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Abstract. Closure operators and closure systems play a significant role in both pure and applied mathematics such that algebra, topology, analysis and computer science. We investigate the relationships between right (resp. left) closure systems and right (resp. left) closure operators on complete generalized residuated lattices. We show that the set induced by a right (resp. left) closure operator is right (resp. left) meet complete.

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Keywords: Generalized residuated lattices, Bi-partially ordered sets, Right (resp. left) closure systems, Right (resp. left) join complete, Right (resp. left) meet complete.

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1. Introduction

Bělohlávek [1, 2, 3, 4] investigate the properties of fuzzy relations and fuzzy closure systems on a complete residuated lattice and developed the fuzzy formal concepts and data analysis as foundation of theoretic computer science. As an Bělohlávek's extension, Fang and Yue [5] introduced strong fuzzy closure systems and strong fuzzy closure operators.

Fuzzy closure operators and fuzzy closure systems can be applied to many different areas as fuzzy Galios connections [6, 7, 8, 9], fuzzy rough sets [10], fuzzy formal concepts [9, 10], decision rules [11] and fuzzy logics [12, 13, 14]. Recently, fuzzy topological structures are introduced on soft sets, octahedron sets and cubic sets [15, 16, 17].

For a non-commutative algebraic structure, Turunen [18] introduced a generalized residuated lattice as a generalization of weak-pseudo-BL-algebras and left continuous pseudo-t-norms [6, 19, 20]. Ko and Kim [21, 22] introduced the notions of right (resp. left) closure operators and right (resp. left) closure systems on a generalized residuated lattice. Moreover, as an extension of Zhang's complete residuated lattices

[23, 24], Ko and Kim [21, 22] investigated the notions of right (left) completeness based on generalized residuated lattices.

In this paper, as the foundations of Alexandrov topologies, fuzzy formal concepts, fuzzy rough sets, fuzzy relation logics, fuzzy equations and decision rules, we examine the relationships between right (resp. left) closure systems and right (resp. left) closure operators (See Section 3). Finally, we show that the set $M = \{A \in L^X \mid C(A) = A\}$, where C is a right (resp. left) closure operator, is a right (resp. left) meet complete (See Section 4).

2. Preliminaries

In this section, we present some preliminary concepts and properties.

Definition 2.1 ([18, 21, 22])**.** A structure $(L, \vee, \wedge, \odot, \to, \Rightarrow, \bot, \top)$ is called a *generalized residuated lattice* if it satisfies the following three conditions:

(GR1) $(L, \vee, \wedge, \top, \bot)$ is bounded where $\top$ is the upper bound and $\bot$ is the universal lower bound,

(GR2) $(L, \odot, \top)$ is a monoid, where $\top$ is the identity,

(GR3) it satisfies a residuation, i.e., $a \odot b \le c$ if and only if $a \le b \to c$ if and only if $b \le a \Rightarrow c$.

In this paper, we always assume that $(L, \wedge, \vee, \odot, \to, \Rightarrow, \top, \bot)$ is a complete generalized residuated lattice.

Lemma 2.2 ([18, 21, 22])**.** *Let $x,\ y,\ z \in L$ and let $\{x_i\}_{i\in\Gamma},\ \{y_i\}_{i\in\Gamma} \subseteq L$. Then the followings hold.*

(1) *If $y \le z$, then $x \odot y \le x \odot z$, $x \to y \le x \to z$, $z \to x \le y \to x$, $x \Rightarrow y \le x \Rightarrow z$ and $z \Rightarrow x \le y \Rightarrow x$.*

(2) $x \to \left(\bigwedge_{i\in\Gamma} y_i\right) = \bigwedge_{i\in\Gamma} (x \to y_i)$, $\left(\bigvee_{i\in\Gamma} x_i\right) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$,

$$\left(\bigvee_{i\in\Gamma} x_i\right) \to \left(\bigvee_{i\in\Gamma} y_i\right) \ge \bigwedge_{i\in\Gamma} (x_i \to y_i), \ \left(\bigwedge_{i\in\Gamma} x_i\right) \to \left(\bigwedge_{i\in\Gamma} y_i\right) \ge \bigwedge_{i\in\Gamma} (x_i \to y_i),$$

$$x \Rightarrow \left(\bigwedge_{i\in\Gamma} y_i\right) = \bigwedge_{i\in\Gamma} (x \Rightarrow y_i), \ \left(\bigvee_{i\in\Gamma} x_i\right) \Rightarrow y = \bigwedge_{i\in\Gamma}(x_i \Rightarrow y),$$

$$\left(\bigvee_{i\in\Gamma} x_i\right) \Rightarrow \left(\bigvee_{i\in\Gamma} y_i\right) \ge \bigwedge_{i\in\Gamma} (x_i \Rightarrow y_i), \ \left(\bigwedge_{i\in\Gamma} x_i\right) \Rightarrow \left(\bigwedge_{i\in\Gamma} y_i\right) \ge \bigwedge_{i\in\Gamma} (x_i \Rightarrow y_i).$$

(3) *$(x \odot y) \to z = x \to (y \to z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.*

(4) *$x \to (y \Rightarrow z) = y \Rightarrow (x \to z)$ and $x \Rightarrow (y \to z) = y \to (x \Rightarrow z)$.*

(5) *$x \odot (x \Rightarrow y) \le y$ and $(x \to y) \odot x \le y$. Moreover, $x \le (x \Rightarrow y) \to y$ and $x \le (x \to y) \Rightarrow y$.*

(6) *$(x \Rightarrow y) \odot z \le x \Rightarrow (y \odot z)$ and $y \odot (x \to z) \le x \to (y \odot z)$.*

(7) *$(x \Rightarrow y) \odot (y \Rightarrow z) \le x \Rightarrow z$ and $(y \to z) \odot (x \to y) \le x \to z$.*

(8) *$(x \Rightarrow z) \le (y \odot x) \Rightarrow (y \odot z)$ and $(x \to z) \le (x \odot y) \to (z \odot y)$.*

(9) *$x \to y \le (y \to z) \Rightarrow (x \to z)$ and $(x \Rightarrow y) \le (y \Rightarrow z) \to (x \Rightarrow z)$.*

(10) *$y \to z \le (x \to y) \to (x \to z)$ and $(y \Rightarrow z) \le (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$.*

(11) *$x \to y = \top$ if and only if $x \le y$. Similarly, $x \Rightarrow y = \top$ if and only if $x \le y$.*

Definition 2.3 ([21, 22]). Let X be a set. A map $e^r_X : X \times X \to L$ is called an *r-partial order* (or *right-partial order*), if it satisfies the following three conditions:
(O1) $e^r_X(x,x) = \top$ for all $x \in X$,
(O2) if $e^r_X(x,y) = e^r_X(y,x) = \top$, where $x, y \in X$, then $x = y$,
(R) $e^r_X(x,y) \odot e^r_X(y,z) \leq e^r_X(x,z)$ for all $x,\ y,\ z \in X$.
A map $e^l_X : X \times X \to L$ is called an *l-partial order* (or *left partial order*), if it satisfies (O1), (O2) and the following condition :
(L) $e^l_X(y,z) \odot e^l_X(x,y) \leq e^l_X(x,z)$ for all $x, y, z \in X$.
The pair (X, e^r_X) is called an *r-partially ordered set* (or *right partially ordered set*).
The pair $\left(X, e^l_X\right)$ is called an *l-partially ordered set* (or *left partially ordered set*).
The triple $\left(X, e^r_X, e^l_X\right)$ is called a *bi-partially ordered set.*

Using Lemma 2.2 (7), one can have the following.

Lemma 2.4. *Let $G \subseteq L^X$. Define $e^r_G : G \times G \to L$ and $e^l_G : G \times G \to L$ by: for any $A,\ B \in X$,*

$$e^r_G(A,B) = \bigwedge_{x \in X} [A(x) \Rightarrow B(x)] \quad \textit{and} \quad e^l_G(A,B) = \bigwedge_{x \in X} [A(x) \to B(x)] \,.$$

Then $\left(G, e^r_G, e^l_G\right)$ is a bi-partially ordered set.

3. Bi-closure operators and bi-closure systems

In this section, we investigate the relationship between right (resp. left) closure systems and right (resp. left) closure operators.

Definition 3.1. A map $C^r : L^X \to L^X$ is called an *r-closure operator* (or *right closure operator*) on X, if it satisfies the following three conditions:
(C1) $A \leq C^r(A)$ for all $A \in L^X$,
(C2) $C^r\left(C^r(A)\right) \leq C^r(A)$ for all $A \in L^X$,
(CR) $e^r_{L^X}(A,B) \leq e^r_{L^X}\left(C^r(A), C^r(B)\right)$ for all $A, B \in L^X$.
A map $C^l : L^X \to L^X$ is called an *l-closure operator* (or *left closure operator*) on X, if it satisfies the following three conditions:
(C1) $A \leq C^l(A)$ for all $A \in L^X$,
(C2) $C^l\left(C^l(A)\right) \leq C^l(A)$ for all $A \in L^X$,
(CL) $e^l_{L^X}(A,B) \leq e^l_{L^X}\left(C^l(A), C^l(B)\right)$ for all $A, B \in L^X$.
The triple $\left(X, C^r, C^l\right)$ is called a *bi-closure space.*

Let $k \in L$ and $A \in L^X$. Define $k \to A : X \to L$ and $k \Rightarrow A : X \to L$ by: for each $x \in X$,

$$(k \to A)(x) = k \to A(x) \ \text{ and } \ (k \Rightarrow A)(x) = k \Rightarrow A(x).$$

Definition 3.2. Let $G^r,\ G^l \subseteq L^X$.
(i) A family G^r is called an *r-closure system* (or *right closure system*) on X, if
 (a) $k \to A \in G^r$ for all $k \in L$ and $A \in G^r$,
 (b) $\bigwedge_{i \in \Gamma} A_i \in G^r$ for all $\{A_i\}_{i \in \Gamma} \subseteq G^r$.
(ii) A family G^l is called an *l-closure system* (or *left closure system*) on X, if
 (a) $k \Rightarrow A \in G^l$ for all $k \in L$ and $A \in G^l$,
 (b) $\bigwedge_{i \in \Gamma} A_i \in G^l$ for all $\{A_i\}_{i \in \Gamma} \subseteq G^l$.
The triple $\left(X, G^r, G^l\right)$ is called a *bi-closure system.*

Remark 3.3. Ko and Kim [21, 22] defined the concepts of Definitions 3.1–3.2 as a sense of completeness and they demonstrated the compatibility of their definitions with Definitions 3.1–3.2.

Lemma 3.4. *Let $k \in L$ and let $A,\ B \in L^X$. Then the following hold.*
(1) $k \leq e^r_{L^X}(k \to A, A)$ *and* $k \leq e^l_{L^X}(k \Rightarrow A, A)$.
(2) *Let* $C^r : L^X \to L^X$ *be an r-closure operator on X. If $A \leq B$, then* $C^r(A) \leq C^r(B)$.
(3) *Let* $C^l : L^X \to L^X$ *be an l-closure operator on X. If $A \leq B$, then* $C^l(A) \leq C^l(B)$.

Proof. (1) By Lemma 2.2 (5), we have

$$e^r_{L^X}(k \to A, A) = \bigwedge_{x \in X} [(k \to A)(x) \Rightarrow A(x)] \geq k.$$

Also, by Lemma 2.2 (5), we get

$$e^l_{L^X}(k \Rightarrow A, A) = \bigwedge_{x \in X} [(k \Rightarrow A)(x) \to A(x)] \geq k.$$

(2) Let $A \leq B$, where $A,\ B \in L^X$. Then

$$\begin{aligned} \top = e^r_{L^X}(A, B) &\leq e^r_{L^X}(C^r(A), C^r(B)) \quad \text{[By (CR)]} \\ &= \textstyle\bigwedge_{x \in X} [C^r(A)(x) \Rightarrow C^r(B)(x)]. \end{aligned}$$

Thus by Lemma 2.2 (11), $C^r(A) \leq C^r(B)$.
(3) It can be similarly proved as in (2). □

Theorem 3.5. (1) *Let* $C^r : L^X \to L^X$ *be an r-closure operator on X. Then* $G^r_{C^r} = \left\{A \in L^X \mid C^r(A) = A\right\}$ *is an r-closure system on X.*
(2) *Let* $C^l : L^X \to L^X$ *be an l-closure operator on X. Then* $G^l_{C^l} = \left\{A \in L^X \mid C^l(A) = A\right\}$ *is an l-closure system on X.*

Proof. (1) Let $k \in L$ and $A \in G^r_{C^r}$. Then

$$\begin{aligned} k &\leq e^r_{L^X}(k \to A, A) \quad \text{[By Lemma 3.4 (1)]} \\ &\leq e^r_{L^X}\left(C^r(k \to A), C^r(A)\right) \quad [\because C^r \text{ is an } r\text{-closure operator}] \\ &= e^r_{L^X}\left(C^r(k \to A), A\right) \quad [\because A \in G^r_{C^r}]. \end{aligned}$$

By residuation, $C^r(k \to A) \leq k \to A$. On the other hand, $k \to A \leq C^r(k \to A)$ by (C1). Thus $k \to A = C^r(k \to A)$. So $k \to A \in G^r_{C^r}$.
Let $\{A_i\}_{i \in \Gamma} \subseteq G^r_{C^r}$. Then

$$\begin{aligned} C^r\left(\textstyle\bigwedge_{i \in \Gamma} A_i\right) &\leq \textstyle\bigwedge_{i \in \Gamma} C^r(A_i) \quad \text{[by Lemma 3.4 (2)]} \\ &= \textstyle\bigwedge_{i \in \Gamma} A_i \quad [\because A_i \in G^r_{C^r}]. \end{aligned}$$

On the other hand, $\bigwedge_{i \in \Gamma} A_i \leq C^r\left(\bigwedge_{i \in \Gamma} A_i\right)$ by (C1). Thus $C^r\left(\bigwedge_{i \in \Gamma} A_i\right) = \bigwedge_{i \in \Gamma} A_i$ and $\bigwedge_{i \in \Gamma} A_i \in G^r_{C^r}$. So $G^r_{C^r}$ is an r-closure system on X.
(2) It can be similarly proved as in (1). □

Lemma 3.6. *Let $A,\ B \in L^X$. Then the following hold.*
(1) $A \leq e^r_{L^X}(A, B) \to B$.
(2) $A \leq e^l_{L^X}(A, B) \Rightarrow B$.
(3) *If $A \leq B$, then* $e^r_{L^X}(A, B) = \top = e^l_{L^X}(A, B)$.

Proof. (1) By residuation, one see that

$$\begin{aligned} A \le e^r_{L^X}(A,B) \to B \ \text{ iff } \ & A(x) \le e^r_{L^X}(A,B) \to B(x) \text{ for all } x \in X \\ \text{iff } \ & A(x) \odot e^r_{L^X}(A,B) \le B(x) \text{ for all } x \in X \\ \text{iff } \ & e^r_{L^X}(A,B) \le A(x) \Rightarrow B(x) \text{ for all } x \in X \\ \text{iff } \ & \bigwedge_{y\in X} [A(y) \Rightarrow B(y)] \le A(x) \Rightarrow B(x) \text{ for all } x \in X. \end{aligned}$$

(2) It can be similarly done as in (1).
(3) It follows by Lemma 2.2 (11). □

Theorem 3.7. (1) *Let G^r be an r-closure system on X. Define $C^r_{G^r} : L^X \to L^X$ by: for each $A \in L^X$,*

$$C^r_{G^r}(A) = \bigwedge \{D \in G^r | A \le D\}.$$

Then $C^r_{G^r}$ is an r-closure operator such that

$$C^r_{G^r}(A) = \bigwedge_{D\in G^r} [e^r_{L^X}(A,D) \to D] \text{ for all } A \in L^X \text{ and } G^r_{C^r_{G^r}} = G^r.$$

(2) *Let G^l be an l-closure system on X. Define $C^l_{G^l} : L^X \to L^X$ by: for each $A \in L^X$,*

$$C^l_{G^l}(A) = \bigwedge \{D \in G^l | A \le D\}.$$

Then $C^l_{G^l}$ is an l-closure operator such that

$$C^l_{G^l}(A) = \bigwedge_{D\in G^l} \left[e^l_{L^X}(A,D) \Rightarrow D\right] \text{ for all } A \in L^X \text{ and } G^l_{C^l_{G^l}} = G^l.$$

(3) *Let $C^r : L^X \to L^X$ be an r-closure operator on X. Then $C^r_{G^r_{C^r}} = C^r$.*
(4) *Let $C^l : L^X \to L^X$ be an l-closure operator on X. Then $C^l_{G^l_{C^l}} = C^l$.*

Proof. (1) *Claim 1: $C^r_{G^r}(A) = \bigwedge_{D\in G^r} \left[e^r_{L^X}(A,D) \to D\right]$ for all $A \in L^X$.* Let $C(A) = \bigwedge_{D\in G^r} \left[e^r_{L^X}(A,D) \to D\right]$, where $A \in L^X$. Note that for all $D \in G^r$, we have $e^r_{L^X}(A,D) \to D \in G^r$ and $A \le e^r_{L^X}(A,D) \to D$ by Lemma 3.6 (1). Then $C^r_{G^r}(A) \le C(A)$. On the other hand, we have

$$\begin{aligned} C(A) &= \textstyle\bigwedge_{D\in G^r} \left[e^r_{L^X}(A,D) \to D\right] \\ &\le e^r_{L^X}(A, C^r_{G^r}(A)) \to C^r_{G^r}(A) \quad [\because C^r_{G^r}(A) \in G^r] \\ &= \top \to C^r_{G^r}(A) \quad [\because A \le C^r_{G^r}(A)], \end{aligned}$$

Thus by residuation, $C(A) \le C^r_{G^r}(A)$. So $C(A) = C^r_{G^r}(A)$.
Claim 2: Let $A \in L^X$ and $D \in G^r$. Then $A \le D$ if and only if $C^r_{G^r}(A) \le D$.
($\Rightarrow$): Suppose $A \le D$. Then $C^r_{G^r}(A) = \bigwedge \{E \in G^r | A \le E\} \le D$.
($\Leftarrow$): Suppose $C^r_{G^r}(A) \le D$. Then $A \le \bigwedge\{E \in G^r | A \le E\} = C^r_{G^r}(A) \le D$.
(C1) By definition, $A \le C^r_{G^r}(A)$ for all $A \in L^X$.
(C2) For all $A \in L^X$,

$$\begin{aligned} C^r_{G^r}(C^r_{G^r}(A)) &= \bigwedge \{D \in G^r | C^r_{G^r}(A) \le D\} \\ &= \bigwedge \{D \in G^r | A \le D\} \quad \text{[By Claim 2]} \\ &= C^r_{G^r}(A). \end{aligned}$$

Claim 3: For all $k \in L$ and $A \in L^X$, $C^r_{G^r}(k \to A) \leq k \to C^r_{G^r}(A)$. Note that

$$\begin{aligned} k \to C^r_{G^r}(A) &= k \to \bigwedge\{D \in G^r | A \leq D\} \\ &= \bigwedge\{k \to D | A \leq D,\, D \in G^r\} \text{ [By Lemma 2.2 (2)]} \\ &\geq \bigwedge\{k \to D | k \to A \leq k \to D,\, k \to D \in G^r\} \\ &\geq C^r_{G^r}(k \to A). \end{aligned}$$

Claim 4: If $A \leq B$, where $A,\ B \in L^X$, then $C^r_{G^r}(A) \leq C^r_{G^r}(B)$. Suppose $A \leq B$. Since $\{D \in G^r | B \leq D\} \subseteq \{D \in G^r | A \leq D\}$, we have $C^r_{G^r}(A) \leq C^r_{G^r}(B)$.

(CR) Let $A, B \in L^X$. Since

$$\begin{aligned} e^r_{L^X}(A,B) \to C^r_{G^r}(B) &\geq C^r_{G^r}\left(e^r_{L^X}(A,B) \to B\right) \quad \text{[By Claim 3]} \\ &\geq C^r_{G^r}(A) \quad \text{[By Lemma 3.6 (1) and Claim 4]}, \end{aligned}$$

we have by residuation that $e^r_{L^X}(A,B) \leq e^r_{L^X}\left(C^r_{G^r}(A), C^r_{G^r}(B)\right)$. Thus $C^r_{G^r}$ is an r-closure operator.

Let $A \in G^r_{C^r_{G^r}}$. Then $A = C^r_{G^r}(A) = \bigwedge\{D \in G^r \mid A \leq D\}$. Thus $A \in G^r$. On the other hand, let $A \in G^r$. Then $C^r_{G^r}(A) = \bigwedge\{D \in G^r \mid A \leq D\} = A$. Thus $A \in G^r_{C^r_{G^r}}$.

(3) *Claim 5: Let $A \in L^X$ and let $D \in G^r_{C^r}$. Then $A \leq D$ if and only if $C^r(A) \leq D$.*

($\Rightarrow$): Suppose $A \leq D$. Then by Lemma 3.4 (2), $C^r(A) \leq C^r(D)$. Since $D \in G^r_{C^r}$, we have $C^r(D) = D$. Thus $C^r(A) \leq D$.

($\Leftarrow$): Suppose $C^r(A) \leq D$. Since $A \leq C^r(A)$, we have $A \leq D$.

Let $A \in L^X$. Since $C^r_{G^r_{C^r}}(A) = \bigwedge\{D \in G^r_{C^r} \mid A \leq D\}$, $C^r(A) \in G^r_{C^r}$ and $A \leq C^r(A)$, we have $C^r_{G^r_{C^r}}(A) \leq C^r(A)$. On the other hand,

$$\begin{aligned} C^r_{G^r_{C^r}}(A) &= \bigwedge\{D \in G^r_{C^r} \mid A \leq D\} \\ &= \bigwedge\{D \in G^r_{C^r} \mid C^r(A) \leq D\} \quad \text{[by Claim 5]} \\ &\geq C^r(A). \end{aligned}$$

(2) and (4) can be similarly proved as in (1) and (3) respectively. □

By Theorem 3.7, we have the following.

Corollary 3.8. *Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set.*
(1) *There is a one to one correspondence between the set of all r-closure operators on X and the set of all r-closure systems on X.*

(2) *There is a one to one correspondence between the set of all l-closure operators on X and the set of all l-closure systems on X.*

Definition 3.9 ([22])**.** Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Define four maps $\downarrow_r, \downarrow_l, \uparrow_r, \uparrow_l \colon L^X \to L^X$ by: for each $S \in L^X$ and each $x \in X$,

$$\begin{aligned} \downarrow_r A(x) &= \bigvee_{y \in X}\left[e^r_X(x,y) \odot A(y)\right], & \downarrow_l A(x) &= \bigvee_{y \in X}\left[A(y) \odot e^l_X(x,y)\right], \\ \uparrow_r A(x) &= \bigvee_{y \in X}\left[A(y) \odot e^r_X(y,x)\right], & \uparrow_l A(x) &= \bigvee_{y \in X}\left[e^l_X(y,x) \odot A(y)\right]. \end{aligned}$$

Definition 3.10. Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Define four maps C^r_1, C^l_1, C^r_2, $C^l_2 : L^X \to L^X$ by: for each $S \in L^X$ and each $x \in X$,

$$\begin{aligned} C^r_1(A)(x) &= \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right], \\ C^l_1(A)(x) &= \bigwedge_{y\in X} \left[\bigwedge_{z\in X} \left[A(z) \to e^l_X(z,y)\right] \Rightarrow e^l_X(x,y)\right], \\ C^r_2(A)(x) &= \bigwedge_{y\in X} \left[\bigwedge_{z\in X} \left[A(z) \Rightarrow e^l_X(y,z)\right] \to e^l_X(y,x)\right], \\ C^l_2(A)(x) &= \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,x)\right]. \end{aligned}$$

Remark 3.11. Let $(L, \vee, \wedge, \to)$ be a frame defined in [1, 18]. Then $C^r_1(A) = C^l_1(A) = (A^u)^l$ and $C^r_2(A) = C^l_2(A) = (A^l)^u$, where $A \in L^X$.

Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Let $x \in X$. Define four maps $(e^r_X)^x$, $\left(e^l_X\right)_x$, $\left(e^l_X\right)^x$, $(e^r_X)_x : X \to L$ by: for each $y \in X$,

$$\begin{aligned} (e^r_X)^x(y) &= e^r_X(y,x), & \left(e^l_X\right)_x(y) &= e^l_X(x,y), \\ \left(e^l_X\right)^x(y) &= e^l_X(y,x), & (e^r_X)_x(y) &= e^r_X(x,y). \end{aligned}$$

Define $\top_x : X \to L$ by: for each $z \in X$, $\top_x(z) = \begin{cases} \top & \text{if } z = x, \\ \bot & \text{if } z \neq x. \end{cases}$

Theorem 3.12. *Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Let A, $B \in L^X$. Then the following hold.*

(1) *$\downarrow_r$ and C^r_1 are r-closure operators with $\downarrow_r(\top_x) = (e^r_X)^x$ and $C^r_1(\top_x) = (e^r_X)^x$. Moreover, $\downarrow_r \leq C^r_1$.*

(2) *$\uparrow_l$ and C^r_2 are r-closure operators with $\uparrow_l(\top_x) = \left(e^l_X\right)_x$ and $C^r_2(\top_x) = \left(e^l_X\right)_x$. Moreover, $\uparrow_l \leq C^r_2$.*

(3) *$\downarrow_l$ and C^l_1 are l-closure operators with $\downarrow_l(\top_x) = \left(e^l_X\right)^x$ and $C^l_1(\top_x) = \left(e^l_X\right)^x$. Moreover, $\downarrow_l \leq C^l_1$.*

(4) *$\uparrow_r$ and C^l_2 are l-closure operators with $\uparrow_r(\top_x) = (e^r_X)_x$ and $C^l_2(\top_x) = (e^r_X)_x$. Moreover, $\uparrow_r \leq C^l_2$.*

Proof. (1) We show that $\downarrow_r$ is an r-closure operator.

(C1) Let $A \in L^X$. Then

$$\downarrow_r A(x) = \bigvee_{y\in X} [e^r_X(x,y) \odot A(y)] \geq e^r_X(x,x) \odot A(x) = \top \odot A(x) = A(x).$$

(C2) Let $A \in L^X$. Then

$$\begin{aligned} \downarrow_r(\downarrow_r A)(x) &= \bigvee_{y\in X} [e^r_X(x,y) \odot \downarrow_r A(y)] \\ &= \bigvee_{y\in X} \left[e^r_X(x,y) \odot \bigvee_{z\in X} [e^r_X(y,z) \odot A(z)]\right] \\ &= \bigvee_{z\in X} \left[\bigvee_{y\in X} [e^r_X(x,y) \odot e^r_X(y,z)] \odot A(z)\right] \\ &= \bigvee_{z\in X} [e^r_X(x,z) \odot A(z)] \\ &= \downarrow_r A(x). \end{aligned}$$

(CR) Let $A, B \in L^X$. Then

$$\begin{aligned} & e^r_{L^X}(\downarrow_r A, \downarrow_r B) \\ &= \bigwedge_{x\in X} [\downarrow_r A(x) \Rightarrow \downarrow_r B(x)] \\ &= \bigwedge_{x\in X} \left[\bigvee_{y\in X} [e^r_X(x,y) \odot A(y)] \Rightarrow \bigvee_{y\in X} [e^r_X(x,y) \odot B(y)]\right] \\ &\geq \bigwedge_{x\in X} \bigwedge_{y\in X} [[e^r_X(x,y) \odot A(y)] \Rightarrow [e^r_X(x,y) \odot B(y)]] \text{ [By Lemma 2.2 (2)]} \\ &\geq \bigwedge_{y\in X} [A(y) \Rightarrow B(y)] \text{ [By Lemma 2.2 (8)]} \end{aligned}$$

$= e^r_{L^X}(A, B)$.

Thus $\downarrow_r$ is an r-closure operator.

Moreover,

$$\begin{aligned} \downarrow_r \top_z(x) &= \bigvee_{y\in X} (e^r_X(x,y) \odot \top_z(y)) \\ &= e^r_X(x,z) \\ &= (e^r_X)^z(x). \end{aligned}$$

We show that C^r_1 is an r-closure operator.

(C1) Let $A \in L^X$. Then

$$\begin{aligned} C^r_1(A)(x) &= \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right] \\ &\geq \bigwedge_{y\in X} [[A(x) \Rightarrow e^r_X(x,y)] \to e^r_X(x,y)] \quad \text{[By Lemma 2.2 (1)]} \\ &\geq A(x) \quad \text{[By Lemma 2.2 (5)]}. \end{aligned}$$

(C2) Let $A \in L^X$.

Claim 1: $\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] = \bigwedge_{z\in X} [C^r_1(A)(z) \Rightarrow e^r_X(z,y)]$.

Since $A(z) \leq C^r_1(A)(z)$ by (C1), we have by Lemma 2.2 (1) that

$$\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \geq \bigwedge_{z\in X} [C^r_1(A)(z) \Rightarrow e^r_X(z,y)].$$

On the other hand, note that

$$\begin{aligned} &\bigwedge_{w\in X} [C^r_1(A)(w) \Rightarrow e^r_X(w,y)] \\ &= \bigwedge_{w\in X} \left[\bigwedge_{p\in X} \left[\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,p)] \to e^r_X(w,p)\right] \Rightarrow e^r_X(w,y)\right] \\ &\geq \bigwedge_{w\in X} \left[\left[\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \to e^r_X(w,y)\right] \Rightarrow e^r_X(w,y)\right] \\ &\quad \text{[By Lemma 2.2 (1)} \\ &\geq \bigwedge_{w\in X} \bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \quad \text{[By Lemma 2.2 (5)]} \\ &= \bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)]. \end{aligned}$$

Then Claim 1 is proved.

Finally, we have

$$\begin{aligned} C^r_1(C^r_1(A))(x) &= \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [C^r_1(A)(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right] \\ &= \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right] \quad \text{[by Claim 1]} \\ &= C^r_1(A)(x). \end{aligned}$$

Thus $C^r_1(C^r_1(A)) = C^r_1(A)$ for all $A \in L^X$.

(CR) Let $A,\ B \in L^X$. Then we have

$$\begin{aligned} &e^r_{L^X}(C^r_1(A), C^r_1(B)) \\ &= \bigwedge_{x\in X} [C^r_1(A)(x) \Rightarrow C^r_1(B)(x)] \\ &= \bigwedge_{x\in X} \Big[\bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right] \\ &\qquad \Rightarrow \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [B(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right]\Big] \\ &\geq \bigwedge_{x\in X} \bigwedge_{y\in X} \Big[\left[\bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right] \\ &\qquad \Rightarrow \left[\bigwedge_{z\in X} [B(z) \Rightarrow e^r_X(z,y)] \to e^r_X(x,y)\right]\Big] \\ &\quad \text{[By Lemma 2.2 (2)]} \\ &\geq \bigwedge_{x\in X} \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [B(z) \Rightarrow e^r_X(z,y)] \to \bigwedge_{z\in X} [A(z) \Rightarrow e^r_X(z,y)]\right] \\ &\quad \text{[By Lemma 2.2 (9)]} \\ &\geq \bigwedge_{x\in X} \bigwedge_{y\in X} \bigwedge_{z\in X} [[B(z) \Rightarrow e^r_X(z,y)] \to [A(z) \Rightarrow e^r_X(z,y)]] \end{aligned}$$

$$\begin{aligned}
&\text{[By Lemma 2.2 (2)]}\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\bigwedge_{z\in X}[A(z)\Rightarrow B(z)] \ \text{[By Lemma 2.2 (9)]}\\
&= \bigwedge_{z\in X}[A(z)\Rightarrow B(z)]\\
&= e^r_{L^X}(A,B).
\end{aligned}$$

Thus C^r_1 is an r-closure operator.

Moreover,

$$\begin{aligned}
C^r_1(\top_w)(x) &= \bigwedge_{y\in X}\left[\bigwedge_{z\in X}[\top_w(z)\Rightarrow e^r_X(z,y)]\to e^r_X(x,y)\right]\\
&= \bigwedge_{y\in X}[e^r_X(w,y)\to e^r_X(x,y)]\\
&= e^r_X(x,w)\\
&= (e^r_X)^w(x).
\end{aligned}$$

We show $\downarrow_r\leq C^r_1$. Let $A\in L^X$. Since

$$\begin{aligned}
e^r_X(x,y)\odot A(y)\odot \bigwedge_{z\in X}[A(z)\Rightarrow e^r_X(z,w)] &\leq e^r_X(x,y)\odot A(y)\odot [A(y)\Rightarrow e^r_X(y,w)]\\
&\leq e^r_X(x,y)\odot e^r_X(y,w) \ \text{(by Lemma 2.2(5))}\\
&\leq e^r_X(x,w) \quad \text{(by (R))},
\end{aligned}$$

we have by residuation that

$$e^r_X(x,y)\odot A(y)\leq \bigwedge_{z\in X}[A(z)\Rightarrow e^r_X(z,w)]\to e^r_X(x,w)\quad \text{for all } y,w\in X,$$

which implies that

$$\bigvee_{y\in X}[e^r_X(x,y)\odot A(y)]\leq \bigwedge_{w\in X}\left[\bigwedge_{z\in X}[A(z)\Rightarrow e^r_X(z,w)]\to e^r_X(x,w)\right].$$

So $\downarrow_r(A)\leq C^r_1(A)$ for all $A\in L^X$

(4) We show that $\uparrow_r$ is an l-closure operator.

(C1) Let $A\in L^X$. Then

$$\uparrow_r A(x)=\bigvee_{y\in X}[A(y)\odot e^r_X(y,x)]\geq A(x)\odot e^r_X(x,x)=A(x)\odot\top=A(x).$$

(C2) Let $A\in L^X$. Then

$$\begin{aligned}
\uparrow_r(\uparrow_r A)(x) &= \bigvee_{y\in X}[\uparrow_r A(y)\odot e^r_X(y,x)]\\
&= \bigvee_{y\in X}\left[\bigvee_{z\in X}[A(z)\odot e^r_X(z,y)]\odot e^r_X(y,x)\right]\\
&= \bigvee_{z\in X}\left[A(z)\odot \bigvee_{y\in X}[e^r_X(z,y)\odot e^r_X(y,x)]\right]\\
&= \bigvee_{z\in X}[A(z)\odot e^r_X(z,x)]\\
&= \uparrow_r A(x).
\end{aligned}$$

(CL) Let $A,\ B\in L^X$. Then we get

$$\begin{aligned}
&e^l_{L^X}(\uparrow_r A,\uparrow_r B)\\
&= \bigwedge_{x\in X}[\uparrow_r A(x)\to\uparrow_r B(x)]\\
&= \bigwedge_{x\in X}\left[\bigvee_{y\in X}[A(y)\odot e^r_X(y,x)]\to\bigvee_{y\in X}[B(y)\odot e^r_X(y,x)]\right]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}[[A(y)\odot e^r_X(y,x)]\to[B(y)\odot e^r_X(y,x)]] \ \text{[By Lemma 2.2 (2)]}\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}[A(y)\to B(y)] \ \text{[By Lemma 2.2 (8)]}\\
&= \bigwedge_{y\in X}[A(y)\to B(y)]
\end{aligned}$$

$= e^l_{L^X}(A, B)$.
Thus $\uparrow_r$ is an l-closure operator.

Moreover,

$$\begin{array}{ll} \uparrow_r \top_z(x) & = \bigvee_{y\in X} [\top_z(y) \odot e^r_X(y,x)] \\ & = e^r_X(z,x) \\ & = (e^r_X)_z(x). \end{array}$$

We show that C^l_2 is an l-closure operator.

(C1) Let $A \in L^X$. Then

$$\begin{array}{ll} C^l_2(A)(x) & = \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,x)\right] \\ & \geq \bigwedge_{y\in X} [[A(x) \to e^r_X(y,x)] \Rightarrow e^r_X(y,x)] \text{ [By Lemma 2.2 (1)]} \\ & \geq \bigwedge_{y\in X} A(x) \text{ [By Lemma 2.2 (5)]} \\ & = A(x). \end{array}$$

(C2) Let $A \in L^X$.

Claim 2: $\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] = \bigwedge_{z\in X} \left[C^l_2(A)(z) \to e^r_X(y,z)\right]$.

Since $A(z) \leq C^l_2(A)(z) \geq$ by (C1), we have by Lemma 2.2 (1) that

$$\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \geq \bigwedge_{z\in X} \left[C^l_2(A)(z) \to e^r_X(y,z)\right].$$

On the other hand, note that

$$\begin{array}{l} \bigwedge_{w\in X} \left[C^l_2(A)(w) \to e^r_X(y,w)\right] \\ = \bigwedge_{w\in X} \left[\bigwedge_{p\in X} \left[\bigwedge_{z\in X} [A(z) \to e^r_X(p,z)] \Rightarrow e^r_X(p,w)\right] \to e^r_X(y,w)\right] \\ \geq \bigwedge_{w\in X} \left[\left[\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,w)\right] \to e^r_X(y,w)\right] \\ \quad \text{[By Lemma 2.2 (1)]} \\ \geq \bigwedge_{w\in X} \bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \text{ [By Lemma 2.2 (5)]} \\ = \bigwedge_{z\in X} [A(z) \to e^r_X(y,z)]. \end{array}$$

Then Claim 2 is proved.

Finally, we have

$$\begin{array}{ll} C^l_2\left(C^l_2(A)\right)(x) & = \bigwedge_{y\in X} \left[\bigwedge_{z\in X} \left[C^l_2(A)(z) \to e^r_X(y,z)\right] \to e^r_X(y,x)\right] \\ & = \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,x)\right] \text{ [By Claim 2]} \\ & = C^l_2(A)(x). \end{array}$$

Thus $C^l_2\left(C^l_2(A)\right) = C^l_2(A)$ for all $A \in L^X$.

(CL) Let $A,\ B \in L^X$. Then we have

$$\begin{array}{l} e^l_{L^X}\left(C^l_2(A), C^l_2(B)\right) \\ = \bigwedge_{x\in X} \left[C^l_2(A)(x) \to C^l_2(B)(x)\right] \\ = \bigwedge_{x\in X} \Big[\bigwedge_{y\in X} \left[\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,x)\right] \\ \qquad \to \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [B(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,x)\right] \Big] \\ = \bigwedge_{x\in X} \bigwedge_{y\in X} \Big[\left[\bigwedge_{z\in X} [A(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,x)\right] \\ \qquad \to \left[\bigwedge_{z\in X} [B(z) \to e^r_X(y,z)] \Rightarrow e^r_X(y,x)\right] \Big] \\ \quad \text{[By Lemma 2.2 (2)]} \\ \geq \bigwedge_{x\in X} \bigwedge_{y\in X} \left[\bigwedge_{z\in X} [B(z) \to e^r_X(y,z)] \Rightarrow \bigwedge_{z\in X} [A(z) \to e^r_X(y,z)]\right] \\ \quad \text{[By Lemma 2.2 (9)]} \end{array}$$

$\geq \bigwedge_{x\in X} \bigwedge_{y\in X} \bigwedge_{z\in X} [[B(z) \to e^r_X(z,y)] \Rightarrow [A(z) \to e^r_X(z,y)]]$
[By Lemma 2.2 (2)]
$\geq \bigwedge_{x\in X} \bigwedge_{y\in X} \bigwedge_{z\in X} [A(z) \to B(z)]$ [By Lemma 2.2 (9)]
$= \bigwedge_{z\in X} [A(z) \to B(z)]$
$= e^l_{L^X}(A,B)$.

Thus C^l_2 is an l-closure operator.

Moreover,

$$\begin{aligned} C^l_2\left(\top_w\right)(x) &= \bigwedge_{y\in X} \left(\bigwedge_{z\in X} \left(\top_w(z) \to e^r_X(y,z)\right) \Rightarrow e^r_X(y,x)\right) \\ &= \bigwedge_{y\in X} \left(e^r_X(y,w) \Rightarrow e^r_X(y,x)\right) \\ &= e^r_X(w,x) \text{ [By (R)]} \\ &= \left(e^r_X\right)_w(x). \end{aligned}$$

We show $\uparrow_r \leq C^l_2$. Let $A \in L^X$. Since
$\bigwedge_{z\in X} [A(z) \to e^r_X(w,z)] \odot A(y) \odot e^r_X(y,x)$
$\leq [A(y) \to e^r_X(w,y)] \odot A(y) \odot e^r_X(y,x)$
$\leq e^r_X(w,y) \odot e^r_X(y,x)$ [By Lemma 2.2 (5)]
$\leq e^r_X(w,x)$ [By (R)],

we have by residuation that

$$A(y) \odot e^r_X(y,x) \leq \bigwedge_{z\in X} [A(z) \to e^r_X(w,z)] \Rightarrow e^r_X(w,x) \quad \text{for all } y,w \in X,$$

which implies that

$$\bigvee_{y\in X} [A(y) \odot e^r_X(y,x)] \leq \bigwedge_{w\in X} \left[\bigwedge_{z\in X} [A(z) \to e^r_X(w,z)] \Rightarrow e^r_X(w,x)\right].$$

So $\uparrow_r A \leq C^l_2(A)$ for all $A \in L^X$.

(2) and (3) can be similarly proved. □

4. Various completeness

In this section, we show that the set $M = \left\{A \in L^X \mid C(A) = A\right\}$, where C is a right (resp. left) closure operator, is a right (resp. left) meet complete.

Definition 4.1 ([22])**.** Let (X, e^r_X) be an r-partially ordered set. Let $A \in L^X$.

(i) A point x_0 is called an *r-join* (or *right-join*) of A, denoted by $x_0 = \sqcup_r A$, if it satisfies the following conditions:

(RJ1) $A(x) \leq e^r_X(x, x_0)$ for all $x \in X$,

(RJ2) $\bigwedge_{x\in X} [A(x) \Rightarrow e^r_X(x,y)] \leq e^r_X(x_0, y)$ for all $y \in X$.

(ii) A point x_1 is called an *r-meet* (or *right-meet*) of A, denoted by $x_1 = \sqcap_r A$, if it satisfies the following conditions:

(RM1) $A(x) \leq e^r_X(x_1, x)$ for all $x \in X$,

(RM2) $\bigwedge_{x\in X} [A(x) \to e^r_X(y,x)] \leq e^r_X(y, x_1)$ for all $y \in X$.

Let $\left(X, e^l_X\right)$ be an l-partially ordered set. Let $A \in L^X$.

(iii) A point x_0 is called an *l-join* (or *left-join*) of A, denoted by $x_0 = \sqcup_l A$, if it satisfies

(LJ1) $A(x) \leq e^l_X(x, x_0)$ for all $x \in X$,

(LJ2) $\bigwedge_{x\in X} \left[A(x) \to e^l_X(x,y)\right] \leq e^l_X(x_0, y)$ for all $y \in X$.

(iv) A point x_1 is called an *l-meet* (or *left-meet*) of A, denoted by $x_1 = \sqcap_l A$, if it satisfies

(LM1) $A(x) \le e^l_X(x_1, x)$ for all $x \in X$,

(LM2) $\bigwedge_{x\in X}\left[A(x) \Rightarrow e^l_X(y,x)\right] \le e^l_X(y, x_1)$ for all $y \in X$.

(v) An r-partially ordered set (X, e^r_X) is *r-join complete* (resp. *r-meet complete*), if there exists $\sqcup_r A$ (resp. $\sqcap_r A$) for all $A \in L^X$.

(vi) An r-partially ordered set (X, e^r_X) is *r-complete* if is r-join complete and r-meet complete.

(vii) An l-partially ordered set $\left(X, e^l_X\right)$ is *l-join complete*(resp. *l-meet complete*) if there exists $\sqcup_l A$ (resp. $\sqcap_l A$) for all $A \in L^X$.

(viii) An l-partially ordered set $\left(X, e^l_X\right)$ is *l-complete* if it is l-join complete and l-meet complete.

Lemma 4.2 ([22]). *Let $\left(X, e^r_X, e^l_X\right)$ be a bi-partially ordered set. Let $x_0, x_1 \in X$. Let $A \in L^X$. Then the following hold.*

(1) $x_0 = \sqcup_r A$ if and only if $\bigwedge_{x\in X}\left[A(x) \Rightarrow e^r_X(x,y)\right] = e^r_X(x_0, y)$ for all $y \in X$.

(2) $x_1 = \sqcap_r A$ if and only if $\bigwedge_{x\in X}\left[A(x) \to e^r_X(y,x)\right] = e^r_X(y, x_1)$ for all $y \in X$.

(3) $x_0 = \sqcup_l A$ if and only if $\bigwedge_{x\in X}\left[A(x) \to e^l_X(x,y)\right] = e^l_X(x_0, y)$ for all $y \in X$.

(4) $x_1 = \sqcap_l A$ if and only if $\bigwedge_{x\in X}\left[A(x) \Rightarrow e^l_X(y,x)\right] = e^l_X(y, x_1)$ for all $y \in X$.

(5) $\sqcup_r A$, $\sqcap_r A$, $\sqcup_l A$ and $\sqcap_l A$ are unique if each exists.

Let $k \in L$. Let $A \in L^X$. Define two maps $k \to A$, $k \Rightarrow A : X \to L$ by

$$(k \to A)(x) = k \to A(x), \quad (k \Rightarrow A)(x) = k \Rightarrow A(x).$$

Theorem 4.3. (1) *Let $C^r : L^X \to L^X$ be an r-closure operator. Let $G^r_{C^r} = \{A \in L^X \mid C^r(A) = A\}$. Then $\left(G^r_{C^r}, e^r_{G^r_{C^r}}\right)$ is r-meet complete, where*

$$\sqcap_r \Psi = \bigwedge_{A\in G^r_{C^r}} [\Psi(A) \to A] \quad \text{for all } \Psi \in L^{L^X}.$$

(2) *Let $C^l : L^X \to L^X$ be an l-closure operator. Let $G^l_{C^l} = \{A \in L^X \mid C^l(A) = A\}$. Then $\left(G^l_{C^l}, e^l_{G^l_{C^l}}\right)$ is l-meet complete, where*

$$\sqcap_l \Psi = \bigwedge_{A\in G^l_{C^l}} [\Psi(A) \Rightarrow A] \quad \text{for all } \Psi \in L^{L^X}.$$

Proof. (1) Let $\Psi : G^r_{C^r} \to L$ be a map. Note that for all $B \in G^r_{C^r}$,

$$\begin{aligned}
&\bigwedge_{A\in G^r_{C^r}}\left[\Psi(A) \to e^r_{G^r_{C^r}}(B, A)\right]\\
&= \bigwedge_{A\in G^r_{C^r}}\left[\Psi(A) \to \bigwedge_{x\in X}[B(x) \Rightarrow A(x)]\right]\\
&= \bigwedge_{A\in G^r_{C^r}}\bigwedge_{x\in X}[\Psi(A) \to [B(x) \Rightarrow A(x)]] \text{ [By Lemma 2.2 (2)]}\\
&= \bigwedge_{A\in G^r_{C^r}}\bigwedge_{x\in X}[B(x) \Rightarrow [\Psi(A) \to A(x)]] \text{ [By Lemma 2.2 (4)]}\\
&= \bigwedge_{x\in X}\left[B(x) \Rightarrow \bigwedge_{A\in G^r_{C^r}}[\Psi(A) \to A(x)]\right] \text{ [By Lemma 2.2 (2)]}\\
&= e^r_{G^r_{C^r}}\left(B, \bigwedge_{A\in G^r_{C^r}}[\Psi(A) \to A]\right).
\end{aligned}$$

By Lemma 4.2, $\sqcap_r \Psi = \bigwedge_{A\in G^r_{C^r}}[\Psi(A) \to A]$.

(2) Let $\Psi : G^l_{C^l} \to L$ be a map. Note that for all $B \in G^l_{C^l}$,

$$\begin{aligned}
&\bigwedge_{A\in G^l_{C^l}}\left[\Psi(A)\Rightarrow e^l_{G^l_{C^l}}(B,A)\right]\\
&=\bigwedge_{A\in G^l_{C^l}}\left[\Psi(A)\Rightarrow \bigwedge_{x\in X}[B(x)\to A(x)]\right]\\
&=\bigwedge_{A\in G^l_{C^l}}\bigwedge_{x\in X}[\Psi(A)\Rightarrow [B(x)\to A(x)]] \ \text{[By Lemma 2.2 (2)]}\\
&=\bigwedge_{A\in G^l_{C^l}}\bigwedge_{x\in X}[B(x)\to [\Psi(A)\Rightarrow A(x)]] \ \text{[By Lemma 2.2 (4)]}\\
&=\bigwedge_{x\in X}\left[B(x)\to \bigwedge_{A\in G^l_{C^l}}[\Psi(A)\Rightarrow A(x)]\right] \ \text{[By Lemma 2.2 (2)]}\\
&=e^l_{G^l_{C^l}}\left(B,\bigwedge_{A\in G^l_{C^l}}[\Psi(A)\Rightarrow A]\right).
\end{aligned}$$

By Lemma 4.2, $\sqcap_l\Psi=\bigwedge_{A\in G^l_{C^l}}[\Psi(A)\Rightarrow A]$. □

By Theorems 3.12 and 4.3, we have the following.

Corollary 4.4. (1) *The pair* $\left(G^r_{C^r}, e^r_{G^r_{C^r}}\right)$ *is* r*-meet complete where* $C^r=\downarrow_r$, C^r_1, $\uparrow_l$ *or* C^r_2.

(2) *The pair* $\left(G^l_{C^l}, e^l_{G^l_{C^l}}\right)$ *is* l*-meet complete where* $C^l=\downarrow_l$, C^l_1, $\uparrow_r$ *or* C^l_2.

Remark 4.5. Ko and Kim [22] proved that the followings hold:

(1) the pair $\left(G^r_{C^r}, e^r_{G^r_{C^r}}\right)$ is r-complete, where $C^r=\downarrow_r$ or $\uparrow_l$,

(2) the pair $\left(G^l_{C^l}, e^l_{G^l_{C^l}}\right)$ is l-complete, where $C^l=\downarrow_l$ or $\uparrow_r$.

5. Conclusion

We have investigated the relationships among right (resp. left) closure systems, right (resp. left) closure operators and right (resp. left) meet complete lattices on generalized complete residuated lattices.

In the future, fuzzy rough sets, information systems and decision rules may be investigated on generalized complete residuated lattices.

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