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New approach for closure spaces by relations via ideals

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Abstract. The main aim of this paper is to integrate the relationships among rough set theory and topology. We introduce the interior and closure operators with respect to an ideal defined on an approximation space, generating an ideal approximation space. Some topological notions such as the subspace, continuous functions, lower separation axioms, and connectedness in ideal approximation spaces are defined and studied. Some examples are given to confirm the implications.

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1. Introduction

Pawlak achieved the theory of rough sets for the first time in 1982 [1, 2]. This theory is an extension of the set theory for the investigation of intelligent systems identified by insufficient and incomplete information [3, 4, 5]. The upper and lower approximation operators are introduced under an equivalence relation given in the universe, which is not sufficient in some situations. From this time, many mathematicians, logicians, and researchers were interested in studying the theory and the extensions of the results and applications. These applications appear in wide fields such as data mining, machine learning, and expert systems [6, 7, 8, 9, 10, 11, 12, 13, 14]. This theory depends on a certain topological structure. It should be noted that the notion of a topological rough set is a very important generalisation of a rough set since it narrows the gap between topological researchers and those who are attentive to the application of topology theory.

Kuratowski [15] and Vaidyanathaswamy [16] introduced and investigated the notion of an ideal on topological spaces. The concept of an ideal is fundamental in

topological spaces and plays an important role in studying topological problems. Jankovic and Hamlett [17] introduced further properties of ideals given on topological spaces.
Studying the interaction between topology and generalised rough set theory applying the notion of an ideal was the main target for many articles such as [11, 18, 19, 20]. Novel rough models called "ideal approximation spaces" have been introduced. In fact, these models enlarge the lower approximation and shrink the upper approximation of subsets, which means they increase their accuracy values. Some researchers followed this course of study and addressed some phenomena as presented in [19, 21, 22, 23, 24] Moreover, extensions of topology have been applied to provide new rough paradigms using certain topological structures and concepts like infra-topology, supra-topology, maximal and minimal neighbourhoods to deal with rough set notions and address some real-life problems [25, 26, 27, 28]. Recently, many authors studied some topological notions such as closure spaces, separation axioms, continuity, and connectedness in ideal approximation spaces [29, 30].
In this direction, we have dedicated this paper to generalise some topological concepts given in [31] with respect to the ideal closure spaces defined in [32, 33] and illustrates the relationship between them. This paper is organised as follows: In Section 2, we give a review of closure spaces with all the definitions related to this work. Section 3 is devoted to studying accumulation points, dense sets, and nowhere dense sets with respect to these definitions, and we gave some examples. In Section 4, we introduce and study subspaces of such spaces under a subideal defined on the given ideal. Separation axioms with respect to these ideal approximation spaces are reformulated via relational concepts in Section 5. We reformulate and study connectedness in these ideal approximation spaces in Section 6. Finally, some remarks and a conclusion are given.

2. Preliminaries

A *relation* R from a universe X to a universe X (a relation on X) is a subset of $X \times X$. The formula $(x, y) \in R$ is abbreviated as xRy and means that x is in relation R with y. Also, the aftersets of $x \in X$ is $xR = \{y : xRy\}$ and the forersets of $x \in X$ is $Rx = \{y : yRx\}$.

Definition 2.1 ([31])**.** Let R be any binary relation on X. Then a set $\langle x \rangle R$ is the intersection of all aftersets containing x, i.e.,

$$\langle x \rangle R = \begin{cases} \cap_{x \in yR}(yR) & \text{if } \exists y : x \in yR, \\ \phi & \text{otherwise.} \end{cases}$$

Also, $R \langle x \rangle$ is the intersection of all foresets containing x, i.e.,

$$R \langle x \rangle = \begin{cases} \cap_{x \in yR}(Ry) & \text{if } \exists y : x \in Ry, \\ \phi & \text{otherwise.} \end{cases}$$

Definition 2.2 ([31])**.** Let R be binary relation on X. For any subset A of X, a pair of *lower* and *upper approximations*, $\underset{\sim}{R}(A)$ and $\tilde{R}(A)$ are defined by:

$$\underset{\sim}{R}(A) = \{x \in A \ : \langle x \rangle \, R \subseteq A\}, \tag{2.1}$$

$$\tilde{R}(A) = A \cup \{x \in X \ : \langle x \rangle \, R \cap A \neq \phi\}. \tag{2.2}$$

Theorem 2.3 ([34])**.** *The upper approximation defined by* (2.2) *has the following properties:*

(1) $\tilde{R}(\phi) = \phi$,
(2) $\underset{\sim}{R}(A) \subseteq A \subseteq \tilde{R}(A)$ *for* $A \subseteq X$,
(3) $\tilde{R}(A \cup B) = \tilde{R}(A) \cup \tilde{R}(B) \ \forall A, \ B \subseteq X$,
(4) $\tilde{R}(\tilde{R}(A)) = \tilde{R}(A) \ \forall A \subseteq X$,
(5) $\tilde{R}(A) = (\underset{\sim}{R}(A^c))^c \ \forall A \subseteq X$*, where* A^c *denotes the complement of* A*.*

The operator $\tilde{R}(A)$ on $P(X)$ is called a *closure operator* and $(X, \tilde{R})$ is called a *closure space*. Moreover, it induces a topology on X denoted by τ_R and defined by

$$\tau_R = \{A \subseteq X : \tilde{R}(A^c) = A^c\}.$$

Definition 2.4 ([17])**.** Let X be a non-empty set. Then $\mathcal{I} \subseteq P(X)$ is called an *ideal* on X, if it satisfies the following conditions:

(i) $\phi \in \mathcal{I}$,
(ii) $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$,
(iii) $A, \ B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$.

Definition 2.5 ([32])**.** Let R be a binary relation on X and $\mathcal{I}$ be an ideal defined on X and $A \subseteq X$. Then the *lower* and *upper approximations of* A *by* $\mathcal{I}$, denoted by $\underline{R}(A)$ and $\overline{R}(A)$, of A are defined as follows:

$$\underline{R}(A) = \{x \in A \ : \langle x \rangle \, R \cap A^c \in \mathcal{I}\}, \tag{2.3}$$

$$\overline{R}(A) = A \cup \{x \in X \ : \langle x \rangle \, R \cap A \notin \mathcal{I}\}. \tag{2.4}$$

Theorem 2.6 ([32])**.** *The upper approximation defined by* (2.4) *has the following properties: for any* $A, \ B \subseteq X$*,*

(1) $\overline{R}(A) = (\underline{R}(A^c))^c$,
(2) $\overline{R}(\phi) = \phi$,
(3) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$,
(4) $A \subseteq B$ *implies* $\overline{R}(A) \subseteq \overline{R}(B)$,
(5) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$,
(6) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$,
(7) $\overline{R}(\overline{R}(A)) = \overline{R}(A)$.

Also, the operator $\overline{R}(A)$ on $P(X)$ defined by (2.4), induced a topology on X denoted by τ_R^* and defined as $\tau_R^* = \{A \subseteq X : \overline{R}(A^c) = A^c\}$.

Definition 2.7 ([33])**.** Let R be a binary relation on X and $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then the *lower* and *upper approximations*, $\underline{\underline{R}}(A)$ and $\overline{\overline{R}}(A)$ of A are defined by:

$$\underline{\underline{R}}(A) = \{x \in A : R\langle x\rangle R \cap A^c \in \mathcal{I}\}, \tag{2.5}$$

$$\overline{\overline{R}}(A) = A \cup \{x \in X : R\langle x\rangle R \cap A \notin \mathcal{I}\}, \tag{2.6}$$

where

$$R\langle x\rangle R = R\langle x\rangle \cap \langle x\rangle R. \tag{2.7}$$

Theorem 2.8 ([33])**.** *The upper approximation defined by (2.6) has the following properties: for any $A,\ B \subseteq X$,*

1. $\overline{\overline{R}}(A) = (\underline{\underline{R}}(A^c))^c$,
2. $\overline{\overline{R}}(\phi) = \phi$,
3. $\underset{\sim}{R}(A) \subseteq \underline{R}(A) \subseteq \underline{\underline{R}}(A) \subseteq A \subseteq \overline{\overline{R}}(A) \subseteq \overline{R}(A) \subseteq \widetilde{R}(A)$,
4. $A \subseteq B$ *implies* $\overline{\overline{R}}(A) \subseteq \overline{\overline{R}}(B)$,
5. $\overline{\overline{R}}(A \cap B) \subseteq \overline{\overline{R}}(A) \cap \overline{\overline{R}}(B)$,
6. $\overline{\overline{R}}(A \cup B) = \overline{\overline{R}}(A) \cup \overline{\overline{R}}(B)$,
7. $\overline{\overline{R}}(\overline{\overline{R}}(A)) = \overline{R}(A)$.

Also, the operator $\overline{\overline{R}}(A)$ on $P(X)$ defined by (2.6), induced a topology on X denoted by τ_R^{**} and defined as $\tau_R^{**} = \{A \subseteq X : \overline{\overline{R}}(A^c) = A^c\}$. It is clear that $\tau_R \subseteq \tau_R^* \subseteq \tau_R^{**}$.

Lemma 2.9 ([31, 35])**.** *Let R be a binary relation on X.*

1. *If $x \in < y > R$, then $< x > R \subseteq < y > R$.*
2. *If $x \in R < y > R$, then $R < x > R \subseteq R < y > R$.*

Definition 2.10 ([31])**.** Let R be a binary relation on X. Then a point $x \in X$ is called an *accumulation point* of A, if $(\langle x\rangle R - \{x\}) \cap A \neq \phi$. The set of all accumulation points of A is denoted by $d(A)$, i.e.,

$$d(A) = \{x \in X : (< x > R - \{x\}) \cap A \neq \phi\}.$$

Definition 2.11 ([31])**.** Let $R_Y \subseteq R$ and $Y \subseteq X$. Then $(Y, \widetilde{R_Y})$ is called a *closure subspace* of a closure space $(X, \widetilde{R})$, if $\langle x\rangle R_Y = \langle x\rangle R \cap Y$ for all $x \in Y$.

3. Closure Spaces by relation via ideals

Lemma 3.1. *Let $(X, R, \mathcal{I})$ be an ideal approximation space. Then*

1. $\underline{R}(< x > R) =< x > R$,
2. $\underline{\underline{R}}(R < x > R) = R < x > R$.

Proof. (1) From Theorem 2.6 (3), it is obvious that $\underline{R}(\langle x\rangle R) \subseteq \langle x\rangle R$. Conversely, we prove that $< x > R \subseteq \underline{R}(< x > R)$. Let $y \in \langle x\rangle R$. Then by Lemma 2.9 (1), $\langle y\rangle R \subseteq \langle x\rangle R$. Thus $\langle y\rangle R \cap (\langle x\rangle R)^c = \phi$. So $\langle y\rangle R \cap (\langle x\rangle R)^c \in \mathcal{I}$. Hence $y \in \underline{R}(\langle x\rangle R)$. Therefore $\langle x\rangle R \subseteq \underline{R}(\langle x\rangle R)$.

(2) Similar to (1) by using Lemma 2.9 (2). □

Corollary 3.2. *Let $(X, R, \mathcal{I})$ be an ideal approximation space. Then*
(1) $\overline{R}((< x > R)^c) = (< x > R)^c$,
(2) $\overline{\overline{R}}((R < x > R)^c) = (R < x > R)^c$.

Proof. Straightforward by Theorem2.6 (1) and Theorem2.8 (1). □

Proposition 3.3. *Let $(X, R, \mathcal{I})$ be an ideal approximation space. For $x \neq y \in X$,*
(1) $x \in \overline{R}(\{y\})$ *iff* $\langle x\rangle R \cap \{y\} \notin \mathcal{I}$ *and* $x \notin \overline{R}(\{y\})$ *iff* $\langle x\rangle R \cap \{y\} \in \mathcal{I}$,
(2) $x \in \overline{\overline{R}}(\{y\})$ *iff* $R\langle x\rangle R \cap \{y\} \notin \mathcal{I}$ *and* $x \notin \overline{\overline{R}}(\{y\})$ *iff* $R\langle x\rangle R \cap \{y\} \in \mathcal{I}$.

Proof. (1) Let $x \in \overline{R}(\{y\})$. Then $x \in (\{y\} \cup \{z \in X : \langle z\rangle R \cap \{y\} \notin \mathcal{I}\})$. Thus $\langle x\rangle R \cap \{y\} \notin \mathcal{I}$. Conversely, let $\langle x\rangle R \cap \{y\} \notin \mathcal{I}$. Then by Definition 2.5, $x \in \overline{R}(\{y\})$.
The proof of the second part is similar.
(2) Similar to (1). □

Proposition 3.4. *Let $(X, R, \mathcal{I})$ be an ideal approximation space and $< x > R \in \mathcal{I}$. Then we have:*
(1) $\underline{R}(\{x\}) = \{x\} = \overline{R}(\{x\})$,
(2) $\underline{\underline{R}}(\{x\}) = \{x\} = \overline{\overline{R}}(\{x\})$.

Proof. (1) Let $\langle x\rangle R \in \mathcal{I}$. Then $\langle x\rangle R \cap (\{x\})^c \in \mathcal{I}$. Thus $x \in \underline{R}(\{x\})$. So $\underline{R}(\{x\}) = \{x\}$. Also, $\langle x\rangle R \in \mathcal{I}$ implies that $\langle x\rangle R \cap \{y\} \in \mathcal{I}$ for all $y \in X$. Hence $\overline{R}(\{x\}) = \{x\}$.
(2) Similar to (1). □

Definition 3.5. Let $(X, R, \mathcal{I})$ be an ideal approximation space and $A \subseteq X$. Then a point $x \in X$ is said to be:
(i) a $*$-*ideal accumulation point* of A, if $(\langle x\rangle R - \{x\}) \cap A \notin \mathcal{I}$.

The set of all $*$-ideal accumulation points of A is denoted by $d^*(A)$, i.e.,

$$d^*(A) = \{x \in X : (\langle x\rangle R - \{x\}) \cap A \notin \mathcal{I}\}.$$

(ii) a $**$-*ideal accumulation point* of A, if $(R\langle x\rangle R - \{x\}) \cap A \notin \mathcal{I}$.

The set of all $**$-ideal accumulation points of A is denoted by $d^{**}(A)$, i.e.,

$$d^{**}(A) = \{x \in X : (R\langle x\rangle R - \{x\}) \cap A \notin \mathcal{I}\}.$$

Lemma 3.6. *Let $(X, R, \mathcal{I})$ be an ideal approximation space and $A \subseteq X$. Then*
(1) $\overline{R}(A) = A \cup d^*(A)$,
(2) $\overline{R}(A) = A$ *iff* $d^*(A) \subseteq A$,
(3) $\overline{\overline{R}}(A) = A \cup d^{**}(A)$,
(4) $\overline{\overline{R}}(A) = A$ *iff* $d^{**}(A) \subseteq A$.

Proof. (1) Let $x \in \overline{R}(A)$. Then $x \in (A \cup \{y \in X : \langle y\rangle R \cap A \notin \mathcal{I}\})$. Thus we have either $x \in A$, i.e,

$$x \in A \cup d^*(A) \tag{3.1}$$

or $x \notin A$. So $x \in \{y \in X : \langle y\rangle R \cap A \notin \mathcal{I}\}$. In the latter case, we have $(\langle x\rangle R - \{x\}) \cap A \notin \mathcal{I}$. Hence $x \in d^*(A)$, i.e,

$$x \in A \cup d^*(A) \tag{3.2}$$

From (3.1) and (3.2), $\overline{R}(A) \subseteq A \cup d^*(A)$. Conversely, let $x \in A \cup d^*(A)$. Then we have either $x \in A$, i.e,

$$x \in \overline{R}(A) \tag{3.3}$$

or $x \notin A$. Thus $x \in d^*(A)$. So $(\langle x \rangle R - \{x\}) \cap A \notin \mathcal{I}$. Hence $x \in \overline{R}(A)$, i.e,

$$x \in \overline{R}(A) \tag{3.4}$$

From (3.3) and (3.4), $A \cup d^*(A) \subseteq \overline{R}(A)$. Therefore $\overline{R}(A) = A \cup d^*(A)$.

(2) Let $x \notin A$, i.e., $x \notin \overline{R}(A)$. Then clearly, $\langle x \rangle R \cap A \in \mathcal{I}$. Thus $(\langle x \rangle R - \{x\}) \cap A \in \mathcal{I}$ and $x \notin d^*(A)$. Conversely, let $d^*(A) \subseteq A$. Then by (1), $d^*(A) \cup A = \overline{R}(A) = A$.

(3) Similar to (1).

(4) Similar to (2). □

Theorem 3.7. *Let $(X, R, \mathcal{I})$ be an ideal approximation space and $x \in X, A \subseteq X$. If $\langle x \rangle R \cap A \in \mathcal{I}$, then*

(1) $\langle x \rangle R \cap \overline{R}(A) \in \mathcal{I}$,

(2) $R \langle x \rangle R \cap \overline{\overline{R}}(A) \in \mathcal{I}$.

Proof. (1) Suppose $\langle x \rangle R \cap A \in \mathcal{I}$. It is clear that $(\langle x \rangle R - \{x\}) \cap A \in \mathcal{I}$. Then $x \notin d^*(A)$. Thus $\langle x \rangle R \cap d^*(A) = \phi$. So $\langle x \rangle R \cap d^*(A) \in \mathcal{I}$. Hence $\langle x \rangle R \cap (A \cup d^*(A)) \in \mathcal{I}$. Therefore $\langle x \rangle R \cap \overline{R}(A) \in \mathcal{I}$.

(2) Similar to (1). □

Lemma 3.8. *Let $(X, R, \mathcal{I})$ be an ideal approximation space and $A, B \subseteq X$. Then*

(1) *if $A \subseteq B$, then $d^*(A) \subseteq d^*(B)$ and $d^{**}(A) \subseteq d^{**}(B)$,*

(2) *$d^*(A \cup B) = d^*(A) \cup d^*(A)$ and $d^{**}(A \cup B) = d^{**}(A) \cup d^{**}(A)$,*

(3) *$d^*(A \cap B) \subseteq d^*(A) \cap d^*(A)$ and $d^{**}(A \cap B) \subseteq d^{**}(A) \cap d^{**}(A)$,*

(4) *$d^*(A \cup d^*(A)) \subseteq A \cup d^*(A)$ and $d^{**}(A \cup d^{**}(A)) \subseteq A \cup d^{**}(A)$.*

Proof. (1) Suppose $A \subseteq B$ and let $x \in d^*(A)$. Then $(\langle x \rangle R - \{x\}) \cap A \notin \mathcal{I}$. Thus $(\langle x \rangle R - \{x\}) \cap B \notin \mathcal{I}$. So $x \in d^*(B)$. The proof of the second part is similar.

(2) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (1), we have

$$d^*(A) \cup d^*(B) \subseteq d^*(A \cup B).$$

Conversely, let $x \notin (d^*(A) \cup d^*(B))$. Then $x \notin d^*(A)$ and $x \notin d^*(B)$. Thus $(\langle x \rangle R - \{x\}) \cap A \in \mathcal{I}$ and $(\langle x \rangle R - \{x\}) \cap B \in \mathcal{I}$. So $(\langle x \rangle R - \{x\}) \cap (A \cup B) \in \mathcal{I}$. Hence $x \in d^*(A \cup B)$. The proof of the second part is similar.

(3) Similar to (2).

(4) Let $x \notin A \cup d^*(A)$. It is obvious that $x \notin A$ and $(\langle x \rangle R - \{x\}) \cap A \in \mathcal{I}$. Then $\langle x \rangle R \cap A \in \mathcal{I}$. Thus $x \notin \overline{R}(A)$. So $x \notin \overline{R}(\overline{R}(A))$. Hence $x \notin d^*(\overline{R}(A)) = d^*(A \cup d^*(A))$. Therefore $d^*(A \cup d^*(A)) \subseteq A \cup d^*(A)$. The proof of the second part is similar. □

Corollary 3.9. *Let $(X, R, \mathcal{I})$ be any ideal approximation space and $A \subseteq X$. Then*

$$d^{**}(A) \subseteq d^*(A) \subseteq d(A).$$

Proof. Let $x \notin d(A)$. Then $(\langle x \rangle R - \{x\}) \cap A = \phi$. Thus $(\langle x \rangle R - \{x\}) \cap A \in \mathcal{I}$. So $x \notin d^*(A)$ and $(R \langle x \rangle R - \{x\}) \cap A \in \mathcal{I}$, where $R \langle x \rangle R \subseteq < x > R$. Hence $x \notin d^{**}(A)$. Therefore $d^{**}(A) \subseteq d^*(A) \subseteq d(A)$. □

Remark 3.10. The following example shows that the converse of Corollary 3.9 is not true in general.

Example 3.11. Let $X = \{a, b, c\}$, $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ and $\mathcal{I} = \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Then $\langle a\rangle R = \{a, b, c\}$, $\langle b\rangle R = \{b, c\}$, $\langle c\rangle R = \{c\}$. Also, $R\langle a\rangle = \{a\}$, $R\langle b\rangle = \{a, b\}$, $R\langle c\rangle = \{a, b, c\}$. Thus $R\langle a\rangle R = \{a\}$, $R\langle b\rangle R = \{b\}$, $R\langle c\rangle R = \{c\}$. Suppose $A = \{b, c\}$. Then we have

$$(< a > R - \{a\}) \cap A = \{b, c\} \neq \phi,$$

$$(< b > R - \{b\}) \cap A = \{c\} \neq \phi,$$

$$(< c > R - \{c\}) \cap A = \phi.$$

Thus $a \in d(A)$, $b \in d(A)$, $c \notin d(A)$. So $d(A) = \{a, b\}$. On the other hand, we get

$$(< a > R - \{a\}) \cap A = \{b, c\} \notin \mathcal{I},$$

$$(< b > R - \{b\}) \cap A = \{c\} \in \mathcal{I},$$

$$(< c > R - \{c\}) \cap A = \phi \in \mathcal{I}.$$

Then $a \in d^*(A)$, $b \notin d^*(A)$ $c \notin d^*(A)$. Thus $d^*(A) = \{a\}$. Also, we have

$$(R < a > R - \{a\}) \cap A = \phi \in \mathcal{I},$$

$$(R < b > R - \{b\}) \cap A = \phi \in \mathcal{I},$$

$$(R < c > R - \{c\}) \cap A = \phi \in \mathcal{I}.$$

Then $a \notin d^{**}(A)$, $b \notin d^{**}(A)$, $c \notin d^{**}(A)$. Thus $d^{**}(A) = \phi$. So $d(A) \nsubseteq d^*(A) \nsubseteq d^{**}(A)$.

Definition 3.12. Let $(X, R, \mathcal{I})$ be any ideal approximation space and $A \subseteq X$. Then A is said to be:

(i) *dense*, if $\widetilde{R}(A) = X$,
(ii) $*$*-ideal dense*, if $\overline{R}(A) = X$,
(iii) $**$*-ideal dense*, if $\overline{\overline{R}}(A) = X$,
(iv) *nowhere dense*, if $\underset{\sim}{R}(\widetilde{R}(A)) = \phi$,
(v) $*$*-ideal nowhere dense*, if $\underset{\sim}{R}(\overline{R}(A)) = \phi$,
(vi) $**$*-ideal nowhere dense*, if $\underset{\sim}{R}(\overline{\overline{R}}(A)) = \phi$.

Corollary 3.13. *Let $(X, R, \mathcal{I})$ be any ideal approximation space and $A \subseteq X$. Then*

(1) $**$*-ideal dense* $\Rightarrow$ $*$*-ideal dense* $\Rightarrow$ *dense,*
(2) *nowhere dense* $\Rightarrow$ $*$*-ideal nowhere dense* $\Rightarrow$ $**$*-ideal nowhere dense.*

Proof. (1) Immediately by Theorem 2.8 (3).

(2) Suppose A is nowhere dense. Then $\underset{\sim}{R}(\widetilde{R}(A)) = \phi$. Thus $\underset{\sim}{R}(\overline{R}(A)) = \phi$ and $\underset{\sim}{R}(\overline{\overline{R}}(A)) = \phi$. So A is $*$-ideal nowhere dense and $**$-ideal nowhere dense. □

Example 3.14. (1) Let $X=\{a,b,c,d\}, R=\{(a,a),(a,b),(b,b),(b,c),(c,c),(d,d),(d,b)\}$, $\mathcal{I}=\{\phi,\{b\}\}$ and $A=\{b,c\}$. Then we have

$$\langle a\rangle R=\{a,b\},\ \langle b\rangle R=\{b\},\ \langle c\rangle R=\{c\},\ \langle d\rangle R=\{b,d\},$$

$$R\langle a\rangle=\{a\},\ R\langle b\rangle=\{b\},\ R\langle c\rangle=\{b,c\},\ R\langle d\rangle=\{d\}.$$

Thus we get

$$R\langle a\rangle R=\{a\},\ R\langle b\rangle R=\{b\},\ R\langle c\rangle R=\{c\},\ R\langle d\rangle R=\{d\}.$$

Suppose $A=\{b,c\}$. Then $\widetilde{R}(A)=A\cup\{x\in X\ :\langle x\rangle R\cap A\neq\phi\}=X$. Thus A is dense. But $\{x\in X\ :\langle x\rangle R\cap A\notin\mathcal{I}\}=\{c\}$ and $\overline{R}(A)=\{b,c\}\neq X$. So A is not $*$-ideal dense. On the other hand, suppose $\mathcal{I}=\{\phi,\{a\}\}$. Then $\overline{R}(A)=X$. Thus A is $*$-ideal dense. But $\overline{\overline{R}}(A)=A\cup\{x\in X\text{ ‘ }:R\langle x\rangle R\cap A\notin\mathcal{I}\}=A\neq X$. So A is not $**$-ideal dense.

(2) Let $X=\{a,b,c,d\}$, $R=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,d),(c,a),(c,b),(c,d),(d,d)\}$, $\mathcal{I}=\{\phi,\{a\}\}\}$. Then we have

$$\langle a\rangle R=\{a,b\},\ \langle b\rangle R=\{a,b\},\ \langle c\rangle R=\{a,b,c\},\ \langle d\rangle R=\{d\},$$

$$R\langle a\rangle=\{a\},\ R\langle b\rangle=\{b,c\},\ R\langle c\rangle=\{b,c\},\ R\langle d\rangle=\{b,c,d\}.$$

Thus we get

$$R\langle a\rangle R=\{a\},\ R\langle b\rangle R=\{b\},\ R\langle c\rangle R=\{b,c\},\ R\langle d\rangle R=\{d\}.$$

Suppose $A=\{a\}$. Then $\overline{R}(A)=\{a\}$ and $\widetilde{R}(A)=\{a,b,c\}$. Thus $\underset{\sim}{R}(\overline{R}(A))=\phi$. So A is $*$-ideal nowhere dense. But $\underset{\sim}{R}(\widetilde{R}(A))=\{a,b,c\}\neq\phi$. Hence A is not nowhere dense. Also, suppose $A=\{b\}$. Then $\overline{R}(A)=\{a,b,c\}$ and $\overline{\overline{R}}(A)=\{b,c\}$. Thus $\underset{\sim}{R}(\overline{\overline{R}}(A))=\phi$. So A is $**$-ideal nowhere dense. But $\underset{\sim}{R}(\overline{R}(A))=\{a,b,c\}\neq\phi$. So A is not $*$-ideal nowhere dense.

Corollary 3.15. *Let $(X,R,\mathcal{I})$ be an ideal approximation space.*

1. *If A is dense, then $(\widetilde{R}(A))^c$ is nowhere dense.*
2. *If A is $*$-dense, then $(\overline{R}(A))^c$ is $*$-nowhere dense.*
3. *If A is $**$-dense, then $(\overline{\overline{R}}(A))^c$ is $**$-nowhere dense .*

Proof. (1) Suppose A is dense. Then $\widetilde{R}(A)=X$. Thus $(\widetilde{R}(A))^c=\phi$ and $\widetilde{R}(\widetilde{R}(A))^c)=\phi$. So $\underset{\sim}{R}(\widetilde{R}(\widetilde{R}(A))^c))=\phi$. Hence $(\widetilde{R}(A))^c$ is nowhere dense.

(2) Suppose A is $*$-dense. Then $\overline{R}(A)=X$. Thus $(\overline{R}(A))^c=\phi$. So $\overline{R}((\overline{R}(A))^c)=\phi$ and $\underset{\sim}{R}(\overline{R}((\overline{R}(A))^c))=\phi$. Hence $(\overline{R}(A))^c$ is $*$-nowhere dense.

(3) Similar to (2). $\square$

4. Ideal approximation subspace

Lemma 4.1. *If $\mathcal{I}$ is an ideal on X and $Y \subseteq X$, then $\mathcal{I}_Y = \{A \cap Y : A \in \mathcal{I}\}$ is an ideal on Y.*

Proof. (i) It is obvious that $\phi \in \mathcal{I}_Y$.

(ii) Suppose $B \subseteq A,\ A \in \mathcal{I}_Y$. Then there exists $C \in \mathcal{I}$ such that $A = C \cap Y$. Since $\mathcal{I}$ is an ideal and $B \subseteq C$, $B \in \mathcal{I}$. Thus $B = B \cap Y \in \mathcal{I}_Y$.

(iii) Suppose $A,\ B \in \mathcal{I}_Y$. Then there exists $C_1, C_2 \in \mathcal{I}$ such that $A = C_1 \cap Y, B = C_2 \cap Y$. Thus $A \cup B = (C_1 \cup C_2) \cap A \in \mathcal{I}_Y$. So $\mathcal{I}_Y$ is an ideal on Y. □

Definition 4.2. Let $Y \subseteq X, R_Y = R \cap (Y \times Y) \subseteq R$ and $\mathcal{I}_Y \subseteq \mathcal{I}$. Then $(Y, \overline{R_Y}, \mathcal{I}_Y)$ is called an *ideal closure subspace* of an ideal closure space $(X, \overline{R}, \mathcal{I})$, if $\langle x \rangle R_Y = \langle x \rangle R \cap Y$ for all $x \in Y$.

Lemma 4.3. *Let $(Y, \overline{R_Y}, \mathcal{I}_Y)$ be an ideal closure subspace of an ideal closure space $(X, \overline{R}, \mathcal{I})$. Then $\langle x \rangle R_Y = \langle x \rangle R \cap Y$ for all $x \in Y$ iff $\overline{R_Y}(A) = \overline{R}(A) \cap Y$ for all $A \subseteq Y$.*

Proof. Suppose $\langle x \rangle R_Y = \langle x \rangle R \cap Y$ for all $x \in Y$. We want to show that $\overline{R_Y}(A) = \overline{R}(A) \cap Y$ for all $A \subseteq Y$. Then

$$\begin{aligned}
\overline{R}(A) \cap Y &= (A \cup \{x \in X : \langle x \rangle R \cap A \notin \mathcal{I}\}) \cap Y \\
&= (A \cap Y) \cup (\{x \in Y : \langle x \rangle R \cap A \notin \mathcal{I}\}) \\
&= A \cup \{x \in Y : \langle x \rangle R_Y \cap A \notin \mathcal{I}_A\} \\
&= \overline{R_Y}(A).
\end{aligned}$$

Conversely, suppose that $\overline{R_Y}(A) = \overline{R}(A) \cap Y$ for all $A \subseteq Y$. Then

$$\begin{aligned}
\overline{R_Y}(A) &= A \cup \{x \in Y : \langle x \rangle R_Y \cap A \notin \mathcal{I}_Y\} \\
&= (A \cup \{x \in X : \langle x \rangle R \cap A \notin \mathcal{I}\}) \cap Y \\
&= (A \cap Y) \cup (\{x \in Y : \langle x \rangle R \cap A \notin \mathcal{I}\}) \\
&= A \cup \{x \in Y : (\langle x \rangle R \cap Y) \cap A \notin \mathcal{I}\}.
\end{aligned}$$

Thus we have $\langle x \rangle R_Y = \langle x \rangle R \cap Y$ for all $x \in Y$. □

Corollary 4.4. *Let $(X, \overline{R}, \mathcal{I})$ be an ideal closure space and $Y \subseteq X$. Then $(Y, \overline{R_Y}, \mathcal{I}_Y)$ is an ideal closure subspace iff $\overline{R_Y}(A) = \overline{R}(A) \cap Y$ for all $A \subseteq Y$.*

Proof. The proof is straightforward from Definition 4.2 and Lemma 4.3. □

The following Lemma show that the ideal closure subspace $(Y, \overline{R_Y}, \mathcal{I}_Y)$ is a topological space.

Lemma 4.5. *An ideal closure subspace $(Y, \overline{R_Y}, \mathcal{I}_Y)$ of an ideal closure space $(X, \overline{R}, \mathcal{I})$ is a topological space.*

Proof. We want only show that the closure operator $\overline{R_Y}$ is idempotent. Let $A \subseteq Y$. Then

$$\begin{array}{rcl}\overline{R_Y}(\overline{R_Y}(A)) & = & \overline{R_Y}(\overline{R}(A) \cap Y) \\ & = & \overline{R}(\overline{R}(A) \cap Y) \cap Y \\ & \subseteq & \overline{R}(\overline{R}(A)) \cap \overline{R}(Y) \cap Y \\ & \subseteq & \overline{R}(A) \cap Y \\ & \subseteq & \overline{R_Y}(A).\end{array}$$

Note that $\overline{R_Y}(A) \subseteq \overline{R_Y}(\overline{R_Y}(A))$. Then $\overline{R_Y}(\overline{R_Y}(A)) = \overline{R_Y}(A)$. Thus $\overline{R_Y}$ is idempotent. □

Theorem 4.6. *Let* $(Y, \overline{R_Y}, \mathcal{I}_Y)$ *be an ideal closure subspace of an ideal closure space* $(X, \overline{R}, \mathcal{I})$ *and* $A \subseteq Y$. *Then*

(1) $d^*_Y(A) = d^*(A) \cap Y$,
(2) $\underline{R}(A) \cap Y \subseteq \underline{R_Y}(A)$ *and* $\underline{R}(A) \cap Y \neq \underline{R_Y}(A)$.

Proof. (1) Let $x \in d^*_Y(A)$ for each $x \in Y$. Then $(\langle x \rangle R_Y - \{x\}) \cap A \notin \mathcal{I}_Y$. Thus $((< x > R - \{x\}) \cap A) \cap Y \notin \mathcal{I}_Y$. So $(\langle x \rangle R - \{x\}) \cap A \notin \mathcal{I}$. Hence, $x \in d^*(A)$ and $x \in d^*(A) \cap Y$, i.e.,

$$d^*_Y(A) \subseteq d^*(A) \cap Y. \tag{4.1}$$

Conversely, let $x \in d^*(A) \cap Y$. Then. $x \in d^*(A)$. Thus $(\langle x \rangle R - \{x\}) \cap A) \notin \mathcal{I}$, i.e., $(((\langle x \rangle R - \{x\}) \cap A) \cap Y \notin \mathcal{I}_Y$. So $(\langle x \rangle R_Y - \{x\}) \cap A \notin \mathcal{I}_Y$. Hence $x \in d^*_Y(A)$, i.e.,

$$d^*(A) \cap Y \subseteq d^*_Y(A). \tag{4.2}$$

Therefore from 4.1 and 4.2, we have $d^*_Y(A) = d^*(A) \cap Y$.

(2) Let $x \in \underline{R}(A) \cap Y$. Then $x \in \underline{R}(A)$. Thus $\langle x \rangle R \cap A^c \in \mathcal{I}$. So $(\langle x \rangle R \cap Y) \cap A^c) \in \mathcal{I}_Y$, i.e., $\langle x \rangle R_Y \cap A^c \in \mathcal{I}_Y$. Hence $x \in \underline{R_Y}(A)$. Therefore $\underline{R}(A) \cap Y \subseteq \underline{R_Y}(A)$.

On the other hand, Let $X = \{a, b, c, d\}$, $R = \{(a, a), (a, c), (a, d), (b, b), (b, c), (c, c), (d, d)\}$, $\mathcal{I} = \{\phi, \{a\}, \{d\}, \{a, d\}\}$, $Y = \{a, b\}$ and $A = \{a\}$. Then we have

$$\langle a \rangle R = \{a, c, d\},\ \langle b \rangle R = \{b, c\},\ \langle c \rangle R = \{c\},\ \langle d \rangle R = \{d\}.$$

Thus $\underline{R}(A) = \{a : \langle a \rangle R \cap \{a\}^c \in \mathcal{I}\} = \phi$, $R_Y = \{(a, a), (b, b)\}$ and $\mathcal{I}_Y = \{\phi, \{a\}\}$. So $\langle a \rangle R_Y = \{a\}$, $\langle b \rangle R_Y = \{b\}$. Hence $\underline{R_Y}(A) = \{a : \langle a \rangle R_Y \cap \{a\}^c \in \mathcal{I}_Y\} = \{a\}$. Hence $\underline{R}(A) \cap Y \neq \underline{R_Y}(A)$. □

5. Lower separation axioms in ideal approximation spaces

Definition 5.1. (i) An approximation space (X, R) is called a T_0*-space*, if $\forall x \neq y \in X$, there exists $A \subseteq X$ such that

$$x \in \underset{\sim}{R}(A),\ y \notin A \text{ or } y \in \underset{\sim}{R}(A),\ x \notin A.$$

(ii) An ideal approximation space $(X, R, \mathcal{I})$ is called a T^*_0*-space*, if $\forall x \neq y \in X$ there exists $A \subseteq X$ such that

$$x \in \underline{R}(A),\ y \notin A \text{ or } y \in \underline{R}(A),\ x \notin A.$$

(iii) An ideal approximation space $(X, R, \mathcal{I})$ is called a T_0^{**}*-space*, if $\forall x \neq y \in X$ there exists $A \subseteq X$ such that

$$x \in \underline{\underline{R}}(A),\ y \notin A \text{ or } y \in \underline{\underline{R}}(A),\ x \notin A.$$

Proposition 5.2. *For an ideal approximation space $(X, R, \mathcal{I})$, the following are equivalent:*

(1) *X is a T_0^*-space,*
(2) *$\overline{R}(\{x\}) \neq \overline{R}(\{y\})$ for each $x \neq y \in X$,*
(3) *$(Y, R_Y, \mathcal{I}_Y)$ is a T_0^*-space for each $Y \subseteq X$.*

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds and let $x \neq y \in X$. Then there exists $A \subseteq X$ such that $x \in \underline{R}(A), y \notin A$. Thus $\langle x \rangle R \cap A^c \in \mathcal{I}$, $y \in A^c$. So $\langle x \rangle R \cap \{y\} \in \mathcal{I}$. Hence by Proposition 3.3 (1), $x \notin \overline{R}(\{y\})$. By the same way, we can prove that $y \notin \overline{R}(\{x\})$. Therefore $\overline{R}(\{x\}) \neq \overline{R}(\{y\})$.

(2) $\Rightarrow$ (3): Suppose (2) holds, and let $Y \subseteq X$ and $x \neq y \in Y$. Then $x \neq y \in X$ and by (2), $x \notin \overline{R}(\{y\})$ or $y \notin \overline{R}(\{x\})$. Thus by Proposition 3.3 (1), $\langle x \rangle R \cap \{y\} \in \mathcal{I}$ or $\langle y \rangle R \cap \{x\} \in \mathcal{I}$, i.e.,

$$x \in \underline{R_Y}(\{y\}^c),\ y \notin \{y\}^c \text{ or } y \in \underline{R_Y}(\{x\}^c),\ x \notin \{x\}^c.$$

So $(Y, R_Y, \mathcal{I}_Y)$ is a T_0^*-space.

(3) $\Rightarrow$ (1): Suppose (3) holds and let $x \neq y \in X$. Then there exists $Y \subseteq X$ such that $x \neq y \in Y$. By (3), there exists $A \subseteq Y$ such that

$$x \in \underline{R_Y}(A),\ y \notin A \text{ or } y \in \underline{R_Y}(A),\ x \notin A.$$

Then $\langle x \rangle R_Y \cap A^c \in \mathcal{I}_Y,\ y \notin A^c$ or $\langle y \rangle R_Y \cap A^c \in \mathcal{I}_Y,\ x \notin A^c$. Thus we get

$$\langle x \rangle R \cap A^c \in \mathcal{I},\ y \notin A^c \text{ or } \langle y \rangle R \cap A^c \in \mathcal{I},\ x \notin A^c.$$

So $\langle x \rangle R \cap \{y\} \in \mathcal{I}$ or $\langle y \rangle R \cap \{x\} \in \mathcal{I}$. Hence we have

$$x \in \underline{R}(\{y\}^c),\ y \notin \{y\}^c \text{ or } y \in \underline{R}(\{x\}^c),\ x \notin \{x\}^c.$$

Therefore X is a T_0^*-space. □

Corollary 5.3. *For an approximation space (X, R), the following are equivalent:*

(1) *X is a T_0-space,*
(2) *for each $x \neq y \in X$, either $x \notin \langle y \rangle R$ or $y \notin \langle x \rangle R$,*
(3) *$\tilde{R}(\{x\}) \neq \tilde{R}(\{y\})$ for each $x \neq y \in X$,*
(4) *(Y, R_Y) is a T_0-space for each $Y \subseteq X$.*

Corollary 5.4. *For an ideal approximation space $(X, R, \mathcal{I})$, the following are equivalent:*

(1) *X is a T_0^{**}-space,*
(2) *$\overline{\overline{R}}(\{x\}) \neq \overline{\overline{R}}(\{y\})$ for each $x \neq y \in X$.*
(3) *$(Y, R_Y, \mathcal{I}_Y)$ is a T_0^{**}-space for each $Y \subseteq X$.*

Definition 5.5. (i) An approximation space (X, R) is called a T_1*-space*, if $\forall x \neq y \in X$, there exists $A,\ B \subseteq X$ such that

$$x \in \underset{\sim}{R}(A),\ y \notin A \text{ and } y \in \underset{\sim}{R}(B),\ x \notin B.$$

(ii) An ideal approximation space $(X, R, \mathcal{I})$ is called a T_1^*-*space*, if $\forall x \neq y \in X$, there exists $A,\ B \subseteq X$ such that

$$x \in \underline{R}(A),\ y \notin A \text{ and } y \in \underline{R}(B),\ x \notin B.$$

(iii) An ideal approximation space $(X, R, \mathcal{I})$ is called a T_1^{**}-*space*, if $\forall x \neq y \in X$, there exists $A,\ B \subseteq X$ such that

$$x \in \underline{\underline{R}}(A),\ y \notin A \text{ and } y \in \underline{\underline{R}}(B),\ x \notin B.$$

Proposition 5.6. *For an ideal approximation space $(X, R, \mathcal{I})$, the following are equivalent:*
1. *X is a T_1^*-space,*
2. *$\overline{R}(\{x\}) = \{x\}$ for each $x \in X$,*
3. *$d^*(\{x\}) = \phi$ for each $x \in X$,*
4. *$(Y, R_Y, \mathcal{I}_Y)$ is a T_1^*-space for each $Y \subseteq X$.*

Proof. (1) $\Rightarrow$ (2): Suppose $(X, R, \mathcal{I})$ is T_1^*-space and let $x \in X$. Then for $y \in X - \{x\}$, $x \neq y$ and $\exists A \subseteq X$ such that $y \in \underline{R}(A)$, $x \notin A$. Thus $\langle y \rangle R \cap A^c \in \mathcal{I}$, $x \in A^c$. So $\langle y \rangle R \cap \{x\} \in \mathcal{I}$, i.e., $y \notin \overline{R}(\{x\})$. Hence

$$\overline{R}(\{x\}) = \{x\}.$$

(2) $\Rightarrow$ (3): Suppose (2) holds and let $x \in X$. Then $\overline{R}(\{x\}) = \{x\} \cup d^*(\{x\})$ but $x \notin d^*(\{x\})$. Thus

$$d^*(\{x\}) = \phi.$$

(3) $\Rightarrow$ (4): Suppose (3) holds and let $x \neq y \in Y$ for each $Y \subseteq X$. Then clearly, $x \neq y \in X$. By (3), $d_Y^*(\{x\}) = d_Y^*(\{y\}) = \phi$. By Theorem 4.6 (1), $d^*(\{x\}) = d^*(\{y\}) = \phi$. Thus $\overline{R_Y}(\{x\}) = \{x\}$ and $\overline{R_Y}(\{y\}) = \{y\}$, i.e., $\underline{R_Y}(\{x\}^c) = \{x\}^c$ and $\underline{R_Y}(\{y\}^c) = \{y\}^c$. So there exist $\{x\}^c$ and $\{y\}^c \subseteq Y$ such that

$$y \in \underline{R_Y}(\{x\}^c),\ x \notin \{x\}^c \text{ and } x \in \underline{R_Y}(\{y\}^c),\ y \notin \{y\}^c.$$

Hence $(Y, R_Y, \mathcal{I}_Y)$ is a T_1^*-space.

(4) $\Rightarrow$ (1): Suppose (4) holds and and let $x \neq y \in X$. Then clearly, there exists $Y \subseteq X$ such that $x \neq y \in Y$. By (4), there exist $A,\ B \subseteq Y$ such that

$$x \in \underline{R_Y}(A),\ y \notin A) \text{ and } y \in \underline{R_Y}(A), x \notin A.$$

Thus $\langle x \rangle R \cap \{y\} \in \mathcal{I}$ and $\langle y \rangle R \cap \{x\} \in \mathcal{I}$. So we have

$$x \in \underline{R}(\{y\}^c),\ y \notin \{y\}^c \text{ and } y \in \underline{R}(\{x\}^c),\ x \notin \{x\}^c).$$

Hence X is a T_1^*-space. □

Corollary 5.7. *For an approximation space (X, R), the following are equivalent:*
1. *X is a T_1-space,*
2. *for each $x \neq y \in X$, $x \notin \langle y \rangle R$ and $y \notin \langle x \rangle R$,*
3. *$\widetilde{R}(\{x\}) = \{x\}$ for each $x \in X$,*
4. *$d(\{x\}) = \phi$ for each $x \in X$,*
5. *(Y, R_Y) is a T_1-space for each $Y \subseteq X$.*

Corollary 5.8. *For an ideal approximation space $(X, R, \mathcal{I})$, the following are equivalent:*

(1) X *is a* T_1^{**}*-space,*
(2) $\overline{\overline{R}}(\{x\}) = \{x\}$ *for each* $x \in X$,
(3) $d^{**}(\{x\}) = \phi$ *for each* $x \in X$,
(4) $(Y, R_Y, \mathcal{I}_Y)$ *is a* T_1^{**}*-space for each* $Y \subseteq X$.

Definition 5.9. (i) An approximation space (X, R) is called an R_0*-space*, if it satisfies the following condition: for any $x \neq y \in X$,

$$\widetilde{R}(\{x\}) = \widetilde{R}(\{y\}) \text{ or } \widetilde{R}(\{x\}) \cap \widetilde{R}(\{y\}) = \phi.$$

(ii) An ideal approximation space $(X, R, \mathcal{I})$ is called an R_0^**-space*, if it satisfies the following condition: for any $x \neq y \in X$,

$$\overline{R}(\{x\}) = \overline{R}(\{y\}) \text{ or } \overline{R}(\{x\}) \cap \overline{R}(\{y\}) = \phi.$$

(iii) An ideal approximation space $(X, R, \mathcal{I})$ is called an R_0^{**}*-space*, if it satisfies the following condition: for any $x \neq y \in X$,

$$\overline{\overline{R}}(\{x\}) = \overline{\overline{R}}(\{y\}) \text{ or } \overline{\overline{R}}(\{x\}) \cap \overline{\overline{R}}(\{y\}) = \phi.$$

Proposition 5.10. *For an ideal approximation space* $(X, R, \mathcal{I})$*, the following are equivalent:*

(1) X *is an* R_0^**-space,*
(2) *if* $x \in \overline{R}(\{y\})$*, then* $y \in \overline{R}(\{x\})$ *for all* $x \neq y \in X$.

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds, and et x and y be two distinct points in $(X, R, \mathcal{I})$. Then clearly, $\overline{R}(\{x\}) = \overline{R}(\{y\})$ or $\overline{R}(\{x\}) \cap \overline{R}(\{y\}) = \phi$.

If $\overline{R}(\{x\}) = \overline{R}(\{y\})$, then $y \in \overline{R}(\{x\})$ and $x \in \overline{R}(\{y\})$.

If $\overline{R}(\{x\}) \cap \overline{R}(\{y\}) = \phi$, then $\{x\} \cap \overline{R}(\{y\}) = \phi$ and $\{y\} \cap \overline{R}(\{x\}) = \phi$. Thus $x \notin \overline{R}(\{y\})$ and $y \notin \overline{R}(\{x\}))$. So $x \notin \overline{R}(\{y\})$ and $y \notin \overline{R}(\{x\})$. Hence in either case, (2) holds.

(2) $\Rightarrow$ (1): Suppose (2) holds and let $x \neq y \in X$. The we have

$$\text{either } x \in \overline{R}(\{y\}) \text{ and } y \in \overline{R}(\{x\}) \text{ or } x \notin \overline{R}(\{y\}) \text{ and } y \notin \overline{R}(\{x\}).$$

If $x \in \overline{R}(\{y\})$ and $y \in \overline{R}(\{x\})$, then

$$\overline{R}(\{x\}) = \overline{R}(\{y\}). \tag{5.1}$$

If $x \notin \overline{R}(\{y\})$ and $y \notin \overline{R}(\{x\})$, then

$$\overline{R}(\{x\}) \cap \overline{R}(\{y\}) = \phi. \tag{5.2}$$

From (5.1) and (5.2), the proof is complete. □

Corollary 5.11. *For an approximation space* (X, R)*, the following are equivalent:*

(1) X *is an* R_0*-space,*
(2) *if* $x \in \langle y \rangle R$*, then* $y \in \langle x \rangle R$ *for all* $x \neq y \in X$.

Corollary 5.12. *For an ideal approximation space* $(X, R, \mathcal{I})$*, the following are equivalent:*

(1) X *is an* R_0^{**}*-space,*
(2) *if* $x \in \overline{\overline{R}}(\{y\})$*, then* $y \in \overline{\overline{R}}(\{x\})$ *for all* $x \neq y \in X$.

Definition 5.13. (i) An approximation space (X, R) is called a T_2-*space*, if $\forall x \neq y \in X$, there exist $A,\ B \subseteq X$ such that

$$x \in \underset{\sim}{R}(A),\ y \in \underset{\sim}{R}(B) \text{ and } A \cap B = \phi.$$

(ii) An ideal approximation space $(X, R, \mathcal{I})$ is called a T_2^*-*space*, if $\forall x \neq y \in X$, there exist $A,\ B \subseteq X$ such that

$$x \in \underline{R}(A),\ y \in \underline{R}(B) \text{ and } A \cap B = \phi.$$

(iii) An ideal approximation space $(X, R, \mathcal{I})$ is called a T_2^{**}-*space*, if $\forall x \neq y \in X$, there exist $A,\ B \subseteq X$ such that

$$x \in \underline{\underline{R}}(A),\ y \in \underline{\underline{R}}(B) \text{ and } A \cap B = \phi.$$

Theorem 5.14. *For an ideal approximation space $(X, R, \mathcal{I})$, the following are equivalent:*

(1) *X is a T_2^*-space,*
(2) *$\exists A \subseteq X : x \in \underline{R}(A), y \in (\overline{R}(A))^c$ for all $x \neq y \in X$.*

Proof. (1) $\Rightarrow$ (2): Suppose X is a T_2^*-space and let $\forall x \neq y \in X$. Then there exist $A,\ B \subseteq X$ such that $x \in \underline{R}(A),\ y \in \underline{R}(B)$ and $A \cap B = \phi$. Thus $\langle y \rangle R \cap B^c \in \mathcal{I}$ and $A \subseteq B^c$. So $(\langle y \rangle R - \{x\}) \cap A \in \mathcal{I}$, i.e., $y \notin d^*(A)$. Hence $\underline{R}(B) \cap d^*(A) = \phi$ and $\underline{R}(B) \cap A = \phi$, i.e., $\underline{R}(B) \cap \overline{R}(A) = \phi$. Therefore $x \in \underline{R}(A), y \in \underline{R}(B) \subseteq (\overline{R}(A))^c$.

(2) $\Rightarrow$ (1): Suppose (2) holds and let $x \neq y \in X$. Then by (2), there exists $A \subseteq X$ such that $x \in \underline{R}(A),\ y \in (\overline{R}(A))^c$. Let $B = (\overline{R}(A))^c$. Then $B = \underline{R}(A^c)$ from Theorem 2.6 (1 and thus $\underline{R}(B) = \underline{R}(\underline{R}(A^c)) = \underline{R}(A^c) = B$. Also $A \cap B = A \cap \underline{R}(A^c) \subseteq A \cap A^c = \phi$. So X is a T_2^*-space. □

Corollary 5.15. *For an approximation space (X, R), the following are equivalent:*

(1) *X is a T_2-space,*
(2) *$\exists A \subseteq X : x \in \underset{\sim}{R}(A),\ y \in (\widetilde{R}(A))^c$ for all $x \neq y \in X$.*

Corollary 5.16. *For an ideal approximation space $(X, R, \mathcal{I})$, the following are equivalent:*

(1) *X is a T_2^{**}-space,*
(2) *$\exists A \subseteq X : x \in \underline{\underline{R}}(A),\ y \in (\overline{\overline{R}}(A))^c$ for all $x \neq y \in X$.*

Corollary 5.17. *For an ideal approximation space $(X, R, \mathcal{I})$, the following are holds:*

(1) $T_1 = R_0 + T_0$,
(2) $T_1^* = R_0^* + T_0^*$,
(3) $T_1^{**} = R_0^{**} + T_0^{**}$.

Proof. Immediately derived from Definition 5.9, Proposition 5.2, 5.6 and Corollary 5.3, 5.4, 5.7 and 5.8. □

Remark 5.18. From Definition 5.1, 5.5 and 5.13 we have the following implication.

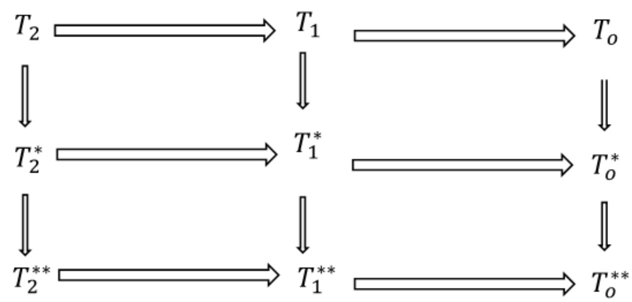

Figure 1. Implication

We introduce the following examples to show that the implication is not reversible. Also, examples show that $R_0 \not\Leftrightarrow T_0$, $R_0 \not\Leftrightarrow R_0^*$, $R_0^* \not\Leftrightarrow T_0^*$, $R_0^* \not\Leftrightarrow R_0^{**}$ and $R_0^{**} \not\Leftrightarrow T_0^{**}$.

Example 5.19. (1) Let $X = \{a, b, c\},\ R = \{(a, a), (a, b), (b, b), (c, c)\}$. Then $\langle a\rangle R = \{a, b\},\ \langle b\rangle R = \{b\},\ \langle c\rangle R = \{c\}$. Thus $\langle a\rangle R \cap \{c\} = \langle b\rangle R \cap \{a\} = \langle b\rangle R \cap \{c\} = \phi$. So $\widetilde{R}(\{a\}) \neq \widetilde{R}(\{b\}) \neq \widetilde{R}(\{c\})$. Hence X is a T_0-space. But $(\langle a\rangle R - \{a\}) \cap \{b\} = \{b\} \neq \phi$. Then $d(\{b\}) \neq \phi$. Thus X is not a T_1-space. Also, $b \in \langle a\rangle R$ but $a \notin \langle b\rangle R$. Thus X is not an R_0-space.

(2) In (1), if $\mathcal{I} = \{\phi, \{a\}, \{c\}, \{a, c\}\}$, then $\langle a\rangle R \cap \{c\} =< b > R \cap \{a\} = \langle b\rangle R \cap \{c\} = \phi \in \mathcal{I}$. Thus $\overline{R}(\{a\}) \neq \overline{R}(\{b\}) \neq \overline{R}(\{c\})$. So X is a T_0^*-space. But $(\langle a\rangle R - \{a\}) \cap \{b\} \notin \mathcal{I}$. Then $d^*(\{b\}) \neq \phi$. Thus X is not a T_1^*-space. Also, $a \in \overline{R}(\{b\})$ but $b \notin \overline{R}(\{a\})$. Then X is not an R_0^*-space.

(3) Let $X = \{a, b, c\}, R = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $\langle a\rangle R = \{a, b\},\ \langle b\rangle R = \{a, b\},\ \langle c\rangle R = \{c\}$. Thus we have

(i) for $a \neq b,\ b \in \langle a\rangle R$ and $a \in \langle b\rangle R$,

(ii) for $b \neq c,\ b \notin \langle c\rangle R$ and $c \notin \langle b\rangle R$,

(iii) for $a \neq c,\ a \notin \langle c\rangle R$ and $c \notin \langle a\rangle R$.

So X is an R_0-space. But $b \in \langle a\rangle R$ and $a \in \langle b\rangle R$. Then X is not a T_0-space.

(4) From (3), we have the following cases.

If $a \neq b$, then $\langle a\rangle R \cap \{b\} \notin \mathcal{I}$ and $\langle b\rangle R \cap \{a\} \notin \mathcal{I}$. Thus $a \in \overline{R}(\{b\})$ and $b \in \overline{R}(\{a\})$.

If $b \neq c$, then $\langle b\rangle R \cap \{c\} = \langle c\rangle R \cap \{b\} = \phi \in \mathcal{I}$. Thus $b \notin \overline{R}(\{c\})$ and $c \notin \overline{R}(\{b\})$.

If $a \neq c$, then $\langle a\rangle R \cap \{c\} = \langle c\rangle R \cap \{a\} = \phi \in \mathcal{I}$. Thus $a \notin \overline{R}(\{c\})$ and $c \notin \overline{R}(\{a\})$. So X is an R_0^*-space. But $\overline{R}(\{a\}) = \overline{R}(\{b\}) = \{a, b\}$. Then X is not a T_0^*-space.

(5) From (3), $(\langle b\rangle R - \{b\}) \cap \{a\} = \{a\} \neq \phi$. Then $d(\{a\}) \neq \phi$. Thus X is not a T_1-space. But if $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$, then $\langle b\rangle R \cap \{a\} \in \mathcal{I}$ and $\langle c\rangle R \cap \{a\} = \phi \in \mathcal{I}$. Thus $\overline{R}(\{a\}) = \{a\}$, $\langle a\rangle R \cap \{b\} \in \mathcal{I}$ and $\langle c\rangle R \cap \{b\} = \phi \in \mathcal{I}$. So $\overline{R}(\{b\}) = \{b\}$, $\langle b\rangle R \cap \{c\} \in \mathcal{I}$ and $\langle a\rangle R \cap \{c\} = \phi \in \mathcal{I}$. Hence $\overline{R}(\{c\}) = \{c\}$. Therefore X is a T_1^*-space.

(6) From (5), X is T_0^* but is not T_0.

(7) In (1), suppose $\mathcal{I} = \{\phi, \{b\}\}$. Then we have the following cases:

If $a \neq b,\ \langle a\rangle R \cap \{b\} \in \mathcal{I}$ and $\langle b\rangle R \cap \{a\} \in \mathcal{I}$, Then $a \notin \overline{R}(\{b\})$ and $b \notin \overline{R}(\{a\})$.

If $b \neq c,\ \langle b\rangle R \cap \{c\} = \langle c\rangle R \cap \{b\} = \phi \in \mathcal{I}$, then $b \notin \overline{R}(\{c\})$ and $c \notin \overline{R}(\{b\})$.

If $a \neq c$, $\langle a \rangle R \cap \{c\} = \langle c \rangle R \cap \{a\} = \phi \in \mathcal{I}$ then $a \notin \overline{R}(\{c\})$ and $c \notin \overline{R}(\{a\})$.

Thus X is an R_0^*-space. But $b \in \langle a \rangle R$ and $a \notin \langle b \rangle R$. Then X is not an R_0-space.

(8) In (3), X is not an R_0-space. But $b \in \overline{R}(\{a\})$ and $a \notin \overline{R}(\{b\})$. Then X is not an R_0^*-space.

Example 5.20. (1) Let $X = \{a, b, c\}, R = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$ and $\mathcal{I} = \{\phi, \{b\}\}$. Then $\langle a \rangle R = \{a, b\}$, $\langle b \rangle R = \{a, b\}$, $\langle c \rangle R = \{c\}$. Also, $R \langle a \rangle = \{a, b\}$, $R \langle b \rangle = \{a, b\}$, $R \langle c \rangle = \{c\}$. Thus $R \langle a \rangle R = \{a, b\}$, $R \langle b \rangle R = \{a, b\}$, $R \langle c \rangle R = \{c\}$. So $R \langle a \rangle R \cap \{b\} \in \mathcal{I}$, $R \langle a \rangle R \cap \{c\} = R \langle b \rangle R \cap \{c\} = \phi \in \mathcal{I}$. Hence $\overline{\overline{R}}(\{a\}) \neq \overline{\overline{R}}(\{b\}) \neq \overline{\overline{R}}(\{c\})$. Therefore X is a T_0^{**}-space. But $(R \langle b \rangle R - \{b\}) \cap \{a\} \notin \mathcal{I}$. Then $d^*(\{a\}) \neq \phi$. Thus X is not a T_1^{**}-space. Also, $a \in \overline{\overline{R}}(\{b\})$ but $b \notin \overline{\overline{R}}(\{a\})$. Then X is not an R_0^{**}-space.

(2) In (1), suppose $\mathcal{I} = \{\phi, \{c\}\}$. Then we have the following cases.

For $a \neq b$, $R \langle a \rangle R \cap \{b\} \notin \mathcal{I}$ and $R \langle b \rangle R \cap \{a\} \notin \mathcal{I}$, then $a \in \overline{\overline{R}}(\{b\})$ and $b \in \overline{\overline{R}}(\{a\})$.

For $b \neq c$, $R \langle b \rangle R \cap \{c\} = R \langle c \rangle R \cap \{b\} = \phi \in \mathcal{I}$, then $b \notin \overline{\overline{R}}(\{c\})$ and $c \notin \overline{\overline{R}}(\{b\})$.

For $a \neq c$, $R \langle a \rangle R \cap \{c\} = R \langle c \rangle R \cap \{a\} = \phi \in \mathcal{I}$, then $a \notin \overline{\overline{R}}(\{c\})$ and $c \notin \overline{\overline{R}}(\{a\})$.

Thus, X is an R_0^{**}-space. But $\overline{\overline{R}}(\{a\}) = \overline{\overline{R}}(\{b\}) = \{a, b\}$. Then X is not a $T_0^{**}-$space.

(3) From (2), X is an R_0^{**}-space. But $\langle b \rangle R \cap \{c\} \notin \mathcal{I}$ and $\langle c \rangle R \cap \{b\} \in \mathcal{I}$ Then $b \in \overline{R}(\{c\})$ but $c \notin \overline{R}(\{b\})$. Thus X is not an R_0^{**}-space.

(4) Let $X = \{a, b, c\}$, $R = \{(a, a), (a, b), (b, c), c, c)\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $\langle a \rangle R = \{a, b\}$, $\langle b \rangle R = \{a, b\}$, $\langle c \rangle R = \{c\}$. Also, $R \langle a \rangle = \{a\}$, $R \langle b \rangle = \{b, c\}$, $R \langle c \rangle = \{b, c\}$. Thus $R \langle a \rangle R = \{a\}$, $R \langle b \rangle R = \{b\}$, $R \langle c \rangle R = \{c\}$. So we have

(i) $R \langle b \rangle R \cap \{a\} = R \langle c \rangle R \cap \{a\} = \phi \in \mathcal{I}$, i.e., $\overline{\overline{R}}(\{a\}) = \{a\}$.
(ii) $R \langle a \rangle R \cap \{b\} = R \langle c \rangle R \cap \{b\} = \phi \in \mathcal{I}$, i.e., $\overline{\overline{R}}(\{b\}) = \{b\}$.
(iii) $R \langle b \rangle R \cap \{c\} = R \langle a \rangle R \cap \{c\} = \phi \in \mathcal{I}$, i.e., $\overline{\overline{R}}(\{c\}) = \{c\}$.

Hence X is a T_1^{**}-space. But $\langle b \rangle R \cap \{a\} = \{a\} \notin \mathcal{I}$. Then $\overline{R}(\{a\}) \neq \{a\}$. Thus X is not a T_1^*-space.

(5) From (4), X is T_0^{**}. But $\overline{R}(\{a\}) = \overline{R}(\{b\}) = \{a, b\}$. Then X is not a T_0^*-space.

(6) Let $X = \{a, b, c\}$, $R = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, c)\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Then $\langle a \rangle R = \{a, b\}$, $\langle b \rangle R = \{a, b\}$, $\langle c \rangle R = \{c\}$. Also, $R \langle a \rangle = \{a, b\}$, $R \langle b \rangle = \{b\}$, $R \langle c \rangle = \{b, c\}$. Thus $R \langle a \rangle R = \{a, b\}$, $R \langle b \rangle R = \{b\}$, $R \langle c \rangle R = \{c\}$. So we have the following cases.

(i) For $a \neq b$, $\langle a \rangle R \cap \{b\} \notin \mathcal{I}$ and $\langle b \rangle R \cap \{a\} \notin \mathcal{I}$, i.e., $a \in \overline{R}(\{b\})$ and $b \in \overline{R}(\{a\})$.
(ii) For $b \neq c$, $\langle b \rangle R \cap \{c\} = \langle c \rangle R \cap \{b\} = \phi \in \mathcal{I}$, i.e., $b \notin \overline{R}(\{c\})$ and $c \notin \overline{R}(\{b\})$.
(iii) For $a \neq c$, $\langle a \rangle R \cap \{c\} = \langle c \rangle R \cap \{a\} = \phi \in \mathcal{I}$, i.e., $a \notin \overline{R}(\{c\})$ and $c \notin \overline{R}(\{a\})$.

Hence X is an R_0^*-space. But $R \langle a \rangle R \cap \{b\} \notin \mathcal{I}$ and $R \langle b \rangle R \cap \{a\} \in \mathcal{I}$. Then $a \in \overline{\overline{R}}(\{b\})$ but $b \notin \overline{\overline{R}}(\{a\})$. Thus X is not an R_0^{**}-space.

Example 5.21. (1) Let X be an infinite set and $R = X \times X$. If $\mathcal{I}_f$ is an ideal of finite subsets of X, then

$$\underset{\sim}{R}(A) = \begin{cases} A & \text{if } A^c \text{ is finite,} \\ \phi & \text{otherwise.} \end{cases}$$

Thus $\forall x \neq y \in X$, we have

$$x \in \underset{\sim}{R}(\{y\}^c) = \{y\}^c,\ y \notin \{y\}^c \text{ and } y \in \underset{\sim}{R}(\{x\}^c) = \{x\}^c,\ x \notin \{x\}^c.$$

So X is a T_1-space. But X is not a T_2 space, since if $x \in \underset{\sim}{R}(A),\ y \in \underset{\sim}{R}(B)$ and $A \cap B = \phi$, then $\underset{\sim}{R}(A) \cap \underset{\sim}{R}(B) = \phi$ and $\underset{\sim}{R}(A) \subseteq (\underset{\sim}{R}(B))^c$ which is impossible because $\underset{\sim}{R}(A)$ is infinite and $(\underset{\sim}{R}(B))^c$ is finite.

(2) In (1), we have

$$\underline{R}(A) = \underline{\underline{R}}(A) = \begin{cases} A & \text{if } A^c \in \mathcal{I}_f, \\ \phi & \text{otherwise.} \end{cases}$$

Then $\forall x \neq y \in X$, we have

$$x \in \underline{R}(\{y\}^c) = \underline{\underline{R}}(\{y\}^c) = \{y\}^c,\ y \notin \{y\}^c \text{ and } y \in \underline{R}(\{x\}^c) = \underline{\underline{R}}(\{x\}^c) = \{x\}^c,\ x \notin \{x\}^c.$$

Thus X is T_1^* and T_1^{**}. But X is neither T_2^* nor T_2^{**}.

(3) In Example 5.19 (1), if $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$, then $\underline{R}(\{a\}) = \{a\},\ \underline{R}(\{b\}) = \{b\}$ and $\underline{R}(\{c\}) = \{c\}$. Thus X is a T_2^*-space. But X is not a T_2-space, since it is not T_1.

(4) In Example 5.20 (4), we have $\underline{\underline{R}}(\{a\}) = \{a\}, \underline{\underline{R}}(\{b\}) = \{b\}$ and $\underline{\underline{R}}(\{c\}) = \{c\}$. Then X is a T_2^{**}-space. But, X is not a T_2^*-space, since it is not T_1^*.

Definition 5.22. Let (X, R_1) and (Y, R_2) are approximation spaces and let $\mathcal{I}$ be an ideal on X. Then

(i) a function $f : (X, R_1) \longrightarrow (Y, R_2)$ is said to be *continuous*, if $\underset{\sim}{R_1}(f^{-1}(V)) \supseteq f^{-1}(\underset{\sim}{R_2}(V))$, i.e., $\widetilde{R_1}(f^{-1}(V)) \subseteq f^{-1}(\widetilde{R_2}(V))$ for all $V \in Y$.

(ii) a function $f : (X, R_1, \mathcal{I}) \longrightarrow (Y, R_2)$ is said to be $*$-*continuous* (resp. $**$-*continuous*), if $\underline{R_1}(f^{-1}(V)) \supseteq f^{-1}(\underset{\sim}{R_2}(V))$ (resp. $\underline{\underline{R_1}}(f^{-1}(V)) \supseteq f^{-1}(\underset{\sim}{R_2}(V))$), i.e., $\overline{R_1}(f^{-1}(V)) \subseteq f^{-1}(\widetilde{R_2}(V)$ (resp. $\overline{\overline{R_1}}(f^{-1}(V)) \subseteq f^{-1}(\widetilde{R_2}(V))$ for all $V \in Y$.

Remark 5.23. From Theorem 2.8 (3), we have the following diagram:

$$\text{Continuous} \Longrightarrow \text{*-continuous} \Longrightarrow \text{**-continuous}.$$

Next example show that the implication in the diagram is not reversible.

Example 5.24. Let $X = \{a, b, c\},\ R_1 = \{(a, a), (a, b), (a, c), (b, b), (b, c)\}$ and let $Y = \{1, 2, 3\}$, $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$. Then $\langle a \rangle R_1 = \{a, b, c\},\ \langle b \rangle R_1 = \{b, c\},\ \langle c \rangle R_1 = \{b, c\}$. Also, $R_1 \langle a \rangle = \{a\},\ R_1 \langle b \rangle = \{a, b\},\ R_1 \langle c \rangle = \phi$. Thus

$$R_1 \langle a \rangle R_1 = \{a\},\ R_1 \langle b \rangle R_1 = \{b\},\ R_1 \langle c \rangle R_1 = \phi.$$

On the other hand, $\langle 1 \rangle R_2 = \{1, 2\},\ \langle 2 \rangle R_2 = \{1, 2\},\ \langle 3 \rangle R_2 = \{3\}$. Also, $R_2 \langle 1 \rangle = \{1, 2\},\ R_2 \langle 2 \rangle = \{1, 2\},\ R_2 \langle 3 \rangle = \{3\}$. So

$$R_2 \langle 1 \rangle R_2 = \{1, 2\},\ R_2 \langle 2 \rangle R_2 = \{1, 2\},\ R_2 \langle 3 \rangle R_2 = \{3\}.$$

Let $f : (X, R_1, \mathcal{I}) \longrightarrow (Y, R_2)$ be the mapping given by $f(a) = f(b) = 1,\ f(c) = 3$.

(1) Consider $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then we have

$$\underline{R_1}(f^{-1}(\{1\})) = \{a, b\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{1\})) = \phi,$$

$$\underline{R_1}(f^{-1}(\{2\})) = \phi \supseteq f^{-1}(\underset{\sim}{R_2}(\{2\})) = \phi,$$

$$\underline{R_1}(f^{-1}(\{3\})) = \{c\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{3\})) = \{c\},$$

$$\underline{R_1}(f^{-1}(\{1, 2\})) = \{a, b\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{1, 2\})) = \{a, b\},$$

$$\underline{R_1}(f^{-1}(\{1, 3\})) = X \supseteq f^{-1}(\underset{\sim}{R_2}(\{1, 3\})) = \{c\},$$

$$\underline{R_1}(f^{-1}(\{2, 3\})) = \{c\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{2, 3\})) = \{c\}.$$

Thus f is $*$-continuous. But f is not continuous, since $\underset{\sim}{R_1}(f^{-1}(\{3\})) = \phi \not\supseteq f^{-1}(\underset{\sim}{R_2}(\{3\})) = \{c\}$.

(2) Consider $\mathcal{I} = \{\phi, \{a\}\}$. Then we get

$$\underline{\underline{R_1}}(f^{-1}(\{1\})) = \{a, b\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{1\})) = \phi,$$

$$\underline{\underline{R_1}}(f^{-1}(\{2\})) = \phi \supseteq f^{-1}(\underset{\sim}{R_2}(\{2\})) = \phi,$$

$$\underline{\underline{R_1}}(f^{-1}(\{3\})) = \{c\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{3\})) = \{c\},$$

$$\underline{\underline{R_1}}(f^{-1}(\{1, 2\})) = \{a, b\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{1, 2\})) = \{a, b\},$$

$$\underline{\underline{R_1}}(f^{-1}(\{1, 3\})) = X \supseteq f^{-1}(\underset{\sim}{R_2}(\{1, 3\})) = \{c\},$$

$$\underline{\underline{R_1}}(f^{-1}(\{2, 3\})) = \{c\} \supseteq f^{-1}(\underset{\sim}{R_2}(\{2, 3\})) = \{c\}.$$

Thus f is $**$-continuous. But f is not $*$-continuous, since $\underline{R_1}(f^{-1}(\{1, 2\})) = \phi \not\supseteq f^{-1}(\underset{\sim}{R_2}(\{1, 2\})) = \{a, b\}$.

Theorem 5.25. *Let $f : (X, R_1) \longrightarrow (Y, R_2)$ be an injective continuous function. Then $(X, R_1, \mathcal{I})$ is a T_i^*-space, if (Y, R_2) is a T_i-space for $i = 0,\ 1,\ 2$.*

Proof. Suppose (Y, R_2) is a T_i-space for $i = 0,\ 1,\ 2$ and let $x \neq y$ in X. We proof for $i = 2$. Since f is injective, $f(x) \neq f(y)$ in Y. Then by the hypothesis, there exist $V,\ W \subseteq Y$ such that $f(x) \in \underset{\sim}{R_1}(V),\ f(y) \in \underset{\sim}{R_2}(W)$ and $V \cap W = \phi$, i.e., $x \in f^{-1}(\underset{\sim}{R_2}(V)),\ y \in f^{-1}(\underset{\sim}{R_2}(W))$ and $f^{-1}(V) \cap f^{-1}(W) = \phi$. Since f is continuous, $x \in \underset{\sim}{R_1}(f^{-1}(V)),\ y \in \underset{\sim}{R_2}(f^{-1}(W))$. Thus $x \in \underline{R_1}(f^{-1}(V)),\ y \in \underline{R_1}(f^{-1}(W))$ i.e., there exist $A = f^{-1}(V),\ B = f^{-1}(W)$ in X such that $x \in \underline{R_1}(A),\ y \in \underline{R_1}(B)$ and $A \cap B = \phi$. So $(X, R_1, \mathcal{I})$ is a T_2^*-space. For $i = 0,\ 1$, the proofs are similar. $\square$

Corollary 5.26. *Let $f : (X, R_1) \longrightarrow (Y, R_2)$ be an injective continuous function. Then $(X, R_1, \mathcal{I})$ is a T_i^{**}-space, if (Y, R_2) is a T_i-space for $i = 0,\ 1,\ 2$.*

6. Connectedness in ideal approximation spaces

Definition 6.1. Let (X, R) be an approximation space. Then (i) $A,\ B \subseteq X$ are called *separated sets*, if $\widetilde{R}(A) \cap B = A \cap \widetilde{R}(B) = \phi$,

(ii) $Y \subseteq X$ is called a *disconnected set*, if there exist separated sets $A\ B \subseteq X$ such that $Y \subseteq A \cup B$. Y is said to be *connected*, if it is not disconnected.

(iii) (X, R) is called a *disconnected space*, if there exist separated sets $A,\ B \subseteq X$ such that $A \cup B = X$. (X, R) is called a *connected space*, if it is not disconnected space.

Definition 6.2. Let $(X, R, \mathcal{I})$ be an ideal approximation space. Then (i) $A,\ B \subseteq X$ are called $*$-*separated* (resp. $**$-*separated*) *sets*, if $\overline{R}(A) \cap B = A \cap \overline{R}(B) = \phi$ (resp. $\overline{\overline{R}}(A) \cap B = A \cap \overline{\overline{R}}(B) = \phi$).

(ii) $Y \subseteq X$ is called a $*$-*disconnected* (resp. $**$-*disconnected*) *set*, if there exist $*$-separated (resp. $**$-separated) sets $A,\ B \subseteq X$ such that $Y \subseteq A \cup B$. Y is said to be $*$-*connected* (resp. $**$-*connected*), if it is not $*$-disconnected (resp. $**$-disconnected).

(iii) $(X, R, \mathcal{I})$ is called a $*$-*disconnected* (resp. $**$-*disconnected*) *space*, if there exists $*$-separated (resp. $**$-separated) sets $A,\ B \subseteq X$ such that $A \cup B = X$. $(X, R, \mathcal{I})$ is called a $*$-*connected* (resp. $**$-*connected*) *space*, if it is not a $*$-disconnected (resp. $**$-disconnected) space.

Remark 6.3. We have the following diagrams:

$$\text{separated} \Longrightarrow \text{*-separated} \Longrightarrow \text{**-separated}.$$

And then

$$\text{**-connected} \Longrightarrow \text{*-connected} \Longrightarrow \text{connected}.$$

Next examples show that the Implication in the diagrams is not reversible.

Example 6.4. Let $X = \{a, b, c, d\}, R = \{(a, a), (a, b), (b, b), (b, c), (c, c), (d, d), (d, b)\}$. Then $\langle a \rangle R = \{a, b\},\ \langle b \rangle R = \{b\},\ \langle c \rangle R = \{c\},\ \langle d \rangle R = \{b, d\}$. Also, $R \langle a \rangle = \{a\},\ R \langle b \rangle = \{b\},\ R \langle c \rangle = \{b, c\},\ R \langle d \rangle = \{d\}$. Thus $R \langle a \rangle R = \{a\},\ R \langle b \rangle R = \{b\},\ R \langle c \rangle R = \{c\},\ R \langle d \rangle R = \{d\}$.

(1) Consider $\mathcal{I} = \{\phi, \{b\}\}$ and $A = \{a, c\},\ B = \{b, d\}$. Then we have

$$\widetilde{R}(A) = A \cup \{x \in X\ : \langle x \rangle R \cap A \neq \phi\} = \{a, c\} \text{ and } \widetilde{R}(B) = \{a, b, d\},$$

$$\overline{R}(A) = A \cup \{x \in X\ : \langle x \rangle R \cap A \notin \mathcal{I}\} = \{a, c\} \text{ and } \overline{R}(B) = \{b, d\}.$$

Thus $\overline{R}(A) \cap B = A \cap \overline{R}(B) = \phi$ but $A \cap \widetilde{R}(B) = \{a\} \neq \phi$. So $A,\ B$ are $*$-separated sets but are not separated sets.

(2) Consider $\mathcal{I} = \{\phi, \{d\}\}$ and $A = \{b\},\ B = \{a, d\}$. Then we get

$$\overline{R}(A) = A \cup \{x \in X\ : \langle x \rangle R \cap A \notin \mathcal{I}\} = \{a, b, d\} \text{ and } \overline{R}(B) = \{a, d\},$$

$$\overline{\overline{R}}(A) = A \cup \{x \in X\ : R \langle x \rangle R \cap A \notin \mathcal{I}\} = \{b\} \text{ and } \overline{\overline{R}}(B) = \{a, d\}.$$

Thus $\overline{\overline{R}}(A) \cap B = A \cap \overline{\overline{R}}(B) = \phi$ but $\overline{R}(A) \cap B = \{a\} \neq \phi$. So $A,\ B$ are $**$-separated sets but are not $*$-separated sets.

Example 6.5. Let $X = \{a, b, c\},\ R = \{(a, a), (a, b), (a, c), (b, b), (b, c)\}$. Then $\langle a\rangle R = \{a, b, c\},\ \langle b\rangle R = \{b, c\},\ \langle c\rangle R = \{b, c\}$. Also, $R\langle a\rangle = \{a\},\ R\langle b\rangle = \{a, b\},\ R\langle c\rangle = \phi$. Thus $R\langle a\rangle R = \{a\},\ R\langle b\rangle R = \{b\},\ R\langle c\rangle R = \phi$.

(1) Consider $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then we have

$$\widetilde{R}(\{b\}) = \widetilde{R}(\{c\}) = \widetilde{R}(\{b, c\}) = \widetilde{R}(\{a, b\}) = \widetilde{R}(\{a, c\}) = X,\ \widetilde{R}(\{a\}) = \{a\}.$$

Thus X is a connected space. But we get

$$X = \{a\} \cup \{b, c\},\ \overline{R}(\{a\}) \cap \{b, c\} = \{a\} \cap \overline{R}(\{b, c\}) = \phi.$$

So X is not a $*$-connected space.

(2) Consider $\mathcal{I} = \{\phi, \{a\}\}$. Then we get

$$\overline{R}(\{b\}) = \overline{R}(\{c\}) = \overline{R}(\{b, c\}) = \overline{R}(\{a, b\}) = \overline{R}(\{a, c\}) = X,\ \overline{R}(\{a\}) = \{a\}.$$

Thus X is a $*$-connected space. But we have

$$X = \{a\} \cup \{b, c\},\ \overline{\overline{R}}(\{a\}) \cap \{b, c\} = \{a\} \cap \overline{\overline{R}}(\{b, c\}) = \phi.$$

So X is not a $**$-connected space.

Proposition 6.6. *Let $(X, R, \mathcal{I})$ be an ideal approximation space. Then the following are equivalent:*

(1) *X is $*$-connected,*
(2) *for each $A,\ B \subseteq X$ with $A \cap B = \phi,\ \underline{R}(A) = A,\ \underline{R}(B) = B$ and $A \cup B = X$, $A = \phi$ or $B = \phi$,*
(3) *for each $A,\ B \subseteq X$ with $A \cap B = \phi,\ \overline{R}(A) = A,\ \overline{R}(B) = B$ and $A \cup B = X$, $A = \phi$ or $B = \phi$.*

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds and let $A,\ B \subseteq X$ with $\underline{R}(A) = A,\ \underline{R}(B) = B$ such that $A \cap B = \phi$ and $A \cup B = X$. Then

$$\overline{R}(A) \subseteq \overline{R}(B^c) = (\underline{R}(B))^c = B^c,$$
$$\overline{R}(B) \subseteq \overline{R}(A^c) = (\underline{R}(A))^c = A^c.$$

Thus $\overline{R}(A) \cap B = A \cap \overline{R}(B) = \phi$. So $A,\ B$ are $*$-separated sets. Since $A \cup B = X$, $A = \phi$ or $B = \phi$ by (1).

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) Clear. □

Corollary 6.7. *Let (X, R) be an approximation space. Then, the following are equivalent:*

(1) *X is connected,*
(2) *for each $A,\ B \subseteq X$ with $A \cap B = \phi,\ \underset{\sim}{R}(A) = A,\ \underset{\sim}{R}(B) = B$ and $A \cup B = X$, $A = \phi$ or $B = \phi$,*
(3) *for each $A,\ B \subseteq X$ with $A \cap B = \phi,\ \widetilde{R}(A) = A,\ \widetilde{R}(B) = B$ and $A \cup B = X$, $A = \phi$ or $B = \phi$.*

Corollary 6.8. *Let $(X, R, \mathcal{I})$ be an ideal approximation space. Then, the following are equivalent:*

(1) *X is $**$-ideal connected,*
(2) *for each $A,\ B \subseteq X$ with $A \cap B = \phi,\ \underline{\underline{R}}(A) = A,\ \underline{\underline{R}}(B) = B$ and $A \cup B = X$, $A = \phi$ or $B = \phi$,*

(3) *for each* $A,\ B \subseteq X$ *with* $A \cap B = \phi$, $\overline{\overline{R}}(A) = A$, $\overline{\overline{R}}(B) = B$ *and* $A \cup B = X$, $A = \phi$ *or* $B = \phi$.

Theorem 6.9. *Let* $(X, R, \mathcal{I})$ *be an ideal approximation space and* $M \subseteq X$ *is* $*$*-connected. If* $A,\ B \subseteq X$ *are* $*$*-separated sets with* $M \subseteq A \cup B$*, then either* $M \subseteq A$ *or* $M \subseteq B$.

Proof. Suppose $A,\ B$ are $*$-separated sets with $M \subseteq A \cup B$. Then we have

$$\overline{R}(A) \cap B = A \cap \overline{R}(B) = \phi,\ M = (M \cap A) \cup (M \cap B).$$

On the other hand, we get

$$\overline{R}(M \cap A) \cap (M \cap B) \subseteq \overline{R}(M) \cap \overline{R}(A) \cap (M \cap B) = \overline{R}(M) \cap M \cap \overline{R}(A) \cap B = M \cap \phi = \phi,$$

$$\overline{R}(M \cap B) \cap (M \cap A) \subseteq \overline{R}(M) \cap \overline{R}(B) \cap (M \cap A) = \overline{R}(M) \cap M \cap \overline{R}(B) \cap A = M \cap \phi = \phi.$$

Thus $M \cap A$ and $M \cap B$ are $*$-separated sets with $M = (M \cap A) \cup (M \cap B)$. But M is $*$-connected implies that $M \subseteq A$ or $M \subseteq B$. □

Corollary 6.10. *Let* (X, R) *be an ideal approximation space and* $M \subseteq X$ *is connected. If* $A,\ B \subseteq X$ *are separated sets with* $M \subseteq A \cup B$*, then either* $M \subseteq A$ *or* $M \subseteq B$.

Corollary 6.11. *Let* $(X, R, \mathcal{I})$ *be an ideal approximation space and* $M \subseteq X$ *is* $**$*-ideal connected. If* $A,\ B \subseteq X$ *are* $**$*-ideal separated sets with* $M \subseteq A \cup B$*, then either* $M \subseteq A$ *or* $M \subseteq B$.

Theorem 6.12. *Let* $f : (X, R_1, \mathcal{I}) \longrightarrow (Y, R_2)$ *be an* $*$*-continuous function. Then* $f(A) \subseteq Y$ *is connected set, if* A *is* $*$*-connected in* X.

Proof. Suppose A is $*$-connected in X. Assume that $f(A)$ is disconnected. Then there exist two separated sets $U,\ V \subseteq Y$ with $f(A) \subseteq U \cup V$, i.e., $\widetilde{R_2}(U) \cap V = U \cap \widetilde{R_2}(V) = \phi$. Since f is $*$-continuous, $A \subseteq f^{-1}(U) \cup f^{-1}(V)$. Thus we have

$$\overline{R_1}(f^{-1}(U)) \cap f^{-1}(V) \subseteq f^{-1}(\widetilde{R_2}(U)) \cap f^{-1}(V) = f^{-1}(\widetilde{R_2}(U) \cap V) = f^{-1}(\phi) = \phi,$$

$$\overline{R_1}(f^{-1}(V)) \cap f^{-1}(U) \subseteq f^{-1}(\widetilde{R_2}(V)) \cap f^{-1}(U) = f^{-1}(\widetilde{R_2}(V) \cap U) = f^{-1}(\phi) = \phi.$$

So $f^{-1}(U)$ and $f^{-1}(V)$ are $*$-separated sets in X, i.e., $A \subseteq f^{-1}(U) \cup f^{-1}(V)$. Hence A is $*-$disconnected, which contradicts that A is $*-$connected. Therefore $f(A)$ is a connected set. □

Corollary 6.13. *Let* $f : (X, R_1, \mathcal{I}) \longrightarrow (Y, R_2)$ *be a* $**$*-continuous function. Then* $f(A) \subseteq Y$ *is connected set, if* A *is* $**$*-connected in* X.

7. Conclusions

Topology is an important branch whose concepts are used in a variety of real-world applications. The generalisation of rough set theory depending on ideal concepts has led to topological rough set approaches, which are used in several fields. The present paper relied on generalising some definitions of closure space defined by [31] in ideal approximation spaces. So accumulation points, subspaces, and lower separation axioms of such spaces are defined and studied. Moreover, connectedness in these spaces is defined, which enables us to make more generalisations and studies. The obtained results are newly presented and could enrich topology theory. In a proposed future work, a new generalisation of rough sets based on an ideal $\mathcal{I}$ will be introduced on soft approximation spaces (See [36, 37, 38, 39]).

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