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Abstract. In this paper, we define an interval-valued continuity and study its several properties. Next, we define separation axioms T_0, T_1, T_2, T_3, T_4 of some types in an interval-valued topological space, discuss some relationships among them and give various examples. Finally, we introduce the concept of interval-valued fuzzy subspaces and obtain some of its properties.

2020 AMS Classification: 54A40

Keywords: Interval-valued set, Interval-valued topological space, Interval-valued continuity, T_0 (T_1, T_2, T_3, T_4), Interval-valued subspace.

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1. Introduction

In 1975, Zadeh [1] introduced the concept of interval-valued fuzzy sets as the generalization of an fuzzy set proposed by himself [2] (Refer to [3]). After then, Biswas [16] applied it first to group theory (For further researches, refer to [5, 6, 7]). Jun [8] dealt with interval-valued subalgebras and ideals in BCK-algebras. Also Maet al. [9] discussed with some kinds of interval-valued ideals in BCI-algebras and Barbhuiya [10] studied interval-valued ideals in BCK-algebras. Mondal and Samanta [11] investigated topological structures based on interval-valued fuzzy sets (For further researches, refer to [12, 13, 14]). Ju and Yuan [15] defined similarity measures on interval-valued fuzzy sets and applied them to pattern recognitions. Roy and Biswas [16] dealt with medical diagnoses by using interval-valued fuzzy relations and Ju [17] applied interval-valued fuzzy relations to medical diagnoses.

As a tool for approximating undefinable or complex concepts, Yao [18] proposed an interval set (which was called an interval-valued set by Kim et al. [19]) which is the generalization of an ordinary set and the special case of an interval-valued fuzzy set. In particular, Kim et al. [19] studied interval-valued neighborhoods of two types and some interval-valued closures and interiors. Recently, Cheong et al. [20] introduced the concept of interval-valued relations and investigated it in the sense of a category theory.

It is well-known [21] that there are six basic separation axioms which are used in a classical topological space (X, T):

Axiom $\mathbf{A}_0$. For $x \neq y \in X$, there is $U \in T$ such that either $x \in U$ but $y \notin U$ or $x \notin U$ but $y \in U$.

Axiom $\mathbf{A}_1$. For $x \neq y \in X$, there are $U,\ V \in T$ such that $x \in U$, $y \notin U$, $x \notin V$ and $y \in V$.

Axiom $\mathbf{A}_2$. For $x \neq y \in X$, there are $U,\ V \in T$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Axiom $\mathbf{A}_3$. For each $x \in X$ and each closed set C in X such that $x \notin C$, there are $U,\ V \in T$ such that $x \in U$, $C \subset V$ and $U \cap V = \varnothing$.

Axiom $\mathbf{A}_4$. For each closed set C in X and each $x \in C^c$, there is a continuous mapping $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in C$.

Axiom $\mathbf{A}_5$. For any closed sets C_i in X $(i = 1,\ 2)$ with $C_1 \cap C_2 = \varnothing$, there are $U_i \in T$ such that $U_1 \cap U_2 = \varnothing$ and $C_i \subset U_1$ $(i = 1,\ 2)$.

Axiom $\mathbf{A}_6$. For any subsets M_i of X $(i = 1,\ 2)$ such that $(M_1 \cap \bar{M}_2) \cup (\bar{M}_1 \cap M_2) = \varnothing$, there are $U_i \in T$ $(i = 1,\ 2)$ such that $M_i \subset U_1$ $(i = 1,\ 2)$.

The aim of our research is to deal with various interval valued continuities, and define T_0, T_1, T_2, T_3, T_4 separation axioms in interval-valued topological spaces from the point of view mentioned above and study various properties of them. In order to accomplish our aim, this paper is composed of six sections. In Section 2, we recall some definitions of interval-valued sets introduced by Yao [18] and three results obtained by Kim et al. [19]. In Section 3, we define an interval-valued continuity and find its several properties. In Section 4, we introduce T_0, T_1, T_2 separation axioms in interval-valued topological spaces and obtain some of their properties, and give some examples. In Section 5, we propose the notions of T_3 and T_4 separation axioms in interval-valued topological spaces and find two characterizations of each concept. In Section 6, we define an interval-valued subspace and deal with some of its properties. In Section 7, Since there is a typo in the Definition 4.11 in [19], we correct it.

2. Preliminaries

In this section, we recall basic concepts and three results related to interval-valued sets introduced by Yao [18] and Kim et al. [19].

Definition 2.1 ([18, 19])**.** Let X be an non-empty set. Then the form

$$[A^-, A^+] = \{B \subset X : A^- \subset B \subset A^+\}$$

is called an interval-valued set (briefly, IVS) or interval set in X, if $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In this case, A^- [resp. A^+] represents the set of minimum [resp. maximum] memberships of elements of X to A. In fact, A^- [resp. A^+] is a minimum [resp. maximum] subset of X agreeing or approving for a certain opinion, view, suggestion or policy. $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the *interval-valued empty* [resp.

whole] *set* in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$]. We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in IVS(X)$, then

$$\chi_A = [\chi_{A^-}, \chi_{A^+}]$$

is an interval-valued fuzzy set in X introduced by Zadeh [1]. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

Definition 2.2 ([18, 19])**.** Let X be a non-empty set and let $A,\ B \in IVS(X)$. Then
(i) we say that A *contained in* B, denoted by $A \subset B$, if $A^- \subset B^-$ and $A^+ \subset B^+$,
(ii) we say that A *equal to* B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,
(iii) the *complement* of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = [(A^+)^c, (A^-)^c],$$

(iv) the *union* of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = [A^- \cup B^-, A^+ \cup B^+],$$

(v) the *intersection* of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = [A^- \cap B^-, A^+ \cap B^+].$$

The followings are (i1), (i2), (i3), (k1), (k2) and (k3) in [18].

Result 2.3. *Let X be a non-empty set and let $A,\ B,\ C \in IVS(X)$. Then*
(1) $\widetilde{\varnothing} \subset A \subset \widetilde{X}$,
(2) *if $A \subset B$ and $B \subset C$, then $A \subset C$,*
(3) *$A \subset A \cup B$ and $B \subset A \cup B$,*
(4) *$A \cap B \subset A$ and $A \cap B \subset B$,*
(5) *$A \subset B$ if and only if $A \cap B = A$,*
(6) *$A \subset B$ if and only if $A \cup B = B$.*

The followings are (I1)–(I8) in [18].

Result 2.4. *Let X be a non-empty set and let $A,\ B,\ C \in IVS(X)$. Then*
(1) (Idempotent laws) $A \cup A = A,\ A \cap A = A$,
(2) (Commutative laws) $A \cup B = B \cup A,\ A \cap B = B \cap A$,
(3) (Associative laws) $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws) $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$,
(6) (DeMorgan's laws) $(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8_a) $A \cup \widetilde{\varnothing} = A,\ A \cap \widetilde{\varnothing} = \widetilde{\varnothing}$,
(8_b) $A \cup \widetilde{X} = \widetilde{X},\ A \cap \widetilde{X} = A$,
(8_c) $\widetilde{X}^c = \widetilde{\varnothing},\ \widetilde{\varnothing}^c = \widetilde{X}$,
(8_d) $A \cup A^c \neq \widetilde{X},\ A \cap A^c \neq \widetilde{\varnothing}$ *in general* (See Example 3.7 in [19]).

Definition 2.5 ([19]). Let $(A_j)_{j\in J}$ be a family of members of $IVS(X)$. Then

(i) the *intersection* of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$, is an IVS in X defined by:

$$\bigcap_{j\in J} A_j = [\bigcap_{j\in J} A_j^-, \bigcap_{j\in J} A_j^+],$$

(ii) the *union* of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} \widetilde{A}_j$, is an IVS in X defined by:

$$\bigcup_{j\in J} A_j = [\bigcup_{j\in J} A_j^-, \bigcup_{j\in J} A_j^+].$$

Result 2.6 (Proposition 3.9, [19]). *Let $A \in [X]$ and let $(A_j)_{j\in J}$ be a family of members of $IVS(X)$. Then*

(1) $(\bigcap_{j\in J} A_j)^c = \bigcup_{j\in J} A_j^c$, $(\bigcup_{j\in J} A_j)^c = \bigcap_{j\in J} A_j^c$,

(2) $A \cap (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}(A \cap A_j)$, $A \cup (\bigcap_{j\in J} A_j) = \bigcap_{j\in J}(A \cup A_j)$.

Definition 2.7 ([19]). Let X be a non-empty set, let $a \in X$ and let $A \in IVS(X)$. Then the form $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an interval-valued [resp. vanishing] point in X and denoted by $a_{_1}$ [resp. $a_{_0}$]. We will denote the set of all interval-valued points in X as $IV_P(X)$.

(i) We say that $a_{_1}$ *belongs to* A, denoted by $a_{_1} \in A$, if $a \in A^-$.

(ii) We say that $a_{_0}$ *belongs to* A, denoted by $a_{_0} \in A$, if $a \in A^+$.

Result 2.8 (Proposition 3.11, [19]). *Let X be a non-empty set and let $A \in IVS(X)$. Then*

$$A = A_{_1} \cup A_{_0},$$

where $A_{_1} = \bigcup_{a_{_1}\in A} a_{_1}$ and $A_{_0} = \bigcup_{a_{_0}\in A} a_{_0}$.

In fact, $A_{_1} = [A^-, A^-]$ and $A_{_0} = [\varnothing, A^+]$

For a set X, let $IVS^*(X) = \{A \in IVS(X) : A^- = A^+\}$. Then from the above Result, $A = A_{_1}$ for each $A \in IVS^*(X)$.

Result 2.9 (Theorem 3.14, [19]). *Let $(A_j)_{j\in J} \subset IVS(X)$ and let $a \in X$.*

(1) *$a_{_1} \in \bigcap A_j$ [resp. $a_{_0} \in \bigcap A_j$] if and only if $a_{_1} \in A_j$ [resp. $a_{_0} \in A_j$] for each $j \in J$.*

(2) *$a_{_1} \in \bigcup A_j$ [resp. $a_{_0} \in \bigcup A_j$] if and only if there exists $j \in J$ such that $a_{_1} \in A_j$ [resp. $a_{_0} \in A_j$.*

Result 2.10 (Theorem 3.15, [19]). *Let $A,\ B \in IVS(X)$. Then*

(1) *$A \subset B$ if and only if $a_{_1} \in A \Rightarrow a_{_1} \in B$ [resp. $a_{_0} \in A \Rightarrow a_{_0} \in B$] for each $a \in X$.*

(2) *$A = B$ if and only if $a_{_1} \in A \Leftrightarrow a_{_1} \in B$ [resp. $a_{_0} \in A \Leftrightarrow a_{_0} \in B$] for each $a \in X$.*

Definition 2.11 ([19]). Let X be a non-empty set and let τ be a non-empty family of IVSs on X. Then τ is called an *interval-valued topology* (briefly, IVT) on X, if it satisfies the following axioms:

(IVO_1) $\widetilde{\varnothing},\ \widetilde{X} \in \tau$,

(IVO_2) $A \cap B \in \tau$ for any $A,\ B \in \tau$,

(IVO_3) $\bigcup_{j\in J} A_j \in \tau$ for any family $(A_j)_{j\in J}$ of members of τ.

In this case, the pair (X, τ) is called an *interval-valued topological space* (briefly, IVTS) and each member of τ is called an *open interval-valued set* (briefly, OIVS) in X. A IVS A is called a *closed interval-valued set* (briefly, CIVS) in X, if $A^c \in \tau$.

It is obvious that $\{\widetilde{\varnothing}, \widetilde{X}\}$ is an IVT on X, and will be called the *interval-valued indiscrete topology* on X and denoted by $\tau_{_{IV,0}}$. Also $IVS(X)$ is an IVT on X, and will be called the *interval-valued discrete topology* on X and denoted by $\tau_{_{IV,1}}$. The pair $(X, \tau_{_{IV,0}})$ [resp. $(X, \tau_{_{IV,1}})$] will be called the *interval-valued indiscrete* [resp. *discrete*] *space*.

We will denote the set of all IVTs on X as $IVT(X)$. For an IVTS (X, τ), we will denote the set of all CIVSs in X with respect to τ as τ^c.

Definition 2.12 ([19]). Let X be a non-empty set and let $\tau_1,\ \tau_2 \in IVT(X)$. Then we say that τ_1 is contained in τ_2 or τ_1 is coarser than τ_2 or τ_2 is finer than τ_1, if $\tau_1 \subset \tau_2$, i.e., $A \in \tau_2$ for each $A \in \tau_1$.

It is obvious that $\tau_{_{IV,0}} \subset \tau \subset \tau_{_{IV,1}}$ for each $\tau \in IVT(X)$.

3. Interval-valued continuities

In this section, we define an interval-valued continuity and find its various properties. First of all, we recall the image and the preimage of an IVS under a mapping, and the interval-valued closure and interior.

Definition 3.1 ([19]). Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping and let $A \in IVS(X),\ B \in IVS(Y)$.

(i) The *image of A under f*, denoted by $f(A)$, is an IVS in Y defined as:

$$f(A) = [f(A^-), f(A^+)].$$

(ii) The *preimage of B under f*, denoted by $f^{-1}(B)$, is an IVS in X defined as:

$$f^{-1}(B) = [f^{-1}(B^-), f^{-1}(B^+)].$$

It is obvious that $f(a_{_1}) = f(a)_{_1}$ and $f(a_{_0}) = f(a)_{_0}$ for each $a \in X$.

Result 3.2 (Proposition 3.17, [19]). *Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping, let $A,\ A_1,\ A_2 \in IVS(X),\ (A_j)_{j\in J} \subset IVS(X)$ and let $B,\ B_1,\ B_2 \in IVS(Y),\ (A_j)_{j\in J} \subset IVS(Y)$. Then*

(1) *if $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$,*
(2) *if $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$,*
(3) *$A \subset f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$,*
(4) *$f(f^{-1}(B)) \subset B$ and if f is surjective, $f(f^{-1}(B)) = B$,*
(5) *$f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$,*
(6) *$f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$,*
(7) *$f(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} f(A_j)$,*
(8) *$f(\bigcap_{j\in J} A_j) \subset \bigcap_{j\in J} f(A_j)$ and if f is injective, then $f(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} f(A_j)$,*
(9) *if f is surjective, then $f(A)^c \subset f(A^c)$.*
(10) *$f^{-1}(B^c) = f^{-1}(B)^c$.*
(11) *$f^{-1}(\widetilde{\varnothing}) = \widetilde{\varnothing}$, $f^{-1}(\widetilde{X}) = \widetilde{X}$,*

(12) *$f(\widetilde{\varnothing})=\widetilde{\varnothing}$ and if f is surjective, then $f(\widetilde{X})=\widetilde{X}$,*
(13) *if $g : Y \to Z$ is a mapping, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for each $C \in IVS(Z)$.*

Definition 3.3 ([19])**.** Let (X,τ) be an IVTS and let $A \in IVS(X)$.
(i) The *interval-valued closure* of A w.r.t. τ, denoted by $IVcl(A)$, is an IVS in X defined as:

$$IVcl(A)=\bigcap\{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) The *interval-valued interior* of A w.r.t. τ, denoted by $IVint(A)$, is an IVS in X defined as:

$$IVint(A)=\bigcup\{G : G \in \tau \text{ and } G \subset A\}.$$

It is obvious that $IVcl(A)$ [resp. $IVint(A)$] is the smallest IVCS in X containing A [resp. the largest IVOS in X contained in A.

Result 3.4 (Proposition 6.4, [19])**.** *Let (X,τ) be an IVTS and let $A \in IVS(X)$. Then*

$$IVint(A^c)=(IVcl(A))^c \text{ and } IVcl(A^c)=(IVint(A))^c.$$

Result 3.5 (Theorem 6.8, [19])**.** *Let X be an IVTS and let $A \in IVS(X)$. Then*
(1) *$A \in CIVS(X)$ if and only if $A = IVcl(A)$,*
(2) *$A \in OIVS(X)$ if and only if $A = IVint(A)$.*

Definition 3.6. Let (X,τ), (Y,δ) be IVTSs and let $f : X \to Y$ be a mapping. Then f is said to be *interval-valued continuous*, if $f^{-1}(V) \in \tau$ for each $V \in \delta$.

Proposition 3.7. *Let X, Y, Z be IVTSs and let $f : X \to Y$ and $g : Y \to Z$ be mappings.*
(1) *The identity mapping $id : X \to X$ is interval-valued continuous.*
(2) *If f, g are interval-valued continuous, then $g \circ f$ is interval-valued continuous.*

Proof. From Definition 3.6 and Result 3.2 (13), the proofs are easy. □

Remark 3.8. Let $\mathbf{IV_{Top}}$ be the collection of all IVTSs and all interval-valued mappings between them. Then we can easily see that $\mathbf{IV_{Top}}$ forms a concrete category from Proposition 3.7.

Definition 3.9 ([19])**.** Let (X,τ) be an IVTS, $a \in X$ and let $N \in IVS(X)$. Then
(i) N is called an *interval-valued neighborhood* (briefly, IVN) of a_1, if there exists $U \in \tau$ such that

$$a_1 \in U \subset N, \text{ i.e., } a \in U^- \subset N^-,$$

(ii) N is called an *interval-valued vanishing neighborhood* (briefly, IVVN) of a_0, if there exists $U \in \tau$ such that

$$a_0 \in U \subset N, \text{ i.e., } a \in U^+ \subset N^+.$$

We will denote the set of all IVNs [resp. IVVNs] of a_1 [resp. a_0] by $N(a_1)$ [resp. $N(a_0)$].

Definition 3.10. Let $X,\ Y$ be IVTSs, let $a \in X$ and let $f : X \to Y$ be a mapping. Then f is said to be:

(i) *interval-valued point-wise continuous* (briefly, IVPC) *at* $a_{_1}$, if $f^{-1}(V) \in N(a_{_1})$ for each $V \in N(f(a)_{_1})$,

(ii) *interval-valued vanishing point-wise continuous* (briefly, IVVPC) *at* $a_{_0}$, if $f^{-1}(V) \in N(a_{_0})$ for each $V \in N(f(a)_{_0})$.

Theorem 3.11. *Let* $(X, \tau),\ (Y, \delta)$ *be two IVTSs. Then a mapping* $f : X \to Y$ *is interval-valued continuous if and only if it is IVPC at each* $a_{_1}$ *and IVVPC at each* $a_{_0}$*.*

Proof. Suppose f is interval-valued continuous and let $V \in N(f(a)_{_1})$ for any $a_{_1}$. Then there is $U \in \delta$ such that $f(a)_{_1} \in U \subset V$. Thus Result 3.2 (2), we have

$$a_{_1} \in f^{-1}(U) \subset f^{-1}(V) \text{ and } f^{-1}(U) \in \tau.$$

So f is IVPC at $a_{_1}$. Similarly, the second part is proved.

Conversely, suppose the necessary condition hold and let $V \in \delta$ such that $f(a)_{_1} \in V$ and $f(a)_{_0} \in V$ for any $a_{_1},\ a_{_0}$. Then by the hypotheses and Result 2.9, there are $U_{_1},\ U_{_0} \in \tau$ such that $f(a)_{_1} \in U_{_1} \subset V_{_1},\ f(a)_{_0} \in U_{_0} \subset V_{_0}$ and $U = U_{_1} \cup U_{_0}$, $V = V_{_1} \cup V_{_0}$. Thus Result 3.2 (2), we get

$$a_{_1} \in f^{-1}(U_{_1}) \subset f^{-1}(V_{_1}) \text{ and } a_{_0} \in f^{-1}(U_{_0}) \subset f^{-1}(V_{_0}).$$

So by Result 3.2 (5), we have

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V_{_1}) \cup f^{-1}(V_{_0}) \\ &= \left(\bigcup_{a_1 \in f^{-1}(V_1)} f^{-1}(U_{_1})\right) \cup \left(\bigcup_{a_0 \in f^{-1}(V_0)} f^{-1}(U_{_0})\right). \end{aligned}$$

Hence $f^{-1}(V) \in \tau$. Therefore f is interval-valued continuous. □

Definition 3.12 ([19])**.** Let (X, τ) be an IVTS.

(i) A subfamily β of τ is called an *interval-valued base* (briefly, IVB) for τ, if for each $A \in \tau$, $A = \widetilde{\varnothing}$ or there is $\beta^{'} \subset \beta$ such that $A = \bigcup \beta^{'}$.

(ii) A subfamily σ of τ is called an *interval-valued subbase* (briefly, IVSB) for τ, if the family $\beta = \{\bigcap \sigma^{'} : \sigma^{'}$ is a finite subset of $\sigma\}$ is an IVB for τ.

There are other equivalent formulations of cubic crisp continuity that are useful at various times, and its proof is almost similar to classical case.

Theorem 3.13. *Let* $(X, \tau),\ (Y, \delta)$ *be IVTSs, let* $f : X \to Y$ *be a mapping and let* $\beta,\ \sigma$ *be a base and subbase for* τ*, respectively. Then the followings are equivalent:*

(1) f *is interval-valued continuous,*
(2) $f^{-1}(C) \in \tau^c$ *for each* $C \in \delta^c$*,*
(3) $f(IVcl(A)) \subset IVcl(f(A))$ *for each* $A \in IVS(X)$*,*
(4) $IVcl(f^{-1}(B) \subset f^{-1}(IVcl(B))$ *for each* $B \in IVS(Y)$*,*
(5) $f^{-1}(B) \in \tau$ *for each* $B \in \beta$*,*
(6) $f^{-1}(S) \in \tau$ *for each* $S \in \sigma$*.*

Definition 3.14. Let $(X, \tau),\ (Y, \delta)$ be IVTSs. Then a mapping $f : X \to Y$ is said to be *interval-valued open* [resp. *closed*], if $f(A) \in \delta$ for each $A \in \tau$ [resp. $f(C) \in \delta^c$ for each $C \in \tau^c$].

From Result 3.2 (13) and Definition 3.14, we have the following.

Proposition 3.15. *Let X, Y, Z be IVTSs and let $f : X \to Y$ and $g : Y \to Z$ be mappings. If f, g are interval-valued open [resp. closed], then so is $g \circ f$.*

We give a necessary and sufficient condition for a mapping to be cubic crisp open.

Theorem 3.16. *Let (X, τ), (Y, δ) be IVTSs and let $f : X \to Y$. Then f is interval-valued open if and only if $f(IVint(A)) \subset IVint(f(A))$ for each $A \in IVS(X)$.*

Proof. Suppose f is interval-valued open and let $A \in IVS(X)$. Since $IVint(A) \in \tau$, $f(IVint(A)) \in \delta$ by the hypothesis. Since $IVint(A) \subset A$, $f(IVint(A)) \subset f(A)$ by Result 3.2 (1). On the other hand, $IVint(f(A))$ is the largest OIVS in X contained in $f(A)$. Then we have $f(IVint(A)) \subset IVint(f(A))$.

Conversely, suppose the necessary condition holds and let $U \in \tau$. Then by Result 3.5 (2), $U = IVint(U)$. Thus by the hypothesis, $f(U) = f(IVint(U)) \subset IVint(f(U))$. On the other hand, it is obvious that $IVint(f(U)) \subset f(U)$. So $f(U) = IVint(f(U))$. Hence $f(U) \in \delta$. Therefore f is interval-valued open. □

Proposition 3.17. *Let (X, τ), (Y, δ) be IVTSs and let $f : X \to Y$ be a mapping. If f is interval-valued continuous, then $IVint(f(A))) \subset f(IVint(A))$ for each $A \in IVS(X)$.*

Proof. Suppose f is interval-valued continuous and let $A \in IVS(X)$. Since $f(IVint(A)) \in \delta$, $f^{-1}(f(IVint(A))) \in \tau$ by the hypothesis. Since f is injective, from Result 3.2 (3), we have

$$f^{-1}(f(IVint(A))) \subset f^{-1}(f(A)) = A.$$

On the other hand, $IVint(A)$ is the largest OIVS in X contained in A. Then $f^{-1}(f(IVint(A))) \subset IVint(A)$. Thus $IVintf(A)) \subset f(IVint(A))$. □

The following is the immediate result of Theorem 3.16 and Proposition 3.17.

Corollary 3.18. *Let X, Y be IVTSs and let $f : X \to Y$ be a mapping. If f is interval-valued continuous, open and injective, then $f(IVint(A)) = IVint(f(A))$ for each $A \in IVS(X)$.*

The following gives a necessary and sufficient condition for a mapping to be interval-valued closed.

Theorem 3.19. *Let (X, τ), (Y, δ) be IVTSs and let $f : X \to Y$ be a mapping. Then f is interval-valued closed if and only if $IVcl(f(A)) \subset f(IVcl(A))$ for each $A \in IVS(X)$.*

Proof. Suppose f is interval-valued closed and let $A \in IVS(X)$. Then clearly, $A \subset IVcl(A)$. Since $IVcl(A) \in \tau^c$, $f(IVcl(A)) \in \delta^c$ by the hypothesis. Thus $IVcl(f(A)) \subset f(IVcl(A))$.

Conversely, suppose the necessary condition holds and let $C \in \tau^c$. Since $C = IVcl(C)$, we have

$$IVcl(f(C)) \subset f(IVcl(C)) = f(C) \subset IVcl(f(C)).$$

Then $f(C) = IVcl(f(C))$. Thus $f(C) \in \delta^c$. So f is interval-valued closed. □

Theorem 3.20. *X, Y be IVTSs and let $f : X \to Y$ be a mapping. Then f is interval-valued continuous and closed if and only if $f(IVcl(A)) = IVcl(f(A))$ for each $A \in IVS(X)$.*

Proof. Let $A \in IVS(X)$. Then from Theorem 3.13 (3), we have

$$f \text{ is interval–valued continuous if and only } f(IVcl(A)) \subset IVcl(f(A)).$$

Also, by Theorem 3.19, $IVcl(f(A)) \subset f(IVcl(A))$. Thus the result holds. □

Definition 3.21. Let X, Y be IVTSs and let $f : X \to Y$ be a mapping. Then f is called an *interval-valued homeomorphism*, if it is bijective, interval-valued continuous and open.

We would like very often to know if there is an IVT on a set X such that a mapping or a family of mappings of X into an IVTS Y is interval-valued continuous. The following Propositions answer this question.

Proposition 3.22. *Let X be a set, let (Y, δ) be an IVTS and let $f : X \to Y$ be a mapping. Then there is a coarsest IVT τ on X such that f is interval-valued continuous.*

Proof. Let $\tau = \{f^{-1}(V) \in IVS(X) : V \in \delta\}$. Then we can easily check that τ satisfies the conditions (IVO_1), (IVO_2) and (IVO_3). Thus τ is an IVT on X. By the definition of τ, it is clear that $f : (X, \tau) \to (Y, \delta)$ is interval-valued continuous. It is easy to prove that τ is the coarsest IVT on X such that $f : (X, \tau) \to (Y, \delta)$ is interval-valued continuous. □

Proposition 3.23. *Let X be a set, let (Y, δ) be an IVTS and let $(f_j : X \to Y)_{j \in J}$ be be a family of mappings, where J is an index set. Then there is a coarsest IVT τ on X such that f_j is interval-valued continuous for each $j \in J$.*

Proof. Let $\sigma = \{f_j^{-1}(V) \in IVS(X) : V \in \delta,\ j \in J\}$. Then we can easily check that τ is the IVT on X having σ as its IVSB. Thus τ is the coarsest IVT on X such that $f_j : (X, \tau) \to (Y, \delta)$ is interval-valued continuous for each $j \in J$. □

The following gives the dual of Proposition 3.22.

Proposition 3.24. *Let (X, τ) be an IVTS, let Y be a set and let $f : X \to Y$ be be a mapping. Then there is a finest IVT δ on Y such that f is interval-valued continuous.*

Proof. Let $\delta = \{V \in IVS(X) : f^{-1}(V) \in \tau\}$. Then we can easily check that δ is the is the finest IVT on Y such that $f : (X, \tau) \to (Y, \delta)$ is interval-valued continuous. □

Definition 3.25. Let (X, τ) be an IVTS, let Y be a set and let $f : X \to Y$ be be a sujective mapping. Then $\delta = \{V \in IVS(X) : f^{-1}(V) \in \tau\}$ is called the *interval-valued quotient topology* (briefly, IVQT) *on* Y *induced by* f. The pair (Y, δ) is called an *interval-valued quotient space* (briefly, IVQS) and f is called an *interval-valued quotient mapping* (briefly, IVQM).

From Proposition 3.24, it is obvious that $\delta \in IVT(Y)$. Moreover, it is easy to see that if (Y, δ) is an IVQS of (X, τ) with IVQM f, then for an IVS C in Y, $C \in \delta^c$ if and only if $f^{-1}(C) \in \tau^c$.

Let (X, τ), (Y, η) be IVTSs and let $f : X \to Y$ be be a sujective mapping. Then the following gives conditions on f such that $\eta = \delta$, where δ is the IVQT on Y induced by f.

Proposition 3.26. *Let (X, τ), (Y, η) be IVTSs, let $f : (X, \tau) \to (Y, \eta)$ be be an interval-valued continuous surjective mapping and let δ is the IVQT on Y induced by f. If f is interval-valued open or closed, then $\eta = \delta$.*

Proof. Suppose f is interval-valued open and let let δ is the IVQT on Y induced by f. Then clearly by Proposition 3.24, δ is the finest IVT on Y for which f is interval-valued continuous. Thus $\eta \subset \delta$. Let $\mathcal{U} \in \delta$. Then clearly $f^{-1}(U) \in \delta$ by the definition of δ. Since f is interval-valued open and surjective, $U = f(f^{-1}(U)) \in \eta$. Thus $\delta \subset \eta$. So $\eta = \delta$.

The proof that if f is interval-valued closed, then $\eta = \delta$ is similar. □

Proposition 3.27. *The composition of two IVQMs is an IVQM.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ and $g : (Y, \delta) \to (Z, \gamma)$ be two IVQMs. Let η be the IVQM on Z induced by $g \circ f$. We prove that $\eta = \gamma$. Let $V \in \gamma$. Since $g : (Y, \delta) \to (Z, \gamma)$ is an IVQM, $g^{-1}(V) \in \delta$. Since $f : (X, \tau) \to (Y, \delta)$ is an IVQM, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \in \tau$. Then $V \in \eta$. Thus $\gamma \subset \eta$. Moreover, we can easily show that $\eta \subset \gamma$. Thus $\eta = \gamma$. So $g \circ f$ is an IVQM. □

The following is a basic result about IVQS.

Theorem 3.28. *Let (X, τ), (Z, η) be two IVTSs, let Y be a set, let $f : X \to Y$ be a surjective mapping and let δ be the IVQT on Y induced by f. Then $g : (X, \tau) \to (Z, \eta)$ is interval-valued continuous if and only if $g \circ f : (X, \tau) \to (Z, \eta)$ is interval-valued continuous*

Proof. Suppose g is interval-valued continuous. Since $f : (X, \tau) \to (Y, \delta)$ is interval-valued continuous, $g \circ f : (X, \tau) \to (Z, \eta)$ is interval-valued continuous by Proposition 3.7 (2).

Suppose $g \circ f$ is interval-valued continuous and let $V \in \eta$. Then clearly, $(g \circ f)^{-1}(V) \in \tau$ and $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$. Thus by the definition of δ, $g^{-1}(V) \in \delta$. So g is interval-valued continuous. □

In order to consider the product of IVTSs, we give the definition for the product of two IVSs.

Definition 3.29. Let X, Y be sets and let $A \in IVS(X)$, $B \in IVS(Y)$. Then the interval-valued product set (briefly, IVPS) of A and B, denoted by $A \times B$, is an IVS in $X \times Y$ defined as:

$$A \times B = [A^- \times B^-, A^+ \times B^+].$$

Example 3.30. Let $X = \{a, b, c\}$, $Y = \{0, 1, 2\}$, $A = [\{a\}, \{a, b\}] \in IVS(X)$ and let $B = [\{0\}, \{0, 2\}] \in IVS(Y)$. Then we can easily check that

$$A \times B = [\{(a, 0)\}, \{(a, 0), (a, 2), (b, 0), (b, 2)\}].$$

Result 3.31 (Theorem 4.12, [19])**.** *Let X be a non-empty set and let $\beta \subset IVS(X)$. Then β is an IVB for an IVT τ on X if and only if it satisfies the followings:*

(1) $\widetilde{X} = \bigcup \beta$,

(2) *if $B_1, B_2 \in \beta$ and $a_{_1} \in B_1 \cap B_2$ [resp. $a_{_0} \in B_1 \cap B_2$], then there exists $B \in \beta$ such that $a_{_1} \in B \subset B_1 \cap B_2$ [resp. $a_{_0} \in B \subset B_1 \cap B_2$].*

Result 3.32 (Proposition 4.14, [19]). *Let X be a non-empty set and let $\sigma \subset IVS(X)$ such that $\widetilde{X} = \bigcup \sigma$. Then there exists a unique IVT τ on X such that σ is an IVSB for τ.*

Proposition 3.33. *Let (X, τ_1), (Y, τ_2) be two IVTSs and let $\beta = \{U \times V : U \in \tau_1,\ V \in \tau_2\}$. Then β is an IVB for an IVT τ on $X \times Y$.*

In this case, τ is called the interval-valued product topology (briefly, IVPT) on $X \times Y$ and the pair $(X \times Y, \tau)$ is called an interval-valued product space (briefly, IVPS) of X and Y.

Proof. It is obvious that $\widetilde{X} \in \tau_1$, $\widetilde{Y} \in \tau_2$. Then $\widetilde{X \times Y} = \widetilde{X} \times \widetilde{Y} \in \beta$. Thus $\widetilde{X \times Y} = \bigcup \beta$. So Result 3.31 (1) holds.

Now suppose $B_1 = U_1 \times V_1$, $B_2 = U_2 \times V_2 \in \beta$, where U_1, $U_2 \in \tau_1$ and V_1, $V_2 \in \tau_2$. For any $(a, b) \in X \times Y$, let $(a, b)_1$, $(a, b)_0 \in B_1 \cap B_2$. Then we have

$$B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \times U_2) \cap (V_1 \times V_2). \tag{3.1}$$

Since U_1, $U_2 \in \tau_1$ and V_1, $V_2 \in \tau_2$, $U_1 \times U_2 \in \tau_1$ and $V_1 \times V_2 \in \tau_2$. Thus $B_1 \cap B_2 \in \beta$. So Result 3.31 (2) holds. Hence β is an IVB for an IVT τ on $X \times Y$. □

Proposition 3.34. *Let (X_1, τ_1), (X_2, τ_2) be two IVTSs and let $(X_1 \times X_2, \tau)$ be the IVPS. Then the projections $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ are interval-valued continuous. Furthermore, τ is the coarsest IVT for which both projections are interval-valued continuous.*

Proof. Let $U_i \in \tau_i$. Then we have

$$\pi_1^{-1}(U_1) = [U_1^- \times X_2, U_1^+ \times X_2] = U_1 \times \widetilde{X_2}, \tag{3.2}$$

$$\pi_2^{-1}(U_2) = [X_1 \times U_2^-, X_2 \times U_2^+] = \widetilde{X_1} \times U_2. \tag{3.3}$$

Thus $\pi_1^{-1}(U_1)$, $\pi_2^{-1}(U_2) \in \tau$. So by Definition 3.6, π_1 and π_2 are interval-valued continuous.

Let δ be an IVT on $X_1 \times X_2$ such that π_1 and π_2 are interval-valued continuous. Let β be the IVB given in Proposition 3.33 for τ and let $B \in \beta$. Then there are $U_1 \in \tau_1$ and $U_2 \in \tau_2$ such that $B = U_1 \times U_2$. Since π_1 and π_2 are interval-valued continuous with respect to δ , $\pi_1^{-1}(U_1)$, $\pi_2^{-1}(U_2) \in \delta$. On the other hand,

$$\begin{aligned}
\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) &= (U_1 \times \widetilde{X_2})) \cap \widetilde{X_1} \times U_2 \text{ [By (3.2) and (3.3)]} \\
&= [U_1^- \times X_2, U_1^+ \times X_2] \cap [X_1 \times U_2^-, X_1 \times U_2^+] \\
&= [U_1^- \times U_2^-, U_1^+ \times U_2^+] \\
&= U_1 \times U_2.
\end{aligned}$$

Thus we have

$$\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) = U_1 \times U_2. \tag{3.4}$$

By (3.4), $B \in \delta$. So $\beta \subset \delta$. Since β is the IVB for τ, $\tau \subset \delta$. Hence τ is coarser than δ. □

Proposition 3.35. *Let (X_1, τ_1), (X_2, τ_2) be two IVTSs, and let $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ be the projections. Then the family of IVSs*

$$\sigma = \{\pi_1^{-1}(U) \in IVS(X_1 \times X_2) : U \in \tau_1\} \cup \{\pi_2^{-1}(V) \in IVS(X_1 \times X_2) : V \in \tau_2\}$$

is an IVSB for the IVPT τ on $X_1 \times X_2$.

Proof. It is clear that $\widetilde{X_1 \times X_2} = \widetilde{X_1} \times \widetilde{X_2} = \bigcup \sigma$. Then by Result 3.32, there is a unique IVT $\tau^{'}$ on $X_1 \times X_2$ such that σ is an IVSB for $\tau^{'}$. Since $\sigma \subset \tau$, arbitrary unions of finite intersections of members of σ is also belong to τ. Thus $\tau^{'} \subset \tau$. Let β be the base for τ given in Proposition 3.33 and let $U \times V \in \beta$. Then by (3.4), $U \times V$ is the intersection of two members of σ. Thus $U \times V \in \tau^{'}$. So $\tau \subset \tau^{'}$. Hence $\tau = \tau^{'}$. This completes the proof. □

Proposition 3.35 is used to provide a characterization of interval-valued continuity for mappings for which the range is an IVPS.

Theorem 3.36. *Let (X, τ), (Y_1, δ_1), (Y_2, δ_2) be IVTSs and let $f : X \to Y_1 \times Y_2$ be a mapping. Then f is interval-valued continuous if and only if $\pi_i \circ f$ is interval-valued continuous for each $i = 1,\ 2$.*

Proof. Suppose f is interval-valued continuous. Then by Proposition 3.34, π_i is interval-valued continuous for each $i = 1,\ 2$. Thus by Proposition 3.7 (2), $\pi_i \circ f$ is interval-valued continuous for each $i = 1,\ 2$.

Conversely, suppose $\pi_i \circ f$ is interval-valued continuous for each $i = 1,\ 2$. Let σ be the IVSB for the IVPT on $Y_1 \times Y_2$ given in Proposition 3.35 and let $\pi_i^{-1}(U_i) \in \sigma$. Then clearly, $f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i)$. Thus by the hypothesis, $(\pi_i \circ f)^{-1}(U_i) \in \tau$, i.e., $f^{-1}(\pi_i^{-1}(U_i)) \in \tau$. So by Theorem 3.13 (6), f is interval-valued continuous. □

The following is an immediate result of Theorem 3.36.

Corollary 3.37. *Let (X, τ), (Y_1, δ_1), (Y_2, δ_2) be IVTSs, let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be mappings. Let $f : X \to Y_1 \times Y_2$ be the mapping defined by: for each $x \in X$,*

$$f(x) = (f_1(x), f_2(x)).$$

Then f is interval-valued continuous if and only if f_1 and f_2 are interval-valued continuous.

The following is a generalization of Proposition 3.35.

Proposition 3.38. *Let $((X_j, \tau_j))_{j \in J}$ be an index family of IVTSs. For each $j \in J$, let $\sigma_j = \{\pi_j^{-1}(U_j) \in IVS(X) : U_j \in \tau_j\}$ and let $\sigma = \bigcup_{j \in J} \sigma_j$, where $X = \Pi_{j \in J} X_j$ and $\pi_j : X \to X_j$ is the projection associated by j. Then σ is an IVSB for the IVT τ on X.*

In this case, τ is called an *interval-valued product topology* (briefly, IVPT) on X. The pair (X, τ) is called the *interval-valued product space* (briefly, IVPS) of $((X_j, \tau_j))_{j \in J}$.

Proof. The proof is similar to Proposition 3.35. □

The following is a generalization of Proposition 3.34.

Proposition 3.39. *Let $((X_j, \tau_j))_{j \in J}$ be an index family of IVTSs and let τ be the IVPT on $X = \Pi_{j \in J} X_j$. Let $\pi_j : X \to X_j$ is the projection associated by j for each $j \in J$. Then π_j is interval-valued continuous for each $j \in J$. Moreover, τ is the coarsest IVT for which π_j is interval-valued continuous.*

Proof. The proof is similar to Proposition 3.34. □

From the above Proposition, we can easily see that the concrete category $\mathbf{IV_{Top}}$ has the initial structure.

The following is a generalization of Theorem 3.36.

Theorem 3.40. *Let $((Y_j, \tau_j))_{j\in J}$ be an index family of IVTSs and let δ be the IVPT on $Y = \Pi_{j\in J} Y_j$. Let (X, τ) be an IVTS and let $f : X \to Y$ be a mapping. Then f is interval-valued continuous if and only if $\pi_j \circ f$ is interval-valued continuous for each $j \in J$.*

Proof. The proof is similar to Proposition 3.36. □

4. T_0-, T_1-, T_2-spaces in interval-valued topological spaces

In this section, we define T_0, T_1 and T_2 separation axioms in IVTSs, and discuss with their properties and give some examples.

Definition 4.1. Let (X, τ) be an IVTS. Then (X, τ) is said to be:

(i) T_0(i), if $\forall x,\ y \in X\ (x \neq y)\ \exists U \in \tau$ such that either $x_1 \in U,\ y_1 \notin U$ or $y_1 \in U,\ x_1 \notin U$, i.e., either $x \in U^-,\ y \notin U^-$ or $y \in U^-,\ x \notin U^-$,

(ii) T_0(ii), if $\forall x,\ y \in X\ (x \neq y)\ \exists U \in \tau$ such that either $x_0 \in U,\ y_0 \notin U$ or $y_0 \in U,\ x_0 \notin U$, i.e., either $x \in U^+,\ y \notin U^+$ or $y \in U^+,\ x \notin U^+$,

(iii) T_0(iii), if $\forall x,\ y \in X\ (x \neq y)\ \exists U \in \tau$ such that either $x_1 \in U \subset y_1^c$ or $y_1 \in U \subset x_1^c$, i.e., either $x \in U^-,\ U^+ \subset \{y\}^c$ or $y \in U^-,\ U^+ \subset \{x\}^c$,

(iv) T_0(iv), if $\forall x,\ y \in X\ (x \neq y)\ \exists U \in \tau$ such that either $x_0 \in U \subset y_0^c$ or $y_0 \in U \subset x_0^c$, i.e., either $x \in U^+,\ U^- \subset \{y\}^c$ or $y \in U^+,\ U^- \subset \{x\}^c$,

(v) T_0(v), if $\forall x,\ y \in X\ (x \neq y)\ \exists U \in \tau$ such that either $x_1 \notin U$ or $y_1 \notin U$, i.e., either $x \notin U^-$ or $y \notin U^-$,

(vi) T_0(vi), if $\forall x,\ y \in X\ (x \neq y)\ \exists U \in \tau$ such that either $x_0 \notin U$ or $y_0 \notin U$, i.e., either $x \notin U^+$ or $y \notin U^+$.

From Definition 4.1, we can easily see that the following result holds.

Proposition 4.2. *Let (X, τ) be an IVTS. Then the following implications hold:*

$$\begin{tikzcd}
T_0(\mathrm{v}) & & T_0(\mathrm{vi}) \arrow[ll] \\
T_0(\mathrm{i}) \arrow[u] & T_0(\mathrm{i})+T_0(\mathrm{ii}) \arrow[l] \arrow[r] \arrow[d, leftrightarrow] & T_0(\mathrm{ii}) \arrow[u] \arrow[d] \\
 & T_0(\mathrm{iii}) & T_0(\mathrm{iv})
\end{tikzcd}$$

Proof. We prove only that $\mathrm{T}_0(\mathrm{i})+\mathrm{T}_0(\mathrm{ii}) \longleftrightarrow \mathrm{T}_0(\mathrm{iii})$. The remainder's proof is easy.

Suppose X is T_0(i) and T_0(ii). Then we can easily show that X is T_0(iii).

Suppose X is T_0(iii) and let $x \neq y \in X$. Then there is $U \in \tau$ such that

$$\text{either } x \in U^- \subset U^+ \subset \{y\}^c \text{ or } y \in U^- \subset U^+ \subset \{x\}^c.$$

Thus either $x_1 \in U$, $y \notin U^-$ and $x_0 \in U$, $y \notin U^+$ or $y_1 \in U$, $x \notin U^-$ and $y_0 \in U$, $x \notin U^+$. So we get

$$\text{either } x_1 \in U,\ y_1 \notin U \text{ or } y_1 \in U,\ x_1 \notin U$$

and

$$\text{either } x_0 \in U,\ y_0 \notin U \text{ or } y_0 \in U,\ x_0 \notin U.$$

Hence X is T_0(i) and T_0(ii). □

Remark 4.3. (1) Let (X, τ) be an IVTS such that $\tau \subset IVS^*(X)$. Then in Definition 4.1, T_0(i), T_0(ii), T_0(iii) and T_0(iv) are coincide, and so are T_0(v) and T_0(vi).

(2) The converses of the above Proposition do not hold, in general (It is seen the Example 4.4).

Example 4.4. (1) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, \widetilde{X}\},$$

where $A = [\{a, b\}, X], B = [\{b, c\}, X], C = [\{b\}, X]$.
Then (X, τ) is T_0(i) but not T_0(ii). Also (X, τ) is T_0(v) but not T_0(vi).

(2) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, \widetilde{X}\},$$

where $A = [\varnothing, \{a, c\}], B = [\varnothing, \{b, c\}], C = [\varnothing, \{c\}]$.
Then (X, τ) is T_0(ii) but not T_0(i). Also (X, τ) is T_0(vi) but not T_0(v).

(3) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, \widetilde{X}\},$$

where $A = [\{a\}, \{a, c\}], B = [\{b\}, \{b, c\}], C = [\{a\}, \{a\}]$,
$D = [\{a, b\}, X],\ E = [\varnothing, \{c\}]$.
Then (X, τ) is T_0(iv) but not T_0(iii).

(4) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, \widetilde{X}\},$$

where $A = [\{a, b\}, X], B = [\{a, c\}, X], C = [\{a\}, X], D = [\{b\}, X]$.
Then (X, τ) is T_0(iv) but not T_0(ii).

(5) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, H, I, \widetilde{X}\},$$

where $A = [\{a\}, \{a, b\}], B = [\{c\}, \{a, c\}], C = [\{b\}, X], D = [\{a, b\}, X], E = [\{a, c\}, X]$,
$F = [\{b, c\}, X], G = [\varnothing, \{a\}], H = [\varnothing, \{a, c\}], I = [\varnothing, \{a, b\}]$.
Then (X, τ) is T_0(i) but not T_0(iii).

Remark 4.5. Let (X, τ) be an ordinary space such that τ is not indiscrete and let τ^1, τ^2 be two IVTs on X (See Remark 4.2 (2) in [19]) given by:

$$\tau^1 = \{[G, G] : G \in \tau\},\ \tau^2 = \{\widetilde{\varnothing}, \widetilde{X}\} \bigcup \{[\varnothing, G] : G \in \tau\}.$$

If (X, τ) is T_0, then (X, τ^1) is T_0(i), T_0(ii), T_0(iii), T_0(iv), T_0(iv), T_0(v) and T_0(vi), and (X, τ^2) is T_0(ii), T_0(iv) and T_0(vi).

From Remark 4.2 (1) in [19]), it is obvious that for each IVTS (X, τ), there are two ordinary topologies τ^- and τ^+ defined as follows:

$$\tau^- = \{U^- \in 2^X : U \in \tau\},\ \tau^+ = \{U^+ \in 2^X : U \in \tau\}.$$

From Definition 4.1 and the above comments, we have the following.

Theorem 4.6. *Let (X, τ) be an IVTS.*
(1) (X, τ) *is* T_0(i) *if and only if* (X, τ^-) *is* T_0.
(2) (X, τ) *is* T_0(ii) *if and only if* (X, τ^+) *is* T_0.

Definition 4.7. An ITS X is said to be a:
(i) T_1(i)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_1} \in U, y_{_1} \notin U \text{ and } x_{_1} \notin V,\ y_{_1} \in V,$$

$$\text{i.e.}, x \in U^-,\ y \notin U^- \text{ and } x \notin V^-,\ y \in V^-,$$

(ii) T_1(ii)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_0} \in U, y_{_0} \notin U \text{ and } x_{_0} \notin V,\ y_{_0} \in V,$$

$$\text{i.e.}, x \in U^+,\ y \notin U^+ \text{ and } x \notin V^+,\ y \in V^+,$$

(iii) T_1(iii)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_1} \in U \subset y^c_{_1} \text{ and } y_{_1} \in V \subset x^c_{_1},$$

$$\text{i.e.}, x \in U^-,\ U^+ \subset \{y\}^c \text{ and } y \in V^-,\ V^+ \subset \{x\}^c,$$

(iv) T_1(iv)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_0} \in U \subset y^c_{_0} \text{ and } y_{_0} \in V \subset x^c_{_0},$$

$$\text{i.e.}, x \in U^+,\ U^- \subset \{y\}^c \text{ and } y \in V^+,\ V^- \subset \{x\}^c,$$

(v) T_1(v)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$y_{_1} \notin U \text{ and } x_{_1} \notin V, \text{ i.e., } y \notin U^- \text{ and } x \notin V^-,$$

(vi) T_1(vi)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$y_{_0} \notin U \text{ and } x_{_0} \notin V, \text{ i.e., } y \notin U^+ \text{ and } x \notin V^+$$

(vii) T_1(vii)-space, if $\forall x \in X,\ x_{_1} \in \tau^c$,
(viii) T_1(viii)-space, if $\forall x \in X,\ x_{_0} \in \tau^c$.

From Definition 4.1, we can easily see that the following result holds.

Proposition 4.8. *Let (X, τ) be an ITS. Then the following implications are true:*

$$\begin{tikzcd}
T_1(\mathrm{v}) & & T_1(\mathrm{vi}) \arrow[ll] \\
T_1(\mathrm{i}) \arrow[u] & T_1(\mathrm{i})+T_1(\mathrm{ii}) \arrow[l] \arrow[r] \arrow[d, leftrightarrow] & T_1(\mathrm{ii}) \arrow[u] \arrow[d] \\
T_1(\mathrm{vii}) & T_1(\mathrm{iii}) \arrow[l] & T_1(\mathrm{iv})
\end{tikzcd}$$

Remark 4.9. (1) Let (X,τ) be an IVTS such that $\tau \subset IVS^*(X)$. Then in Definition 4.7, T_1(i), T_1(ii), T_1(iii) and T_1(iv) are coincide, and so are T_1(v) and T_1(vi), T_1(vii) and T_1(viii) respectively.

(2) The converses of the above Proposition do not hold, in general (This situation is illustrated in the Example 4.10).

Example 4.10. (1) Let $X = \{a,b,c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, \widetilde{X}\},$$

where $A = [\{a,c\}, X],\ B = [\{b\}, X],\ C = [\{a\}, X],\ D = [\{c\}, X],$
$E = [\{a,b\}, X],\ F = [\{b,c\}, X],\ G = [\varnothing, X].$
Then (X,τ) is T_1(i) but not T_1(ii).

(2) Let $X = \{a,b\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, \widetilde{X}\},$$

where $A = [\varnothing, \{b,c\}],\ B = [\varnothing, \{a,c\}].$
Then (X,τ) is T_1(v) but not T_1(i).

(3) Let $X = \{a,b,c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, \widetilde{X}\},$$

where $A = [\varnothing, \{c\}],\ B = [\{c\}, \{c\}],\ C = [\varnothing, \{a\}],\ D = [\{c\}, \{a,c\}],$
$E = [\{a,c\}, \{a,c\}],\ F = [\varnothing, \{a,c\}].$
Then (X,τ) is T_1(Vi) but not T_1(ii).

(4) Let $X = \{a,b,c\}$ and consider the family σ of IVSs in X given by:

$$\sigma = \{A, B, C, D, E, F, G, H, I, \widetilde{X}\},$$

where $A = [\{a\}, \{a,b\}],\ B = [\{b\}, \{b,c\}],\ C = [\{a\}, \{a\}],\ D = [\varnothing, \{a,c\}],$
$E = [\{a,b\}, X],\ F = [\varnothing, \{b\}],\ G = [\varnothing, \{a\}],\ H = [\{a\}, X],\ I = [\{a\}, \{a,c\}].$
Let τ be the IVT on X having the IVSB σ. Then (X,τ) is T_1(iV) but not T_1(iii).

(5) Let $X = \{a,b,c,d\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, \widetilde{X}\},$$

where $A = [\{a\}, X],\ B = [\{b\}, X],\ C = [\{c\}, X],\ D = [\{a,b\}, X],$
$E = [\{b,c\}, X],\ F = [\{a,b,c\}, X],\ G = [\varnothing, X].$
Then (X,τ) is T_1(V) but not T_1(vi).

(6) Let $X = \{a,b,c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, H, I, \widetilde{X}\},$$

where $A = [\{a\}, \{a,b\}],\ B = [\{b\}, X],\ C = [\{c\}, X],\ D = [\{a,b\}, X],$
$E = [\{a,c\}, X],\ F = [\{b,c\}, X],\ G = [\varnothing, \{a,b\}],\ H = [\varnothing, X],\ I = [\{a\}, X].$
Then (X,τ) is T_1(i) but not T_1(iii).

(7) Let $X = \{a,b,c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, \widetilde{X}\},$$

where $A = [\{a,c\}, X],\ B = [\{b,c\}, X],\ C = [\{b\}, X],\ D = [\{a,b\}, X],$
$E = [\{c\}, X],\ F = [\{a\}, X],\ G = [\varnothing, X].$
Then (X,τ) is T_1(iv) but not T_1(ii).

(8) Let X be the set of all natural numbers and the IVSs A_n in X given by:

$$A_1 = [\{2, 3, 4, \cdots\}, X],$$
$$A_2 = [\{3, 4, 5, \cdots\}, X \setminus \{1\}],$$
$$\cdots \ \cdots \ \cdots$$
$$A_n = [\{n+1, n+2, n+3, \cdots\}, X \setminus \{1, 2, 3, \cdots, n-1\}].$$

Let $\tau = \{\widetilde{\varnothing}, \widetilde{X}\} \bigcup \{A_n : n = 1, 2, 3, \cdots\}$. Then we can easily check that $\tau \in IVT(X)$ and (X, τ) is T_1(vi) but not T_1(ii).

(9) Let $X = \{a, b, c\}$ and let τ be the IVT on X given by:

$$\tau = \{\widetilde{\varnothing}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, \widetilde{X}\},$$

where $A_1 = [\{a, b\}, \{a, b\}],\ A_2 = [\{b, c\}, \{b, c\}],\ A_3 = [\{c\}, \{a, c\}],\ A_4 = [\{b\}, \{b\}],$
$A_5 = [\varnothing, \{a\}],\ A_6 = [\{c\}, \{c\}],\ A_7 = [\{b, c\}, X].$

Then we can easily see that (X, τ) is T_1(ii) but not T_1(i). On the other hand,

$$\bigcup \{U \in \tau : x_{_0} \in U \subset b^c_{_0}\} = A_3 \cup A_5 \cup A_6 = [\{c\}, \{a, c\}] \neq b^c_{_0}.$$

Thus $b^c_{_0} \notin \tau$. So $b_{_0} \notin CIVS(X)$. Hence (X, τ) is not T_1(Viii).

The followings are immediate results of Definition 4.7.

Theorem 4.11. *Let (X, τ) be an IVTS.*
(1) *(X, τ) is T_1(i) if and only if (X, τ^-) is T_1.*
(2) *(X, τ) is T_1(ii) if and only if (X, τ^+) is T_1.*

The followings are immediate results of Definitions 4.1 and 4.7.

Proposition 4.12. *Let X be an IVTS.*
(1) *If X is T_1(i), then it is T_0(i).*
(2) *If X is T_1(ii), then it is T_0(ii).*
(3) *If X is T_1(iii), then it is T_0(iii).*
(4) *If X is T_1(iv), then it is T_0(iv).*
(5) *If X is T_1(v), then it is T_0(v).*
(6) *If X is T_1(vi), then it is T_0(vi).*

Each converse of the above Proposition is not true, in general (See Example 4.13).

Example 4.13. (1) Let $X = \{a, b, c\}$ and consider the IVT τ defined by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, \widetilde{X}\},$$

where $A = [\{a, b\}, \{a, b\}],\ B = [\{b, c\}, \{b, c\}],\ C = [\{c\}, \{a, c\}],\ D = [\{b\}, \{b\}],$
$E = [\varnothing, \{a\}],\ F = [\{c\}, \{c\}].$

Then (X, τ) is a T_0(i) but not T_1(i).

(2) Let $X = \{a, b, c\}$ and consider the IVT τ defined by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, \widetilde{X}\},$$

where $A = [\{a, c\}, X],\ B = [\{b, c\}, X],\ C = [\{b\}, X],\ D = [\{a, b\}, X],$
$E = [\{c\}, X],\ F = [\{a\}, X], G = [\varnothing, X].$

Then (X, τ) is a T_0(ii) but not T_1(ii).

(3) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, \widetilde{X}\},$$

where $A = [\{a, b\}, X],\ B = [\{b, c\}, X],\ C = [\{b\}, X]$.
Then (X, τ) is T_0(iii) but not T_1(iii).

(4) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, \widetilde{X}\},$$

where $A = [\{a, b\}, X],\ B = [\{c\}, X],\ C = [\varnothing, X]$.
Then (X, τ) is T_0(iv) but not T_1(iv).

(5) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, \widetilde{X}\},$$

where $A = [\{a\}, \{a, c\}],\ B = [\{b\}, \{a, b\}],\ C = [\{a, c\}, X]$,
$D = [\varnothing, \{a\}],\ E = [\varnothing, \{a, b\}],\ F = [\{a, b\}, X]$.
Then (X, τ) is T_0(v) but not T_1(v).

(6) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, \widetilde{X}\},$$

where $A = [\varnothing, \{a, b\}],\ B = [\varnothing, \{b\}],\ C = [\varnothing, \{a, c\}]$,
$D = [\varnothing, \{a\}],\ E = [\varnothing, \{b\}],\ F = [\varnothing, X]$.
Then (X, τ) is T_0(vi) but not T_1(vi).

Definition 4.14. Let X be a non-empty set. Then the *interval-valued diagonal* Δ_X of X is an IVS in $X \times X$ defined as follows:

$$\Delta_X = [\{(x_1, x_2) : x_1 = x_2\}, \{(x_1, x_2) : x_1 = x_2\}].$$

Remark 4.15. For a set X, let $IVS^*(X) = \{A \in IVS(X) : A^- = A^+\}$. Then from Result 2.9, $A = A_{_1}$ for each $A \in IVS^*(X)$. From Remark 4.2 (1), we can easily see that if $\tau \in IVT(X)$ such that $\tau \subset IVS^*(X)$ and $A \in \tau$, then $A^- \in \tau^-$. Moreover, it is obvious that $\Delta_X \in IVS^*(X)$.

Definition 4.16. An ITS (X, τ) is said to be a:

(i) T_2(i)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_1} \in U,\ y_{_1} \in V \text{ and } U \cap V = \widetilde{\varnothing},$$

$$\text{i.e., } x \in U^-,\ y \in V^- \text{ and } U^+ \cap V^+ = \varnothing,$$

(ii) T_2(ii)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_0} \in U,\ y_{_0} \in V \text{ and } U \cap V = \widetilde{\varnothing},$$

$$\text{i.e., } x \in U^+,\ y \in V^+ \text{ and } U^+ \cap V^+ = \varnothing,$$

(iii) T_2(iii)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_1} \in U,\ y_{_1} \in V \text{ and } U \subset V^c,$$

$$\text{i.e., } x \in U^-,\ y \in V^- \text{ and } U^- \subset {V^+}^c,\ U^+ \subset {V^-}^c,$$

(iv) T_2(iv)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_0} \in U,\ x_{_0} \in V \text{ and } U \subset V^c,$$

$$\text{i.e., } x \in U^+,\ y \in V^+ \text{ and } U^- \subset {V^+}^c,\ U^+ \subset {V^-}^c,$$

(v) T_2(v)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_1} \in U \subset y_{_1}^c,\ y_{_1} \in V \subset x_{_1}^c \text{ and } U \cap V = \widetilde{\varnothing},$$

$$\text{i.e., } x \in U^- \subset U^+ \subset \{y\}^c,\ y \in V^- \subset V^+ \subset \{x\}^c \text{ and } U^+ \cap V^+ = \varnothing,$$

(vi) T_2(vi)-space, if $\forall x,\ y \in X\ (x \neq y)\ \exists U,\ V \in \tau$ such that

$$x_{_0} \in U \subset y_{_0}^c,\ y_{_0} \in V \subset x_{_0}^c \text{ and } U \cap V = \widetilde{\varnothing},$$

$$\text{i.e., } x \in U^+,\ U^- \subset \{y\}^c,\ y \in V^+,\ V^- \subset \{x\}^c \text{ and } U^+ \cap V^+ = \varnothing,$$

(vii) T_2(vii)-space, if Δ_X is a CIVS in the IVTS $(X \times X, \tau_{X \times X})$.

Proposition 4.17. *Let (X, τ) be an IVTS. Then the following implications are true:*

$$\begin{tikzcd}
 & T_2(v) \arrow[r] \arrow[d] & T_2(vi) \arrow[d] \\
T_2(vii) & T_2(i) \arrow[l] \arrow[r] \arrow[d] & T_2(ii) \arrow[d] \\
 & T_2(iii) \arrow[r] & T_2(iv)
\end{tikzcd}$$

Proof. We prove only the case of T_2(i) $\Longrightarrow$ T_2(vii). The proofs of remainder's implications are easy from Definition 4.16. Let $\tau_{_{X\times X}}$ be the IVPT on $X \times X$ and let $(a,b)_{_1} \in \Delta_X^c$. Then by the definition of Δ_X, $a \neq b$. Since (X, τ) is T_2(i), there are $U_{a_{_1}},\ V_{b_{_1}} \in \tau$ such that $a_{_1} \in U,\ b_{_1} \in V$ and $U \cap V = \widetilde{\varnothing}$. Thus $a \in U^-_{a_{_1}}$ and $b \in V^-_{b_{_1}}$, i.e., $(a,b) \in U^-_{a_{_1}} \times V^-_{b_{_1}}$. So we get

$$(a,b)_{_1} \in [U^-_{a_{_1}} \times V^-_{b_{_1}}, U^+_{a_{_1}} \times V^+_{b_{_1}}] = U_{a_{_1}} \times V_{b_{_1}}.$$

In fact, from Remark 4.15, $U^-_{a_{_1}} \times V^-_{b_{_1}} = U^+_{a_{_1}} \times V^+_{b_{_1}}]$. It is clear that $U_{a_{_1}} \times V_{b_{_1}} \in \tau_{_{X\times X}}$. Furthermore, we can easily check that

$$U_{a_{_1}} \times V_{a_{_1}} \subset \Delta_X^c \Longleftrightarrow U^-_{a_{_1}} \times V^-_{b_{_1}} \subset \{(x,y) : x = y\}.$$

This implies that $\Delta_X^c = \bigcup_{(a,b)_{_1} \in \Delta_X^c} \left(U_{a_{_1}} \times V_{b_{_1}}\right)$. Hence $\Delta_X^c \in \tau_{_{X\times X}}$, i.e., $\Delta_X \in \tau^c_{_{X\times X}}$. Therefore (X, τ) is T_2(Vii). □

Remark 4.18. (1) Let (X, τ) be an IVTS such that $\tau \subset IVS^*(X)$. Then in Definition 4.16, T_1(i), T_1(ii), T_1(iii), T_1(iv), $_1$(v) and T_1(vi) are coincide.

(2) Let (X, τ) be an ordinary space such that τ is not indiscrete. It is obvious that if (X, τ) is T_2, then (X, τ^1) is T_2(i), T_2(ii), T_2(iii), T_2(iv), T_2(v) and T_2(vi), and (X, τ^2) is T_2(ii), T_2(iv) and T_2(vi).

(3) The converses of Proposition 4.17 does not hold in general (See Example 4.19).

Example 4.19. (1) Let $X = \{a, b\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, \widetilde{X}\},$$

where $A = [\varnothing, \{a\}],\ B = [\varnothing, \{b\}]$. Then (X, τ) is T_2(ii) but neither T_2(i) nor T_2(v).

(2) Let $X = \{a, b, c\}$ and consider the family σ of IVSs τ on X given by:

$$\sigma = \{A, B, C, D\},$$

where $A = [\varnothing, \{a\}],\ B = [\{b\}, \{b, c\}],\ C = [\{a\}, \{a, b\}],\ D = [\varnothing, \{c\}]$.
Let τ be the IVT on X generated by the IVSB σ. Then we can easily check that (X, τ) is T_2(iv) but neither T_2(iii) nor T_2(i).

(3) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, H, I, J, K, \widetilde{X}\},$$

where $A = [\varnothing, \{a, c\}],\ B = [\varnothing, \{b\}],\ C = [\{a\}, \{a\}],\ D = [\varnothing, \{b, c\}]$,
$E = [\varnothing, \{c\}],\ F = [\varnothing, \{a, b\}],\ G = [\{a\}, \{a, b\}],\ H = [\{a\}, X]$,
$I = [\{a\}, \{a, c\}],\ J = [\varnothing, \{a\}],\ K = [\varnothing, X]$.
Then (X, τ) is T_2(vi) but not T_2(v).

(4) Let $X = \{a, b, c, d\}$ and consider the family σ of IVSs τ on X given by:

$$\sigma = \{A, B, C, D, E, F, G, H, I, J, K, L\},$$

where $A = [\{a\}, \{a, c, d\}],\ B = [\{b\}, \{b, c\}],\ C = [\{b\}, \{a, b, d\}]$,
$D = [\{c\}, \{c, d\}],\ E = [\{a\}, \{a, b, c\}],\ F = [\{d\}, \{b, c, d\}]$,
$G = [\{b\}, \{a, b, c\}],\ H = [\{d\}, \{a, c, d\}], I = [\{c\}, \{a, b, c\}]$,
$J = [\{d\}, \{a, b, d\}],\ K = [\{a\}, \{a, b, d\}],\ L = [\{c\}, \{b, c, d\}]$,
Let τ be the IVT on X generated by the IVSB σ. Then we can easily check that (X, τ) is T_2(iii) but not T_2(i).

(5) Let $X = \{a, b\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, \widetilde{X}\},$$

where $A = [\{b\}, X],\ B = [\varnothing, \{b\}]$. Then (X, τ) is T_2(iv) but not T_2(ii).

The followings are immediate results of Definition 4.16.

Proposition 4.20. *Let (X, τ) be an IVTS.*
(1) *If (X, τ) is T_2(i), then (X, τ^-) is T_2.*
(2) *If (X, τ) is T_2(ii), then (X, τ^+) is T_2.*

The converse of the above Proposition does not true in general (See Example 4.21).

Example 4.21. (1) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, G, \widetilde{X}\},$$

where $A = [\{a\}, X],\ B = [\{b\}, X],\ C = [\{c\}, X],\ D = [\{a, b\}, X]$,
$E = [\{a, c\}, X],\ F = [\{b, c\}, X],\ G = [\varnothing, X]$.
Then clearly, $\tau^- = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Thus we can easily see that (X, τ^-) is T_2 but not (X, τ) T_2(i).

(2) Let $X=\{a,b,c,d\}$ and consider the IVT τ on X given by:

$$\tau=\{\widetilde{\varnothing},A,B,C,D,E,E,F,G,H,I,J,K,L,M,N,O,\widetilde{X}\},$$

where $A=[\varnothing,\{a\}],\ B=[\varnothing,\{c\}],\ C=[\varnothing,\{d\}],\ D=[\{b\},\{b,d\}],$
$E=[\varnothing,\{a,c\}],\ F=[\varnothing,\{a,d\}],\ G=[\{b\},\{a,b,d\}),$
$H=[\varnothing,\{c,d\}],\ I=[\{b\},\{b,c,d\}].$
Then clearly,

$$\tau^{+}=\{\varnothing,\{a\},\{c\},\{d\},\{a,c\},\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{b,c,d\},X\}.$$

Thus we can easily see that (X,τ_2) is T_2 but not (X,τ) T_2(ii).

The followings are immediate results of Definitions 4.7 and 4.16.

Proposition 4.22. *Let (X,τ) be an IVTS.*
(1) *If (X,τ) is T_2(i), then it is T_1(iii).*
(2) *If (X,τ) is T_2(ii), then it is T_1(ii).*
(3) *If (X,τ) is T_2(iii), then it is T_1(iii).*
(4) *If (X,τ) is T_2(iv), then it is T_1(iv).*
(5) *If (X,τ) is T_2(v), then it is T_1(iii).*
(6) *If (X,τ) is T_2(vi), then it is T_1(vi).*

Proposition 4.23. *Let (X,τ) be an IVTS.*
(1) *If (X,τ) is T_2(i), then it is T_1(i).*
(2) *If (X,τ) is T_2(v), then it is T_1(v).*

The converses of the above two Propositions are not true in general (See Example 4.24).

Example 4.24. (1) Let $X=\{a,b,c\}$ and consider the IVT τ on X given by:

$$\tau=\{\widetilde{\varnothing},A,B,C,D,E,F,G,H,\widetilde{X}\},$$

where $A=[\{a\},\{a,c\}],\ B=[\{b\},\{a,b\}],\ C=[\{c\},\{b,c\}],\ D=[\{a,b\},X],$
$E=[\varnothing,\{a\}],\ F=[\varnothing,\{b\}],\ G=[\varnothing,\{c\}],\ H=[\{a,c\},X].$
Then clearly, (X,τ) is T_1(i) but not T_2(i).

(2) Let $X=\{a,b,c\}$ and consider the IVT τ on X given by:

$$\tau=\{\widetilde{\varnothing},A,B,C,D,E,F,G,\widetilde{X}\},$$

where $A=[\{a\},\{a,c\}],\ B=[\{b\},\{a,b\}],\ C=[\{c\},\{b,c\}],\ D=[\{a,b\},X],$
$E=[\varnothing,\{a\}],\ F=[\varnothing,\{b\}],\ G=[\varnothing,\{c\}],\ H=[\varnothing,\{c\}],$
$I=[\varnothing,\{a\}],\ J=[\{a\},\{a\}],\ K=[\{b\},\{b\}],\ L=[\varnothing,\{b\}],$
$M=[\{c\},\{c\}],\ N=[\varnothing,\{a,b\}],\ O=[\{a,b\},X],\ P=[\{a,c\},\{a,c\}],$
$Q=[\{a\},X],\ S=[\{a,b\},X],\ T=[\{c\},X],\ U=[\{a,b\},\{a,b\}].$
Then clearly, (X,τ) is T_1(iii) but not T_2(i). Also it is neither T_2(iii) nor T_2(v).

(3) Let $X=\{a,b,c\}$ and consider the IVT τ on X given by:

$$\tau=\{\widetilde{\varnothing},A,B,C,D,E,\widetilde{X}\},$$

where $A=[\varnothing,\{b,c\}],\ B=[\varnothing,\{a,b\}],\ C=[\varnothing,\{c\}],\ D=[\varnothing,\{b\}],\ E=[\varnothing,X].$
Then (X,τ) is T_1(ii) but not T_2(ii).

(4) Let (X,τ) be the IVTS given in Example 4.10 (7). Then we can easily check that (X,τ) is T_1(iv) but not T_2(iv).

(5) Let (X,τ) be the IVTS given in Example 4.10 (5). Then clearly, (X,τ) is T_1(v) but not T_2(v).

(6) Let (X,τ) be the IVTS given in Example 4.10 (8). Then clearly, (X,τ) is T_1(vi) but not T_2(vi).

Proposition 4.25. *Let (X,τ) be* T_2(i) *and let $x \in X$. Then*
$x_{_1} = \bigcap\{N \in CIVS(X) : N \in N(x_{_1})\}$ *and thus* $x_{_1} = IVcl(x_{_1})$.

Proof. Suppose (X,τ) is T_2(i) and for each $x \in X$, let

$$\bigcap\{N \in CIVS(X) : N \in N(x_{_1})\} = C = [C^-, C^+].$$

Assume that there is a distinct IVP $y_{_1} \in C$, i.e., $y \in C^-$.

Case 1: Suppose $\{x\} \subsetneqq C^-$. Then there is $y \in C^-$ such that $x \neq y$. Since (X,τ) is T_2(i), there are $U,\ V \in \tau$ such that $x_{_1} \in U$, $y_{_1} \in V$ and $U \cap V = \widetilde{\varnothing}$. Thus $x_{_1} \in U \subset V^c$. So V^c is a closed interval-valued neighborhood of $x_{_1}$. From our assumption, $y_{_1} \in V^c$. Since $y_{_1} \in V$ and $y_{_1} \in V^c$, $y \in V^- \cap V^{+^c} = \varnothing$. This is a contradiction. Hence $C = \{x_{_1}\}$.

Case 2: Suppose $\{x\} \subsetneqq C^+$ and $C^- = \{x\}$. Then we have the same result in Case 1. This completes the proof. □

Definition 4.26. Let X, Y be two non-empty sets and let $f : X \to Y$ be a mapping. Then the *graph* of f, denoted by $G(f)$, is an IVS in $X \times X$ defined as follows:

$$G(f) = [\{(x, f(x)) : x \in X\}, \{(x, f(x)) : x \in X\}].$$

Proposition 4.27. *Let (X,τ), (Y,δ) be IVTSs such that (Y,δ) is* T_2(i). *If $f : (X,\tau) \to Y,\delta)$ is interval-valued continuous, then $G(f)$ is a CIVS in $(X \times Y, \tau \times \delta)$.*

Proof. It is obvious that $G(f) \in IVS^*(X \times Y)$. Let $(a,b)_{_1} \in [G(f)]^c$. Then clearly, $(a,b) \in \{(x, f(x)) : x \in X\}^c$, i.e., $b \neq f(a)$. Since (Y,δ) is T_2(i), there are $U_{_{b_1}}$, $V_{_{f(a_1)}} \in \delta$ such that

$$b_{_1} \in U_{_{b_1}},\ f(a_{_1}) \in V_{_{f(a_1)}} \text{ and } U_{_{b_1}} \cap V_{_{f(a_1)}} = \widetilde{\varnothing}.$$

Since f is interval-valued continuous, we get

$$f^{-1}(V_{_{f(a_1)}}) = [f^{-1}(V^-_{_{f(a_1)}}), f^{-1}(V^+_{_{f(a_1)}})] \in \tau.$$

It is clear that $f^{-1}(V_{_{f(a_1)}}) \times U_{_{b_1}} \in \tau \times \delta$, where $\tau \times \delta$ denote the IVPT on $X \times Y$. Since $G(f) \in IVS^*(X \times Y)$, we can easily see that

$$G(f)^c = \bigcup_{(a,b)_1 \in G(f)^c} \left(f^{-1}(V_{_{f(a_1)}}) \times U_{a_1}\right).$$

Thus $G(f)^c \in \tau \times \delta$. So $G(f) \in (\tau \times \delta)^c$. □

Proposition 4.28. *Let (X,τ), (Y,δ) be IVTSs such that (Y,δ) is* T_2(i) *and let C be the IVS in $X \times X$ given by:*

$$C = [\{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}, \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}].$$

If $f : (X,\tau) \to (Y,\delta)$ is interval-valued continuous, then C is a CIVS in the product space $(X \times X, \tau \times \tau)$.

Proof. Assume that $(a,b)_1 \in C^c$. Then clearly, $(a,b) \notin C^-$, i.e., $f(a) \neq f(b)$. Since (Y,δ) is T_2(i), there are $U,\ V \in \delta$ such that $f(a_1) = f(a)_1 \in U$, $f(b_1) = f(b)_1 \in V$ and $U \cap V = \widetilde{\varnothing}$. Since f is interval-valued continuous, $f^{-1}(U),\ f^{-1}(V) \in \tau$, $a_1 \in f^{-1}(U)$ and $b_1 \in f^{-1}(V)$. Thus $(a,b)_1 \in f^{-1}(U) \times f^{-1}(V) \subset C^c$. Since $C \in IVS^*(X \times X)$, from Remark 4.15, $C^c = C^c_1$. So $C^c \in \tau \times \tau$. Hence C is a CIVS in $(X \times X, \tau \times \tau)$. □

Proposition 4.29. *Let (X,τ), (Y,δ) be IVTSs, let C be the IVS in $X \times X$ given in Proposition 4.28 and let $f : X \to Y$ be interval-valued open and surjective. If $C \in (\tau \times \tau)^c$, then (Y,δ) is T_2(i).*

Proof. Suppose $C \in (\tau \times \tau)^c$ and let $c,\ d \in Y$ such that $c \neq d$. Since f is surjective, there are $a,\ b \in X$ such that $f(a) = c$ and $f(b) = d$. Then clearly, $a \neq b$, i.e., $(a,b)_1 \notin C$ by the definition of C. By the hypothesis, $C^c \in \tau \times \tau$. Thus there are $U,\ V \in \tau$ such that $a_1 \in U$, $b_1 \in V$ and $(U \times V) \cap C = \widetilde{\varnothing}$. Since f is interval-valued open, $f(U),\ f(V) \in \delta$. Furthermore, $f(a_1) \in f(U)$, $f(b_1) \in f(V)$ and $f(U) \cap f(V) = \widetilde{\varnothing}$. So (Y,δ) is T_2(i). □

From Propositions 4.28 and 4.29, we obtain the following Corollary.

Corollary 4.30. *Let (X,τ), (Y,δ) be IVTSs, let C be the IVS in $X \times X$ given in Proposition 4.28 and let $f : X \to Y$ be interval-valued open and surjective. Then (Y,δ) is T_2(i) if and only if $C \in (\tau \times \tau)^c$.*

Proposition 4.31. *Let (X,τ), (Y,δ) be IVTSs.*
(1) *If (X,τ) and (Y,δ) are T_1(i), then so is $(X \times Y, \tau \times \delta)$.*
(2) *If (X,τ) and (Y,δ) are T_1(ii), then so is $(X \times Y, \tau \times \delta)$.*

Proof. (1) Suppose (X,τ) and (Y,δ) are T_1(i) and let $(x_1,y_1),\ (x_2,y_2) \in X \times Y$ such that $(x_1,y_1) \neq (x_2,y_2)$, say $x_1 \neq x_2$. Since (X,τ) is T_1(i), there are $U,\ V \in \tau$ such that $x_{1_1} \in U,\ x_{2_1} \notin U$ and $x_{2_1} \in V,\ x_{1_1} \notin V$. Then we have

$$U \times \widetilde{Y} = [U^- \times Y, U^+ \times Y],\ V \times \widetilde{Y} = [V^- \times Y, V^+ \times Y] \in \tau \times \delta$$

satisfying the following properties:

$$(x_1,y_1)_1 \in U \times \widetilde{Y},\ (x_2,y_2)_1 \notin U \times \widetilde{Y}$$

and

$$(x_1,y_1)_1 \notin V \times \widetilde{Y},\ (x_2,y_2)_1 \in V \times \widetilde{Y}.$$

Similarly, we can prove the case $y_1 \neq y_2$. Thus $(X \times Y, \tau \times \delta)$ is T_1(i).
(2) The proof is similar to (1). □

Proposition 4.32. *Let (X,τ), (Y,δ) be IVTSs.*
(1) *If (X,τ) and (Y,δ) are T_2(i), then so is $(X \times Y, \tau \times \delta)$.*
(2) *If (X,τ) and (Y,δ) are T_2(ii), then so is $(X \times Y, \tau \times \delta)$.*
(3) *If (X,τ) and (Y,δ) are T_2(iii), then so is $(X \times Y, \tau \times \delta)$.*
(4) *If (X,τ) and (Y,δ) are T_2(vii), then so is $(X \times Y, \tau \times \delta)$.*

Proof. (1) Suppose (X,τ) and (Y,δ) are T_2(i) and let $(x_1,y_1),\ (x_2,y_2)\in X\times Y$ such that $(x_1,y_1)\neq(x_2,y_2)$, say $x_1\neq x_2$. Since (X,τ) is T_2(i), there are $U,\ V\in\tau$ such that $x_{1_1}\in U,\ x_{2_1}\in V$ and $U\cap V=\widetilde{\varnothing}$. Then we have

$$U\times\widetilde{Y}=[U^-\times Y,U^+\times Y],\ V\times\widetilde{Y}=[V^-\times Y,V^+\times Y]\in\tau\times\delta$$

such that $(x_1,y_1)_1\in U\times\widetilde{Y}$ and $(x_2,y_2)_1\in V\times\widetilde{Y}$. On the other hand,

$$\begin{aligned}U\times\widetilde{Y}\cap V\times\widetilde{Y}&=[(U^-\times Y)\cap(V^-\times Y),(U^+\times Y)\cap(V^+\times Y)]\\&=[(U^-\cap V^-)\times(Y\times Y),(U^+\cap V^+)\times(Y\times Y)]\\&=[\varnothing\times Y,\varnothing\times Y]\ [\text{Since } U\cap V=\widetilde{\varnothing}]\\&=\widetilde{\varnothing}.\end{aligned}$$

A similar argument holds in case $y_1\neq y_1$. Thus $(X\times Y,\tau\times\delta)$ is T_2(i).

(2) The proof is similar to (1).

(3) Suppose (X,τ) and (Y,δ) are T_2(iii) and let $(x_1,y_1),\ (x_2,y_2)\in X\times Y$ such that $(x_1,y_1)\neq(x_2,y_2)$, say $x_1\neq x_2$. Since (X,τ) is T_2(iii), there are $U,\ V\in\tau$ such that $x_{1_1}\in U,\ x_{2_1}\in V$ and $U\subset V^c$. Then we have

$$U\times\widetilde{Y}=[U^-\times Y,U^+\times Y],\ V\times\widetilde{Y}=[V^-\times Y,V^+\times Y]\in\tau\times\delta$$

such that $(x_1,y_1)_1\in U\times\widetilde{Y}$ and $(x_2,y_2)_1\in V\times\widetilde{Y}$. It is easy to prove that $U\times\widetilde{Y}\subset(V\times\widetilde{Y})^c$ holds. A similar argument holds in case $y_1\neq y_1$. Thus $(X\times Y,\tau\times\delta)$ is T_2(iii).

(4) We will show that $\Delta^c_{X\times Y}\in((\tau\times\delta)\times(\tau\times\delta))^c$. It is clear that $\Delta_{X\times Y}\in IVS^*((X\times Y)\times(X\times Y))$. Then it is sufficient to show that for each $((a_1,b_1),(a_2,b_2))_1\in\Delta^c_{X\times Y}$, there is U in $(\tau\times\delta)\times(\tau\times\delta)$ such that

$$((a_1,b_1),(a_2,b_2))_1\in U\subset\Delta^c_{X\times Y}.$$

Let $((a_1,b_1),(a_2,b_2))_1\in\Delta^c_{X\times Y}$. Then clearly, $(a_1,b_1)\neq(a_2,b_2)$, i.e., $a_1\neq a_2$ or $b_1\neq b_2$. Thus we can consider three possible cases:

(i) $a_1\neq a_2,\ b_1=b_2$; (ii) $a_1=a_2,\ b_1\neq b_2$; (iii) $a_1\neq a_2,\ b_1\neq b_2$.

We prove only (iii). The remainder's proof are similar. Suppose $a_1\neq a_2,\ b_1\neq b_2$. Then clearly, $(a_1,a_2)_1\in\Delta^c_X\in(\tau\times\tau)$ and $(b_1,b_2)_1\in\Delta^c_Y\in(\delta\times\delta)$. Thus there are $U_1,\ U_2\in\tau$ and $V_1,\ V_2\in\delta$ such that

$$(a_1,a_2)_1\in U_1\times U_2\subset\Delta^c_X \text{ and } (b_1,b_2)_1\in V_1\times V_2\subset\Delta^c_Y.$$

Now we show that the following holds:

$$((a_1,b_1),(a_2,b_2))_1\in(U_1\times V_1)\times(U_2\times V_2)\subset\Delta^c_{X\times Y}. \tag{4.1}$$

(4.1) can be proved in two steps.

Step 1: First, we show that $((a_1,b_1),(a_2,b_2))_1\in(U_1\times V_1)\times(U_2\times V_2)$. From Definition 2.7, it is obvious that

$$\begin{aligned}&((a_1,b_1),(a_2,b_2))_{IVP}\in(U_1\times V_1)\times(U_2\times V_2)\\\Longleftrightarrow\ &((a_1,b_1),(a_2,b_2))\in(U_1\times V_1)^-\times(U_2\times V_2)^-\\\Longleftrightarrow\ &((a_1,b_1),(a_2,b_2))\in(U_1^-\times V_1^-)\times(U_2^-\times V_2^-).\end{aligned}$$

Then $a_1\in U_1^-,\ a_2\in U_2^-,\ b_1\in V_1^-,\ b_2\in V_2^-$. Thus $(a_1,b_1)\in U_1^-\times V_1^-$ and $(a_2,b_2)\in U_{12}^-\times V_2^-$. So the result holds.

Step 2: Second, we prove that $(U_1 \times V_1) \times (U_2 \times V_2) \subset \Delta^c_{X \times Y}$. In order to show the inclusion, we must prove that one of the followings hold:

$$(U_1 \times V_1)^- \times (U_2 \times V_2)^- \subset \{((u_1, v_1, (u_2, v_2)) : (u_1, v_1 \neq (u_2, v_2\} \tag{4.2}$$

or

$$(U_1^- \times V_1^-) \times (U_2^- \times V_2^-) \subset \{((u_1, v_1, (u_2, v_2)) : (u_1, v_1 \neq (u_2, v_2\}. \tag{4.3}$$

Since $U_1 \times U_2 \subset \Delta^c_X$ and $V_1 \times V_2 \subset \Delta^c_Y$, we have

$$U_1^- \times U_2^- \subset \{(u_1, u_2) \in X \times X : u_1 \neq u_2\}$$

and

$$V_1^- \times V_2^- \subset \{(v_1, v_2) \in Y \times Y : v_1 \neq v_2\}.$$

Then (4.2) holds. Similarly, (4.3) can be proved. Thus by (4.1), we get

$$\Delta^c_{X \times Y} \in ((\tau \times \delta) \times (\tau \times \delta)).$$

So $\Delta_{X \times Y} \in ((\tau \times \delta) \times (\tau \times \delta))^c$. Hence $(X \times Y, \tau \times \delta)$ is T_2(vii). □

The following is the converse of Proposition 4.32.

Proposition 4.33. *Let (X, τ), (Y, δ) be IVTSs.*
(1) *If $(X \times Y, \tau \times \delta)$ is T_2(i), then so are (X, τ) and (Y, δ).*
(2) *If $(X \times Y, \tau \times \delta)$ is T_2(ii), then so are (X, τ) and (Y, δ).*
(3) *If $(X \times Y, \tau \times \delta)$ is T_2(iii), then so are (X, τ) and (Y, δ).*

Proof. The proofs of (1) and (2) are easy. we prove only (3). Suppose $(X \times Y, \tau \times \delta)$ is T_2(iii) and let $a,\ b \in X$ such that $a \neq b$. Let us take a fixed $y \in Y$. Then clearly, $(a, y) \neq (b, y) \in X \times Y$. Since $(X \times Y, \tau \times \delta)$ is T_2(iii), there are $U_1 \times V_1,\ U_2 \times V_2 \in \tau \times \delta$ such that $(a, y)_1 \in U_1 \times V_1$, $(b, y)_1 \in U_2 \times V_2$ and $U_1 \times V_1 \subset (U_2 \times V_2)^c$. Thus we have $(a, y) \in U_1^- \times V_1^-$, $(b, y) \in U_2^- \times V_2^-$ and

$$U_1^- \times V_1^- \subset (U_2^+ \times V_2^+)^c = {U_2^+}^c \times {V_2^+}^c,\ U_1^+ \times V_1^+ \subset (U_2^- \times V_2^-)^c = {U_2^-}^c \times {V_2^-}^c. \tag{4.4}$$

So we get $a \in U_1^-$, $y \in V_1^-$, $b \in U_2^-$, $y \in V_2^-$ and

$$(U_1^- \times V_1^-) \cap (U_2^+ \times V_2^+) = \varnothing,\ (U_1^+ \times V_1^+) \cap (U_2^- \times V_2^-) = \varnothing. \tag{4.5}$$

Moreover, we have

$$(U_1^- \times V_1^-) \cap (U_2^+ \times V_2^+) = (U_1^- \cap U_2^+) \times (V_1^- \cap V_2^+), \tag{4.6}$$

$$(U_1^+ \times V_1^+) \cap (U_2^- \times V_2^-) = (U_1^+ \cap U_2^-) \times (V_1^+ \cap V_2^-). \tag{4.7}$$

From (4.5), (4.6) and (4.7), we have

$$(U_1^- \cap U_2^+) \times (V_1^- \cap V_2^+) = \varnothing,\ (U_1^+ \cap U_2^-) \times (V_1^+ \cap V_2^-) = \varnothing. \tag{4.8}$$

Since $y \in V_1^-$, $y \in V_2^-$ and $V_2^- \subset V_2^+$, $y \in V_1^- \cap V_2^+$. Similarly, we get $y \in V_1^+ \cap V_2^-$. From (4.4), $U_1^- \subset {U_2^+}^c$ and $U_1^+ \subset {U_2^-}^c$, i.e., $U_1 \subset U_2^c$. Since $a \in U_1^-$ and $b \in U_2^-$, $a_{IVP} \in U_1$ and $b_{IVP} \in U_2$. Hence (X, τ) is T_2(iii). Similarly, the second part is proved. □

5. T_3-, T_4-spaces in interval-valued topological spaces

The properties that are studied in the previous section describe the separation of pairs of interval-valued points of two types by IVOSs. In this section, in order to describe the separation of an Interval-valued point from a CIVS by OIVSs, we define two types T_3 and T_4 separation axioms in interval-valued topological spaces, and obtain some properties.

Definition 5.1. (i) A T_1(i)-space (X, τ) is called a T_3(i)-space, if it satisfies the following axiom:

[The regular axiom (i)]: for any $F \in \tau^c$ such that $x_{_1} \in F^c$, there exist $U,\ V \in \tau$ such that $F \subset U$, $x_{_1} \in V$ and $U \cap V = \widetilde{\varnothing}$.

(ii) A T_1(ii)-space (X, τ) is called a T_3(ii)-space, if the following conditions:

[The regular axiom (ii)]: for any $F \in \tau^c$ such that $x_{_0} \in F^c$, there exist $U,\ V \in \tau$ such that $F \subset U$, $x_{_0} \in V$ and $U \cap V = \widetilde{\varnothing}$.

It is obvious that if $X$is T_3(i), then it is T_2(i).

Example 5.2. (1) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A, B, C, D, E, F, \widetilde{X}\},$$

where $A = [\{a\}, \{a\}],\ B = [\{b\}, \{b\}],\ C = [\{c\}, \{c\}],\ D = [\{a, b\}, \{a, b\}],$
$E = (\{a, c\}, \{a, c\}],\ F = (\{b, c\}, \{b, c\}].$
Then clearly, (X, τ) is T_1(i). In fact, it is T_2(i) and $\tau^c = \tau$. But, we can easily check that (X, τ) is not T_3(i).

(2) Let $X = \{a, b, c, d\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, \widetilde{X}\},$$

where $A_1 = [\{a\}, \{a, d\}],\ A_2 = [\{b\}, \{a, b\}],\ A_3 = [\{c\}, \{b, c\}],\ A_4 = [\{d\}, \{c, d\}],$
$A_5 = [\varnothing, \{a\}],\ A_6 = [\varnothing, \{b\}],\ A_7 = [\varnothing, \{c\}],\ A_8 = [\varnothing, \{d\}],$
$A_9 = [\{a, b\}, \{a, b, d\}],\ A_{10} = [\{a, d\}, \{a, c, d\}],\ A_{11} = [\{c, d\}, \{b, c, d\}],$
$A_{12} = [\{b, c\}, \{a, b, c\}], A_{13} = [\{a, c\}, X],\ A_{14} = [\{b, d\}, X].$
Then clearly, (X, τ) is T_1(i) and

$$\tau^c = \{\widetilde{\varnothing}, A^c_1, A^c_2, A^c_3, A^c_4, A^c_5, A^c_6, A^c_7, A^c_8, A^c_9, A^c_{10}, A^c_{11}, A^c_{12}, A^c_{13}, A^c_{14}, \widetilde{X}\},$$

where $A^c_1 = [\{b, c\}, \{b, c, d\}],\ A^c_2 = [\{c, d\}, \{a, c, d\}],\ A^c_3 = [\{a, d\}, \{a, b, d\}],$
$A^c_4 = [\{a, b\}, \{a, b, c\}],\ A^c_5 = [\{b, c, d\}, X],\ A^c_6 = [\{a, c, d\}, X],$
$A^c_7 = [\{a, b, d\}, X],\ A^c_8 = [\{a, b, c\}, X],\ A^c_9 = [\{c\}, \{c, d\}],$
$A^c_{10} = [\{b\}, \{b, c\}],\ A^c_{11} = [\{a\}, \{a, b\}],\ A^c_{12} = [\{d\}, \{a, d\}],$
$A^c_{13} = [\varnothing, \{b, d\}],\ A^c_{14} = [\phi, \{a, c\}].$
Thus we can easily see that (X, τ) is T_3(i).

(3) Let $X = \{a, b, c, d\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, \widetilde{X}\},$$

where $A_1 = [\{a, b\}, \{a, b, c\}],\ A_2 = [\{c, d\}, \{a, c, d\}],\ A_3 = [\{a, d\}, \{a, b, d\}],$
$A_4 = [\{b, c\}, \{b, c, d\}],\ A_5 = [\varnothing, \{a, c\}],\ A_6 = [\{a\}, \{a, b\}],\ A_7 = [\{b\}, \{b, c\}],$
$A_8 = [\varnothing, \{a, d\}],\ A_9 = [\{c\}, \{c, d\}],\ A_{10} = [\varnothing, \{b, d\}],\ A_{11} = [\{a, b, d\}, X],$

$A_{12} = [\{a, b, c\}, X], A_{13} = [\{a, c, d\}, X],\ A_{14} = [\{b, c, d\}, X)$.
Then clearly, (X, τ) is T_1(ii) and

$$\tau^c = \{\widetilde{\varnothing}, A_1^c, A_2^c, A_3^c, A_4^c, A_5^c, A_6^c, A_7^c, A_8^c, A_9^c, A_{10}^c, A_{11}^c, A_{12}^c, A_{13}^c, A_{14}^c, \widetilde{X}\},$$

where $A_1^c = [\{d\}, \{c, d\}],\ A_2^c = [\{b\}, \{a, b\}],\ A_3^c = [\{c\}, \{b, c\}],\ A_4^c = (\{a\}, \{a, d\}]$,
$A_5^c = [\{b, d\}, X],\ A_6^c = [\{c, d\}, \{b, c, d\}],\ A_7^c = [\{a, d\}, \{a, c, d\}]$,
$A_8^c = [\{b, c\}, X],\ A_9^c = (\{a, b\}, \{a, b, d\}],\ A_{10}^c = [\{a, c\}, X],\ A_{11}^c = [\varnothing, \{c\}]$,
$A_{12}^c = [\varnothing, \{d\}],\ A_{13}^c = [\varnothing, \{b\}],\ A_{14}^c = [\varnothing, \{a\}]$.
Thus we can easily see that (X, τ) is T_3(ii).

Remark 5.3. (1) If $\tau \subset IVS^*(X)$, then two concepts in Definition 5.1 are coincident.

(2) Let (X, τ_o) be an ordinary topological space. Then clearly, $\tau = \{[A, A] : A \in \tau_o\} \in IVT(X)$ and $\tau \subset IVS_*(X)$. Thus we can easily check that if (X, τ_o) is T_3, then (X, τ) is both T_3(i) and T_3(ii) by (1).

(3) Let (X, τ) be an ordinary space such that τ is not indiscrete let τ^1, τ^2 be two IVTs on X given in Remark 4.5. Then we can easily see that if (X, τ) is T_3, then (X, τ^1) is T_3(i) and (X, τ^2) is T_3(ii).

Theorem 5.4. *Let (X, τ) be an IVTS such that $\tau \subset IVS^*(X)$. Then*
(1) *(X, τ) is T_3(i) if and only if (X, τ^-) is T_3.*
(2) *(X, τ) is T_3(i) if and only if (X, τ^+) is T_3.*

Proof. (1) By Theorem 4.11 (1), (X, τ) is T_1(i) if and only if (X, τ_1) is T_1. Then it is sufficient to prove that (X, τ) satisfies the regular axiom (i) if and only if (X, τ^-) is regular.

Suppose (X, τ) satisfies the regular axiom (i) and let A be any closed set in (X, τ^-) such that $x \in A^c$. Then clearly, $A^c \in \tau^-$. By the definition of τ^-, there is $W \in \tau$ such that $W = [A^c, W^+]$. Let $F = W^c$. Then clearly, $F = [W^{+^c}, A] \in \tau^c$. Since A is a CIVS in (X, τ^-), $W^{+^c} = A$. Thus $F = [A, A] \in \tau^c$ and $W = F^c = [A^c, A^c] \in \tau$. Since $x \in A^c$, $x_{_1} \in F^c$. By the hypothesis, there are $U,\ V \in \tau$ such that $F \subset U,\ x_{_1} \in V$ and $U \cap V = \widetilde{\varnothing}$. Since $\tau \subset IVS^*(X)$, $F^- = A \subset U^-,\ x \in V^-$ and $U^- \cap V^- = \varnothing$. By the definition of τ^-, it is clear that $U^-,\ V^- \in \tau^-$. So (X, τ^-) is regular. Hence (X, τ^-) is T_3.

Conversely, suppose (X, τ^-) is regular and let $F \in \tau^c$ such that $x_{_1} \in F^c$. Then clearly, $F^c = [F^{+^c}, F^{-^c}] \in \tau^c$. By the definition of τ^-, F^- is closed in (X, τ^-) and $x \in F^{+^c} \in \tau^-$. Since $\tau \subset IVS^*(X)$, $F_F^c = F^-$. By the hypothesis, there are $U,\ V \in \tau^-$ such that $F^- \subset U,\ x \in V$ and $U \cap V = \varnothing$. Thus $F^{-^c} \supset U^c$ and $U^c \cup V^c = X$. Let $A = [U, U],\ B = [V, V]$. Then clearly, $A,\ B \in \tau,\ F \subset A,\ x_{_1} \in B$ and $A \cap B = \widetilde{\varnothing}$. Thus (X, τ) is T_3(i).

(2) The proof is similar to (1). □

Remark 5.5. If the condition "$\tau \subset IVS^*(X)$" is taken off, then Theorem 5.4 does not hold in general (See Example 5.6).

Example 5.6. (1) Let (X, τ) be the T_3(i)-space given in Example 5.2 (2). Then clearly, $\tau \not\subset IVS^*(X)$ and

$$\tau^- = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{a, c\}, \{b, c\}, \{c, d\}, X\}.$$

Thus we can easily see that (X, τ^-) is not T_3.

(2) Let (X, τ) be the T_3(ii)-space given in Example 5.2 (3). Then clearly, $\tau \not\subset IVS^*(X)$ and

$$\tau^+ = \{\varnothing, \{a,b,c\}, \{a,c,d\}, \{a,b,d\}, \{b,c,d\}, \{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}, \{a,c\}, \{b,d\}, X\}.$$

Thus we can easily see that (X, τ^+) is not T_3.

The following two Theorems characterize T_3(i)-space and T_3(ii)-space

Theorem 5.7. *Let (X, τ) be an IVTS.*

(1) If X is T_1(i), then it is T_3(i) if and only if for each $x \in X$ and each $U \in N(x_{_1})$, there is $V \in N(x_{_1})$ such that $IVcl(V) \subset U$.

(2) If X is T_1(ii), then it is T_3(ii) if and only if for each $x \in X$ and each $U \in N(x_{_0})$, there is $V \in N(x_{_0})$ such that $IVcl(V) \subset U$.

Proof. (1) Suppose X is T_3(i). Let $x \in X$ and let $U \in N(x_{_1})$. Then clearly, $U^c \in \tau^c$ and $x_{_1} \notin U^c$. By hypothesis, there are $V,\ W \in \tau$ such that $x_{_1} \in V$, $U^c \subset W$ and $V \cap W = \widetilde{\varnothing}$. Thus $V \subset W^c$ and $W^c \in \tau^c$. So $IVcl(V) \subset W^c$. Since $U^c \subset W$, $W^c \subset U$. Hence $V \in N(x_{_1})$ such that $IVcl(V) \subset U$.

Suppose X is T_1(i) satisfying the necessary condition. Let $F \in \tau^c$ such that $x_{_1} \in F^c$. Then clearly, $F^c \in N(x_{_1})$. Thus by the hypothesis, there exists $V \in N(x_{_1})$ such that $IVcl(V) \subset F^c$. Since $V \subset IVcl(V)$, $V^- \subset (Icl(V))^-$ and $V^+ \supset (Icl(V))^+$. So we get

$$V^- \cap (Icl(V))^{+^c} \subset V^- \cap V^{-^c} = \varnothing \text{ and } V^+ \cap (Icl(V))^{-^c} \supset V^+ \cap V^{+^c} = \varnothing.$$

Hence $V \cap (Icl(V))^c = \widetilde{\varnothing}$, $x_{_1} \in V$ and $F \subset (IVcl(V))^c$. Therefore X is T_3(i).

(2) The proof is similar to (1). □

Theorem 5.8. *Let (X, τ) be an IVTS.*

(1) If X is T_1(i), then it is T_3(i) if and only if for each $x \in X$ and each $F \in \tau^c$ such that $x_{_1} \in F^c$, there exist $U,\ V \in \tau$ such that $F \subset U$, $x_{_1} \in V$ and $IVcl(U) \cap IVcl(V) = \widetilde{\varnothing}$.

(2) If X is T_1(ii), then it is T_3(ii) if and only if for each $x \in X$ and each $F \in CIVS(X)$ such that $x_{_0} \in F^c$, there exist $U,\ V \in \tau$ such that $F \subset U$, $x_{_0} \in V$ and $IVcl(U) \cap IVcl(V) = \widetilde{\varnothing}$.

Proof. Suppose X is T_3(i). Let $x \in X$ and let $F \in \tau^c$ such that $x_{_1} \in F^c$. Then $F^c \in N(x_{_1})$. Thus by Theorem 5.7 (1), there is $W \in N(x_{_1})$ such that $IVcl(W) \subset F^c$. Again by Theorem 5.7 (1), There is $V \in N(x_{_1})$ such that $IVcl(V) \subset W$. Let $U = (IVcl(W))^c \in \tau$. Since $IVcl(W) \subset F^c$, $F \subset (IVcl(W))^c = U$. So we have

$$IVcl(V) \cap IVcl(U) \subset W \cap IVcl((IVcl(W))^c) = \widetilde{\varnothing}.$$

Hence U and V are the desired OIVSs in X.

The converse can be easily proved.

(2) The proof is similar to (1). □

Now we deal with the separation of two OIVSs.

Definition 5.9. A T_1(i)-space (X, τ) is called an *interval-valued normal space* or a *T_4-space*, if for any $A,\ B \in \tau^c$ such that $A \cap B = \widetilde{\varnothing}$, there exist $U,\ V \in \tau$ such that $U \cap V = \widetilde{\varnothing}$, $A \subset U$ and $B \subset V$.

It is obvious that if X is T_4(i), then it is T_3(i). But the converse is no true in general (See Example 5.10 (2)).

Example 5.10. (1) Let $X = \{a, b, c\}$ and consider the IVT τ on X given by:

$$\tau = \{\widetilde{\varnothing}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, \widetilde{X}\},$$

where $A_1 = [\{a\}, \{a, c\}]$, $A_2 = [\{b\}, \{b\}]$, $A_3 = [\{a, c\}, \{a, c\}]$, $A_4 = [\{a, b\}, \{a, b\}]$,
$A_5 = [\{c\}, \{c\}]$, $A_6 = [\{b, c\}, X]$, $A_7 = [\{a, b\}, X]$, $A_8 = [\{a\}, \{a\}]$,
$A_9 = [\varnothing, \{c\}]$, $A_{10} = [\varnothing, \{a, c\}]$, $A_{11} = [\{c\}, \{a, c\}]$, $A_{12} = [\{a\}, \{a, c\}]$
$A_{13} = [\{b\}, \{a, b\}]$, $A_{14} = [\{b\}, X]$, $A_{15} = [\{a, b\}, X]$, $A_{16} = [\{b, c\}, \{b, c\}]$.
Then clearly, (X, τ) is T_1(i) and

$$\tau^c = \{\widetilde{\varnothing}, A^c_1, A^c_2, A^c_3, A^c_4, A^c_5, A^c_6, A^c_7, A^c_8, A^c_9, A^c_{10}, A^c_1, A^c_{12}, A^c_{13}, A^c_{14}, A^c_{15}, A^c_{16}, \widetilde{X}\},$$

where $A^c_1 = [\{b\}, \{b, c\}]$, $A^c_2 = [\{a, c\}, \{a, c\}]$, $A^c_3 = [\{b\}, \{b\}]$, $A^c_4 = [\{c\}, \{c\}]$,
$A^c_5 = [\{a, b\}, \{a, b\}]$, $A^c_6 = [\varnothing, \{a\}]$, $A^c_7 = [\varnothing, \{c\}]$, $A^c_8 = [\{b, c\}, \{b, c\}]$,
$A^c_9 = [\{a, b\}, X]$, $A^c_{10} = [\{b\}, X]$, $A^c_{11} = [\{b\}, \{a, b\}]$, $A^c_{12} = [\{b\}, \{b, c\}]$,
$A^c_{13} = [\{c\}, \{a, c\}]$, $A^c_{14} = [\varnothing, \{a, c\}]$, $A^c_{15} = [\varnothing, \{c\}]$, $A^c_{16} = [\{a\}, \{a\}]$.
Thus we can easily see that (X, τ) is T_4.

(2) For each $a \in \mathbb{R}$, let $U_a = [(a, \infty), (a, \infty)]$ and $\tau = \{U_a : a \in \mathbb{R}\}$. Then clearly, τ is an IVT on $\mathbb{R}$. In fact, we can easily check that $(\mathbb{R}, \tau)$ is $\mathrm{T}_4(i)$. On the other hand, consider $C = [(-\infty, 0], (-\infty, 0]] \in IVS(\mathbb{R})$. Then clearly, C is a CIVS in $\mathbb{R}$) and $1_{_1} \in C^c$. Let $U \in \tau$ such that $C \subset U$. Then $U = \mathbb{R}$. Thus $1_{_1} \in U$. So $(\mathbb{R}, \tau)$ is not T_3(i).

The followings characterize T_4-space.

Theorem 5.11. *Let (X, τ) be an IVTS such that (X, τ) is T_1(i). Then (X, τ) is T_4 if and only if for each $F \in IVC(X)$ and each $U \in \tau$ such that $F \subset U$, there exists $V \in \tau$ such that $F \subset V$ and $IVcl(V) \subset U$.*

Proof. Suppose (X, τ) is T_4. Let $F \in \tau^c$ and let $U \in \tau$ such that $F \subset U$. Then clearly, $F^- \subset U^-$ and $F^+ \subset U^+$. Thus $F^- \cap {U^+}^c = \varnothing$ and $F^+ \cap {U^-}^c = \varnothing$, i.e., $F \cap U^c = \widetilde{\varnothing}$. Since $U^c \in \tau^c$, by the hypothesis, there exist $V,\ W \in \tau$ such that $V \cap W = \widetilde{\varnothing}$, $F \subset V$ and $U^c \subset W$. Since $V \cap W = \widetilde{\varnothing}$, $V \subset W^c$ and $W^c \in \tau^c$. So $IVcl(V) \subset W^c$. Since $U^c \subset W$, $W^c \subset U$. Hence $IVcl(V) \subset U$.

Conversely, suppose the necessary condition holds. Let $A,\ B \in \tau^c$ such that $A \cap B = \widetilde{\varnothing}$. Then clearly, $A \subset B^c$ and $B^c \in \tau$. Thus by the hypothesis, there exists $V \in \tau$ such that $A \subset V$ and $IVcl(V) \subset B^c$. Let $U = (IVcl(V))^c$. Then clearly, $U \in \tau$, $B \subset U$ and $U \cap V = \widetilde{\varnothing}$. Thus (X, τ) is T_4. □

Theorem 5.12. *Let (X, τ) be an IVTS such that (X, τ) is T_1(i). Then (X, τ) is T_4 if and only if for any $A,\ B \in \tau^c$ such that $A \cap B = \widetilde{\varnothing}$, there exists $U,\ V \in \tau$ such that $A \subset U$, $B \subset V$ and $IVcl(U) \cap IVcl(V) = \widetilde{\varnothing}$.*

Proof. Suppose (X, τ) is T_4. Let $A,\ B \in \tau^c$ such that $A \cap B = \widetilde{\varnothing}$. Then by the hypothesis, there exist $G,\ H \in \tau$ such that $G \cap H = \widetilde{\varnothing}$, $A \subset G$ and $B \subset H$. Thus by Theorem 5.11, there exist $U,\ V \in \tau$ such that $A \subset U$, $IVcl(U) \subset A$ and $B \subset V$, $IVcl(V) \subset B$. So $IVcl(U) \cap IVcl(V) \subset A \cap B = \widetilde{\varnothing}$. Hence $IVcl(U) \cap IVcl(V) = \widetilde{\varnothing}$.

The proof of the converse is easy. □

Proposition 5.13. *Let (X, τ) be T_4, let (Y, δ) be an IVTS and let $f : X \to Y$ be interval-valued continuous, closed and surjective. Then (Y, δ) is T_4.*

Proof. Let $C,\ D \in \delta^c$ such that $C \cap D = \widetilde{\varnothing}$. Then clearly, $f^{-1}(C),\ f^{-1}(D) \in \tau^c$ such that $f^{-1}(C) \cap f^{-1}(D) = \widetilde{\varnothing}$. Since X is T_4, there are $U,\ V \in \tau$ such that $f^{-1}(C) \subset U$ and $f^{-1}(D) \subset V$. Since f is closed, $f(U^c),\ f(V^c) \in \delta^c$. Let $M = [f(U^c)]^c,\ N = [f(V^c)]^c$. Then clearly, $M,\ N \in \delta$. Thus we have

$f^{-1}(M) = f^{-1}([f(U^c)]^c = f^{-1}(f(U^c))^c$ [By Result 3.2 (10)]

$= (U^c)^c$ [By Result 3.2 (4), since f is surjective]

$= U$. [By Result 2.4 (7)]

Similarly, we get $f^{-1}(N) = V$. So $C \subset M$ and $D \subset N$. Moreover, we can easily check that $M \cap N = \widetilde{\varnothing}$. Hence (Y, δ) is T_4. □

6. Interval-valued subspaces

In this section, we define an interval-valued subspace of an IVTS and dealt with some of its properties.

Proposition 6.1. *(X, τ) be an IVTS and let $A \in IVS(X)$. Then the family τ_A of IVSs in X given by:*

$$\tau_A = \{U \cap A : U \in \tau\}$$

is an IVT on A.

In this case, τ_A is called the *interval-valued relative topology on A* and the pair (A, τ_A) is called an *interval-valued subspace* of (X, τ). The members of τ_A are called *relatively open sets* or simply *open sets in A.*

Proof. Since $\widetilde{\varnothing},\ \widetilde{X} \in \tau,\ \widetilde{\varnothing} \cap A = \widetilde{\varnothing},\ \widetilde{X} \cap A = A \in \tau_A$. Then τ_A holds the axiom (IVO_1). Let $U \cap A,\ V \cap A \in \tau_A$. Then clearly, $(U \cap A) \cap (V \cap A) = (U \cap V) \cap A$ and $U \cap V \in \tau$. Thus $(U \cap A) \cap (V \cap A) \in \tau_A$. So τ_A holds the axiom (IVO_2). Finally, let $(U_j \cap A)_{j \in J}$ be any family of members of τ_A. Then clearly, $\bigcup_{j \in J}(U_j \cap A) = \left(\bigcup_{j \in J} U_j\right) \cap A$ and $\bigcup_{j \in J} U_j \in \tau$. Thus $\bigcup_{j \in J}(U_j \cap A) \in \tau_A$. So τ_A holds the axiom (IVO_3). Hence τ_A is an IVT on A. □

Example 6.2. Let $\tau = \{U \in IVS(\mathbb{R}) : 0 \in U^- \textit{ or } U = \widetilde{\varnothing}\}$. Then we can easily check that τ is an IVT on $\mathbb{R}$. Let $A = [1, 2]_{IVI}$ (See Definition 7.1) and let $x_1 \in A$. Then clearly, $0_1 \cup x_1 = [\{0, x\}, \{0, x\}] \in \tau$ and $(0_1 \cup x_1) \cap A = x_1$. Thus $x_1 \in \tau_A$. So τ_A is the interval-valued discrete topology.

Remark 6.3. (1) Let (X, τ) be an interval-valued discrete space and let $A \in IVS(X)$. Then τ_A is the discrete topology on A.

(2) Let (X, τ) be an interval-valued indiscrete space and let $A \in IVS(X)$. Then τ_A is the indiscrete topology on A.

(3) Let (X, τ) be an IVTS and let $A,\ B \in IVS(X)$ such that $A \subset B$. Then $\tau_A = \tau_{B_A}$.

(4) Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then $(A^-, {\tau_A}^-)$ and $(A^+, {\tau_A}^+)$ are classical subspaces of (X, τ^-) and (X, τ^+), respectively.

The following is an immediate result of Definition 3.12 and Proposition 6.1.

Proposition 6.4. *Let (X,τ) be an IVTS, let $A \in IVS(X)$ and let β be a base for τ. Then $\beta_A = \{B \cap A : B \in \beta\}$ is a base for τ_A.*

The following gives a special situation in which every member of the interval-valued relative topology is also a number of the IVT on X.

Proposition 6.5. *Let (X,τ) be an IVTS and let $A \in \tau$. If $V \in \tau_A$, then $V \in \tau$.*

Proof. Suppose $V \in \tau_A$. Then by Proposition 6.1, there is $U \in \tau$ such that $V = U \cap A$. Since $A \in \tau$, $U \cap A \in \tau$. Thus $V \in \tau$. □

Theorem 6.6. *Let (A,τ_A) be an interval-valued subspace of an IVTS (X,τ) and let $C \in \tau^c$. Then C is closed in (A,τ_A) if and only if there is $D \in \tau^c$ such that $C = D \cap A$.*

Proof. Suppose C is closed in (A,τ_A). Then by Proposition 6.1, there is $U \in \tau$ such that $A - C = U \cap A$, where $A - C = [A^- \cap {C^+}^c, A^+ \cap {C^-}^c]$. Thus we have

$$C = A - (A - C) = A - (U \cap A) = A \cap (A - U) = A \cap U^c.$$

Since $U^c \in \tau^c$, the necessary condition holds.

Suppose $C \subset A$ and there is $D \in \tau^c$ such that $C = D \cap A$. Then clearly, $D^c \in \tau$. Moreover, $A - C = A - (D \cap A) = A \cap (A - D) = A \cap D^c$. Thus $A - C \in \tau_A$. So C is closed in (A,τ_A). □

The following provides a criterion for an interval-valued closed subset of an interval-valued subspace to be closed in the IVTS.

Proposition 6.7. *Let (X,τ) be an IVTS an let $A \in \tau^c$. If C is closed in (A,τ_A), then $C \in \tau^c$.*

When dealing with subspaces of an IVTS, we need to exercise care in taking closures of an IVS because the closure in the interval-valued subspace may be quite different from the closure in the IVTS. The following gives a criterion for dealing with this situation.

Proposition 6.8. *Let (A,τ_A) be an interval-valued subspace of an IVTS (X,τ) and let $B \subset A$. Then $IVcl_{(A,\tau_A)}(B) = A \cap IVcl(B)$, where $IVcl_{(A,\tau_A)}(B)$ denotes the interval-valued closure in (A,τ_A).*

Proof. The proof is similar to one of classical case. □

Theorem 6.9. *Let (A,τ_A) be an interval-valued subspace of an IVTS (X,τ) and let $U \subset A$.*

(1) *Suppose $a_1 \in A$. Then $U \in N_{\tau_A}(a_1)$ if and only if there is $V \in N(a_1)$ such that $U = V \cap A$, where $N_{\tau_A}(a_1)$ denotes the set of all neighborhoods of a_1 with respect to τ_A.*

(2) *Suppose $a_0 \in A$. Then $U \in N_{\tau_A}(a_0)$ if and only if there is $V \in N(a_0)$ such that $U = V \cap A$, where $N_{\tau_A}(a_0)$ denotes the set of all neighborhoods of a_0 with respect to τ_A.*

Proof. The proof is similar to one of classical case. □

Proposition 6.10. (1) *Every interval-valued subspace of a* T_0(j)*-space* (X, τ)*is a* T_0(j)*-space for each* j=i, ii, iii, iv, v, vi.

(2) *Every interval-valued subspace of a* T_1(j)*-space* (X, τ) *is a* T_1(j)*-space for each* j=i, ii, iii, iv, v, vi, vii, viii.

(3) *Every interval-valued subspace of a* T_2(j)*-space* (X, τ) *is a* T_2(j)*-space for each* j=i, ii, iii, iv, v, vi.

Proof. (1) We prove only the cases T_0(i), T_0(iv) and T_0(vi). The remainder's proofs are similar.

Suppose A is an interval-valued subspace of a T_0(i)-space (X, τ) and let $a_{_1},\ b_{_1} \in A$ such that $a \neq b$. Then clearly, $a,\ b \in A^-$ and there is $U \in \tau$ such that either $a \in U^-,\ b \notin U^-$ or $b \in U^-,\ a \notin U^-$. Thus $U \cap A \in \tau_{_A}$ such that either $a \in U^- \cap A^-$, $b \notin U^- \cap A^-$ or $b \in U^- \cap A^-$, $a \notin U^- \cap A^-$. So either $a_{_1} \in U \cap A,\ b_{_1} \notin U \cap A$ or $b_{_1} \in U \cap A,\ a_{_1} \notin U \cap A$. Hence A is T_0(i).

Suppose A is an interval-valued subspace of a T_0(iv)-space (X, τ) and let $a_{_0},\ b_{_0} \in A$ such that $a \neq b$. Then clearly, $a,\ b \in A^+$ and there is $U \in \tau$ such that either $a \in U^+,\ U^- \subset \{b\}^c$ or $b \in U^+,\ U^- \subset \{a\}^c$. Thus $U \cap A \in \tau_{_A}$ such that either $a \in U^+ \cap A^+,\ U^- \cap A^- \subset \{b\}^c$ or $b \in U^+ \cap A^+,\ U^- \cap A^- \subset \{a\}^c$. So either $a_{_0} \in U \cap A \subset b^c_{_0}$ or $b_{_0} \in U \cap A \subset a^c_{_0}$. Hence A is T_0(iv).

Suppose A is an interval-valued subspace of a T_0(V)-space (X, τ) and let $a_{_1},\ b_{_1} \in A$ such that $a \neq b$. Then clearly, $a,\ b \in A^-$ and there is $U \in \tau$ such that either $a \notin U^-$ or $b \notin U^-$. Thus $U \cap A \in \tau_{_A}$ such that either $a \notin U^- \cap A^-$ or $b \notin U^- \cap A^-$. So either $a_{_1} \notin U \cap A$ or $b_{_1} \notin U \cap A$. Hence A is T_0(v).

(2) We prove only the cases T_1(ii) and T_1(vii). The remainder's proofs are similar.

Suppose A is an interval-valued subspace of a T_1(ii)-space (X, τ) and let $a_{_0},\ b_{_0} \in A$ such that $a \neq b$. Then $a,\ b \in A^+$ and there are $U,\ V \in \tau$ such that

$$a \in U^+,\ b \notin U^+ \text{ and } a \notin V^+,\ b \in V^+.$$

Thus $U \cap A,\ V \cap A \in \tau_{_A}$. Furthermore, we have

$$a \in U^+ \cap A^+,\ b \notin U^+ \cap A^+ \text{ and } a \notin V^+ \cap A^+,\ b \in V^+ \cap A^+.$$

So we get

$$a_{_0} \in U \cap A,\ b_{_0} \notin U \cap A \text{ and } b_{_0} \notin V \cap A,\ a_{_0} \in V \cap A.$$

Hence A is T_1(ii).

Suppose A is an interval-valued subspace of a T_1(Vii)-space (X, τ) and let $a_{_1} \in A$. Then clearly, $a_{_1} \in \tau^c$. Since $a_{_1} \in A$, $a_{_1} \cap A = a_{_1}$. Thus by Theorem 6.6, $a_{_1}$ is closed in $(A, \tau_{_A})$. So A is T_1(Vii).

(3) The proofs are similar to (2). □

Proposition 6.11. *Every interval-valued subspace of a* T_3(j)*-space* (X, τ) *is a* T_3(j)*-space for each* j=i, ii.

Proof. Suppose A is an interval-valued subspace of a T_3(i)-space (X, τ). Then by Proposition 6.10 (2), $(A, \tau_{_A})$ is T_1(i). Let C be closed in $(A, \tau_{_A})$ such that $a_{_1} \in (A - C)$. Then by Theorem 6.6, there is $D \in \tau^c$ such that $C = D \cap A$. Since $a_{_1} \in (A - C)$, $a_{_1} \in D^c$. Since X is T_3(i), there are $U,\ V \in OIVS(X)$ such that

$D \subset U$, $a_1 \in V$ and $U \cap V = \widetilde{\varnothing}$. Thus $U \cap A$ and $V \cap A$ are open in (A, τ_A). Moreover, we can easily see that

$$C \subset U \cap A,\ a_1 \in V \cap A \text{ and } (U \cap A) \cap (V \cap A) = \widetilde{\varnothing}.$$

So A is T_3(i). The remainder's proof is similar. □

Proposition 6.12. *Let (X, τ) be a T_4-space and let $C \in CIVS(X)$. Then (C, τ_C) is T_4.*

Proof. Let F_1 and F_2 be closed in (C, τ_C) such that $F_1 \cap F_2 = \widetilde{\varnothing}$. Since $C \in \tau^c$, $F_1,\ F_2 \in \tau^c$ by Proposition 6.7. Since (X, τ) is T_4, there are $U,\ V \in \tau$ such that $F_1 \subset U$, $F_2 \subset V$ and $U \cap V = \widetilde{\varnothing}$. Then we have

$$F_1 \subset U \cap C,\ F_2 \subset V \cap C \text{ and } (U \cap C) \cap (V \cap C) = \widetilde{\varnothing}.$$

It is clear that $U \cap C$ and $V \cap C$ are open in (C, τ_C). Thus (C, τ_C) is T_4. □

Definition 6.13. A T_1(i)-space (X, τ) is said to be *interval-valued completely normal*, if for any $A,\ B \in IVS(X)$ such that $A \cap IVcl(B) = IVcl(A) \cap B = \widetilde{\varnothing}$, there are $U,\ V \in \tau$ such that $A \subset U$ and $B \subset V$.

It is obvious that every interval-valued completely normal space is T_4. The following gives a characterization of an interval-valued completely normality.

Theorem 6.14. *Let (X, τ) be a T_4-space and let $C \in \tau^c$. Then (C, τ_C) is T_4.*

Proof. The proof is similar to one of classical case. □

7. Appendix

In this section, There is a typo in the Definition 4.11 in [19], so we correct it.

Definition 7.1. Let $a,\ b \in \mathbb{R}$ such that $a \leq b$. Then
(i) (the closed interval) $[a, b]_{IVI} = [[a, b], [a, b]]$,
(ii) (the open interval) $(a, b)_{IVI} = [(a, b), (a, b)]$,
(iii) (the half open interval or the half closed interval)

$$(a, b]_{IVI} = [(a, b], (a, b]],\ [a, b)_{IVI} = [[a, b), [a, b)],$$

(iv) (the half interval-valued real line)

$$(-\infty, a]_{IVI} = [(-\infty, a], (-\infty, a]],\ (-\infty, a)_{IVI} = [(-\infty, a), (-\infty, a)],$$
$$[a, \infty)_{IVI} = [[a, \infty), [a, \infty)],\ (a, \infty)_{IVI} = [(a, \infty), (a, \infty)],$$

(v) (the interval-valued real line) $(-\infty, \infty)_{IVI} = [(-\infty, \infty), (-\infty, \infty)] = \widetilde{\mathbb{R}}$.

Definition 7.2. An *interval-valued bipolar valued fuzzy set* A in a nonempty set X is an object having the form

$$A = ([A^{N,-}, A^{N,+}], [A^{P,-}, A^{P,+}]),$$

where $[A^{N,-}, A^{N,+}] : X \to D[-1, 0]$ and $[A^{P,-}, A^{P,+}] : X \to D[0, 1]$ are mappings. For each $x \in X$, $[A^{P,-}, A^{P,+}](x) = [A^{P,-}(x), A^{P,+}(x)]$ (called the *positive interval-valued membership degree*) and $[A^{N,-}, A^{N,+}](x) = [A^{N,-}(x), A^{N,+}(x)]$ (called the *negative interval-valued membership degree*) denotes the satisfaction degree of an element x to the property corresponding to and to some implicit counter property of interval-valued bipolar valued fuzzy set A.

8. Conclusions

In this paper, the topics of interval-valued continuous functions and separation axioms in interval-valued topological spaces have been extensively investigated, a detailed study has been done on these subjects and the obtained theorems are supported with appropriate examples. By accepting the interval-valued continuity results, we were able to continue our research on the **IVTop** concrete category later. As a result, we initially found that the concrete category **IVTop** has the initial structure from Proposition 3.39. One of the most important issues in this article is that there are two new characterizations for each T_3 and T_4 separation axiom in interval-valued topological spaces that are given in Theorems 5.7, 5.8 and Theorems 5.11, 5.12. The relationships and transitions between these newly defined separation axioms are shown with the help of tables, and the non-transition sections are supported with counter examples. As a possible advancement of the research proposed here, it can be extended to solve separation axioms in more comprehensive spaces, such as the hyperspace of interval-valued topological spaces, thereby completing and enriching the research. The results of this article have specific findings that warrant further and deeper exploration.

In the future, we expect that one can investigate hyperspace of interval-valued topological spaces, and topological structures based on interval-valued bipolar valued fuzzy sets (See Definition 7.2).

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