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# Square root fuzzy sub-implicative ideals of $KU$-algebras

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Abstract. In real life problems, square roots are used in finance (rates of return over 2 years), normal distributions (probability density functions), lengths and distances (Pythagorean Theorem), quadratic formula (height of falling objects), radius of circles, simple harmonic motion (pendulums and springs), and standard deviation. There are many reasons from which we inspire to explore further algebraic structure in a fuzzy setting. The following are the key reasons from which we have motivated: It is seen that the use of fuzzy sets is more convenient in real life problem than ordinary sets, and so it is important in the case of algebraic structures. As a result, an effort has been made to further examine square root structure in a fuzzy situation. In this paper,we consider the square root fuzzy sub-implicative (sub-commutative) ideals in $KU$-algebras, and investigate some related properties. We give conditions for a square root fuzzy of ideal to be a square root fuzzy sub-implicative(sub-commutative) ideal. We show that any square root fuzzy sub-implicative (sub-commutative) ideal is a square root fuzzy ideal, but the converse is not true. Using a level set of a fuzzy set in a $KU$-algebra, we give a characterization of a square root fuzzy sub-implicative (sub-commutative) ideal.

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Keywords: $KU$-algebras, Square root fuzzy sub-implicative (sub-commutative) ideal.

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## 1. Introduction

$BCK$-algebras form an important class of logical algebras introduced by Iséki [1, 2, 3] and was extensively investigated by several researchers. It is an important way to research the algebras by its ideals. The notions of ideals in $BCK$-algebras and positive implicative ideals in $BCK$-algebras (i.e., Iséki's implicative ideals) were introduced by Iséki [1, 2, 3]. The notions of commutative (sub-commutative) ideals in $BCK$-algebras, positive implicative and implicative (sub-implmicative) ideals in $BCK$-algebras were introduced by [4, 5, 6, 7, 8, 9, 10, 11]. Zadeh [12]introduced the notion of fuzzy sets. At present, this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, Xi [13] applied this concept to $BCK$-algebras, and he introduced the notion of fuzzy sub-algebras (ideals) of $BCK$-algebras with respect to minimum, and since then Jun et al. [14, 15] studied fuzzy ideals. Prabpayak and Leerawat [16, 17] introduced a new algebraic structure which is called $KU$-algebra. They gave the concept of homomorphisms of $KU$-algebras and investigated some related properties. Mostafa et al. [18] introduced the notion of fuzzy $KU$-ideals of $KU$-algebras and then they investigated several basic properties which are related to fuzzy $KU$-ideals. The idea of sub implicative ideal was introduced by Liu and Meng [6] and they established the concepts of sub implicative ideals and sub commutative ideals in $BCI$-algebras and investigated some of their properties. Jun [19] considered the fuzzification of sub-implicative ideals in $BCI$-algebras and investigated some related properties. Mostafa et al. [20] introduced the notion of sub implicative(sub-commutative) ideals of $KU$-algebras and investigated of their properties.

In this paper, the notion of square root fuzzy sub-implicative (sub-commutative) ideals of $KU$-algebras are introduced and then the several basic properties are investigated.

## 2. Preliminaries

We will recall some known concepts related to $KU$-algebras and fuzzy sets in a nonempty set $X$ from the literature which will be helpful in further study of this article.

**Definition 2.1** ([16, 17])**.** An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *$KU$-algebra*, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,

$(KU_1)$ $(x * y) * [(y * z) * (x * z)] = 0$,
$(KU_2)$ $x * 0 = 0$,
$(KU_3)$ $0 * x = x$,
$(KU_4)$ $x * y = 0$ and $y * x = 0$ imply $x = y$,
$(KU_5)$ $x * x = 0$.

On a $KU$-algebra $(X, *, 0)$, we can define a binary relation $\leq$ on $X$ as follows: for any $x,\ y \in X$,

$$x \leq y \Leftrightarrow y * x = 0.$$

Then we can easily see that a $KU$-algebra $(X, *, 0)$ satisfies the following conditions: for any $x,\ y,\ z \in X$,

(1) $(y * z) * (x * z) \leq x * y$,

(2) $0 \leq x$,
(3) $x \leq y$ and $y \leq x$ imply $x = y$,
(4) $x \leq x$.

For any elements $x$ and $y$ of a $KU$-algebra $X$ and each $n \in \mathbb{N}$, $y * x^n$ define by

$$y * x^n = (\cdots((y * x) * x) * \cdots) * x \ (n \text{ times}).$$

It is clear that $y * x^3 = y * x$ for any $x,\ y \in X$.

**Result 2.2** (See Lemmas 2.5, 2.6 and 2.7, [18]). *In a $KU$-algebra $X$, the following conditions are satisfied: for any $x,\ y,\ z \in X$,*
(1) $x \leq y$ *implies* $y * z \leq x * z$,
(2) $x * (y * z) = y * (x * z)$,
(3) $(y * x) * x \leq y$.

**Proposition 2.3.** *In a $KU$-algebra $X$, the followings hold: for any $x,\ y,\ z \in X$,*
(1) $(z * y) * (z * x) \leq y * x$,
(2) $x \leq y$ *implies* $z * x \leq z * y$,
(3) $y * x \leq x$,
(4) $x * y^2 \leq x$,
(5) $(x * y) * (y * x^2) \leq x * y^2$,
(6) $[z * ((x * y) * (y * x^2))] * [(x * y) * (y * x^2)] \leq z$.

*Proof.* (1) Let $x,\ y,\ z \in X$. Then by Result 2.2 (2), $(KU_1)$ and $(KU_5)$, we have

$$(y * x) * [(z * y) * (z * x)] = (z * y) * [(y * x) * (z * x)] \leq (z * y) * (z * y) = 0.$$

Thus by $(KU_2)$ and $(KU_4)$, $(z * y) * (z * x) \leq y * x$.

(2) Let $x,\ y \in X$. Then by (1), $(z * y) * (z * x) \leq y * x = 0$. Thus by $(KU_3)$ and $(KU_4)$, $(z * y) * (z * x) = 0$. So $z * x \leq z * y$.

(3) The proof follows from Result 2.2 (2), $(KU_5)$ and $(KU_2)$.

(4) Suppose $x \leq y$. Then we get

$$\begin{aligned} x * y^2 &= (x * y) * y \\ &= (x * y) * (0 * y) \ [\text{By } (KU_2)] \\ &\leq 0 * x \ [\text{By } (KU_1)] \\ &= x. \ [\text{By } (KU_3)] \end{aligned}$$

(5) Let $x,\ y \in X$. Then we have

$$\begin{aligned} (x * y) * (y * x^2) &= (x * y) * [(y * x) * x] \\ &= (y * x) * [(x * y) * x] \ [\text{By Result 2.2 (2)}] \\ &\leq (x * y) * y \ [\text{By } (KU_1)] \\ &= x * y^2. \end{aligned}$$

(6) Let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} &[z * ((x * y) * (y * x^2))] * [(x * y) * (y * x^2)] \\ &= [(x * y) * (z * (y * x^2))] * [(x * y) * (y * x^2)] \ [\text{By Result 2.2 (2)}] \\ &\leq [z * (y * x^2)] * (y * x^2) \ [\text{By (2)}] \\ &= [z * (y * x^2)] * [0 * (y * x^2)] \ [\text{By } (KU_3)] \\ &\leq 0 * z \ [\text{By Result 2.2 (2)}] \\ &= z. \ [\text{By } (KU_3)] \end{aligned}$$

□

**Definition 2.4** ([16, 17])**.** Let $I$ be a nonempty subset of a $KU$-algebra $X$. Then $I$ is called an *ideal* of $X$, if it satisfies the following axioms: for any $x,\ y \in X$,
 $(I_1)$ $0 \in I$,
 $(I_2)$ if $x * y \in I$ and $x \in I$, then $y \in I$.

**Definition 2.5** ([18])**.** Let $I$ be a nonempty subset of a $KU$-algebra $X$. Then $I$ is called a *$KU$-ideal* of $X$, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
 $(I_1)$ $0 \in I$,
 $(I_2)$ if $x * (y * z) \in I$ and $y \in I$, then $x * z \in I$.

**Definition 2.6** ([21])**.** A $KU$-algebra $X$ is said to be *implicative*, if for any $x,\ y \in X$,

$$x * y^2 = (x * y) * (y * x^2).$$

**Definition 2.7** ([21])**.** A $KU$-algebra $X$ is said to be *positive implicative*, if for any $x,\ y,\ z \in X$,

$$(z * x) * (z * y) = z * (x * y).$$

**Definition 2.8** ([21])**.** A $KU$-algebra $X$ is said to be *commutative*, if for any $x,\ y \in X$,

$$x \leq y \text{ implies } x * y^2 = x.$$

**Result 2.9** (See Theorem 2.16, [22])**.** *Let $X$ be a $KU$-algebra. Then $X$ is implicative if and only if it is positive implicative and commutative.*

**Definition 2.10** ([20])**.** Let $I$ be a nonempty subset of a $KU$-algebra $X$. Then $I$ is called a *sub-implicative ideal* of $X$, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
 (i) $0 \in I$,
 (ii) $z * [(x * y) * (y * x^2)] \in I$ and $z \in I$ imply $x * y^2 \in I$.

**Definition 2.11** ([20])**.** Let $I$ be a nonempty subset of a $KU$-algebra $X$. Then $I$ is called a *$KU$-positive implicative ideal* of $X$, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
 (i) $0 \in I$,
 (ii) $z * (x * y) \in I$ and $z * x \in I$ imply $z * y^2 \in I$.

**Definition 2.12** ([20])**.** Let $I$ be a nonempty subset of a $KU$-algebra $X$. Then $I$ is called a *sub-commutative ideal* of $X$, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
 (i) $0 \in I$,
 (ii) $z * [(y * x^2) * y^2] \in I$ and $z \in I$ imply $y * x^2 \in I$.

**Definition 2.13** ([18])**.** Let $I$ be a nonempty subset of a $KU$-algebra $X$. Then $I$ is called a *kp-ideal* of $X$, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
 (i) $0 \in I$,
 (ii) $(z * y) * (z * x) \in I$ and $y \in I$ imply $x \in I$.

For a nonempty set $X$, a mapping $A : X \to [0, 1]$ is called a *fuzzy set* in $X$ (See [12]). For any $x,\ y \in [0, 1]$, $x \wedge y$ and $x \vee y$ are defined respectively as follows:

$$x \wedge y = min\{x, y\} \text{ and } x \vee y = max\{x, y\}.$$

**Definition 2.14** ([18])**.** Let $A$ be a fuzzy set in a $KU$-algebra $X$. Then $A$ is called a *fuzzy sub-algebra* of $X$, if for any $x,\ y \in X$,

$$A(x * y) \geq A(x) \wedge A(y).$$

**Definition 2.15** ([18])**.** Let $A$ be a fuzzy set in a $KU$-algebra $X$. Then $A$ is called a *fuzzy ideal* of $X$, if it satisfies the following axioms: for any $x,\ y \in X$,
$(F_1)$ $A(0) \geq A(x)$,
$(F_2)$ $A(y) \geq A(x * y) \wedge A(x)$.

## 3. Square root fuzzy sub-implicative ideals

**Definition 3.1.** Let $X$ be a nonempty set. Then a mapping $\sqrt{A} = (\sqrt{A}^{(+)}, \sqrt{A}^{(-)}) : X \to [0,1] \times [-1,0]$ is called a *square root fuzzy set* (briefly, SRFS) in $X$, where $A$ is a fuzzy set in $X$. In fact, for each $x \in X$,

$$\sqrt{A}(x) = (\sqrt{A(x)}^{(+)}, \sqrt{A(x)}^{(-)}) = (\sqrt{A(x)}, -\sqrt{A(x)}).$$

The *whole square root fuzzy set* and the *empty square root fuzzy set* in $X$, denoted by $\sqrt{1}$ and $\sqrt{0}$, defined by for each $x \in X$,

$$\sqrt{1}(x) = (1, -1),\ \sqrt{0}(x) = (0, 0).$$

We will denote the set of all square root fuzzy set in $X$ as $SRF(X)$.

**Example 3.2.** Let $X = \{0, a, b, c\}$ be a set and consider the mapping $\sqrt{A} = (\sqrt{A}^{(+)}, \sqrt{A}^{(-)}) : X \to [0,1] \times [-1,0]$ defined as follows:

$$A(0) = 0.06,\ A(a) = 0.05,\ A(b) = 0.04,\ A(c) = 0.03.$$

Then we have

$$\sqrt{A}(0) = (\sqrt{A(0)}, -\sqrt{A(0)}) = (0.245, -0.245),$$
$$\sqrt{A}(a) = (\sqrt{A(a)}, -\sqrt{A(a)}) = (0.224, -0.224),$$
$$\sqrt{A}(b) = (\sqrt{A(b)}, -\sqrt{A(b)}) = (0.2, -0.2),$$
$$\sqrt{A}(c) = (\sqrt{A(c)}, -\sqrt{A(c)}) = (0.173, -0.173).$$

Thus $\sqrt{A} \in SRF(X)$.

Now, we list some concepts related to square root fuzzy set (for examples, the inclusion between two square root fuzzy sets ,the complement of a square root fuzzy set, the intersection of square root fuzzy set and the union of a square root fuzzy set.

**Definition 3.3.** Let $X$ be nonempty set and $\sqrt{A},\ \sqrt{B} \in SRF(X)$.
(i) We say that $\sqrt{A}$ *is a subset of* $\sqrt{B}$, denoted by $\sqrt{A} \subset \sqrt{B}$, if for each $x \in X$,

$$\sqrt{A(x)}^{(+)} \leq \sqrt{B(x)}^{(+)},\ \sqrt{A(x)}^{(-)} \geq \sqrt{B(x)}^{(-)}.$$

(ii) The *complement* of $\sqrt{A}$, denoted by $\sqrt{A}^c = (\sqrt{A}^{(+)^c}, \sqrt{A}^{(-)^c})$, is a SRFS in $X$ defined as follows: for each $x \in X$,

$$\sqrt{A(x)}^{(+)^c} = 1 - \sqrt{A(x)}^{(+)},\ \sqrt{A(x)}^{(-)^c} = -1 - \sqrt{A(x)}^{(-)}.$$

(iii) The *intersection* of $\sqrt{A}$ and $\sqrt{B}$, denoted by $\sqrt{A}\cap\sqrt{B}$, is a SRFS in $X$, defined as follows: for each $x \in X$,

$$(\sqrt{A}\cap\sqrt{B})(x) = (\sqrt{A(x)}^{(+)} \wedge \sqrt{B(x)}^{(+)}, \sqrt{A(x)}^{(-)} \vee \sqrt{B(x)}^{(-)}).$$

(iv) The *union* of $\sqrt{A}$ and $\sqrt{B}$, denoted by $\sqrt{A}\cup\sqrt{B}$, is a SRFS in $X$, defined as follows: for each $x \in X$,

$$(\sqrt{A}\cup\sqrt{B})(x) = (\sqrt{A(x)}^{(+)} \vee \sqrt{B(x)}^{(+)}, \sqrt{A(x)}^{(-)} \wedge \sqrt{B(x)}^{(-)}).$$

It is obvious that $\sqrt{A}^{cc} = \sqrt{A}$.

**Definition 3.4.** Let $X$ be a $KU$-algebra and let $\sqrt{0} \neq \sqrt{A} \in SRF(X)$. Then $\sqrt{A}$ is called a *square root fuzzy ideal* (briefly, SRFI) of $X$, if it satisfies the following axioms: for any $x,\ y \in X$,

(SRFI$_1$) $\sqrt{A(0)}^{(+)} \geq \sqrt{A(x)}^{(+)},\ \sqrt{A(0)}^{(-)} \leq \sqrt{A(x)}^{(-)}$,
(SRFI$_2$) $\sqrt{A(y)}^{(+)} \geq \sqrt{A(x*y)}^{(+)} \wedge \sqrt{A(x)}^{(+)}$,
(SRFI$_3$) $\sqrt{A(y)}^{(-)} \leq \sqrt{A(x*y)}^{(-)} \vee \sqrt{A(x)}^{(-)}$.

**Definition 3.5.** Let $X$ be a $KU$-algebra and let $\sqrt{0} \neq \sqrt{A} \in SRF(X)$. Then $\sqrt{A}$ is called a *square root fuzzy sub-implicative ideal* (briefly, SRFSII) of $X$, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,

(SRFI$_1$) $\sqrt{A(0)}^{(+)} \geq \sqrt{A(x)}^{(+)},\ \sqrt{A(0)}^{(-)} \leq \sqrt{A(x)}^{(-)}$,
(SRFSII$_1$) $\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(z*[(x*y)*(y*x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)}$,
(SRFSII$_2$) $\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(z*[(x*y)*(y*x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)}$.

It is obvious that $\sqrt{0}$ and $\sqrt{1}$ are SRFIs and SRFSIIs of $X$ respectively.

**Example 3.6.** Let $X$ be the set with the operation $*$ given by the table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 0 |
| 1 | 1 | 0 | 2 | 0 |
| 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 2 | 1 | 0 |

Table 3.1

Then clearly, $(X, *, 0)$ is a $KU$-algebra. Define the SRFS $\sqrt{A}$ in $X$ as follows:

$$A(0) = 0.7,\ A(1) = 0.05,\ A(2) = 0.04,\ A(3) = 0.03.$$

Then we have

$$\sqrt{A}(0) = (0.837, -0.837),\ \sqrt{A}(1) = (0.707, -0.707),$$

$$\sqrt{A}(2) = (0.632, -0.632),\ \sqrt{A}(3) = (0.548, -9.548).$$

Thus by the routine calculation, we can check that $\sqrt{A}$ is an SRFSII of $X$.

**Proposition 3.7.** *Every SRFSII of a $KU$-algebra $X$ is order reversing*

*Proof.* Let $\sqrt{A}$ be an SRFSII of $X$ and let $x,\ y,\ z \in X$. Suppose $x \leq z$. Then clearly, $z * x = 0$. By (SRFI$_1$), $\sqrt{A(0)}^{(+)} \geq \sqrt{A(x)}^{+}$ and $\sqrt{A(0)}^{(-)} \leq \sqrt{A(x)}^{-}$. By (SRFSII$_1$), we get

$$\sqrt{A(x * y^2)}^{(+)} \geq \sqrt{A(z * [(x * y) * (y * x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)}.$$

Let $y = x$. Then $\sqrt{A(x * x^2)}^{(+)} \geq \sqrt{A(z * [(x * x) * (x * x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)}$. Thus by the axioms ($KU_3$) and ($KU_5$), we have

$$\sqrt{A(x)}^{(+)} \geq \sqrt{A(z * x)}^{(+)} \wedge \sqrt{A(z)}^{+} = \sqrt{A(0)}^{(+)} \wedge \sqrt{A(z)}^{(+)} = \sqrt{A(z)}^{(+)}.$$

On the other hand, by (SRFSII$_2$), we have

$$\sqrt{A(x * y^2)}^{(-)} \leq \sqrt{A(z * [(x * y) * (y * x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Let $y = x$. Then $\sqrt{A(x * x^2)}^{(-)} \leq \sqrt{A(z * [(x * x) * (x * x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)}$. Thus we get

$$\sqrt{A(x)}^{(-)} \leq \sqrt{A(z * x)}^{(-)} \vee \sqrt{A(z)}^{(-)} = \sqrt{A(0)}^{(-)} \vee \sqrt{A(z)}^{(-)} = \sqrt{A(z)}^{(-)}.$$

This completes the proof. □

**Lemma 3.8.** *Let $\sqrt{A}$ be an SRFSII of a KU-algebra $X$ and let $x,\ y,\ z \in X$. If $z * x \leq y$ or $y * x \leq z$, then*

$$\sqrt{A(x)}^{(+)} \geq \sqrt{A(y)}^{(+)} \wedge \sqrt{A(z)}^{(+)},\ \sqrt{A(x)}^{(-)} \leq \sqrt{A(y)}^{(-)} \wedge \sqrt{A(z)}^{(-)}.$$

*Proof.* Suppose $z * x \leq y$. Then by Proposition 3.7, we have

$$\sqrt{A(z * x)}^{(+)} \geq \sqrt{A(y)}^{(+)},\ \sqrt{A(z * x)}^{(-)} \leq \sqrt{A(y)}^{(-)}. \tag{3.1}$$

Let $y = x$. Then by (SRFSII$_1$), we get

$$\sqrt{A(x * x^2)}^{(+)} \geq \sqrt{A(z * [(x * x) * (x * x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)}.$$

Thus by (3.1), we have

$$\sqrt{A(x)}^{(+)} \geq \sqrt{A(z * x)}^{(+)} \wedge \sqrt{A(z)}^{(+)} = \sqrt{A(y)}^{(+)} \wedge \sqrt{A(z)}^{(+)}.$$

Similarly, by (SRFSII$_2$) and (3.1), we get

$$\sqrt{A(x)}^{(-)} \leq \sqrt{A(z * x)}^{(-)} \vee \sqrt{A(z)}^{(-)} = \sqrt{A(y)}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

If $y * x \leq z$, then by the similar arguments, we obtain the same consequences. This completes the proof. □

**Lemma 3.9.** *Let $X$ be a implicative KU-algebra. Then every SRFI of $X$ is an SRFSII of $X$.*

*Proof.* Let $\sqrt{A}$ be an SRFI of $X$ and let $x,\ y \in X$. Then by Definition 3.4, we have: for any $x,\ y \in X$,

(SRFI$_1$) $\sqrt{A(0)}^{(+)} \geq \sqrt{A(x)}^{(+)},\ \sqrt{A(0)}^{(-)} \leq \sqrt{A(x)}^{(-)}$,

(SRFI$_2$) $\sqrt{A(y)}^{(+)} \geq \sqrt{A(x * y)}^{(+)} \wedge \sqrt{A(x)}^{(+)}$,

(SRFI$_3$) $\sqrt{A(y)}^{(-)} \leq \sqrt{A(x*y)}^{(-)} \vee \sqrt{A(x)}^{(-)}$.
By substituting $x*y^2$ for $y$ in (SRFSII$_1$) and (SRFSII$_2$), we get

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A[(z*(x*y^2)]}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A[(z*(x*y^2)]}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Since $X$ is implicative, $x*y^2 = (x*y)*(y*x^2)$. So we have

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(z*[(x*y)*(y*x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(z*[(x*y)*(y*x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Hence $\sqrt{A}$ is an SRFSII of $X$. This completes the proof. □

**Proposition 3.10.** *Let $X$ be a KU-algebra and let $\sqrt{A} \in SRF(X)$. If $\sqrt{A}$ satisfies the conditions* (SRFSII$_1$) *and* (SRFSII$_2$)*, then the following inequalities hold: for any $x,\ y \in X$,*

(SRFSII$_3$) $\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A((x*y)*(y*x^2))}^{(+)}$,
(SRFSII$_4$) $\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A((x*y)*(y*x^2))}^{(-)}$.

*Proof.* Suppose $\sqrt{A}$ satisfies the conditions (SRFSII$_1$) and (SRFSII$_2$) and let $x,\ y,\ z \in X$. Then we have

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(z*[(x*y*(y*x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(z*[(x*y*(y*x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Let $z = 0$. Then by ($F_1$) and ($KU_3$), we get

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(0*[(x*y)*(y*x^2)])}^{(+)} \wedge \sqrt{A(0)}^{(+)} = \sqrt{A((x*y)*(y*x^2))}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(0*[(x*y)*(y*x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)} = \sqrt{A((x*y)*(y*x^2))}^{(-)}.$$

This completes the proof. □

**Proposition 3.11.** *Every SRFSII of a KU-algebra $X$ is an SRFI of $X$ but the converse is not true.*

*Proof.* Let $\sqrt{A}$ be an SRFSII of $X$ and let $x,\ y,\ z \in X$. Let $x = y$. Then by (SRFSII$_1$) and (SRFSII$_2$), we have

$$\sqrt{A(x*x^2)}^{(+)} \geq \sqrt{A(z*[(x*x)*(x*x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x*x^2)}^{(-)} \leq \sqrt{A(z*[(x*x)*(x*x^2)])}^{(-)} \wedge \sqrt{A(z)}^{(-)}.$$

Thus by ($KU_3$) and ($KU_4$), we get

$$\sqrt{A(x)}^{(+)} \geq \sqrt{A(z*x)}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x)}^{(-)} \leq \sqrt{A(z*x)}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

So $\sqrt{A}$ is an SRFI of $X$. □

The following Example shows that the converse of Theorem 3.11 may not be true.

**Example 3.12.** Let $X = \{0, 1, 2, 3, 4\}$ be the set with the operation $*$ given by the table:

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 0 | 1 | 3 | 4 |
| 2 | 0 | 0 | 0 | 3 | 4 |
| 3 | 0 | 0 | 0 | 0 | 4 |
| 4 | 0 | 0 | 0 | 0 | 0 |

Table 3.2

Then clearly, $X$ is a $KU$-algebra. Consider the fuzzy set $A$ in $X$ defined as follows:

$$A(0) = 0.7,\ A(1) = A(2) = A(3) = A(4) = 0.2.$$

Then we have

$$\sqrt{A}(0) = (0.837, -0.837),\ \sqrt{A}(1) = \sqrt{A}(2) = \sqrt{A}(3) = \sqrt{A}(4) = (0.447, -0.447).$$

Thus we can easily check that $\sqrt{A}$ is an SRFI of $X$.

On the other hand, let us take $z = 0,\ x = 1$ and $y = 2$. Then we have

$$\text{L.H.S of (SRFSII}_1) = \sqrt{A(1 * 2^2)}^{(+)} = \sqrt{A(1)}^{(+)} = 0.447,$$

$$\text{R.H.S of (SRFSII}_1) = \sqrt{A(0 * [(1 * 2)(2 * 1^2)])}^{(+)} \wedge \sqrt{A(0)}^{(+)} = \sqrt{A(0)}^{(+)} = 0.837.$$

Thus $\sqrt{A(1 * 2^2)}^{(+)} \not\geq \sqrt{A(0 * [(1 * 2)(2 * 1^2)])}^{(+)} \wedge \sqrt{A(0)}^{(+)}$. So $\sqrt{A}$ is not an SRFSII of $X$.

Now we give a condition for an SRFI to be an SRFSII-ideal.

**Proposition 3.13.** *Let $X$ be a $KU$-algebra and let $\sqrt{A}$ be an SRFI of $X$. If $\sqrt{A}$ satisfies the conditions* (SRFSII$_3$) *and* (SRFSII$_4$)*, then $\sqrt{A}$ is an SRFSII of $X$.*

*Proof.* Suppose $\sqrt{A}$ satisfies the conditions (SRFSII$_3$) and (SRFSII$_4$ , and let $x,\ y,\ z \in X$. Then we have

$$\begin{aligned}\sqrt{A(x * y^2)}^{(+)} &\geq \sqrt{A((x * y) * (y * x^2))}^{(+)} \text{ [By (SRFSII}_3)] \\ &\geq \sqrt{A(z * [(x * y) * (y * x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)} \text{ [By (SRFI}_2)], \\ \sqrt{A(x * y^2)}^{(-)} &\leq \sqrt{A((x * y) * (y * x^2))}^{(-)} \text{ [By (SRFSII}_4)] \\ &\leq \sqrt{A(z * [(x * y) * (y * x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)} \text{ [By (SRFI}_3)].\end{aligned}$$

Thus $\sqrt{A}$ satisfies the conditions (SRFSII$_2$) and (SRFSII$_4$). So $\sqrt{A}$ is an SRFSII of $X$. □

**Theorem 3.14.** *Let $\sqrt{A}$ be an SRFI of a $KU$-algebra $X$. Then the followings are equivalent:*

(1) *$\sqrt{A}$ is an SRFSII of $X$,*

(2) $\sqrt{A(x * y^2)}^{(+)} \geq \sqrt{A((x * y) * (y * x^2))}^{(+)}$,
$\sqrt{A(x * y^2)}^{(-)} \leq \sqrt{A((x * y) * (y * x^2))}^{(-)}$ *for any $x,\ y,\ z \in X$,*

(3) $\sqrt{A(x * y^2)} = (\sqrt{A(z * [(x * y) * (y * x^2)])}^{(+)}, \sqrt{A(z * [(x * y) * (y * x^2)])}^{(-)})$ *for any $x,\ y,\ z \in X$.*

*Proof.* (1)⇒(2): Suppose $\sqrt{A}$ is an SRFSII of $X$ and let $x,\ y \in X$. Then we have

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(0*[(x*y)*(y*x^2)])}^{(+)} \wedge \sqrt{A(0)}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(0*[(x*y)*(y*x^2)])}^{(+)} \vee \sqrt{A(0)}^{(+)}.$$

Thus by ($KU_3$) and ($\text{SRFI}_1$), we get

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A((x*y)*(y*x^2))}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A((x*y)*(y*x^2))}^{(-)}.$$

(2)⇒(3): Suppose (2) holds and let $x,\ y \in X$. Then Proposition 2.3 (5),

$$(x*y)*(y*x^2) \leq (x*y)*y = x*y^2.$$

Thus by Proposition 3.7,

$$\sqrt{A((x*y)*(y*x^2))}^{(+)} \geq \sqrt{A(x*y^2)}^{(+)},$$
$$\sqrt{A((x*y)*(y*x^2))}^{(-)} \leq \sqrt{A(x*y^2)}^{(-)}.$$

So (3) holds.

(3)⇒(1): Suppose (3) holds and let $x,\ y,\ z \in X$. Then it is obvious that the following inequalities:

$$\sqrt{A((x*y)*(x*y^2))}^{(+)} \geq \sqrt{A((x*y)*(x*y^2))}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A((x*y)*(x*y^2))}^{(-)} \leq \sqrt{A((x*y)*(x*y^2))}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Thus by the condition (3), we have

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A((x*y)*(x*y^2))}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A((x*y)*(x*y^2))}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

So $\sqrt{A}$ is an SRFSII of $X$. □

**Definition 3.15.** Let $\sqrt{A}$ be an SRFS in a set $X$ and let $t \in [0,1]$. Then the *$t$-level set* of $\sqrt{A}$, denoted by $\sqrt{A_t}$, is a subset of $X$ defined as follows:

$$\sqrt{A_t} = \{x \in X : \sqrt{A(x)}^{(+)} \geq t,\ \sqrt{A(x)}^{(-)} \geq -t\}.$$

It is obvious that $\sqrt{A_t} \neq \varnothing$ for each $\sqrt{A(0)} \neq \sqrt{A_t} \in SRF(X)$ and each $t \in (0,1]$.

**Theorem 3.16.** *Let $\sqrt{A}$ be an SRFS in a $KU$-algebra $X$. Then $\sqrt{A}$ is an SRFSII of $X$ if and only if $\sqrt{A_t}$ is a sub-implicative ideal of $X$ for each $t \in (0,1]$.*

*Proof.* Suppose $\sqrt{A}$ is an SRFSII of $X$ and let $t \in (0,1]$. Then clearly, $\sqrt{A_t} \neq \varnothing$. Thus there is $x \in X$ such that $\sqrt{A(x)}^{(+)} \geq t,\ \sqrt{A(x)}^{(-)} \geq -t$. By ($\text{SRFI}_1$), we have

$$\sqrt{A(0)}^{(+)} \geq \sqrt{A(x)}^{(+)} \geq t,\ \sqrt{A(0)}^{(-)} \geq \sqrt{A(x)}^{(-)} \geq -t.$$

So $0 \in \sqrt{A_t}$. Hence $\sqrt{A_t}$ satisfies the condition (i) of Definition 2.9.

Now suppose $z*[(x*y)*(y*x^2)] \in \sqrt{A_t}$ and $z \in \sqrt{A_t}$ for any $x,\ y,\ z \in X$. Then we have

$$\sqrt{A(z*[(x*y)*(y*x^2)])}^{(+)} \geq t,\ \sqrt{A(z)}^{(+)} \geq t,$$

$$\sqrt{A(z*[(x*y)*(y*x^2)])}^{(-)} \leq -t,\ \sqrt{A(z)}^{(-)} \geq -t.$$

Thus by (SRFSII$_1$) and (SRFSII$_2$), we get

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(z*[(x*y)*(y*x^2)])}^{(+)} \wedge \sqrt{A(z)}^{(+)} \geq t \wedge t = t,$$

$$\sqrt{A(x*y^2)}^{(+)} \leq \sqrt{A(z*[(x*y)*(y*x^2)])}^{(-)} \vee \sqrt{A(z)}^{(-)} \leq (-t) \vee (-t) = -t.$$

So $x*y^2 \in \sqrt{A_t}$. Hence $\sqrt{A_t}$ satisfies the condition (ii) of Definition 2.9. Therefore $\sqrt{A_t}$ is a sub-implicative ideal of $X$.

Suppose the necessary condition holds and let $\sqrt{A(x)} = t$ for each $x \in X$ and each $t \in (0,1]$. Then clearly, $x \in \sqrt{A_t}$, i.e., $\sqrt{A(x)}^{(+)} = t,\ \sqrt{A(x)}^{(-)} = -t$. Since $0 \in \sqrt{A_t}$, we have $\sqrt{A(0)}^{(+)} \geq \sqrt{A(x)}^{(+)},\ \sqrt{A(0)}^{(-)} \leq \sqrt{A(x)}^{(-)}$.

Now assume that $\sqrt{A}$ does not satisfy the conditions (SRFSII$_1$) and (SRFSII$_2$). Then there are $a,\ b,\ c \in X$ such that

$$\sqrt{A(a*b^2)}^{(+)} < \sqrt{A(c*[(a*b)*(b*a^2)])}^{(+)} \wedge \sqrt{A(c)}^{(+)},$$

$$\sqrt{A(a*b^2)}^{(-)} > \sqrt{A(c*[(a*b)*(b*a^2)])}^{(-)} \vee \sqrt{A(c)}^{(-)}.$$

Let $t_0 = \frac{1}{2}(\sqrt{A(a*b^2)}^{(+)} + \sqrt{A(c*[(a*b)*(b*a^2)])}^{(+)} \wedge \sqrt{A(c)}^{(+)})$ and let $-t_0 = \frac{1}{2}(\sqrt{A(a*b^2)}^{(-)} + \sqrt{A(c*[(a*b)*(b*a^2)])}^{(-)} \vee \sqrt{A(c)}^{(-)})$. Then we get

$$\sqrt{A(a*b^2)}^{(+)} < t_0 < \sqrt{A(c*[(a*b)*(b*a^2)])}^{(+)} \wedge \sqrt{A(c)}^{(+)},$$

$$\sqrt{A(a*b^2)}^{(-)} < -t_0 < \sqrt{A(c*[(a*b)*(b*a^2)])}^{(-)} \vee \sqrt{A(c)}^{(-)}.$$

Thus $c*[(a*b)*(b*a^2)],\ c \in \sqrt{A_{t_0}}$ but $a*b^2 \notin \sqrt{A_{t_0}}$. So $\sqrt{A_{t_0}}$ is not a sub-implicative ideal of $X$. This is a contradiction. Hence $\sqrt{A}$ is an SRFSII of $X$. □

## 4. Square root fuzzy sub-commutative ideals

**Definition 4.1.** Let $X$ be a $KU$-algebra and let $\sqrt{0} \neq \sqrt{A} \in SRF(X)$. Then $\sqrt{A}$ is called a *square root fuzzy sub-commutative ideal* (briefly, SRFSCI) of $X$, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,

(SRFI$_1$) $\sqrt{A(0)}^{(+)} \geq \sqrt{A(x)}^{(+)},\ \sqrt{A(0)}^{(-)} \leq \sqrt{A(x)}^{(-)}$,

(SRFSCI$_1$) $\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(z*[(y*x^2)*y^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)}$,

(SRFSCI$_2$) $\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(z*[(y*x^2)*y^2])}^{(-)} \vee \sqrt{A(z)}^{(-)}$.

It is clear that $\sqrt{0}$ and $\sqrt{1}$ are SRFSCIs of $X$ respectively.

**Proposition 4.2.** *Every SRFSCI of a $KU$-algebra $X$ is order reversing.*

*Proof.* Let $\sqrt{A}$ be an SRFSCI of $X$ and let $x,\ y,\ z \in X$. Suppose $x \leq z$. Then clearly, $z*x = 0$. Moreover, by (SRFSCI$_1$) and (SRFSCI$_2$), we get

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(z*[(y*x^2)*y^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$

$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(z*[(y*x^2)*y^2])}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Let $y = x$. Then we have

$$\begin{aligned}\sqrt{A(x)}^{(+)} &= \sqrt{A(x * x^2)}^{(+)} \text{ [By } (KU_5) \text{ and } (KU_3)] \\ &\geq \sqrt{A(z * [(x * x^2) * x^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)} \\ &= \sqrt{A(0)}^{(+)} \wedge \sqrt{A(z)}^{(+)} \text{ [By } (KU_5), (KU_3) \text{ and } (KU_2)] \\ &= \sqrt{A(z)}^{(+)}.\end{aligned}$$

Similarly, we have $\sqrt{A(x)}^{(+)} \leq \sqrt{A(x)}^{(-)}$. This completes the proof. □

**Lemma 4.3.** *Let $\sqrt{A}$ be an SRFSCI of a KU-algebra $X$ and let $x,\ y,\ z \in X$. If $y * x \leq z$ or $z * x \leq y$, then*

$$\sqrt{A(x)}^{(+)} \geq \sqrt{A(y)}^{(+)} \wedge \sqrt{A(z)}^{(+)},\ \sqrt{A(x)}^{(-)} \geq \sqrt{A(y)}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

*Proof.* Suppose $z * x \leq y$. Then by Proposition 4.2, we have

$$\sqrt{A(z * x)}^{(+)} \geq \sqrt{A(y)}^{(+)},\ \sqrt{A(z * x)}^{(-)} \geq \sqrt{A(y)}^{(-)}. \tag{4.1}$$

On the other hand, by (SRFSC$_1$) and (SRFSC$_2$), we get

$$\sqrt{A(x * y^2)}^{(+)} \geq \sqrt{A(z * [(y * x^2) * y^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x * y^2)}^{(+)} \geq \sqrt{A(z * [(y * x^2) * y^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)}.$$

Let $x = y$. Then we have

$$\begin{aligned}\sqrt{A(x)}^{(+)} &= \sqrt{A(x * x^2)}^{(+)} \text{ [By } (KU_5) \text{ and } (KU_3)] \\ &\geq \sqrt{A(z * [(x * x^2) * x^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)} \\ &= \sqrt{A(z * x)}^{(+)} \wedge \sqrt{A(z)}^{(+)} \\ &= \sqrt{A(y)}^{(+)} \wedge \sqrt{A(z)}^{(+)} \text{ [By (4.1)]},\end{aligned}$$

$$\begin{aligned}\sqrt{A(x)}^{(-)} &= \sqrt{A(x * x^2)}^{(-)} \\ &\leq \sqrt{A(z * [(x * x^2) * x^2])}^{(-)} \vee \sqrt{A(z)}^{(-)} \\ &= \sqrt{A(z * x)}^{(-)} \vee \sqrt{A(z)}^{(-)} \\ &= \sqrt{A(y)}^{(-)} \vee \sqrt{A(z)}^{(-)}.\end{aligned}$$

If $y * x \leq z$, then we obtain the same result by the similar arguments. This completes the proof. □

**Lemma 4.4.** *Let $X$ be a commutative KU-algebra. Then every SRFI of $X$ is an SRFSCI of $X$.*

*Proof.* Let $\sqrt{A}$ be an SRFI of $X$ and let $x,\ y,\ z \in X$. Then

(SRFI$_1$) $\sqrt{A(y)}^{(+)} \geq \sqrt{A(x * y)}^{(+)} \wedge \sqrt{A(x)}^{(+)}$,

(SRFI$_2$) $\sqrt{A(y)}^{(-)} \leq \sqrt{A(x * y)}^{(-)} \vee \sqrt{A(x)}^{(-)}$.

By substituting $y * x^2$ for $y$ and $z$ for $x$ in (SRFI$_1$) and (SRFI$_2$), we obtain

$$\sqrt{A(y * x^2)}^{(+)} \geq \sqrt{A(z * (y * x^2))}^{(+)} \wedge \sqrt{A(z)}^{(+)}, \tag{4.2}$$

$$\sqrt{A(y * x^2)}^{(-)} \leq \sqrt{A(z * (y * x^2))}^{(-)} \vee \sqrt{A(z)}^{(-)}. \tag{4.3}$$

Since $X$ is commutative, we have

$$\sqrt{A(x*y^2)}^{(+)} = \sqrt{A(y*x^2)}^{(+)} \geq \sqrt{A(z*(y*x^2))}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} = \sqrt{A(y*x^2)}^{(-)} \leq \sqrt{A(z*(y*x^2))}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

On the other hand, by ($KU_1$) and ($KU_3$), we get

$$(y*x^2)*y^2 = [((y*x)*x)*y]*y = [((y*x)*x)*y]*(0*y) \leq y*x^2.$$

Thus by Proposition 2.3 (3), $z*[(y*x^2)*y^2] \leq z*(y*x^2)$. By Proposition 4.2, we have

$$\sqrt{A(z*[(y*y^2)*y^2])}^{(+)} \leq \sqrt{A(z*(y*x^2))}^{(+)},$$
$$\sqrt{A(z*[(y*y^2)*y^2])}^{(-)} \geq \sqrt{A(z*(y*x^2))}^{(-)}.$$

So from (4.2) and (4.3), we obtain

$$\sqrt{A(x*y^2)}^{(+)} \geq \sqrt{A(z*[(y*y^2)*y^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$
$$\sqrt{A(x*y^2)}^{(-)} \leq \sqrt{A(z*[(y*y^2)*y^2])}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Hence $\sqrt{A}$ is an SRFSCI of $X$. □

**Proposition 4.5.** *Let $X$ be a $KU$-algebra and let $\sqrt{A} \in SRF(X)$. If $\sqrt{A}$ satisfies the conditions* (SRFSCI$_1$) *and* (SRFSCI$_2$)*, then the following inequalities holds: for any $x,\ y \in X$,*

(SRFSCI$_3$) $\sqrt{A(y*x^2)}^{(+)} \geq \sqrt{A((y*x^2)*y^2)}^{(+)}$,
(SRFSCI$_4$) $\sqrt{A(y*x^2)}^{(-)} \leq \sqrt{A((y*x^2)*y^2)}^{(-)}$.

*Proof.* Suppose $\sqrt{A}$ satisfies the conditions (SRFSCI$_1$) and (SRFSCI$_2$) and let $x,\ y \in X$. Then we have

$$\begin{aligned}\sqrt{A(y*x^2)}^{(+)} &\geq \sqrt{A(0*[(y*x^2)*y^2])}^{(+)} \wedge \sqrt{A(0)}^{(+)} \quad \text{[By (SRFSCI}_1\text{)]}\\ &= \sqrt{A(0*[(y*x^2)*y^2])}^{(+)} \quad \text{[By (SRFI}_1\text{)]}\\ &= \sqrt{A((y*x^2)*y^2)}^{(+)}. \quad \text{[[By (}KU_3\text{)]}\end{aligned}$$

Similarly, we obtain $\sqrt{A(y*x^2)}^{(-)} \leq \sqrt{A((y*x^2)*y^2)}^{(-)}$. This completes the proof. □

**Proposition 4.6.** *Every SRFSCI of a $KU$-algebra $X$ is an SRFI of $X$ but the converse is not true.*

*Proof.* Let $\sqrt{A}$ be an SRFSCI of $X$ and let $x,\ y,\ z \in X$ such that $x = y$. Then we get

$$\begin{aligned}\sqrt{A(x*x^2)}^{(+)} &\geq \sqrt{A(z*[(x*x^2))*x^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)} \quad \text{[By (SRFI}_1\text{)]}\\ &= \sqrt{A(z*x)}^{(+)} \wedge \sqrt{A(z)}^{(+)}\\ &\quad \text{[By (}KU_2\text{), (}KU_3\text{) and (}KU_5\text{)]}\end{aligned}$$

Similarly, we have $\sqrt{A(x*x^2)}^{(-)} \leq \sqrt{A(z*x)}^{(-)} \vee \sqrt{A(z)}^{(-)}$. Thus $\sqrt{A}$ is an SRFI of $X$. □

The following Example shows that the converse of Theorem 4.6 may not be true.

**Example 4.7.** Let $X = \{0, 1, 2, 3, 4\}$ be the $KU$-algebra given in Example 3.12. Consider the SRFS $\sqrt{A}$ in $X$ defined as follows:

$$A(0) = 0.7,\ A(1) = A(2) = A(3) = A(4) = 0.2.$$

Then we can easily see that $\sqrt{A}$ is an SRFI of $X$. On the other hand, et $z = 0,\ x = 1,\ y = 3$. Then we have

$$\text{L.H.S of (SRFSCI}_1) = \sqrt{A(3 * 1^2)}^{(+)} = \sqrt{A(1)}^{(+)} = 0.447,$$

$$\text{R.H.S of (SRFSCI}_1) = \sqrt{A(0 * [(3 * 1^2) * 3^2)])}^{(+)} \wedge \sqrt{A(0)}^{(+)} = \sqrt{A(0)}^{(+)} = 0.837.$$

Thus $\sqrt{A(3 * 1^2)}^{(+)} \ngeq \sqrt{A(0 * [(3 * 1^2) * 3^2)])}^{(+)} \wedge \sqrt{A(0)}^{(+)}$. So $\sqrt{A}$ is not an SRFSCI of $X$.

Now, we give a condition for an SRFI to be an SRFSCI.

**Proposition 4.8.** *Let $X$ be a $KU$-algebra and let $\sqrt{A}$ be an SRFI of $X$. If $\sqrt{A}$ satisfies the conditions* (SRFSCI$_3$) *and* (SRFSCI$_4$)*, then $\sqrt{A}$ is an SRFSCI of $X$.*

*Proof.* Suppose $\sqrt{A}$ satisfies the conditions (SRFSCI$_3$) and (SRFSCI$_4$) and let $x,\ y \in X$. Then clearly, we have

$$\sqrt{A(y * x^2)}^{(+)} \geq \sqrt{A((y * x^2) * y^2)}^{(+)},\ \sqrt{A(y * x^2)}^{(-)} \leq \sqrt{A((y * x^2) * y^2)}^{(-)}.$$

Since $\sqrt{A}$ is an SRFI of $X$, by (SRFI$_2$) and (SRFI$_3$), we get : for each $z \in X$,

$$\sqrt{A(y * x^2)}^{(+)} \geq \sqrt{A((y * x^2) * y^2)}^{(+)} \geq \sqrt{A(z * (y * x^2) * y^2])}^{(+)} \wedge \sqrt{A(z)}^{(+)},$$

$$\sqrt{A(y * x^2)}^{(-)} \leq \sqrt{A((y * x^2) * y^2)}^{(-)} \leq \sqrt{A(z * (y * x^2) * y^2])}^{(-)} \vee \sqrt{A(z)}^{(-)}.$$

Thus $\sqrt{A}$ satisfies the conditions(SRFSCI$_1$) and (SRFSCI$_2$). So $\sqrt{A}$ is an SRFSCI of $X$. □

**Theorem 4.9.** *Let $X$ be a $KU$-algebra and let $\sqrt{A}$ be an SRFI of $X$. Then the followings are equivalent:*
(1) *$\sqrt{A}$ is an SRFSCI,*
(1) $\sqrt{A(y * x^2)}^{(+)} \geq \sqrt{A((y * x^2) * y^2)}^{(+)},\ \sqrt{A(y * x^2)}^{(-)} \leq \sqrt{A((y * x^2) * y^2)}^{(-)}$ *for any $x,\ y \in X$.*

*Proof.* The proof is similar to Theorem 3.14. □

**Theorem 4.10.** *Let $\sqrt{A}$ be an SRFS in a $KU$-algebra $X$. Then $\sqrt{A}$ is an SRFSCI of $X$ if and only if $\sqrt{A_t}$ is a sub-commutative ideal of $X$ for each $t \in (0, 1]$.*

*Proof.* The proof is similar to Theorem 3.16. □

## 5. Homomorphisms of $KU$-algebras and the products of square root fuzzy sets

In this section, we deal with the image and the pre-image of SRFSs, SRFSIIs and SRFSCIs under a homomorphism, and the product of SRFSs, SRFSIIs and SRFSCIs.

**Definition 5.1.** Let $(X, *_X, 0_X)$ and be $(Y, *_Y, 0_Y)$ be $KU$-algebras. Then a mapping $f : X \to Y$ is called a *homomorphism*, if $f(x *_X y) = f(x) *_Y f(y)$ for any $x,\ y \in X$.

It is clear that $f(0_X) = 0_Y$.

**Definition 5.2.** Let $X$ and be $Y$ be nonempty sets, and let $\sqrt{A} \in SRF(X)$, $\sqrt{B} \in SRF(Y)$ and let $f : X \to Y$ be a mapping.

(i) The *image of* $\sqrt{A}$ *under* $f$, denoted by $f(\sqrt{A})$, is a SRFS in $X$ defined as follows: for each $y \in Y$,

$$\sqrt{A}(y) = \begin{cases} \left(\bigvee_{x\in f^{-1}(y)} \sqrt{A(x)}^{(+)}, \bigwedge_{x\in f^{-1}(y)} \sqrt{A(x)}^{(-)}\right) & \text{if } f^{-1}(y) \neq \varnothing \\ \sqrt{0} & \text{otherwise.} \end{cases} \tag{5.1}$$

(ii) The *pre-image of* $\sqrt{B}$ *under* $f$, denoted by $f^{-1}(\sqrt{B})$, is a SRFS in $X$ defined as follows: for each $x \in X$,

$$f^{-1}(\sqrt{B})(x) = \left(\sqrt{B(f(x))}^{(+)}, \sqrt{B(f(x))}^{(-)}\right). \tag{5.2}$$

**Proposition 5.3.** *Let $f : (X, *, 0) \to (Y, *, 0)$ be a homomorphism of $KU$-algebras and let $\sqrt{B} \in SRF(Y)$. If $\sqrt{B}$ is an SRFI of $Y$, then $f^{-1}(\sqrt{B})$ is an SRFI of $X$.*

*Proof.* Suppose $\sqrt{B}$ is an SRFI of $Y$ and let $x \in X$. Since $f$ is a homomorphism, we get

$$f^{-1}(\sqrt{B})^{(+)}(0) = \sqrt{B(f(0))}^{+} = \sqrt{B(0)}^{+} \geq \sqrt{B(f(x))}^{+} = f^{-1}(\sqrt{B})^{(+)}(x),$$
$$f^{-1}(\sqrt{B})^{(-)}(0) = \sqrt{B(f(0))}^{-} = \sqrt{B(0)}^{-} \leq \sqrt{B(f(x))}^{-} = f^{-1}(\sqrt{B})^{(-)}(x).$$

Then $f^{-1}(\sqrt{B})$ satisfies the condition ($SRFI_1$).

Now let $x,\ y \in X$. Then we have

$$\begin{aligned} f^{-1}(\sqrt{B})^{(+)}(x) &= \sqrt{B(f(x))}^{+} \\ &\geq \sqrt{B(f(x) * f(y))}^{+} \wedge \sqrt{B(f(x))}^{+} \\ &= \sqrt{B(f(x * y)}^{+} \wedge \sqrt{B(f(x))}^{+} \\ &\quad \text{[Since } f \text{ is a homomorphism]} \\ &= f^{-1}(\sqrt{B})^{(+)}(x * y) \wedge f^{-1}(\sqrt{B})^{(+)}(x). \end{aligned}$$

Similarly, $f^{-1}(\sqrt{B})^{(-)}(x) \leq f^{-1}(\sqrt{B})^{(-)}(x * y) \vee f^{-1}(\sqrt{B})^{(-)}(x)$. Thus $f^{-1}(\sqrt{B})$ satisfies the conditions ($SRFI_2$) and ($SRFI_3$). So $f^{-1}(\sqrt{B})$ is an SRFI of $X$. □

**Proposition 5.4.** *Let $f : (X, *, 0) \to (Y, *, 0)$ be a homomorphism of $KU$-algebras and let $\sqrt{B} \in SRF(Y)$. If $\sqrt{B}$ is an SRFSII of $Y$, then $f^{-1}(\sqrt{B})$ is an SRFSII of $X$.*

*Proof.* From the proof of Proposition 5.3, it is obvious that $f^{-1}(\sqrt{B})$ satisfies the condition (SRFI$_1$). Let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} & f^{-1}(\sqrt{B})^{(+)}(x * y^2) \\ &= \sqrt{B(f(x * y^2))}^{(+)} \\ &= \sqrt{B(f(x) * f(y)^2)}^{(+)} \text{ [Since } f \text{ is a homomorphism]} \\ &\geq \sqrt{B(f(z) * [(f(x) * f(y)) * (f(y) * f(x)^2)])}^{(+)} \wedge \sqrt{B(f(z))}^{(+)} \\ &= \sqrt{B(f(z * [(x * y) * (y * x^2)])}^{(+)} \wedge \sqrt{B(f(z))}^{(+)} \\ &= f^{-1}(\sqrt{B})^{(+)}(z * [(x * y) * (y * x^2)]) \wedge f^{-1}(\sqrt{B})^{(+)}(z). \end{aligned}$$

Similarly, $f^{-1}(\sqrt{B})^{(-)}(x*y^2) \leq f^{-1}(\sqrt{B})^{(-)}(z*[(x*y)*(y*x^2)]) \vee f^{-1}(\sqrt{B})^{(-)}(z)$. Thus $f^{-1}(\sqrt{B})$ satisfies the conditions (SRFSII$_2$) and (SRFSII$_3$). So $f^{-1}(\sqrt{B})$ is an SRFSII of $X$. □

**Proposition 5.5.** *Let $f : (X, *, 0) \to (Y, *, 0)$ be a homomorphism of $KU$-algebras and let $\sqrt{B} \in SRF(Y)$. If $\sqrt{B}$ is an SRFSCI of $Y$, then $f^{-1}(\sqrt{B})$ is an SRFSCI of $X$.*

*Proof.* It is clear that $f^{-1}(\sqrt{B})$ satisfies the condition (SRFI$_1$). Let $x,\ y,\ z \in X$. Then we have

$$\begin{aligned} f^{-1}(\sqrt{B})^{(+)}(x * y^2) &= \sqrt{B(f(x * y^2))}^{(+)} \\ &= \sqrt{B(f(x) * f(y^2))}^{(+)} \\ &\geq \sqrt{B(f(z) * [(f(y) * f(x^2)) * f(y^2)])}^{(+)} \\ &= \sqrt{B(f(z * [(y * x^2) * y^2])}^{(+)} \\ &= f^{-1}(\sqrt{B})^{(+)}(z * [(y * x^2) * y^2]), \\ f^{-1}(\sqrt{B})^{(-)}(x * y^2) &= \sqrt{B(f(x * y^2))}^{(-)} \\ &= \sqrt{B(f(x) * f(y^2))}^{(-)} \\ &\leq \sqrt{B(f(z) * [(f(y) * f(x^2)) * f(y^2)])}^{(-)} \\ &= \sqrt{B(f(z * [(y * x^2) * y^2])}^{(-)} \\ &= f^{-1}(\sqrt{B})^{(-)}(z * [(y * x^2) * y^2]). \end{aligned}$$

Thus $f^{-1}(\sqrt{B})$ satisfies the conditions (SRFSCI$_2$) and (SRFSCI$_3$). So $f^{-1}(\sqrt{B})$ is an SRFSCI of $X$. □

**Proposition 5.6.** *Let $f : (X, *, 0) \to (Y, *, 0)$ be an epimorphism of $KU$-algebras and let $\sqrt{B} \in SRF(Y)$. If $f^{-1}(\sqrt{B})$ is an SRFI of $X$, then $\sqrt{B}$ is an SRFI of $Y$.*

*Proof.* Let $a \in Y$. Since $f$ is surjective, there is $x \in X$ such that $a = f(x)$. Then we get

$$\begin{aligned} \sqrt{B(0)}^{(+)} &= \sqrt{B(f(0))}^{(+)} \text{ [Since } f \text{ is a homomorphism]} \\ &= f^{-1}(\sqrt{B})^{(+)}(0) \\ &\geq f^{-1}(\sqrt{B})^{(+)}(x) \\ &= \sqrt{B(f(x))}^{(+)} \\ &= \sqrt{B(a)}^{(+)}, \\ \sqrt{B(0)}^{(-)} &= \sqrt{B(f(0))}^{(-)} \end{aligned}$$

$$= f^{-1}(\sqrt{B})^{(-)}(0)$$
$$\leq f^{-1}(\sqrt{B})^{(-)}(x)$$
$$= \sqrt{B(f(x))}^{(-)}$$
$$= \sqrt{B(a)}^{(-)}.$$

Thus $\sqrt{B}$ satisfies the condition ($\text{SRFI}_1$).

Now let $a,\ b \in Y$. Then clearly, there are $x,\ y \in X$ such that $a = f(x),\ b = f(y)$. Thus we have

$$\sqrt{B(b)}^{(+)} = \sqrt{B(f(y))}^{(+)}$$
$$= f^{-1}(\sqrt{B})^{(+)}(y)$$
$$\geq f^{-1}(\sqrt{B})^{(+)}(x * y) \wedge f^{-1}(\sqrt{B})^{(+)}(x)$$
$$= \sqrt{B(f(x * y))}^{(+)} \wedge \sqrt{B(f(x))}^{(+)}$$
$$= \sqrt{B(f(x) * f(y))}^{(+)} \wedge \sqrt{B(f(x))}^{(+)}$$
$$= \sqrt{B(a * b)}^{(+)} \wedge \sqrt{B(a)}^{(+)},$$
$$\sqrt{B(b)}^{(-)} = \sqrt{B(f(y))}^{(-)}$$
$$= f^{-1}(\sqrt{B})^{(-)}(y)$$
$$\leq f^{-1}(\sqrt{B})^{(-)}(x * y) \vee f^{-1}(\sqrt{B})^{(-)}(x)$$
$$= \sqrt{B(f(x * y))}^{(-)} \vee \sqrt{B(f(x))}^{(-)}$$
$$= \sqrt{B(f(x) * f(y))}^{(+)} \vee \sqrt{B(f(x))}^{(-)}$$
$$= \sqrt{B(a * b)}^{(-)} \vee \sqrt{B(a)}^{(-)}.$$

So $\sqrt{B}$ satisfies the conditions ($\text{SRFI}_2$) and ($\text{SRFI}_3$). Hence $\sqrt{B}$ is an SRFI of $X$. □

**Proposition 5.7.** *Let $f : (X, *, 0) \to (Y, *, 0)$ be an epimorphism of KU-algebras and let $\sqrt{B} \in SRF(Y)$. If $f^{-1}(\sqrt{B})$ is an SRFSII of $X$, then $\sqrt{B}$ is an SRFSII of $Y$.*

*Proof.* It is clear that $\sqrt{B}$ satisfies the condition ($\text{SRFI}_1$). Let $x,\ y,\ z \in Y$. Since $f$ is surjective, there are $a,\ b,\ c \in X$ such that $a = f(x),\ b = f(y),\ c = f(z)$. Then

$$\sqrt{B(a * b^2)}^{(+)} = \sqrt{B(f(x) * f(y^2))}^{(+)}$$
$$= \sqrt{B(f(x * y^2))}^{(+)}$$
$$= f^{-1}(\sqrt{B})^{(+)}(x * y^2)$$
$$\geq f^{-1}(\sqrt{B})^{(+)}(z * [(x * y) * (y * x^2)]) \wedge f^{-1}(\sqrt{B})^{(+)}(z)$$
$$= \sqrt{B(f(z * [(x * y) * (y * x^2)]))}^{(+)} \wedge \sqrt{B(f(z))}^{(+)}$$
$$= \sqrt{B(f(z) * [(f(x) * f(y)) * (f(y) * f(x^2))])}^{(+)} \wedge \sqrt{B(f(z))}^{(+)}$$
$$= \sqrt{B(c * [(a * b) * (b * a^2)])}^{(+)} \wedge \sqrt{B(c)}^{(+)},$$
$$\sqrt{B(a * b^2)}^{(-)} = \sqrt{B(f(x) * f(y^2))}^{(-)}$$
$$= \sqrt{B(f(x * y^2))}^{(-)}$$
$$= f^{-1}(\sqrt{B})^{(-)}(x * y^2)$$
$$\leq f^{-1}(\sqrt{B})^{(-)}(z * [(x * y) * (y * x^2)]) \vee f^{-1}(\sqrt{B})^{(-)}(z)$$
$$= \sqrt{B(f(z * [(x * y) * (y * x^2)]))}^{(-)} \vee \sqrt{B(f(z))}^{(-)}$$

$$= \sqrt{B(f(z) * [(f(x) * f(y)) * (f(y) * f(x^2))])}^{(-)} \vee \sqrt{B(f(z))}^{(-)}$$
$$= \sqrt{B(c * [(a * b) * (b * a^2)])}^{(-)} \vee \sqrt{B(c)}^{(-)}.$$

Thus $\sqrt{B}$ satisfies the conditions (SRFSII$_2$) and (SRFSII$_3$). So $\sqrt{B}$ is an SRFSII of $Y$ □

**Proposition 5.8.** *Let $f : (X, *, 0) \to (Y, *, 0)$ be an epimorphism of KU-algebras and let $\sqrt{B} \in SRF(Y)$. If $f^{-1}(\sqrt{B})$ is an SRFSCI of $X$, then $\sqrt{B}$ is an SRFSCI of $Y$.*

*Proof.* It is obvious that $\sqrt{B}$ satisfies the condition (SRFI$_1$). Let $x,\ y,\ z \in Y$. Then there are $a,\ b,\ c \in X$ such that $a = f(x),\ b = f(y),\ c = f(z)$. Thus we have

$$\begin{aligned}
\sqrt{B(a * b^2)}^{(+)} &= \sqrt{B(f(x) * f(y^2))}^{(+)}\\
&= \sqrt{B(f(x * y^2))}^{(+)}\\
&= f^{-1}(\sqrt{B})^{(+)}(x * y^2)\\
&\geq f^{-1}(\sqrt{B})^{(+)}(z * [(y * x^2) * y^2]) \wedge f^{-1}(\sqrt{B})^{(+)}(z)\\
&= \sqrt{B(f(z * [(y * x^2) * y^2]))}^{(+)} \wedge \sqrt{B(f(z))}^{(+)}\\
&= \sqrt{B(f(z) * [(f(y) * f(x^2)) * f(y^2)])}^{(+)} \wedge \sqrt{B(f(z))}^{(+)}\\
&= \sqrt{B(c * [(b * a^2) * a^2])}^{(+)} \wedge \sqrt{B(c)}^{(+)},\\
\sqrt{B(a * b^2)}^{(-)} &= \sqrt{B(f(x) * f(y^2))}^{(-)}\\
&= \sqrt{B(f(x * y^2))}^{(-)}\\
&= f^{-1}(\sqrt{B})^{(-)}(x * y^2)\\
&\leq f^{-1}(\sqrt{B})^{(-)}(z * [(y * x^2) * y^2]) \vee f^{-1}(\sqrt{B})^{(-)}(z)\\
&= \sqrt{B(f(z * [(y * x^2) * y^2]))}^{(-)} \vee \sqrt{B(f(z))}^{(-)}\\
&= \sqrt{B(f(z) * [(f(y) * f(x^2)) * f(y^2)])}^{(-)} \vee \sqrt{B(f(z))}^{(+)}\\
&= \sqrt{B(c * [(b * a^2) * a^2])}^{(-)} \vee \sqrt{B(c)}^{(-)}.
\end{aligned}$$

So $\sqrt{B}$ satisfies the conditions (SRFSCI$_2$) and (SRFSCI$_3$). Hence $\sqrt{B}$ is an SRFSCI of $Y$. □

**Definition 5.9.** Let $X$ be a nonempty set and let $\sqrt{A},\ \sqrt{B} \in SRF(X)$. Then the Cartesian product of $\sqrt{A}$ and $\sqrt{B}$, denoted by $\sqrt{A} \times \sqrt{B}$, is an SRFS in $X \times X$ defined as follows: for each $(x, y) \in X \times X$,

$$(\sqrt{A} \times \sqrt{B})(x, y) = (\sqrt{A(x)}^{(+)} \wedge \sqrt{B(y)}^{(+)}, \sqrt{A(x)}^{(-)} \vee \sqrt{B(y)}^{(-)}).$$

For two $KU$-algebras $(X, *, 0)$ and $(Y, *, 0)$, we define $*$ on $X \times Y$ as follows: for any $(x, y),\ (u, v) \in X \times Y$,

$$(x, y) * (u, v) = (x * u, y * v).$$

Then we can easily see that $(X \times Y, *, (0, 0))$ is a $KU$-algebra.

**Proposition 5.10.** *Let $(X, *, 0)$ be a KU-algebra and let $\sqrt{A},\ \sqrt{B} \in SRF(X)$. If $\sqrt{A}$ and $\sqrt{B}$ are SRFIs of $X$, then $\sqrt{A} \times \sqrt{B}$ is an SRFI of $X \times X$.*

*Proof.* Let $(x, y) \in X \times X$. Then we get

$$(\sqrt{A} \times \sqrt{B})^{(+)}(x, y) = \sqrt{A(x)}^{(+)} \wedge \sqrt{B(y)}^{(+)}$$

$$\begin{aligned}
&\leq \sqrt{A(0)}^{(+)} \wedge \sqrt{B(0)}^{(+)} \\
&= (\sqrt{A} \times \sqrt{B})^{(+)}(0,0), \\
(\sqrt{A} \times \sqrt{B})^{(-)}(x,y) &= \sqrt{A(x)}^{(-)} \vee \sqrt{B(y)}^{(-)} \\
&\geq \sqrt{A(0)}^{(-)} \vee \sqrt{B(0)}^{(-)} \\
&= (\sqrt{A} \times \sqrt{B})^{(-)}(0,0).
\end{aligned}$$

Thus $\sqrt{A} \times \sqrt{B}$ satisfies the condition (SRFI$_1$).

Now let $(x_1, x_2),\ (y_1, y_2) \in X \times X$. Then we have

$$\begin{aligned}
&(\sqrt{A} \times \sqrt{B})^{(+)}(y_1, y_2) \\
&= \sqrt{A(y_1)}^{(+)} \wedge \sqrt{B(y_2)}^{(+)} \\
&\geq (\sqrt{A(x_1 * y_1)}^{(+)} \wedge \sqrt{A(x_1)}^{(+)}) \wedge (\sqrt{B(x_2 * y_2)}^{(+)} \wedge \sqrt{B(x_2)}^{(+)}) \\
&= (\sqrt{A(x_1 * y_1)}^{(+)} \wedge \sqrt{B(x_2 * y_2)}^{(+)}) \wedge (\sqrt{A(x_1)}^{(+)} \wedge \sqrt{B(x_2)}^{(+)}) \\
&= (\sqrt{A} \times \sqrt{B})^{(+)}(x_1 * y_1, x_2 * y_2) \wedge (\sqrt{A} \times \sqrt{B})^{(+)}(x_1, x_2) \\
&= (\sqrt{A} \times \sqrt{B})^{(+)}((x_1, x_2) * (y_1, y_2)) \wedge (\sqrt{A} \times \sqrt{B})^{(+)}(x_1, x_2), \\
&(\sqrt{A} \times \sqrt{B})^{(-)}(y_1, y_2) \\
&= \sqrt{A(y_1)}^{(-)} \vee \sqrt{B(y_2)}^{(-)} \\
&\geq (\sqrt{A(x_1 * y_1)}^{(-)} \vee \sqrt{A(x_1)}^{(-)}) \vee (\sqrt{B(x_2 * y_2)}^{(-)} \vee \sqrt{B(x_2)}^{(-)}) \\
&= (\sqrt{A(x_1 * y_1)}^{(-)} \vee \sqrt{B(x_2 * y_2)}^{(-)}) \vee (\sqrt{A(x_1)}^{(-)} \vee \sqrt{B(x_2)}^{(-)}) \\
&= (\sqrt{A} \times \sqrt{B})^{(-)}(x_1 * y_1, x_2 * y_2) \vee (\sqrt{A} \times \sqrt{B})^{(-)}(x_1, x_2) \\
&= (\sqrt{A} \times \sqrt{B})^{(-)}((x_1, x_2) * (y_1, y_2)) \vee (\sqrt{A} \times \sqrt{B})^{(-)}(x_1, x_2).
\end{aligned}$$

So $\sqrt{A} \times \sqrt{B}$ satisfies the conditions (SRFI$_2$) and (SRFI$_2$). Hence $\sqrt{A} \times \sqrt{B}$ is an SRFI of $X \times X$. □

**Proposition 5.11.** *Let $(X, *, 0)$ be a KU-algebra and let $\sqrt{A},\ \sqrt{B} \in SRF(X)$. If $\sqrt{A}$ and $\sqrt{B}$ are SRFSIIs of $X$, then $\sqrt{A} \times \sqrt{B}$ is an SRFSII of $X \times X$.*

*Proof.* It is clear that $\sqrt{A}\times\sqrt{B}$ satisfies the condition (SRFI$_1$). Let $(x_1, x_2),\ (y_1, y_2),\ (z_1, z_2) \in X \times X$. Then we get

$$\begin{aligned}
&(\sqrt{A} \times \sqrt{B})^{(+)}((x_1, x_2) * (y_1, y_2)^2) \\
&= \sqrt{A(x_1 * y_1^2)}^{(+)} \wedge \sqrt{B(x_2 * y_2^2)}^{(+)} \\
&\geq (\sqrt{A(z_1 * [(x_1 * y_1) * (y_1 * x_1^2)])}^{(+)} \wedge \sqrt{A(z_1)}^{(+)}) \\
&\quad \wedge(\sqrt{B(z_2 * [x_2 * y_2) * (y_2 * x_2^2)])}^{(+)} \wedge \sqrt{B(z_2)}^{(+)}) \\
&= (\sqrt{A(z_1 * [(x_1 * y_1) * (y_1 * x_1^2)])}^{(+)} \wedge \sqrt{B(z_2 * [(x_2 * y_2) * (y_2 * x_2^2)])}^{(+)}) \\
&\quad \wedge(\sqrt{A(z_1)}^{(+)} \wedge \sqrt{B(z_2)}^{(+)}) \\
&= (\sqrt{A} \times \sqrt{B})^{(+)}(z_1 * [(x_1 * y_1) * (y_1 * x_1^2)]), (z_2 * [(x_2 * y_2) * (y_2 * x_2^2)]) \\
&\quad \wedge(\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2) \\
&= (\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2) * [((x_1, x_2) * (y_1, y_2)) * ((y_1, y_2) * (x_1, x_2)^2)]) \\
&\quad \wedge(\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2),
\end{aligned}$$

$$\begin{aligned}
&(\sqrt{A} \times \sqrt{B})^{(-)}((x_1, x_2) * (y_1, y_2)^2) \\
&= \sqrt{A(x_1 * y_1^2)}^{(-)} \vee \sqrt{B(x_2 * y_2^2)}^{(-)} \\
&\geq (\sqrt{A(z_1 * [(x_1 * y_1) * (y_1 * x_1^2)])}^{(-)} \vee \sqrt{A(z_1)}^{(-)})
\end{aligned}$$

$$\vee(\sqrt{B(z_2 * [x_2 * y_2) * (y_2 * x_2^2)])}^{(-)} \vee \sqrt{B(z_2)}^{(-)})$$
$$= (\sqrt{A(z_1 * [(x_1 * y_1) * (y_1 * x_1^2)])}^{(-)} \vee \sqrt{B(z_2 * [(x_2 * y_2) * (y_2 * x_2^2)])}^{(-)})$$
$$\vee(\sqrt{A(z_1)}^{(+)} \vee \sqrt{B(z_2)}^{(-)})$$
$$= (\sqrt{A} \times \sqrt{B})^{(+)}(z_1 * [(x_1 * y_1) * (y_1 * x_1^2)]), (z_2 * [(x_2 * y_2) * (y_2 * x_2^2)])$$
$$\vee(\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2)$$
$$= (\sqrt{A} \times \sqrt{B})^{(-)}(z_1, z_2) * [((x_1, x_2) * (y_1, y_2)) * ((y_1, y_2) * (x_1, x_2)^2)])$$
$$\vee(\sqrt{A} \times \sqrt{B})^{(-)}(z_1, z_2).$$

Thus $\sqrt{A} \times \sqrt{B}$ satisfies the conditions ($SRFSII_2$) and ($SRFSII_2$). So $\sqrt{A} \times \sqrt{B}$ is an SRFSII of $X \times X$. □

**Proposition 5.12.** *Let $(X, *, 0)$ be a KU-algebra and let $\sqrt{A}$, $\sqrt{B} \in SRF(X)$. If $\sqrt{A}$ and $\sqrt{B}$ are SRFSCIs of $X$, then $\sqrt{A} \times \sqrt{B}$ is an SRFSCI of $X \times X$.*

*Proof.* It is obvious that $\sqrt{A}\times\sqrt{B}$ satisfies the condition ($SRFI_1$). Let $(x_1, x_2)$, $(y_1, y_2)$, $(z_1, z_2) \in X \times X$. Then we have

$$(\sqrt{A} \times \sqrt{B})^{(+)}((x_1, x_2) * (y_1, y_2)^2)$$
$$= \sqrt{A(x_1 * y_1^2)}^{(+)} \wedge \sqrt{B(x_2 * y_2^2)}^{(+)}$$
$$\geq (\sqrt{A(z_1 * [(y_1 * x_1^2) * y_1^2)])}^{(+)} \wedge \sqrt{A(z_1)}^{(+)})$$
$$\wedge(\sqrt{B(z_2 * [y_2 * x_2^2) * y_2^2)])}^{(+)} \wedge \sqrt{B(z_2)}^{(+)})$$
$$= (\sqrt{A(z_1 * [(y_1 * x_1^2) * y_1^2)])}^{(+)} \wedge \sqrt{B(z_2 * [(y_2 * x_2^2) * y_2^2)])}^{(+)})$$
$$\wedge(\sqrt{A(z_1)}^{(+)} \wedge \sqrt{B(z_2)}^{(+)})$$
$$= (\sqrt{A} \times \sqrt{B})^{(+)}(z_1 * [(y_1 * x_1^2) * y_1^2)]), (z_2 * [(y_2 * x_2^2) * y_2^2)])$$
$$\wedge(\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2)$$
$$= (\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2) * [((y_1, y_2) * (x_1, x_2)^2) * (y_1, y_2)^2)])$$
$$\wedge(\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2),$$

$$(\sqrt{A} \times \sqrt{B})^{(-)}((x_1, x_2) * (y_1, y_2)^2)$$
$$= \sqrt{A(x_1 * y_1^2)}^{(-)} \vee \sqrt{B(x_2 * y_2^2)}^{(-)}$$
$$\leq (\sqrt{A(z_1 * [(y_1 * x_1^2) * y_1^2)])}^{(-)} \vee \sqrt{A(z_1)}^{(-)})$$
$$\vee(\sqrt{B(z_2 * [y_2 * x_2^2) * y_2^2)])}^{(-)} \vee \sqrt{B(z_2)}^{(-)})$$
$$= (\sqrt{A(z_1 * [(y_1 * x_1^2) * y_1^2)])}^{(-)} \vee \sqrt{B(z_2 * [(y_2 * x_2^2) * y_2^2)])}^{(-)})$$
$$\vee(\sqrt{A(z_1)}^{(-)} \vee \sqrt{B(z_2)}^{(-)})$$
$$= (\sqrt{A} \times \sqrt{B})^{(-)}(z_1 * [(y_1 * x_1^2) * y_1^2)]), (z_2 * [(y_2 * x_2^2) * y_2^2)])$$
$$\vee(\sqrt{A} \times \sqrt{B})^{(-)}(z_1, z_2)$$
$$= (\sqrt{A} \times \sqrt{B})^{(+)}(z_1, z_2) * [((y_1, y_2) * (x_1, x_2)^2) * (y_1, y_2)^2)])$$
$$\vee(\sqrt{A} \times \sqrt{B})^{(-)}(z_1, z_2).$$

Thus $\sqrt{A} \times \sqrt{B}$ satisfies the conditions ($SRFSCI_2$) and ($SRFSCI_2$). So $\sqrt{A} \times \sqrt{B}$ is an SRFSCI of $X \times X$. □

## 6. Conclusions

A fuzzy set theory becomes a strong area of making observations in different area's like medical science, social sciences, engineering, management sciences, artificial intelligence, robotics, computer networks, decision making and so on. We studied square root fuzzy sub-implicative (sub-commutative) ideals of $KU$-algebras, the image and the pre-images of the square root fuzzy sub-implicative (sub-commutative) ideals of $KU$-algebras under homomorphism were defined and how the image and the pre-image of the square root fuzzy sub-implicative (sub-commutative) ideals of $KU$-algebras become the square root fuzzy sub-implicative (sub-commutative) ideals studied. Moreover, the product of the square root fuzzy sub-implicative (sub-commutative) ideals of $KU$-algebras was established. The main purpose of our future work is to investigate the crossing square root structure, crossing square root cubic structure topology and crossing square root cubic structure $m$-polar of $KU$-ideal in $KU$-algebras. In the future, we will focus on square root fuzzy soft sub-implicative (sub-commutative) ideals of $KU$-algebras and it's applications in artificial intelligence and general systems. this method is very effective for data analysis and one may apply for medical diagnosis.

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