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Abstract. In this paper, the concepts of cubic Pythagorean KU-ideals are introduced and several properties are investigated. Also, relations between cubic Pythagorean KU-ideals and cubic Pythagorean ideals are given. The pre-image of cubic Pythagorean KU-ideals under homomorphism of KU-algebras are defined and how the pre-image of cubic Pythagorean KU-ideals under homomorphism of KU-algebras become cubic Pythagorean KU-ideal are studied. Moreover, the Cartesian product of cubic Pythagorean KU-ideals in Cartesian product KU-algebras is given. Finally, novel new correlation coefficient between two cubic Pythagorean fuzzy sets are also studied.

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Keywords: Cubic Pythagorean KU-ideal, The pre-image of cubic Pythagorean KU-ideals under homomorphism of KU-algebras, Cartesian product of cubic Pythagorean KU-ideals, Correlation coefficient between two cubic Pythagorean fuzzy sets.

Corresponding Author: Ola Wageeh Abd El-Baseer (`olawagih@yahoo.com`)
J. G. Lee (`jukolee@wku.ac.kr`)

1. Introduction

Iséki and Tanaka introduced two classes of abstract algebras, BCK-algebras and BCI-algebras [1, 2, 3]. Prabpayak and Leerawat [4, 5] introduced a new algebraic structure which is called *KU-algebra*. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. In 1965, Zadeh [6] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. In 1991, Xi [7] applied the concept of fuzzy sets to BCI, BCK,

MV-algebras. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc. (See [8, 9, 10, 11, 12, 13]). Zadeh [13] made an extension of the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as i-v fuzzy set. Zadeh also constructed a method of approximate inference using his i-v fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov [8] as a generalization of the notion of fuzzy sets. After that many researchers consider the Fuzzifications of ideals and sub-algebras in algebraic structures. Mostafa et al. [10, 11, 14, 15], Ansari et al. [16] and Koam et al. [17] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Jun et al. [18] introduced the notion of a novel extension of cubic sets and its applications in BCK/BCI-algebras (See [19] for the concept of cubic sets). Yager [20] launched a nonstandard fuzzy set referred to as Pythagorean fuzzy set which is the generalization of intuitionistic fuzzy sets. The construct of Pythagorean fuzzy sets can be used to characterize uncertain information more sufficiently and accurately than intuitionistic fuzzy set. The authors in [21, 22] defined a new generalized Pythagorean fuzzy set is called $(3, 2)$-*Fuzzy sets*. In 2020, Fermatean fuzzy sets proposed by Senapati and Yager [15] can handle uncertain information more easily in the process of decision making. They also defined basic operations over the Fermatean fuzzy sets. The main advantages of Fermatean fuzzy sets is that it can describe more uncertainties than Pythagorean fuzzy sets, which can be applied in many decision-making problems. The relevant research can be referred to (SR-Fuzzy sets) (See [23]). Pythagorean fuzzy set is one of the successful extensions of the fuzzy set for handling uncertainties in information. Under this environment, Salih et al. [24] introduced a new type of generalized fuzzy sets called CR-*fuzzy sets* and compared CR-fuzzy sets with Pythagorean fuzzy sets and Fermatean fuzzy sets. The set operations, score function and accuracy function of CR-fuzzy sets will study along with their several properties. Recently, Jun et al. [22] introduced the concept of the $(m; n)$-fuzzy set which is the subclass of intuitionistic fuzzy set, Pythagorean fuzzy set, $(3; 2)$-fuzzy set, Fermatean fuzzy set and n-Pythagorean fuzzy set and compared with them. They introduced some operations for the $(m; n)$-fuzzy set, investigate their properties and applied the $(m; n)$-fuzzy set to BCK-algebras and BCI-algebras. They introduced the $(m; n)$-fuzzy subalgebra in BCK-algebras and BCI-algebras and investigate their properties. Ahn et al. [21] apply the concept of $(2, 3)$-fuzzy sets to BCK-algebras and BCI-algebas. Ibrahim et al. [25] introduced $(3; 2)$-fuzzy sets and applied it to topological spaces. Decision-making problems are very important in establishing foreign policy, national defense policy, economic policy, various election strategies, and prevention policy of the recent worldwide corona virus, etc. So, many mathematicians have dealt with decision-making problems using the algorithms for decision making in three directions: aggregation operators, similarity measures and correlation coefficients based on various fuzzy sets. kinds of fuzzy concepts or fused fuzzy concepts. Mainly, they presented algorithms for decision making in three directions: aggregation operators, similarity measures and correlation coefficients based on various fuzzy sets (See [9, 26, 27]).

In this paper, the concept of cubic Pythagorean KU-ideals is introduced and its several properties are investigated. Also, relations between cubic Pythagorean KU-ideal and cubic Pythagorean ideal are given. The pre-image of cubic Pythagorean KU-ideals under homomorphism of KU-algebras are defined and how the pre-image of cubic Pythagorean KU-ideals under homomorphism of KU-algebras become cubic Pythagorean KU-ideal are studied. Moreover, the Cartesian product of cubic Pythagorean KU-ideals in Cartesian product KU-algebras is given. Finally, novel new correlation coefficient between two cubic Pythagorean fuzzy sets are also studied.

Throughout this paper, J denotes an index set.

2. Preliminaries

Now we review some concepts related to KU-algebra and interval-valued -intuitionistic-fuzzy logic

Definition 2.1 ([4, 5])**.** A *KU-algebra* is a triple $(X, *.0)$, where X is a nonempty set, $*$ is a binary operation on X and 0 is a fixed element of such that the following axioms are satisfied: for all $x,\ y,\ z \in X$,

(KU_1) $(x * y) * [(y * z) * (x * z)] = 0$,
(KU_2) $x * 0 = 0$,
(KU_3) $0 * x = x$,
(KU_4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(KU_5) $x * x = 0$.

On a $KU-$algebra, we can define a binary relation $\leq$ on X as follows: for any $x,\ y \in X$,

$$x \leq y \Longleftrightarrow y * x = 0.$$

Theorem 2.2 ([14])**.** *Let be a KU-algebra. Then the followings hold: for all $x,\ y,\ z \in X$,*

(1) $x * (y * x) = 0$,
(2) *if* $x \leq y$, *then* $y * z \leq x * z$,
(3) $x * (y * z) = y * (x * z)$,
(4) $y * [(y * x) * x] = 0$.

Example 2.3. Let $X = \{0, a, b, c, d\}$ be a set with $*$ defined by the following table:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	c	d
b	0	a	0	c	d
c	0	0	0	0	d
d	0	0	0	0	0

Table 2.1

Then $(X, *, 0)$ is a KU-algebra.

Definition 2.4 ([8]). Le X be a nonempty set and let $I = [0, 1]$. Then a mapping $\overline{A} = (A^{\in}, A^{\notin}) : X \to I \times I$ is called an *intuitionistic fuzzy set*, if $0 \leq A^{\in}(x) = A^{\notin}(x) \leq 1$ for each $x \in X$. In this case, $A^{\in} : X \to I$ and $A^{\in} : X \to I$ denote the degree of membership and the degree of non-membership of $\overline{A}$ respectively.

Remark 2.5. Le X be a nonempty set and let $\overline{A} = (A^{\in}, A^{\notin}) : X \to I \times I$ be a mapping. Then the structure $\overline{A}$ is called:

(i) an *intuitionistic fuzzy set* in X, if $0 \leq A^{\in}(x) + A^{\notin}(x) \leq 1$ (See [8]),
(ii) a Pythgorean fuzzy set in X, if $0 \leq A^{\in^2}(x) + A^{\notin^2}(x) \leq 1$ (See [20]),
(iii) an n-Pythgorean fuzzy set in X, if $0 \leq A^{\in^n}(x) + A^{\notin^n}(x) \leq 1$ (See [28]),
(iv) a Fermatean fuzzy set in X, if $0 \leq A^{\in^3}(x) + A^{\notin^3}(x) \leq 1$ (See [29]),
(v) a $(3; 2)$-fuzzy set in X, if $0 \leq A^{\in^3}(x) + A^{\notin^2}(x) \leq 1$ (See [21, 22]),
(vi) an $(m; n)$-fuzzy set in X, if $0 \leq A^{\in^m}(x) + A^{\notin^n}(x) \leq 1$ (See [22]),
(vii) an SR-fuzzy set in X, if $0 \leq A^{\in^2}(x) + \sqrt{A^{\notin}(x)} \leq 1$ (See [23]),
(viii) a CR-fuzzy set in X, if $0 \leq A^{\in^3}(x) + \sqrt[3]{A^{\notin}(x)} \leq 1$ (See [24]).

For any $a,\ b \in I$, $a \wedge b$ and $a \vee b$ mean that $a \wedge b = min\{a, b\}$ and $a \vee b = max\{a, b\}$. For any family $(a_j)_{j\in J}$ of members of I, the inf and sup of $(a_j)_{j\in J}$ denoted by

$$inf_{j\in J} a_j = \bigwedge_{j\in J} a_j \text{ and } sup_{j\in J} a_j = \bigvee_{j\in J} a_j.$$

Definition 2.6 ([13]). An interval number $\widetilde{a} = [a^-, a^+]$, where $a \leq a^- \leq a^+ \leq 1$. Let $[I]$ denote the family of all closed subintervals of I, i.e.,

$$[I] = \{\widetilde{a} = [a^-, a^+] : a^- \leq a^+ \text{ for any } a^-,\ a^+ \in I\}.$$

We define the operations $\leq$, $=$, $\wedge$ and $\vee$ between two elements of I as follows: for any $\widetilde{a} = [a^-, a^+], \widetilde{b} = [b^-, b^+] \in [I]$,

(i) $\widetilde{a} \leq \widetilde{b}$ iff $a^- \leq b^-$, $a^+ \leq b^+$,
(ii) $\widetilde{a} = \widetilde{b}$ iff $a^- =^-$, $a^+ = b^+$,
(iii) $\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(iv) $\widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+]$.

Here we consider that $\widetilde{0} = [0, 0]$ as least element and $\widetilde{1} = [1, 1]$ as greatest element.

For any family $(\widetilde{a}_j)_{j\in J}$ of members of $[I]$, the inf and sup of $(\widetilde{a}_j)_{j\in J}$, denoted by $\bigwedge_{j\in J} \widetilde{a}_j$ and $\bigvee_{j\in J} \widetilde{a}_j$, are defined as follows:

$$\bigwedge_{j\in J} \widetilde{a}_j = [\bigwedge_{j\in J} a_j^-, \bigwedge_{j\in J} a_j^+] \text{ and } \bigvee_{j\in J} \widetilde{a}_j = [\bigvee_{j\in J} a_j^-, \bigvee_{j\in J} a_j^+].$$

A mapping $\widetilde{A} = [A^-, A^+] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVF-set) in X, if $A^-(x) \leq A^+(x)$ for each $x \in X$ (See [13]).

A mapping $\overline{\widetilde{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin}) : X \to [I] \times [I]$ is called an *interval-valued intuitionistic fuzzy set* (briefly, IVI-set) in X, if $A^{\in,+}(x) + A^{\notin,+}(x)$ for each $x \in X$, where $\widetilde{A}^{\in} = [A^{\in,-}, A^{\in,+}]$ and $\widetilde{A}^{\notin} = [A^{\notin,-}, A^{\notin,+}]$ (See [30]).

A mapping $\mathcal{A} = \left\langle \widetilde{A}, A \right\rangle : X \to [I] \times I$ is called a *cubic set* in X (See [19]).

3. Cubic Pythagorean KU-ideals in KU-algebras

In this section, we will introduce a new notion called cubic Pythgorean KU-ideal in KU-algebras and study its several properties.

Definition 3.1. Let X be a nonempty set. Then a mapping $\mathcal{A} = \left\langle \widetilde{\overline{A}}, \overline{A} \right\rangle : X \to ([I] \times [I]) \times (I \times I)$ is called a *cubic Pythagorean fuzzy set* (briefly, CPS) in X, if it satisfies the following conditions: for each $x \in X$,

(i) $0 \leq A^{\in,-^2}(x) + A^{\notin,-^2}(x) \leq 1$ and $0 \leq A^{\in,+^2}(x) + A^{\notin,+^2}(x) \leq 1$, i.e.,

$$\widetilde{0} \leq (\widetilde{A}^{\in})^2(x) + (\widetilde{A}^{\notin})^2(x) \leq \widetilde{1},$$

(ii) $0 \leq A^{\in^2}(x) + A^{\notin^2}(x) \leq 1$.

We will denote the set of all CPSs in X as $CPS(X)$.

Definition 3.2. Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A} \in CPS(X)$. Then a mapping $\mathcal{A}$ is called a *cubic Pythagorean fuzzy subalgebra* (briefly, CPSA) of X, if it satisfies the following conditions: for any $x,\ y \in X$,

(CPSA$_1$) $(\widetilde{A}^{\in})^2(x * y) \geq (\widetilde{A}^{\in})^2(x) \wedge (\widetilde{A}^{\in})^2(y)$,

(CPSA$_2$) $(\widetilde{A}^{\notin})^2(x * y) \leq (\widetilde{A}^{\notin})^2(x) \vee (\widetilde{A}^{\notin})^2(y)$,

(CPSA$_3$) $A^{\in^2}(x * y) \leq A^{\in^2}(x) \vee A^{\in^2}(y)$

(CPSA$_4$) $A^{\notin^2}(x * y) \geq A^{\notin^2}(x) \wedge A^{\notin^2}(y)$.

Definition 3.3. Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A} \in CPS(X)$. Then a mapping $\mathcal{A}$ is called a *cubic Pythagorean fuzzy ideal* (briefly, CPI) of X, if it satisfies the following conditions: for any $x,\ y \in X$,

(CPI$_1$) $(\widetilde{A}^{\in})^2(0) \geq (\widetilde{A}^{\in})^2(x)$, $(\widetilde{A}^{\notin})^2(0) \leq (\widetilde{A}^{\notin})^2(x)$,

(CPI$_2$) $(\widetilde{A}^{\in})^2(x) \geq (\widetilde{A}^{\in})^2(y * x) \wedge (\widetilde{A}^{\in})^2(y)$,

(CPI$_3$) $(\widetilde{A}^{\notin})^2(x) \leq (\widetilde{A}^{\notin})^2(y * x) \vee (\widetilde{A}^{\notin})^2(y)$,

(CPI$_4$) $A^{\in^2}(0) \leq A^{\in^2}(x)$, $A^{\notin^2}(0) \geq A^{\notin^2}(x)$,

(CPI$_5$) $A^{\in^2}(x) \leq A^{\in^2}(y * x) \vee A^{\in^2}(y)$,

(CPI$_6$) $A^{\notin^2}(x) \geq A^{\notin^2}(y * x) \wedge A^{\notin^2}(y)$.

Definition 3.4. Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A} \in CPS(X)$. Then a mapping $\mathcal{A}$ is called a *cubic Pythagorean fuzzy KU-ideal* (briefly, CPKUI) of X, if it satisfies the following conditions: for any $x,\ y \in X$,

(CPKUI$_1$) $(\widetilde{A}^{\in})^2(0) \geq (\widetilde{A}^{\in})^2(x)$, $(\widetilde{A}^{\notin})^2(0) \leq (\widetilde{A}^{\notin})^2(x)$,

(CPKUI$_2$) $(\widetilde{A}^{\in})^2(z * x) \geq (\widetilde{A}^{\in})^2[z * (y * x)] \wedge (\widetilde{A}^{\in})^2(y)$,

(CPKUI$_3$) $(\widetilde{A}^{\notin})^2(z * x) \leq (\widetilde{A}^{\notin})^2[z * (y * x)] \vee (\widetilde{A}^{\notin})^2(y)$,

(CPKUI$_4$) $A^{\in^2}(0) \leq A^{\in^2}(x)$, $A^{\notin^2}(0) \geq A^{\notin^2}(x)$,

(CPKUI$_5$) $A^{\in^2}(z * x) \leq A^{\in^2}[z * (y * x)] \vee A^{\in^2}(y)$,

(CPKUI$_6$) $A^{\notin^2}(z * x) \geq A^{\notin^2}[z * (y * x)] \wedge A^{\notin^2}(y)$.

Example 3.5. Let $X = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $*$ defined by the following table:

Then clearly, $(X, *, 0)$ is a KU-algebra. Define $\mathcal{A} = \left\langle \widetilde{\overline{A}}, \overline{A} \right\rangle$ as follows: for each

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	3
2	0	0	0	1	4
3	0	0	0	0	3
4	0	0	0	0	0

Table 3.1

$x \in X$,

$$\widetilde{A}(x) = \begin{cases} ([0.2, 0.8], [0.1, 0.5]) & \text{if } x \in \{0, 1\} \\ ([0.1, 0.4], [0.4, 0.8]) & \text{otherwise,} \end{cases}$$

$\overline{A}(0) = (0.7, 0.2)$, $\overline{A}(1) = (0.6, 0.3)$, $\overline{A}(2) = (0.2, 0.5)$, $\overline{A}(3) = (0.2, 0.6)$, $\overline{A}(4) = (0.1, 0.4)$. Then we can easily check that $\mathcal{A}$ is a CPSA of X.

Lemma 3.6. *Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A} \in CPS(X)$. If $\mathcal{A}$ is a CPSA of X, then the followings hold: for each $x \in X$,*

$$(\widetilde{A}^{\in})^2(0) \geq (\widetilde{A}^{\in})^2(x),\ (\widetilde{A}^{\notin})^2(0) \leq (\widetilde{A}^{\notin})^2(x),\ {A^{\in}}^2(0) \leq {A^{\in}}^2(x),\ {A^{\notin}}^2(0) \geq {A^{\notin}}^2(x).$$

Proof. Let $x \in X$. Then from the axiom (KU$_5$) and Definition 3.2, we have

$$(\widetilde{A}^{\in})^2(x * x) = (\widetilde{A}^{\in})^2(0) \geq (\widetilde{A}^{\in})^2(x) \wedge (\widetilde{A}^{\in})^2(x) = (\widetilde{A}^{\in})^2(x),$$
$$(\widetilde{A}^{\notin})^2(x * x) = (\widetilde{A}^{\notin})^2(0) \leq (\widetilde{A}^{\notin})^2(x) \vee (\widetilde{A}^{\notin})^2(x) = (\widetilde{A}^{\in})^2(x),$$
$${A^{\in}}^2(x * x) = {A^{\in}}^2(0) \leq {A^{\in}}^2(x) \vee {A^{\in}}^2(x) = {A^{\in}}^2(x),$$
$${A^{\notin}}^2(x * x) = {A^{\notin}}^2(0) \geq {A^{\notin}}^2(x) \wedge {A^{\notin}}^2(x) = {A^{\notin}}^2(x).$$

□

Lemma 3.7. *Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A}$ be a CPKUI of X. If $x \leq y$ in X, then the followings hold: for each $x,\ y \in X$,*

$$(\widetilde{A}^{\in})^2(x) \geq (\widetilde{A}^{\in})^2(y),\ (\widetilde{A}^{\notin})^2(x) \leq (\widetilde{A}^{\notin})^2(y),\ {A^{\in}}^2(x) \leq {A^{\in}}^2(y),\ {A^{\notin}}^2(x) \geq {A^{\notin}}^2(y).$$

Proof. Let $x,\ y \in X$ such that $x \leq y$. Then clearly, $y * x = 0$. Thus we get

$$\begin{aligned}(\widetilde{A}^{\in})^2(x) &= (\widetilde{A}^{\in})^2(0 * x) \text{ [By the axiom (KU}_3\text{)]} \\ &\geq (\widetilde{A}^{\in})^2[0 * (y * x)] \wedge (\widetilde{A}^{\in})^2(y) \text{ [By the condition (CPKUI}_2\text{)]} \\ &= (\widetilde{A}^{\in})^2(y * x) \wedge (\widetilde{A}^{\in})^2(y) \text{ [By the axiom (KU}_3\text{)]} \\ &= (\widetilde{A}^{\in})^2(0) \wedge (\widetilde{A}^{\in})^2(y) \text{ [Since } y * x = 0\text{]} \\ &= (\widetilde{A}^{\in})^2(y). \text{ [By the condition (CPKUI}_1\text{)]}\end{aligned}$$

Similarly, from the condition (CPKUI$_3$) and the axiom (KU$_3$), we have

$$(\widetilde{A}^{\in})^2(x) \leq (\widetilde{A}^{\in})^2(y).$$

Also, from the conditions (CPKUI$_5$) and (CPKUI$_6$), and the axiom (KU$_3$), we prove that the followings are hold:

$${A^{\in}}^2(x) \leq {A^{\in}}^2(y),\ {A^{\notin}}^2(x) \geq {A^{\notin}}^2(y).$$

This completes the proof. □

Lemma 3.8. *Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A}$ be a CPKUI of X. If $x * y \leq z$ in X, then the followings hold: for each $x,\ y,\ z \in X$,*
(1) $(\widetilde{A}^{\in})^2(x) \geq (\widetilde{A}^{\in})^2(y) \wedge (\widetilde{A}^{\in})^2(z)$,
(2) $(\widetilde{A}^{\notin})^2(x) \leq (\widetilde{A}^{\notin})^2(y) \vee (\widetilde{A}^{\notin})^2(z)$,
(3) ${A^{\in}}^2(x) \leq {A^{\in}}^2(y) \vee {A^{\in}}^2(z)$,
(4) ${A^{\notin}}^2(x) \geq {A^{\notin}}^2(y) \wedge {A^{\notin}}^2(z)$.

Proof. (1) Let $x,\ y,\ z \in X$ such that $y * x \leq z$. Then we get

$$\begin{aligned}(\widetilde{A}^{\in})^2(x) &= (\widetilde{A}^{\in})^2(0 * x) \text{ [By the axiom (KU}_3\text{)]}\\ &\geq (\widetilde{A}^{\in})^2[0 * (y * x)] \wedge (\widetilde{A}^{\in})^2(y) \text{ [By the condition (CPKUI}_2\text{)]}\\ &= (\widetilde{A}^{\in})^2(y * x) \wedge (\widetilde{A}^{\in})^2(y) \text{ [By the axiom (KU}_3\text{)]}\\ &\geq (\widetilde{A}^{\in})^2(z) \wedge (\widetilde{A}^{\in})^2(y). \text{ [By Lemma 3.7]}\end{aligned}$$

(2) The proof is similar to (1) from the axiom (KU$_3$) and the condition (CPKUI$_3$).
(3) The proof follows from the axiom (KU$_3$) and the condition (CPKUI$_5$).
(4) The proof is similar to (1) from the axiom (KU$_3$) and the condition (CPKUI$_6$). □

Proposition 3.9. *Every CPKUI of a KU-algebra X is a CPSA of X.*

Proof. Let X be a KU-algebra, let $\mathcal{A}$ be a CPKUI of X and let $x,\ y \in X$. Then by Theorem 2.2 (2), $x * y \leq x$. Thus we have

$$\begin{aligned}(\widetilde{A}^{\in})^2(y * x) &\geq (\widetilde{A}^{\in})^2(x) \text{ [By Lemma 3.7]}\\ &= (\widetilde{A}^{\in})^2(0 * x) \text{ [By the the axiom (KU}_3\text{)]}\\ &\geq (\widetilde{A}^{\in})^2[0 * (y * x)] \wedge (\widetilde{A}^{\in})^2(y) \text{ [By the condition (CPKUI}_2\text{)]}\\ &= (\widetilde{A}^{\in})^2(y * x) \wedge (\widetilde{A}^{\in})^2(y) \text{ [By the the axiom (KU}_3\text{)]}\\ &\geq (\widetilde{A}^{\in})^2(x) \wedge (\widetilde{A}^{\in})^2(y). \text{ [By Lemma 3.7]}\end{aligned}$$

Similarly, we can show that the followings hold:

$$(\widetilde{A}^{\notin})^2(y * x) \leq (\widetilde{A}^{\notin})^2(x) \vee (\widetilde{A}^{\notin})^2(y),$$

$${A^{\in}}^2(y * x) \leq {A^{\in}}^2(x) \vee {A^{\in}}^2(y),\ {A^{\notin}}^2(y * x) \geq {A^{\notin}}^2(x) \wedge {A^{\notin}}^2(y).$$

So $\mathcal{A}$ is a CPKUI of X. □

Proposition 3.10. *Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A}$ be a CPSA of X such that the conditions (1)–(4) in Lemma 3.8 hold for each $x,\ y,\ z \in X$ with $y * x \leq z$. Then $\mathcal{A}$ is a CPKUI of X.*

Proof. Suppose $\mathcal{A}$ is a CPSA of X such that the conditions (1)–(4) in Lemma 3.8 hold for each $x,\ y,\ z \in X$ with $y * x \leq z$. Let $x \in X$. Then by Lemma 3.6, we get

$$(\widetilde{A}^{\in})^2(0) \geq (\widetilde{A}^{\in})^2(x),\ (\widetilde{A}^{\notin})^2(0) \leq (\widetilde{A}^{\notin})^2(x),\ {A^{\in}}^2(0) \leq {A^{\in}}^2(x),\ {A^{\notin}}^2(0) \geq {A^{\notin}}^2(x).$$

Thus the conditions (CPKUI$_1$) and (CPKUI$_4$) hold.

Let $x,\ y,\ z \in X$. Then by Theorem 2.2, $(y*z)*(x*z) \leq x*y$. Thus $(y*z)*(0*z) \leq 0 * y$, i.e.,

$$(y * z) * z \leq y. \tag{3.1}$$

On the other hand, we get

$$(\widetilde{A}^{\in})^2(x) \geq (\widetilde{A}^{\in})^2(y) \wedge (\widetilde{A}^{\in})^2(z) \text{ [By the hypothesis]}$$

$$\geq (\widetilde{A}^{\in})^2(y) \wedge (\widetilde{A}^{\in})^2(y * x). \text{ [Since } (\widetilde{A}^{\in})^2(z) \geq (\widetilde{A}^{\in})^2(y * x)]$$

So we have

$$(\widetilde{A}^{\in})^2(x) \geq (\widetilde{A}^{\in})^2(y) \wedge (\widetilde{A}^{\in})^2(y * x). \tag{3.2}$$

By substituting $z * x$ for x and $z * (y * x)$ for y in (3.2), we have

$$\begin{aligned}(\widetilde{A}^{\in})^2(z * x) &\geq (\widetilde{A}^{\in})^2[z * (y * x)] \wedge (\widetilde{A}^{\in})^2[(z * (y * x)) * (z * x)] \\ &\geq (\widetilde{A}^{\in})^2[(z * (y * x)] \wedge (\widetilde{A}^{\in})^2[(y * x) * x)] \text{ [By (3.1)]} \\ &\geq (\widetilde{A}^{\in})^2[(z * (y * x)] \wedge (\widetilde{A}^{\in})^2(y).\end{aligned}$$

So the condition (CPKUI_2) holds. Similarly, we show that the conditions (CPKUI_3), (CPKUI_5) and (CPKUI_6) hold. This completes the proof. □

Definition 3.11. Let $\overline{A}$ be a Pythagorean fuzzy set in a KU-algebra X. Then $\overline{A}$ is called a *Pythagorean anti fuzzy KU-ideal* (briefly, PAKUI) of X, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,

(PAKUI_1) $A^{\in^2}(z * x) \leq A^{\in^2}(z * (y * x)) \vee A^{\in^2}(y)$,

(PAKUI_2) $A^{\notin^2}(z * x) \geq A^{\notin^2}(z * (y * x)) \wedge A^{\notin^2}(y)$.

Definition 3.12. Let $\widetilde{\overline{A}}$ be an interval-valued Pythagorean fuzzy set in a KU-algebra X. Then $\widetilde{\overline{A}}$ is called an *interval-valued Pythagorean fuzzy KU-ideal* (briefly, IVPKUI) of X, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,

(IVPKUI_1) $(\widetilde{A}^{\in})^2(z * x) \geq (\widetilde{A}^{\in})^2(z * (y * x)) \vee (\widetilde{A}^{\in})^2(y)$,

(IVPKUI_2) $(\widetilde{A}^{\notin})^2(z * x) \leq (\widetilde{A}^{\notin})^2(z * (y * x)) \wedge (\widetilde{A}^{\notin})^2(y)$, where

$$\widetilde{0} \leq (\widetilde{A}^{\in})^2(x) + (\widetilde{A}^{\notin})^2(x) \leq \widetilde{1}.$$

Theorem 3.13. *Let $(X, *, 0)$ be a KU-algebra and let $\mathcal{A} = \left\langle \widetilde{\overline{A}}, \overline{A} \right\rangle \in CPS(X)$. Then $\mathcal{A}$ is a CPKUI of X if ad only if $\widetilde{\overline{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin})$ is an interval-valued Pythagorean fuzzy KU-ideal and $\overline{A} = (A^{\in}, A^{\notin})$ is a Pythagorean anti fuzzy KU-ideal of X.*

Proof. (⇐): Suppose $\widetilde{\overline{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin})$ is an interval-valued Pythagorean fuzzy KU-ideal and $\overline{A} = (A^{\in}, A^{\notin})$ is a Pythagorean anti fuzzy KU-ideal of X. Then clearly,

$$(\widetilde{A}^{\in})^2(0) \geq (\widetilde{A}^{\in})^2(x),\ (\widetilde{A}^{\notin})^2(0) \leq (\widetilde{A}^{\notin})^2(x),\ A^{\in^2}(0) \leq A^{\in^2}(x),\ A^{\notin^2}(0) \geq A^{\notin^2}(x)$$

for each $x \in X$.

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned}(\widetilde{A}^{\in})^2(z * x) &= [A^{\in,-^2}(z * x), A^{\in,+^2}(z * x)] \\ &\geq [A^{\in,-^2}[z * (y * x)] \wedge A^{\in,-^2}(y), A^{\in,+^2}[z * (y * x)] \wedge A^{\in,+^2}(y)] \\ &\quad \text{[Since } \widetilde{\overline{A}} \text{ is an interval-valued Pythagorean fuzzy } KU\text{-ideal]} \\ &= [A^{\in,-^2}[z * (y * x)], A^{\in,+^2}[z * (y * x)]] \wedge [A^{\in,-^2}(y) \wedge A^{\in,+^2}(y)] \\ &= (\widetilde{A}^{\in})^2[z * (y * x)] \wedge (\widetilde{A}^{\in})^2(y).\end{aligned}$$

Similarly, we have $(\widetilde{A}^{\notin})^2(z*x) \leq (\widetilde{A}^{\notin})^2[z*(y*x)] \vee (\widetilde{A}^{\notin})^2(y)$. Since $\overline{A}$ is a Pythagorea anti fuzzy KU-ideal of X, it is clear that

$$A^{\in^2}(z * x) \leq A^{\in^2}[z * (y * x)] \vee A^{\in^2}(y),\ A^{\notin^2}(z * x) \geq A^{\notin^2}[z * (y * x)] \wedge A^{\in^2}(y).$$

Thus $\mathcal{A}$ is a CPKUI of X.

($\Rightarrow$): Suppose $\mathcal{A}$ is a CPKUI of X and let $x \in X$. Then by (CPKUI$_1$) and (CPKUI$_4$), we have

$$(\widetilde{A}^{\in})^2(0) \geq (\widetilde{A}^{\in})^2(x),\ (\widetilde{A}^{\notin})^2(0) \leq (\widetilde{A}^{\notin})^2(x),\ A^{\in 2}(0) \leq A^{\in 2}(x),\ A^{\notin 2}(0) \geq A^{\notin 2}(x).$$

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned}
&[A^{\in,-2}(z*x), A^{\in,+2}(z*x)] \\
&= (\widetilde{A}^{\in})^2(z*x) \\
&\geq (\widetilde{A}^{\in})^2[z*(y*x)] \wedge (\widetilde{A}^{\in})^2(y) \\
&= [A^{\in,-2}[z*(y*x)], A^{\in,+2}[z*(y*x)] \wedge [A^{\in,-2}(y), A^{\in,+2}(y)] \\
&= [A^{\in,-2}[z*(y*x)] \wedge A^{\in,-2}(y), A^{\in,+2}[z*(y*x)] \wedge A^{\in,+2}(y)].
\end{aligned}$$

Thus we have

$$A^{\in,-2}(z*x) \geq A^{\in,-2}[z*(y*x)] \wedge A^{\in,-2}(y),\ A^{\in,+2}(z*x) \geq A^{\in,+2}[z*(y*x)] \wedge A^{\in,+2}(y).$$

Similarly, we can show that

$$A^{\notin,-2}(z*x) \leq A^{\notin,-2}[z*(y*x)] \vee A^{\notin,-2}(y),\ A^{\notin,+2}(z*x) \leq A^{\notin,+2}[z*(y*x)] \vee A^{\notin,+2}(y)$$

and

$$A^{\in 2}(z*x) \leq A^{\in 2}[z*(y*x)] \vee A^{\in 2}(y),\ A^{\notin 2}(z*x) \geq A^{\notin,+2}[z*(y*x)] \wedge A^{\notin,+2}(y).$$

So $\widetilde{\overline{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin})$ is an interval-valued Pythagorean KU-ideal and $\overline{A} = (A^{\in}, A^{\notin})$ is a Pythagorean anti fuzzy KU-ideal of X. This completes the proof. □

Proposition 3.14. *Let $(X, *, 0)$ be a KU-algebra and let $(\mathcal{A}_j)_{j\in J}$ be a family of CPKLUIs of X. Then $\bigcap_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \widetilde{\overline{A}}_j, \bigcap_{j\in J} \overline{A}_j \right\rangle$ is a CPKUI of X.*

Proof. Let $(\mathcal{A}_j)_{j\in J}$ be a family of CPKLUIs of X and let $x,\ y,\ z \in X$. Note that

$$\bigcap_{j\in J} \mathcal{A}_j = \left\langle (\bigcap_{j\in J} (\widetilde{A}_j^{\in})^2, \bigcup_{j\in J} (\widetilde{A}_j^{\notin})^2), (\bigcup_{j\in J} (A_j^{\in 2}, \bigcap_{j\in J} A_j^{\notin 2}) \right\rangle.$$

Then we have

$$\begin{aligned}
(\textstyle\bigcap_{j\in J} \widetilde{A}_j^{\in})^2)(0) &= \textstyle\bigwedge_{j\in J} (\widetilde{A}_j^{\in})^2(0) \\
&\geq \textstyle\bigwedge_{j\in J} (\widetilde{A}_j^{\in})^2(x) \text{ [By (CPKUI}_1\text{)]} \\
&= (\textstyle\bigcap_{j\in J} \widetilde{A}_j^{\in})^2)(x),
\end{aligned}$$

$$\begin{aligned}
(\textstyle\bigcup_{j\in J} \widetilde{A}_j^{\notin})^2)(0) &= \textstyle\bigvee_{j\in J} (\widetilde{A}_j^{\notin})^2(0) \\
&\leq \textstyle\bigvee_{j\in J} (\widetilde{A}_j^{\notin})^2(x) \\
&= (\textstyle\bigcap_{j\in J} \widetilde{A}_j^{\in})^2)(x),
\end{aligned}$$

$$\begin{aligned}
(\textstyle\bigcap_{j\in J} \widetilde{A}_j^{\in})^2)(z*x) &= \textstyle\bigwedge_{j\in J} (\widetilde{A}_j^{\in})^2(z*x) \\
&\geq \textstyle\bigwedge_{j\in J} [(\widetilde{A}_j^{\in})^2[z*(y*x)] \wedge (\widetilde{A}_j^{\in})^y] \text{ [By (CPKUI}_2\text{)]} \\
&= \textstyle\bigwedge_{j\in J} (\widetilde{A}_j^{\in})^2[z*(y*x)] \wedge \textstyle\bigwedge_{j\in J} (\widetilde{A}_j^{\in})^2(y) \\
&= (\textstyle\bigcap_{j\in J} (\widetilde{A}_j^{\in})^2)[z*(y*x)] \wedge (\textstyle\bigcap_{j\in J} (\widetilde{A}_j^{\in})^2)(y),
\end{aligned}$$

$$(\textstyle\bigcup_{j\in J} \widetilde{A}_j^{\notin})^2)(z*x) = \textstyle\bigvee_{j\in J} (\widetilde{A}_j^{\notin})^2(z*x)$$

$$\begin{aligned}&\leq \bigvee_{j\in J}[(\widetilde{A}_j^{\not\in})^2[z*(y*x)] \vee (\widetilde{A}_j^{\not\in})^y] \text{ [By (CPKUI}_3)]\\ &= \bigvee_{j\in J}(\widetilde{A}_j^{\not\in})^2[z*(y*x)] \vee \bigwedge_{j\in J}(\widetilde{A}_j^{\not\in})^2(y)\\ &= (\bigcup_{j\in J}(\widetilde{A}_j^{\not\in})^2)[z*(y*x)] \vee (\bigcup_{j\in J}(\widetilde{A}_j^{\not\in})^2)(y).\end{aligned}$$

Thus $\bigcap_{j\in J}\mathcal{A}_j$ satisfies the axioms ($CPKUI_1$), ($CPKUI_2$) and ($CPKUI_3$).

Also, we get

$$\begin{aligned}(\bigcup_{j\in J} A_j^{\in^2}(0) &= \bigvee_{j\in J} A_j^{\in^2}(0)\\ &\leq \bigvee_{j\in J} A_j^{\in^2}(x) \text{ [By (CPKUI}_4)]\\ &= (\bigcup_{j\in J} A_j^{\in^2})(x),\end{aligned}$$

$$\begin{aligned}(\bigcap_{j\in J} A_j^{\not\in^2}(0) &= \bigwedge_{j\in J} A_j^{\not\in^2}(0)\\ &\geq \bigwedge_{j\in J} A_j^{\not\in^2}(x)\\ &= (\bigcap_{j\in J} A_j^{\not\in^2})(x),\end{aligned}$$

$$\begin{aligned}(\bigcup_{j\in J} A_j^{\in^2})(z*x) &= \bigvee_{j\in J} A_j^{\in^2}(z*x)\\ &\leq \bigvee_{j\in J}[A_j^{\in^2}[z*(y*x)] \vee A_j^{\in^y}] \text{ [By (CPKUI}_5)]\\ &= (\bigvee_{j\in J} A_j^{\in^2}[z*(y*x)]) \vee \bigvee_{j\in J} A_j^{\in^2}(y))\\ &= \bigcup_{j\in J} A_j^{\in^2}[z*(y*x)] \vee \bigcup_{j\in J} A_j^{\in^2})(y),\end{aligned}$$

$$\begin{aligned}(\bigcap_{j\in J} A_j^{\not\in^2})(z*x) &= \bigwedge_{j\in J} A_j^{\not\in^2}(z*x)\\ &\geq \bigwedge_{j\in J}[A_j^{\not\in^2}[z*(y*x)] \wedge A_j^{\not\in^y}] \text{ [By (CPKUI}_6)]\\ &= \bigwedge_{j\in J} A_j^{\not\in^2}[z*(y*x)] \wedge \bigwedge_{j\in J} A_j^{\not\in^2}(y)\\ &= (\bigcap_{j\in J} A_j^{\not\in^2})[z*(y*x)] \wedge (\bigcup_{j\in J} A_j^{\not\in^2})(y).\end{aligned}$$

So $\bigcap_{j\in J}\mathcal{A}_j$ satisfies the axioms ($CPKUI_4$), ($CPKUI_5$) and ($CPKUI_6$). Hence $\bigcap_{j\in J}\mathcal{A}_j$ is a CPKUI of X. □

4. The pre-images of cubic Pythagorean fuzzy KU-ideals

Definition 4.1. Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be KU-algebras. Then a mapping $f : X \to Y$ is called a *homomorphism*, if $f(x*y) = f(x) *^{'} f(y)$ for each $x \in X$.

Definition 4.2. Let X, Y be two nonempty sets, $f : X \to Y$ be a mapping and let $\mathcal{B} \in CPS(Y)$. Then the *pre-image of* $\mathcal{B}$ *under* f, denoted by $f^{-1}(\mathcal{B})$, is a cubic Pythagorean fuzzy set in X defined as follows: for each $x \in X$.

$$f^{-1}(\mathcal{B})(x) = \mathcal{B}(f(x)) = \left\langle \widetilde{\overline{B}}(f(x)), \overline{B}(f(x)) \right\rangle.$$

In fact, $f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\widetilde{\overline{B}}), f^{-1}(\overline{B}) \right\rangle$.

Proposition 4.3. *Let $f : X \to Y$ be a homomorphism of KU-algebras. If $\mathcal{B}$ is a CPKUI of X, then $f^{-1}(\mathcal{B})$ is a CPKUI of X.*

Proof. Let $x,\ y,\ z \in X$. Then we have

$$\begin{aligned}(f^{-1}(\widetilde{B}^{\in}))^2(0) &= (\widetilde{B}^{\in})^2(0) \text{ [By Definition 4.1]}\\ &= (\widetilde{B}^{\in})^2(f(0)) \text{ [Since } f \text{ is a homomorphism]}\end{aligned}$$

$$\begin{aligned}
&\geq (\widetilde{B}^{\in})^2(f(x)) \text{ [By the hypothesis]}\\
&= (f^{-1}(\widetilde{B}^{\in}))^2(x),
\end{aligned}$$

$$\begin{aligned}
(f^{-1}(\widetilde{B}^{\notin}))^2(0) &= (\widetilde{B}^{\notin})^2(0)\\
&= (\widetilde{B}^{\in})^2(f(0))\\
&\geq (\widetilde{B}^{\in})^2(f(x))\\
&= (f^{-1}(\widetilde{B}^{\in}))^2(x),
\end{aligned}$$

$$\begin{aligned}
(f^{-1}(\widetilde{B}^{\in}))^2(z*x) &= (\widetilde{B}^{\in})^2(f(z*x))\\
&= (\widetilde{B}^{\in})^2(f(z)*f(x))\\
&\geq (\widetilde{B}^{\in})^2[f(z)*(f(y)*f(x))] \wedge (\widetilde{B}^{\in})^2(f(y))\\
&\quad \text{[By (CPKUI}_2\text{)]}\\
&= (\widetilde{B}^{\in})^2[f(z*(y*x))] \wedge (\widetilde{B}^{\in})^2(f(y))\\
&= (f^{-1}(\widetilde{B}^{\in}))^2(z*(y*x)) \wedge (f^{-1}(\widetilde{B}^{\in}))^2(y),
\end{aligned}$$

$$\begin{aligned}
(f^{-1}(\widetilde{B}^{\notin}))^2(z*x) &= (\widetilde{B}^{\notin})^2(f(z*x))\\
&= (\widetilde{B}^{\notin})^2(f(z)*f(x))\\
&\leq (\widetilde{B}^{\notin})^2[f(z)*(f(y)*f(x))] \vee (\widetilde{B}^{\notin})^2(f(y))\\
&\quad \text{[By CPKUI}_3\text{]}\\
&= (\widetilde{B}^{\notin})^2[f(z*(y*x))] \vee (\widetilde{B}^{\notin})^2(f(y))\\
&= (f^{-1}(\widetilde{B}^{\notin}))^2(z*(y*x)) \vee (f^{-1}(\widetilde{B}^{\notin}))^2(y).
\end{aligned}$$

Thus $f^{-1}(\mathcal{B})$ satisfies the axioms (CPKUI_1), (CPKUI_2) and (CPKUI_3).

Also, from Definition 4.1, the hypothesis, axioms (CPKUI_4), (CPKUI_5) and (CPKUI_6), we get

$$(f^{-1}(B^{\in}))^2(0) \leq (f^{-1}(B^{\in}))^2(x),\ (f^{-1}(B^{\notin}))^2(0) \geq (f^{-1}(B^{\notin}))^2(x),$$
$$(f^{-1}(B^{\in}))^2(z*x) \leq (f^{-1}(B^{\in}))^2(z*(y*x)) \vee (f^{-1}(B^{\in}))^2(y),$$
$$(f^{-1}(B^{\notin}))^2(z*x) \geq (f^{-1}(B^{\notin}))^2(z*(y*x)) \wedge (f^{-1}(B^{\notin}))^2(y).$$

So $f^{-1}(\mathcal{B})$ satisfies the axioms (CPKUI_4), (CPKUI_5) and (CPKUI_6). Hence $f^{-1}(\mathcal{B})$ is a CPKUI of X. □

Proposition 4.4. *Let $f : X \to Y$ be a homomorphism of KU-algebras and let $\mathcal{B} \in CPS(Y)$. If $f^{-1}(\mathcal{B})$ is a CPKUI of X, then $\mathcal{B}$ is a CPKUI of Y.*

Proof. Let $a,\ b,\ c \in Y$. Since f is surjective, there are $x,\ y,\ z \in X$ such that

$$a = f(x),\ b = f(y),\ c = f(z).$$

Then we get

$$\begin{aligned}
(\widetilde{B}^{\in})^2(a) &= (\widetilde{B}^{\in})^2(f(x))\\
&= (f^{-1}(\widetilde{B}^{\in}))^2(x) \text{ [By Definition 4.1]}\\
&\leq (f^{-1}(\widetilde{B}^{\in}))^2(0) \text{ [By (CPKUI}_1\text{)]}\\
&= (\widetilde{B}^{\in})^2(f(0))\\
&= (\widetilde{B}^{\in})^2(0), \text{ [Since } f \text{ is a homomorphism]}
\end{aligned}$$

$$(\widetilde{B}^{\notin})^2(a) = (\widetilde{B}^{\notin})^2(f(x))$$

$$
\begin{aligned}
&= (f^{-1}(\widetilde{B}^{\notin}))^2(x)\\
&\geq (f^{-1}(\widetilde{B}^{\notin}))^2(0)\\
&= (\widetilde{B}^{\notin})^2(f(0))\\
&= (\widetilde{B}^{\notin})^2(0),
\end{aligned}
$$

$$
\begin{aligned}
B^{\in 2}(a) &= B^{\in 2}(f(x))\\
&= (f^{-1}(B^{\in}))^2(x) \text{ [By Definition 4.1]}\\
&\geq (f^{-1}(B^{\in}))^2(0) \text{ [By (CPKUI}_4\text{)]}\\
&= B^{\in 2}(f(0))\\
&= B^{\in 2}(0), \text{ [Since } f \text{ is a homomorphism]}
\end{aligned}
$$

Thus $\mathcal{B}$ satisfies the axioms (CPKUI$_1$) and (CPKUI$_4$). On the other hand, we have

$$
\begin{aligned}
(\widetilde{B}^{\in})^2(c*a) &= (\widetilde{B}^{\in})^2(f(z*x))\\
&= (f^{-1}(\widetilde{B}^{\in}))^2(z*x) \text{ [By Definition 4.1]}\\
&\geq (f^{-1}(\widetilde{B}^{\in}))^2(z*(y*x)) \wedge (f^{-1}(\widetilde{B}^{\in}))^2(y) \text{ [By (CPKUI}_2\text{)]}\\
&= (\widetilde{B}^{\in})^2(f(z*(y*x))) \wedge (\widetilde{B}^{\in})^2(f(y))\\
&= (\widetilde{B}^{\in})^2(f(z)*(f(y)*f(x))) \wedge (\widetilde{B}^{\in})^2(f(y))\\
&\quad \text{[Since } f \text{ is a homomorphism]}\\
&= (\widetilde{B}^{\in})^2(c*(b*a)) \wedge (\widetilde{B}^{\in})^2(b),
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{B}^{\notin})^2(c*a) &= (\widetilde{B}^{\notin})^2(f(z*x))\\
&= (f^{-1}(\widetilde{B}^{\notin}))^2(z*x)\\
&\leq (f^{-1}(\widetilde{B}^{\notin}))^2(z*(y*x)) \vee (f^{-1}(\widetilde{B}^{\notin}))^2(y) \text{ [By (CPKUI}_3\text{)]}\\
&= (\widetilde{B}^{\notin})^2(f(z*(y*x))) \vee (\widetilde{B}^{\notin})^2(f(y))\\
&= (\widetilde{B}^{\notin})^2(f(z)*(f(y)*f(x))) \vee (\widetilde{B}^{\in})^2(f(y))\\
&= (\widetilde{B}^{\notin})^2(c*(b*a)) \vee (\widetilde{B}^{\notin})^2(b),
\end{aligned}
$$

$$
\begin{aligned}
B^{\in 2}(c*a) &= B^{\in 2}(f(z*x))\\
&= (f^{-1}(B^{\in}))^2(z*x)\\
&\leq (f^{-1}(B^{\in}))^2(z*(y*x)) \vee (f^{-1}(B^{\in}))^2(y) \text{ [By (CPKUI}_5\text{)]}\\
&= B^{\in 2}(f(z*(y*x))) \vee B^{\in 2}(f(y))\\
&= B^{\in 2}(f(z)*(f(y)*f(x))) \vee B^{\in 2}(f(y))\\
&= B^{\in 2}(c*(b*a)) \vee B^{\in 2}(b),
\end{aligned}
$$

$$
\begin{aligned}
B^{\notin 2}(c*a) &= B^{\notin 2}(f(z*x))\\
&= (f^{-1}(B^{\notin}))^2(z*x)\\
&\geq (f^{-1}(B^{\notin}))^2(z*(y*x)) \wedge (f^{-1}(B^{\notin}))^2(y) \text{ [By (CPKUI}_6\text{)]}\\
&= B^{\notin 2}(f(z*(y*x))) \wedge B^{\notin 2}(f(y))\\
&= B^{\notin 2}(f(z)*(f(y)*f(x))) \wedge B^{\notin 2}(f(y))\\
&= B^{\notin 2}(c*(b*a)) \wedge B^{\notin 2}(b).
\end{aligned}
$$

So $\mathcal{B}$ satisfies the axioms (CPKUI$_2$), (CPKUI$_3$), (CPKUI$_5$) and (CPKUI$_6$). Hence $\mathcal{B}$ is a CPKUI of Y. □

5. The Cartesian product of cubic Pythagorean fuzzy KU-ideals

Definition 5.1. Let X be a nonempty set and let $\mathcal{A},\ \mathcal{B} \in CPS(X)$. Then the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times \mathcal{B}$, is a CPS in $X \times X$ defined as follows:for each $(x, y) \in X \times X$,

$$(\mathcal{A} \times \mathcal{B})(x, y) = \left\langle (\widetilde{\overline{A}} \times \widetilde{\overline{B}})(x, y), (\overline{A} \times \overline{B})(x, y) \right\rangle = \left\langle \widetilde{\overline{A}}(x) \wedge \widetilde{\overline{B}}(y), \overline{A}(x) \vee \overline{B}(y) \right\rangle,$$

where
$$\begin{aligned}\widetilde{\overline{A}}(x) \wedge \widetilde{\overline{B}}(y) &= ([A^{\in,-}(x) \wedge B^{\in,-}(x), [A^{\in,+}(x) \wedge B^{\in,+}(x)],\\ &\quad [A^{\notin,-}(x) \vee B^{\notin,-}(x), [A^{\notin,+}(x) \vee B^{\notin,+}(x)]),\\ \overline{A}(x) \vee \overline{B}(y) &= (A^{\in}(x) \vee B^{\in}(x), A^{\notin}(x) \wedge B^{\notin}(x)).\end{aligned}$$

It is obvious that if $(X, *, 0)$ is a KU-algebra, then $(X \times X, *, (0, 0))$ is a KU-algebra.

Proposition 5.2. *Let $(X, *, 0)$ be a KU-algebra. If $\mathcal{A}$ and $\mathcal{B}$ are CPKUIs of X, then $\mathcal{A} \times \mathcal{B}$ is a CPKUI of $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Then we have

$$\begin{aligned}(\widetilde{A}^{\in} \times \widetilde{B}^{\in})^2(0, 0) &= (\widetilde{A}^{\in})^2(0) \wedge (\widetilde{B}^{\in})^2(0)\\ &\geq (\widetilde{A}^{\in})^2(x) \wedge (\widetilde{B}^{\in})^2(y) \text{ [By (CPKUI}_1\text{)]}\\ &= (\widetilde{A}^{\in} \times \widetilde{B}^{\in})^2(x, y),\end{aligned}$$

$$\begin{aligned}(A^{\in} \times B^{\in})^2(0, 0) &= A^{\in 2}(0) \vee B^{\in 2}(0)\\ &\leq A^{\in 2}(x) \vee B^{\in 2}(y) \text{ [By (CPKUI}_4\text{)]}\\ &= (A^{\in} \times B^{\in})^2(x, y).\end{aligned}$$

Similarly, we can show that

$(\widetilde{A}^{\notin} \times \widetilde{B}^{\notin})^2(0, 0) \leq (\widetilde{A}^{\notin} \times \widetilde{B}^{\notin})^2(x, y)$ and $(A^{\notin} \times B^{\notin})^2(0, 0) \geq (A^{\notin} \times B^{\notin})^2(x, y)$.

Thus $\mathcal{A} \times \mathcal{B}$ satisfies the axioms (CPKUI_1) and (CPKUI_4).

Now let $(x_1, x_2),\ (y_1, y_2),\ (z_1, z_2) \in X \times X$. Then we get

$$\begin{aligned}&(\widetilde{A}^{\in} \times \widetilde{B}^{\in})^2((z_1, z_2) * (x_1, x_2))\\ &= (\widetilde{A}^{\in} \times \widetilde{B}^{\in})^2(z_1 * x_1, z_2 * x_2)\\ &= (\widetilde{A}^{\in})^2(z_1 * x_1) \wedge (\widetilde{B}^{\in})^2(z_2 * x_2)\\ &\geq [(\widetilde{A}^{\in})^2(z_1 * (y_1 * x_1)) \wedge (\widetilde{A}^{\in})^2(y_1)] \wedge [(\widetilde{B}^{\in})^2(z_2 * (y_2 * x_2)) \wedge (\widetilde{B}^{\in})^2(y_2)]\\ &\quad \text{[By (CPKUI}_2\text{)]}\\ &= [(\widetilde{A}^{\in})^2(z_1 * (y_1 * x_1)) \wedge (\widetilde{B}^{\in})^2(z_2 * (y_2 * x_2))] \wedge [(\widetilde{A}^{\in})^2(y_1) \wedge (\widetilde{B}^{\in})^2(y_2)]\\ &= (\widetilde{A}^{\in} \times \widetilde{B}^{\in})^2[(z_1, z_2) * ((y_1, y_2) * (x_1, x_2))] \wedge (\widetilde{A}^{\in} \times \widetilde{B}^{\in})^2(y_1, y_2),\end{aligned}$$

$$\begin{aligned}&(A^{\in} \times B^{\in})^2((z_1, z_2) * (x_1, x_2))\\ &= (A^{\in} \times B^{\in})^2(z_1 * x_1, z_2 * x_2)\\ &= A^{\in 2}(z_1 * x_1) \vee B^{\in 2}(z_2 * x_2)\\ &\leq [A^{\in 2}(z_1 * (y_1 * x_1)) \vee A^{\in 2}(y_1)] \vee [B^{\in 2}(z_2 * (y_2 * x_2)) \vee B^{\in 2}(y_2)]\\ &\quad \text{[By (CPKUI}_5\text{)]}\\ &= [A^{\in 2}(z_1 * (y_1 * x_1)) \vee B^{\in 2}(z_2 * (y_2 * x_2))] \vee [A^{\in 2}(y_1) \vee B^{\in 2}(y_2)]\\ &= (A^{\in} \times B^{\in})^2[(z_1, z_2) * ((y_1, y_2) * (x_1, x_2))] \vee (A^{\in} \times B^{\in})^2(y_1, y_2).\end{aligned}$$

$\mathcal{A} \times \mathcal{B}$ satisfies the axioms (CPKUI_2) and (CPKUI_5). Similarly, we can prove that

$\mathcal{A} \times \mathcal{B}$ satisfies the axioms (CPKUI$_3$) and (CPKUI$_6$). So $\mathcal{A} \times \mathcal{B}$ is a CPKUI of $X \times X$. □

6. Correlation coefficients of cubic Pythagorean fuzzy sets

A correlation plays an important role in statistics, engineering sciences and so on (See [12, 27, 28]. In this section, we propose some correlation coefficients for any two cubic Pythagorean fuzzy sets.

Let $\mathcal{A} = \left\langle \widetilde{\overline{A}}, \overline{A} \right\rangle$, $\mathcal{B} = \left\langle \widetilde{\overline{B}}, \overline{B} \right\rangle$ be two CPSs in the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$. Then

(i) the *informational cubic Pythagorean fuzzy energy* of $\mathcal{A}$ in X, denoted by $E(\mathcal{A})$, is defined by: for each $x_i \in X$,

$$E(\mathcal{A}) = \frac{1}{2}\Sigma_{i=1}^{n}[(\widetilde{\overline{A}}^2(x_i))^2 + (\overline{A}^2(x_i))^2], \tag{6.1}$$

where $(\widetilde{\overline{A}}^2(x_i))^2 = (A^{\in,-}(x_i))^4 + (A^{\in,+}(x_i))^4 + (A^{\notin,-}(x_i))^4 + (A^{\notin,+}(x_i))^4$,
$(\overline{A}^2(x_i))^2 = (A^{\in}(x_i))^4 + (A^{\notin}(x_i))^4$.

(ii) the *correlation* between $\mathcal{A}$ and $\mathcal{B}$, denoted by $C(\mathcal{A}, \mathcal{B})$ is defined by: for each $x_i \in X$,

$$C(\mathcal{A}, \mathcal{B}) = \frac{1}{2}\Sigma_{i=1}^{n}[\widetilde{\overline{A}}^2(x_i) \cdot \widetilde{\overline{B}}^2(x_i) + \overline{A}^2(x_i) \cdot \overline{B}^2(x_i)], \tag{6.2}$$

where
$$\begin{aligned}\widetilde{\overline{A}}^2(x_i) \cdot \widetilde{\overline{B}}^2(x_i) &= (A^{\in,-}(x_i))^4(B^{\in,-}(x_i))^4 + (A^{\in,+}(x_i))^4(B^{\in,+}(x_i))^4 \\ &\quad + (A^{\notin,-}(x_i))^4(B^{\notin,-}(x_i))^4 + (A^{\notin,+}(x_i))^4(B^{\notin,+}(x_i))^4, \\ \overline{A}^2(x_i) \cdot \overline{B}^2(x_i) &= (A^{\in}(x_i))^4(B^{\in}(x_i))^4 + (A^{\notin}(x_i))^4(B^{\notin}(x_i))^4.\end{aligned}$$

Remark 6.1. From (6.1) and (6.2), it is obvious that
(1) $C(\mathcal{A}, \mathcal{A}) = E(\mathcal{A})$,
(2) $C(\mathcal{A}, \mathcal{B}) = C(\mathcal{B}, \mathcal{A})$.

Now by using idea of Xu [31], we can suggest new alternative form of the correlation coefficient between two cubic Pythgorean fuzzy sets $\mathcal{A}$ and $\mathcal{A}$ as follows.

Definition 6.2. Let $\mathcal{A}$, $\mathcal{B} \in CPS(X)$. Then the *correlation coefficient* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\kappa(\mathcal{A}, \mathcal{B})$, is defined as follows:

$$\kappa(\mathcal{A}, \mathcal{B}) = \frac{C(\mathcal{A}, \mathcal{B})}{\sqrt{E(\mathcal{A}) \cdot E(\mathcal{B})}}. \tag{6.3}$$

Proposition 6.3. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse. Then the followings hold: for any $\mathcal{A}$, $\mathcal{B} \in CPS(X)$,*
(1) $\kappa(\mathcal{A}, \mathcal{B}) = \kappa(\mathcal{B}, \mathcal{A})$,
(2) $0 \leq \kappa(\mathcal{A}, \mathcal{B}) \leq 1$,
(3) if $\mathcal{A} = \mathcal{B}$, then $\kappa(\mathcal{A}, \mathcal{B}) = 1$.

Proof. (1) The proof is straightforward.

(2) It is clear that $\kappa(\mathcal{A}, \mathcal{B}) \geq 0$. It is sufficient to prove that $\kappa(\mathcal{A}, \mathcal{B}) \leq 1$. Consider the Cauchy-Schwarz inequality: for any $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n) \in \mathbb{R}$,

$$(\Sigma_{i=1}^{n} x_i y_i)^2 \leq (\Sigma_{i=1}^{n} x_i)(\Sigma_{i=1}^{n} y_i). \tag{6.4}$$

Then we have

$$\begin{aligned}(C(\mathcal{A},\mathcal{B}))^2 &= \tfrac{1}{4}(\Sigma_{i=1}^n[\widetilde{\overline{A}}^2(x_i)\cdot\widetilde{\overline{B}}^2(x_i)+\overline{A}^2(x_i)\cdot\overline{B}^2(x_i)])^2 \text{ [By (6.2)]}\\ &\leq \tfrac{1}{4}(\Sigma_{i=1}^n[\widetilde{\overline{A}}^2(x_i)+\overline{A}^2(x_i)])\cdot(\Sigma_{i=1}^n[\widetilde{\overline{B}}^2(x_i)+\overline{B}^2(x_i)]) \text{ [By (6.4)]}\\ &= \tfrac{1}{2}(\Sigma_{i=1}^n[\widetilde{\overline{A}}^2(x_i)+\overline{A}^2(x_i)])\cdot\tfrac{1}{2}(\Sigma_{i=1}^n[\widetilde{\overline{B}}^2(x_i)+\overline{B}^2(x_i)])\\ &= E(\mathcal{A})\cdot E(\mathcal{B}).\text{ [By (6.1)]}\end{aligned}$$

Thus by (6.3), $\kappa(\mathcal{A},\mathcal{B})\leq 1$. So $0\leq\kappa(\mathcal{A},\mathcal{B})\leq 1$.

(3) The proof is easy. □

Definition 6.4. Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe of discourse and let $\mathcal{A},\ \mathcal{B}\in CPS(X)$. Then $\mathcal{A}$ and $\mathcal{B}$ are said to be:

(i) $\hat{\text{i}}$ndependent, if $\kappa(\mathcal{A},\mathcal{B})<\frac{1}{2}$,

(ii) $\hat{\text{d}}$ependent, if $\kappa(\mathcal{A},\mathcal{B})>\frac{1}{2}$,

(iii) $\hat{\text{n}}$eutral, if $\kappa(\mathcal{A},\mathcal{B})=\frac{1}{2}$.

Example 6.5. Let $X=\{0,1,2,3\}$ be a set with a binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Table 6.1

Then clearly, $(X,*,0)$ is a KU-algebra. Define $\mathcal{A}$ and $\mathcal{B}$ as follows: for each $x\in X$,

$$\widetilde{\overline{A}}(x)=\begin{cases}([0.3,0.7],[0.2,0.5]) & \text{if } x\in\{0,1\}\\ ([0.2,0.3],[0.4,0.5]) & \text{otherwise,}\end{cases}$$

$$\overline{A}(0)=(0.6,0.1),\ \overline{A}(1)=(0.5,0.2),\ \overline{A}(2)=(0.3,0.3),\ \overline{A}(3)=(0.2,0.4),$$

$$\widetilde{\overline{B}}(x)=\begin{cases}([0.4,0.6],[0.3,0.6]) & \text{if } x\in\{0,1\}\\ ([0.3,0.4],[0.5,0.7]) & \text{otherwise,}\end{cases}$$

$$\overline{B}(0)=(0.7,0.3),\ \overline{B}(1)=(0.6,0.4),\ \overline{B}(2)=(0.4,0.5),\ \overline{B}(3)=(0.3,0.5).$$

Then we can easily check that $\mathcal{A},\ \mathcal{B}\in CPS(X)$ and we can calculate the followings:

$$E(\mathcal{A})=0.6473,\ E(\mathcal{B})=1.1913\ C(\mathcal{A},\mathcal{B})=0.0978.$$

Thus $\kappa(\mathcal{A},\mathcal{B})=0.1114<\frac{1}{2}$. So $\mathcal{A}$ and $\mathcal{B}$ are independent.

7. Conclusions

We have studied the cubic Pythagorean of KU-ideal in KU-algebras. Also we discussed a few results of cubic Pythagorean of KU-ideal in KU-algebras. The pre-image of cubic Pythagorean of KU-ideal in KU-algebras under homomorphism are defined and how the pre-image of cubic Pythagorean of KU-ideal in KU-algebras become cubic Pythagorean of KU-ideal are studied. Moreover, the product of cubic Pythagorean of KU-ideal is established. This part is sufficient to study the topological space for cubic Pythagorean KU-ideals of a KU-algebra. Finally, we have studied a new novel correlation coefficient between two cubic Pythagorean fuzzy sets. More applications are needed to be discussed in our further study

References

[1] K. Iséki, On BCI-algebras, Mathematics Seminar Notes 8 (1) 1980 125–130.

[2] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Mathematica Japonica 23 (1) (1978) 1–26.

[3] K. Iséki and S. Tanaka, Ideal theory of BCK-algebras, Mathematica Japonica 21 (4) (1976) 351–366.

[4] C. Prabpayak and U. Leerawat, On ideals and congruence in KU-algebras, Scientia Magna international book series 5 (1) (2009) 54–57.

[5] C. Prabpayak and U. Leerawat, On isomorphisms of KU-algebras, Scientia Magna international book series 5 (3) (2009) 25–31.

[6] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[7] O. G. Xi, Fuzzy BCK-algebras, Math. Japon. 36 (5) (1991) 935–942.

[8] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.

[9] T. Gerstenkorn and J. Manko, Correlation of intuitionistic fuzzy sets, Fuzzy sets Systems 44 (1991) 39–43.

[10] Samy M. Mostafa, Mokhtar A. Abdel Naby and Osama R. Elgendy, Interval-valued fuzzy KU-ideals in KU-algebras, International Mathematical Forum 6 (64) (2011) 3151–3159.

[11] Samy M. Mostafa, Mokhtar A. Abdel Naby and Osama R. Elgendy, Intuitionistic fuzzy KU-ideals in KU-algebras, Int. J. of Mathematical Sciences and Applications 1 (3) (2011) 1379–1384

[12] Naveed Yaqoob, Samy M. Mostafa and Moin A. Ansari, Cubic KU-Ideals of KU-algebras, International Scholarly Research Network ISRN Algebra 2012 (2012) 19 pages.

[13] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning. I, Inform. Sci. 8 (1975) 199–249.

[14] Samy M. Mostafa, Mokhtar A. Abd-Elnaby and Moustafa M. M. Yousef, Fuzzy ideals of KU-Algebras, Int. Math. Forum 6 (63) (2011) 3139–3149.

[15] Samy M. Mostafa, Mokhtar A. Abd-Elnaby and Moustafa M. M. Yousef, Anti-fuzzy KU-ideals of KU-algebras, International Journal of Algebra and Statistics 1 (1) (2012) 92–99.

[16] Moin A. Ansari, Ali N. A. Koam and A. Haider, Intersection soft ideals and their quotients on KU-algebras, AIMS Mathematics 6 (11) (2021) 12077–12084.

[17] Ali N. A. Koam, A. Haider and Moin A. Ansari, On an extension of KU-algebras, AIMS Math 6 (2021) 1249-1257.

[18] Y. B. Jun, A novel extension of cubic sets and its applications in BCK/BCI-algebras Ann. Fuzzy Math. Inform. 14 (5) (2017) 475–486.

[19] Y. B. Jun, C. S. Kim and K.O. Yang, Cubic sets, Ann. Fuzzy Math. Inform. 4 (1) (2012) 83–98.

[20] R. R. Yager , Pythagorean membership grades in multi-criteria decision making, Technical report MII-3301 Machine Intelligence Institute, Iona College, New Rochelle NY 2013.

[21] S. S. Ahn, H. S. Kim, S.Z. Song and Y. B. Jun, The $(2,3)$-fuzzy set and its application in BCK-algebras and BCI-algebras, J. Math. Computer Sci. 27 (2022) 118–130.

[22] Y. B. Jun and K. Hur, The $(m;n)$-fuzzy set and its application in BCK-algebras, Ann. Fuzzy Math. Inform. 24 (1) (2022) 17–29.

[23] T. M. Al-shami, A. I. EL-Maghrabi, H. Z. Ibrahim and A. A. Azzam, SR-fuzzy sets and their weighted aggregated operators in application to decision-making, Hindawi Journal of Function Spaces 2022 (2022) Article ID 3653225 14 pages.

[24] Y. A. Salih and Hariwan Z. Ibrahim, CR-fuzzy sets and their applications, J. Math. Computer Sci. 28 (2023) 171–181.

[25] H. Z. T Ibrahim, M.Al-shamiand and O. G Elbarbary, $(3,2)$-fuzzy sets and their applications to topology and computational optimal choices, Intelligence and Neuroscience 2021 (2021) Article ID 1272266, 14 pages.

[26] Baek Jong Il, Sang Hyeon Han, Samy M. Mostafa and Kul Hur, Correlation coefficient between intuitionistic single-valued neutrosophic sets and its applications, Ann. Fuzzy Math. Inform. 23 (1) (2022) 81–105.
[27] D. A. Chiang and N. P. Lin, Correlation of fuzzy sets, Fuzzy Sets and Systems 102 (1999) 221–226.
[28] A. Bryniarska, The n-Pythagorean fuzzy sets, Symmetry 2020, 12,1772.
[29] T. Senapati and R. R. Yager, Fermatean fuzzy sets, Journal of Ambient Intelligence and Humanized Computing 11 (2020) 663–674.
[30] K. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989) 343–349.
[31] Z. S. Xu, On correlation measures of intuitionistic fuzzy sets, Lect. Notes Comput. Sci. 4224 (2006) 16–24.

Ola W. Abd El-Baseer (olawagih@yahoo.com)
Ain Shams University Faculty of Education Department of Math. Roxy, Cairo, Egypt

J. G. Lee (jukolee@wku.ac.kr)
Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University, Korea

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

Amany M. Menshawy (amaniminshawi@edu.asu.edu.eg)
Ain Shams University Faculty of Education Department of Math. Roxy, Cairo, Egypt

K. Hur (kulhur@wku.ac.kr)
Division of Mathematics and Informational Statistics, Institute of Basic Natural Science, Wonkwang University, Korea

Samy. M. Mostafa (samymostafa@yahoo.com)
Ain Shams University Faculty of Education Department of Math. Roxy, Cairo, Egypt