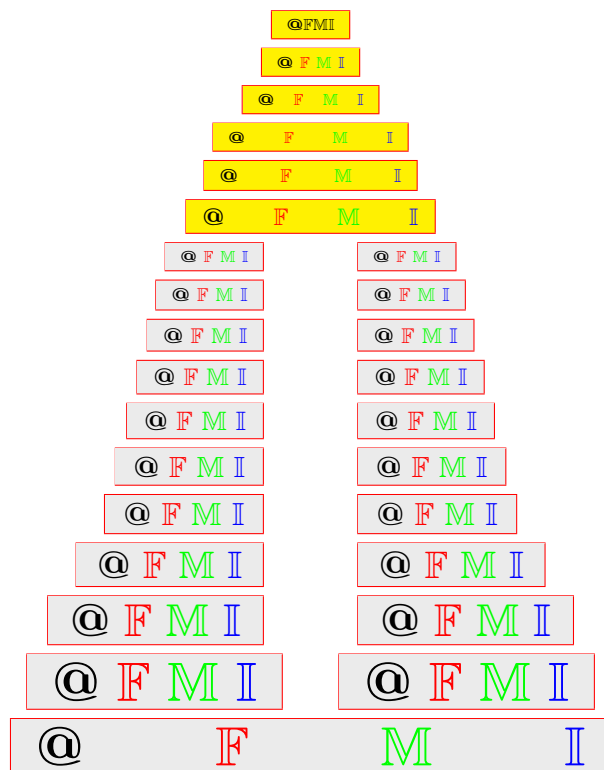


Fixed point theorems for contractive mappings in fuzzy metric space

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ABSTRACT. In this paper, we prove common fixed point theorems for continuous maps in partially ordered fuzzy metric spaces under (ψ, α, β) -weak contraction. In light of the (ψ, α, β) -weak contraction, we first establish the coincident point theorem and then show that if the maps are weakly compatible, this coincident point becomes a fixed point. Our result generalizes the result of Vetro et al. [1]. Examples are provided to demonstrate the results.

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1. INTRODUCTION AND PRELIMINARIES

Zadeh [2] introduced the notion of fuzzy sets in 1965 to provide a precise natural framework for mathematical modelling of non-probabilistic situations with vagueness and uncertainty. Many authors developed it further, including interesting applications of this theory in various fields. Kramosil and Michalek [3] defined fuzzy metric space to use this concept in topology and analysis, the formal definition is as follows:

Definition 1.1 (See [3]). A *fuzzy metric space* (in sense of Kramosil and Michalek) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that the following axioms hold: for any $x, y, z \in X$ and any $s, t > 0$,

- (i) $M(x, y, 0) = 0$,
- (ii) $M(x, y, t) = 1$ iff $x = y$,
- (iii) $M(x, y, t) = M(y, x, t)$,
- (iv) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous,
- (v) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.

We shall refer to these spaces as *KM-fuzzy metric spaces*. This concept was further modified by George and Veeramani [4] as follows.

Definition 1.2 (See [4]). A *fuzzy metric space* (in sense of George and Veeramani) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$ such that the following axioms hold: for any $x, y, z \in X$ and any $s, t > 0$,

- (i) $M(x, y, t) > 0$,
- (ii) $M(x, y, t) = 1$ iff $x = y$,
- (iii) $M(x, y, t) = M(y, x, t)$,
- (iv) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,
- (v) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.

Notice that the axiom (v) is a fuzzy version of triangular inequality. The value $M(x, y, t)$ can be thought of as a degree of nearness between x and y with respect to t and from the axiom (ii), we can relate the values 0 and 1 of a fuzzy metric to the notions of ∞ and 0 of a classical metric respectively.

Example 1.3. Let (X, d) be a metric space. We define mappings $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ and $M : X^2 \times (0, \infty) \rightarrow [0, 1]$ as follows: for any $a, b \in [0, 1]$, any $x, y \in X$ and each $t \in (0, \infty)$,

$$a * b = \min\{a, b\} \text{ and } M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space. It is called the *fuzzy metric space induced* by d .

Singh and Chauhan [5] introduced the concept of compatible maps and proved two common fixed point theorems in the fuzzy metric space.

Definition 1.4 ([5]). Let X be a non-empty set and let $f, g : X \rightarrow X$ be self maps. Then f and g on a fuzzy metric space $(X, M, *)$ are said to be *compatible*, if for all $t > 0$, $\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$ for some $z \in X$.

Jain and Singh [6] studied the notion of compatible map of type(A) in fuzzy metric spaces in 2007. After that, Cho [7] used compatible maps to prove fixed point theorems in fuzzy metric spaces.

Definition 1.5 (See [8]). Two self maps f and g on a fuzzy metric space $(X, M, *)$ are said to be *compatible of type (A)*, if $M(fgx_n, ggx_n, t) \rightarrow 1$ and $M(gfx_n, ffx_n, t) \rightarrow 1 \forall t > 0$, whenever $\{x_n\}$ is a sequence in X such that $gx_n, fx_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.

Example 1.6. Let $X = [0, 1]$ and $(X, M, *)$ be a complete fuzzy metric space. Define $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$, where d is the usual metric on X . Let f and g be two self maps on X defined by: for each $x \in [0, 1]$,

$$fx = \begin{cases} x, & x \in [0, \frac{1}{2}) \\ 1, & x \in (\frac{1}{2}, 1] \end{cases} \text{ and } gx = \begin{cases} 1 - x, & x \in [0, \frac{1}{2}) \\ 1, & x \in (\frac{1}{2}, 1] \end{cases}.$$

Consider the sequence $\{x_n\}$, where $\{x_n\} = \frac{1}{2} + \frac{1}{n}$. Clearly $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 1$. Also $M(fgx_n, ggx_n, t) \rightarrow 1$ and $M(gfx_n, ffx_n, t) \rightarrow 1 \forall t > 0$. Then f and g are compatible maps of type (A).

The concept of compatible maps of type (A) is more general than the concept of compatible maps in fuzzy metric space.

Example 1.7. Let $X = \mathbb{R}$, with the usual metric d . Define $f, g : \mathbb{R} \rightarrow \mathbb{R}$ by: $fx = x$ for all $x \in \mathbb{R}$ and

$$gx = \begin{cases} 1, & \text{if } x \text{ is not an integer} \\ 0, & \text{otherwise} \end{cases}.$$

Then for the sequence $\{x_n\} = \{1 \pm \frac{1}{n+1}\}$, we see that $fx_n \rightarrow 1$; $ffx_n \rightarrow 1$; $gx_n = ffx_n = gfx_n = 1$ but $ggx_n = 0$ as $n \rightarrow \infty$. This shows that the pair (f, g) is compatible but not compatible of type (A).

Jungck and Rhoades [9] proved that weakly compatible maps in metric space are compatible but not vice versa.

Definition 1.8 (See [10]). Let X be a non-empty set and let $f, g : X \rightarrow X$ be self maps. If there exist $x_0, y_0 \in X$ such that $x_0 = gy_0 = fy_0$, then y_0 is called a *point of coincidence* of g and f , and x_0 is called a *point of coincidence of g and f* . The maps g and f are said to be *weakly compatible*, if they commute at their coincidence points.

Khan et al. [11] introduced the idea of altering distance function in 1984. An altering function is a control function that changes the metric distance between two points to solve relatively new classes of fixed point problems. Indeed, certain altering distance function choices yield fixed point results. Since the triangular inequality does not apply, changing distance requires special techniques. Rhoades [12] proved intriguing fixed point theorems for ψ -weak contraction in complete metric space to address this issue. Because they are strictly related to Banach's fixed point theorem and other important results, ψ -weak contractions are important. Inspired by the idea of Rhoades [12], Vetro et al. [1] extended the notion of (ϕ, ψ) -weak contraction, and proved the common fixed point theorem for weakly compatible maps in setting of fuzzy metric space.

In 2010, Choudhary et al. [13] introduced concept of (ψ, α, β) -weak contraction in partial ordered metric space and proved the following result.

Theorem 1.9 (Theorem 2.1, [13]). Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f, g : X \rightarrow X$ be such that $f(X) \subseteq g(X)$, f is g -non-decreasing, $g(X)$ is closed and

$$(1.1) \quad \begin{aligned} \psi(d(fx, fy)) &\leq \alpha(d(gx, gy)) - \beta(d(gx, gy)) \\ &\text{for all } x, y \in X \text{ such that } gx \preceq gy, \end{aligned}$$

where $\psi, \alpha, \beta : [0, \infty) \rightarrow [0, \infty)$ are such that ψ is continuous and monotone non-decreasing, α is continuous, β is lower semi-continuous,

$$(1.2) \quad \psi(t) = 0 \text{ if and only if } t = 0, \alpha(0) = \beta(0) = 0$$

and

$$(1.3) \quad \psi(t) - \alpha(t) + \beta(t) > 0 \text{ for all } t > 0.$$

Also, if any non-decreasing sequence $\{x_n\}$ in X converges to z , then we assume

$$(1.4) \quad x_n \preceq z \text{ for all } n \geq 0$$

If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Motivated by the developments in this area, in this paper we establish a common fixed point theorem using the concept of (ψ, α, β) -weak contraction in fuzzy metric spaces. The formal definition of a newly introduced notion of (ψ, α, β) -weak contraction in fuzzy metric spaces is as follows.

Definition 1.10. Let $(X, M, *)$ be a fuzzy metric space and $g : X \rightarrow X$ be a map. A map $f : X \rightarrow X$ is called a (ψ, α, β) -weak contraction with respect to g , if there exist a function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and altering distance functions α and β such that

$$\psi \left(\frac{1}{M(fx, fy, t)} - 1 \right) \leq \alpha \left(\frac{1}{M(gx, gy, t)} - 1 \right) - \beta \left(\frac{1}{M(gx, gy, t)} - 1 \right)$$

holds for every $x, y \in X$ and each $t > 0$. If the map g is the identity map, then the map $f : X \rightarrow X$ is called a (ψ, α, β) -weak contraction.

Lemma 1.11 (Lemma 2, [14]). If f and g are either compatible, or compatible of type (A), then f and g are weakly compatible.

Definition 1.12 (See [9]). Suppose $(X, \preceq)$ is a partially ordered set and $f, g : X \rightarrow X$ are maps of X to itself. Then f is said to be g -non-decreasing, if for $x, y \in X, gx \preceq gy$ implies $fx \preceq fy$.

2. MAIN RESULTS

Theorem 2.1. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a fuzzy metric M on X such that $(X, M, *)$ is a complete fuzzy metric space. Let $f, g : X \rightarrow X$ be maps such that $f(X) \subseteq g(X)$. Suppose f is g -non-decreasing and continuous, g is continuous and $g(X)$ is closed and

$$(2.1) \quad \psi \left(\frac{1}{M(fx, fy, t)} - 1 \right) \leq \alpha \left(\frac{1}{M(gx, gy, t)} - 1 \right) - \beta \left(\frac{1}{M(gx, gy, t)} - 1 \right)$$

for all $x, y \in X$ such that $gx \preceq gy$, where $\psi, \alpha, \beta : [0, \infty) \rightarrow [0, \infty)$ are functions such that ψ is continuous and monotone non-decreasing, α is continuous, β is lower semi-continuous,

$$(2.2) \quad \psi(s) = 0 \text{ iff } s = 0, \alpha(0) = 0, \beta(0) = 0$$

$$(2.3) \quad \text{and } \psi(s) - \alpha(s) + \beta(s) > 0 \quad \forall s > 0$$

also if any non-decreasing sequence $\{x_n\}$ in X converges to z , then we may assume

$$(2.4) \quad x_n \preceq z \text{ for all } n \geq 0.$$

If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Proof. Suppose there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$. Since $f(X) \subseteq g(X)$, one can have $x_1 \in X$ such that $gx_1 = fx_0$. Then $gx_0 \preceq fx_0 = gx_1$. Since f is g -non-decreasing, we have $fx_0 \preceq fx_1$. In the similar vein, we can construct the sequence $\{x_n\}$ as

$$(2.5) \quad fx_n = gx_{n+1} \quad \forall \quad n \geq 1$$

for which

$$(2.6) \quad gx_0 \preceq fx_0 = gx_1 \preceq fx_1 = gx_2 \preceq fx_2 = gx_3 \cdots .$$

Now, we will show that $\{fx_n\}$ is a Cauchy sequence. If any two consecutive terms in the sequence $\{x_n\}$ are equal, i.e., if $fx_n = fx_{n+1}$ for some n , then $fx_n = gx_{n+1} = fx_{n+1}$. This shows that x_{n+1} is a coincident point and thus nothing to prove. So let us assume that

$$(2.7) \quad M(fx_{n-1}, fx_n, t) \neq 1 \quad \forall \quad n \geq 1.$$

Now, we shall show that $\left\{ \frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \right\}$ is a monotonic decreasing sequence of real numbers. Suppose to the contrary that, for some n ,

$$\left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1 \right) < \left(\frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \right).$$

Taking $x = x_n$ and $y = x_{n+1}$ in (2.1), using (2.5) and (2.6), and the monotone property of ψ , we have

$$(2.8) \quad \psi \left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1 \right) \leq \psi \left(\frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \right).$$

This gives

$$(2.9) \quad \begin{aligned} \psi \left(\frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \right) &\leq \alpha \left(\frac{1}{M(gx_n, gx_{n+1}, t)} - 1 \right) - \beta \left(\frac{1}{M(gx_n, gx_{n+1}, t)} - 1 \right) \\ &= \alpha \left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1 \right) - \beta \left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1 \right). \end{aligned}$$

The by using (2.3) along with the definition of fuzzy metric space, we get

$$M(fx_{n-1}, fx_n, t) = 1.$$

This is a contradiction to (2.7). Thus for all $n \geq 1$,

$$\frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \leq \frac{1}{M(fx_{n-1}, fx_n, t)} - 1.$$

It follows that

$$M(fx_{n-1}, fx_n, t) \leq M(fx_n, fx_{n+1}, t)$$

for all n , which implies that $M(fx_n, fx_{n+1}, t)$ is a non-decreasing sequence of positive real numbers in $(0, 1]$. So the sequence is convergent.

Let $\gamma(t) = \lim_{n \rightarrow \infty} M(fx_n, fx_{n+1}, t)$. Our aim is to show that $\gamma(t) = 1$ for all $t > 0$. If not, then there corresponds some $t > 0$ such that $\gamma(t) < 1$. To do this, taking

$n \rightarrow \infty$ and using lower semi-continuity of β and continuity of ψ and α in the inequality (2.9), we obtain

$$\psi\left(\frac{1}{\gamma(t)} - 1\right) - \alpha\left(\frac{1}{\gamma(t)} - 1\right) + \beta\left(\frac{1}{\gamma(t)} - 1\right) \leq 0.$$

This is a contradiction as $\psi(s) - \alpha(s) + \beta(s) > 0 \forall s > 0$. Then $\gamma(t) = 1$ implies $M(fx_n, fx_{n+1}, t) \rightarrow 1$ as $n \rightarrow \infty$. Thus $\{fx_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a complete fuzzy metric space, $\{fx_n\}$ converges in X . Since $g(X)$ is closed, by (2.5), there exists $z \in X$ such that

$$(2.10) \quad \lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} fx_n = gz.$$

Now, we shall show that z is a coincidence point of f and g . From (2.6), we get $\{gx_n\}$ is a non-decreasing sequence in X , by using (2.10) and condition of our theorem, we have

$$(2.11) \quad gx_n \preceq gz.$$

Putting $x = x_n$ and $y = z$ in (2.1). Then by virtue of (2.11), we have

$$\psi\left(\frac{1}{M(fx_n, fz, t)} - 1\right) \leq \alpha\left(\frac{1}{M(gx_n, gz, t)} - 1\right) - \beta\left(\frac{1}{M(gx_n, gz, t)} - 1\right).$$

Taking $n \rightarrow \infty$ in the above inequality along with using (2.2), (2.6) and (2.10), we get $M(gz, fz, t) = 1$, i.e.,

$$(2.12) \quad fz = gz.$$

This completes the proof. $\square$

Theorem 2.2. *If in Theorem 2.1, it is additionally assumed that*

$$(2.13) \quad gz \preceq ggz,$$

where z is the same as in (2.4) and f, g are weakly compatible, then f and g have a common fixed point in X .

Proof. Following the proof of Theorem 2.1, we obtain (2.10), that gives us a non-decreasing sequence $\{gx_n\}$ converging to gz . Then by (2.13), we have $gz \preceq ggz$. Since f and g are weakly compatible and by (2.12), one has $f gz = g fz$.

If we set

$$(2.14) \quad w = gz = fz,$$

then in view of (2.13)

$$gz \preceq ggz = gw.$$

Also,

$$(2.15) \quad fw = f gz = g fz = gw.$$

If $z = w$, then z is a common fixed point and if $z \neq w$, then by (2.1), we have

$$\begin{aligned} \psi\left(\frac{1}{M(gz, gw, t)} - 1\right) &= \psi\left(\frac{1}{M(fz, fw, t)} - 1\right) \\ &\leq \alpha\left(\frac{1}{M(gz, gw, t)} - 1\right) - \beta\left(\frac{1}{M(gz, gw, t)} - 1\right). \end{aligned}$$

From (2.3), $gz = gw$. Thus by (2.14) and (2.15), one has $w = gw = fw$. So f and g have a common fixed point. $\square$

Remark 2.3. One can notice that Theorem 2.1 is a generalized version of Theorem 1.9 from metric space to fuzzy metric space which is obtained by choosing $M(x, y, t)$ to be standard fuzzy metric and $t = 1$ in Theorem 2.1.

The following example illustrates Theorem 2.1.

Example 2.4. Let $X = [0, 1]$. Define a partial order ' $\preceq$ ' on X as $x \preceq y$ if and only if $x \geq y$ for $x, y \in X$. For $x, y \in X$, take the usual metric d defined by $d(x, y) = |x - y|$ and a fuzzy metric M defined by $M(x, y, t) = \frac{t}{t + d(x, y)}$. Let f, g be two self maps on X such that for all $x \in [0, 1]$,

$$fx = \frac{1}{6}x - \frac{1}{6}x^2 \text{ and } gx = x - \frac{1}{3}x^2.$$

Let $\psi, \alpha, \beta : [0, \infty) \rightarrow [0, \infty)$ be defined as $\psi(s) = s$,

$$\alpha x = \begin{cases} \frac{x^2}{2}, & \text{if } 0 \leq x \leq 2 \\ x, & \text{if } x > 2 \end{cases} \text{ and } \beta x = \begin{cases} \frac{x(x-1)}{2}, & \text{if } 0 \leq x \leq 2 \\ \frac{x}{2}, & \text{if } x > 2 \end{cases}.$$

Then for all $x, y \in [0, 1]$,

$$d(f(x), f(y)) = \left| \frac{1}{6}(x - y) - \frac{1}{6}(x^2 - y^2) \right| \text{ and}$$

$$d(g(x), g(y)) = \left| (x - y) - \frac{1}{3}(x^2 - y^2) \right|.$$

Clearly, $\psi(s) = 0$ iff $s = 0$. Also, for $x = 0$, $\alpha(0) = 0$ and $\beta(0) = 0$, and to verify the equation (2.3), we have the following cases:

Case I When $0 \leq s \leq 2$,

$$\psi(s) - \alpha(s) + \beta(s) = s - \frac{s^2}{2} + \frac{s^2}{2} - \frac{s}{2} > 0 \text{ for } s > 0.$$

Case II When $s > 2$,

$$\psi(s) - \alpha(s) + \beta(s) = s - s + \frac{s}{2} = \frac{s}{2} > 0.$$

It is clear that $\frac{1}{M(gx, gy, t)} - 1 = \frac{|gx - gy|}{t} \geq 0$ for all $t \in (0, \infty)$. To verify the inequality (2.1), again two cases arise:

Case I When $0 \leq \left(\frac{1}{M(gx, gy, t)} - 1 \right) \leq 2$,

$$\begin{aligned} & \alpha \left(\frac{1}{M(gx, gy, t)} - 1 \right) - \beta \left(\frac{1}{M(gx, gy, t)} - 1 \right) \\ &= \alpha \left(\frac{d(gx, gy)}{t} \right) - \beta \left(\frac{d(gx, gy)}{t} \right) \\ &= \frac{1}{t} \left(\frac{1}{2}(x - y) - \frac{1}{6}(x^2 - y^2) \right) \geq \frac{1}{t} \left(\frac{1}{6}(x - y) - \frac{1}{6}(x^2 - y^2) \right) \\ &= \psi \left(\frac{1}{M(fx, fy, t)} - 1 \right), \text{ i.e.,} \end{aligned}$$

$$\psi\left(\frac{1}{M(fx, fy, t)} - 1\right) \leq \alpha\left(\frac{1}{M(gx, gy, t)} - 1\right) - \beta\left(\frac{1}{M(gx, gy, t)} - 1\right) \text{ holds.}$$

Case II When $\frac{1}{M(gx, gy, t)} - 1 > 2$,

$$\begin{aligned} & \alpha\left(\frac{1}{M(gx, gy, t)} - 1\right) - \beta\left(\frac{1}{M(gx, gy, t)} - 1\right) \\ &= \alpha\left(\frac{d(gx, gy)}{t}\right) - \beta\left(\frac{d(gx, gy)}{t}\right) \\ &= \frac{1}{t} \left(\frac{1}{2}(x - y) - \frac{1}{6}(x^2 - y^2) \right) \geq \frac{1}{t} \left(\frac{1}{6}(x - y) - \frac{1}{6}(x^2 - y^2) \right) \\ &= \psi\left(\frac{1}{M(fx, fy, t)} - 1\right). \end{aligned}$$

Then the inequality (2.1) is satisfied. Thus with any choice of x_0 in $(0, 1)$, all the conditions of Theorem 2.1 are satisfied. Also, f and g are weakly compatible. Further g also satisfies (2.13). So Theorem 2.2 is also applicable on this example and $z = 0$ is a coincidence point as well as common fixed point of f and g .

Example 2.5. Let $X = \mathbb{N} \cup \{0\}$. We define a partial order ' $\preceq$ ' in X as $x \preceq y$ if and only if $x \geq y$ and $(y - x)$ is divisible by 2 for all $x, y \in \{2, 3, 4, \dots\}$ and $1 \preceq 0$, $2 \preceq 1$. We define metric d on X and a fuzzy metric M on $X^2 \times (0, \infty)$ define as follows: for any $x, y \in X$ and each $t \in (0, \infty)$,

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases} \text{ and } M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then we can easily prove that $(X, M, *)$ is a complete fuzzy metric space.

Let $f, g : X \rightarrow X$ be defined as

$$fx = \begin{cases} \frac{x}{3}, & \text{if } x > 3 \\ 0, & \text{if } x \in \{0, 1, 2, 3\} \end{cases} \text{ and } gx = \begin{cases} x - 1, & \text{if } x > 1 \\ 0, & \text{if } x \in \{0, 1\} \end{cases}.$$

Let $\psi, \alpha, \beta : [0, \infty) \rightarrow [0, \infty)$ be defined as $\psi(s) = s$, $\alpha(s) = s$, $\beta(s) = \frac{s}{6} \forall s \geq 0$. Here

$$\psi(s) - \alpha(s) + \beta(s) > 0 \quad s > 0$$

Without loss of generality assume that $x > y$ and verify the inequality (2.1).

If $x = 2$, then $y = 0$ as $y - x$ is divisible by 2. Thus $fx = 0$, $fy = 0$, $gx = x - 1$ and $gy = 0$. So the result is true.

If $x = 3$, then $y = 1$. Thus $fx = 0$, $fy = 0$, $gx = x - 1$ and $gy = 0$. Here also the following inequality holds:

$$\psi\left(\frac{1}{M(fx, fy, t)} - 1\right) \leq \alpha\left(\frac{1}{M(gx, gy, t)} - 1\right) - \beta\left(\frac{1}{M(gx, gy, t)} - 1\right).$$

So only two cases arise:

Case I If x is even and $x > 2$, then $y \geq 2$ and y has to be even. Thus $fx = \frac{x}{3}$, $fy = 0$ or $\frac{y}{3}$. So $d(fx, fy) = \frac{|x-y|}{3}$ or $\frac{x}{3}$. Also, $gx = x - 1$ and $gy = y - 1$. So $d(gx, gy) = |x - y|$. Furthermore, we have

$$\psi\left(\frac{1}{M(fx, fy, t)} - 1\right) = \frac{|x - y|}{3t} \text{ or } \frac{x}{3t} \quad (y = 2)$$

and

$$\alpha \left(\frac{1}{M(gx, gy, t)} - 1 \right) - \beta \left(\frac{1}{M(gx, gy, t)} - 1 \right) = \frac{|x-y|}{t} - \frac{|x-y|}{6t} = \frac{5}{6} \frac{|x-y|}{t}.$$

So $\frac{|x-y|}{3t} < \frac{5}{6} \frac{|x-y|}{t}$. Hence the condition holds.

Now, for $x > 2$ and $y = 2$, clearly, $fy = 0$ and $\frac{x}{3t} < \frac{5}{6} \frac{|x-y|}{t}$, i.e.,

$$\psi \left(\frac{1}{M(fx, fy, t)} - 1 \right) \leq \alpha \left(\frac{1}{M(gx, gy, t)} - 1 \right) - \beta \left(\frac{1}{M(gx, gy, t)} - 1 \right) \text{ holds.}$$

Case II If x is odd and $x > 3$, then $y \geq 3$. Thus

$$fx = \frac{x}{3}, fy = 0 \text{ or } \frac{y}{3} \text{ and } gx = x-1, gy = y-1.$$

Again, in the similar manner, we calculate

$$\psi \left(\frac{1}{M(fx, fy, t)} - 1 \right) = \frac{|x-y|}{3t} \text{ or } \frac{x}{3t} (y=2)$$

or

$$\alpha \left(\frac{1}{M(gx, gy, t)} - 1 \right) - \beta \left(\frac{1}{M(gx, gy, t)} - 1 \right) = \frac{5}{6} \frac{|x-y|}{t}.$$

So the following inequality holds:

$$\psi \left(\frac{1}{M(fx, fy, t)} - 1 \right) \leq \alpha \left(\frac{1}{M(gx, gy, t)} - 1 \right) - \beta \left(\frac{1}{M(gx, gy, t)} - 1 \right).$$

Moreover, $fx = 0$ and $gx = 0$ at $x = 0$. Hence '0' is a common fixed point of f and g .

Remark 2.6. One may notice that in Theorem 2.1 if g is assumed to be piecewise continuous, then the result does not hold at the point of discontinuity of the function g .

Remark 2.7. By considering Lemma 1.11, the conclusion of Theorem 2.2 still holds if f and g are compatible, compatible of type (A).

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REFERENCES

- [1] C. Vetro, D. Gopal and M. Imdad, Common fixed point theorems for (ϕ, φ) -weak contractions in fuzzy metric spaces, Indian J. Math. 52 (2010) 573–590.
- [2] L. A. Zadeh, Fuzzy sets, Information and Control 89 (1965) 338–353.
- [3] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975) 336–344.
- [4] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.
- [5] B. Singh and M. S. Chouhan, Common fixed points of compatible maps in fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 471–475.
- [6] A. Jain and B. Singh, A fixed point theorem for compatible mappings of type (A) in fuzzy metric space, Acta Ciencia Indica, XXXIII (2) (2007) 339–346.

- [7] S. H. Cho, On common fixed point theorems in fuzzy metric spaces, J. Appl. Math. Comput. 20 (1-2) (2006) 523–533.
- [8] G. Jungck, P. P. Murthy and Y. J. Cho, Compatible mappings of type (A) and common fixed points Math. Japon. 38 (1993) 381–390.
- [9] G. Jungck and B. E. Rhoades, Fixed point for set valued function without continuity, Indian J. Pure Appl. Math. 29 (3) (1998) 227–238.
- [10] B. Singh and S. Jain, Weak compatibility and fixed point theorems in fuzzy metric spaces, hGanita 56 (2) (2005) 167–176.
- [11] M. S. Khan, M. Swaleh and S. Sessa, Fixed points theorems by altering distances between the points, Bull. Aust. Math. Soc. 30 (1984) 1–9.
- [12] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (2001) 2683–2693.
- [13] B. S. Choudhury and A. Kundu, (ψ, α, β) -weak contractions in partially ordered metric spaces, Appl. Math. Lett. 25 (2012) 6–10.
- [14] A. Djoudi and L. Nisse, Gregus type fixed points for weakly compatible maps, Bull. Belg. Math. Soc. Simon Stevin 10 (2003) 369–378.

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