fuzzy Baire sets in fuzzy fraction dense spaces are not fuzzy dense sets. Also the conditions under which fuzzy Baire sets are generated from fuzzy nowhere dense sets, fuzzy dense and fuzzy open sets in fuzzy fraction dense and fuzzy DG_δ-spaces are obtained. A condition under which fuzzy Baire sets are generated from fuzzy co-σ-boundary sets in fuzzy weakly Baire spaces is also established. It is shown that existence of a fuzzy co-σ-boundary set in fuzzy weakly Baire spaces ensures the existence of a pair of disjoint fuzzy Baire sets. It is established that fuzzy open sets in fuzzy hyperconnected and fuzzy nodef spaces are fuzzy Baire sets. This work may be extended to study various forms of fuzzy Baireness in fuzzy soft topological spaces.

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G. Thangaraj (g.thangaraj@rediffmail.com)
Department of Mathematics, Thiruvalluvar University, Vellore-632 115, Tamilnadu, India

N. Raji (way2rajii@gmail.com)
Department of Mathemathics, K.M.G. College of Arts and Science, Gudiyattam-632602,Tamilnadu, India