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Abstract. In this paper, we study embedding maps by using distance functions based on complete co-residuated lattices. We construct two embedding maps from the distance space to the two Alexandrov topologies induced by the distance space. We study the various maps induced by two maps and give their examples. Moreover, as the topological representation, we investigate the embedding map. We give an example for an information.

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1. Introduction

Ward and Dilworth [1][1] introduced a complete residuated lattice which is an important mathematical tool as algebraic structures for many valued logics (See [2, 3, 4, 5, 6]. Bělohlávek [2, 3] investigated information systems and decision rules on complete residuated lattices. Höhle and Rodabaugh [5] introduced L-fuzzy topologies with algebraic structure L (cqm, quantales, MV-algebra). Zheng and Wang [7] introduced a complete co-residuated lattice as a generalization of t-conorm. Junsheng and Qing [8] investigated a $(\odot, \&)$-generalized fuzzy rough set on $(L, \odot, \&)$ where $(L, \&)$ is a complete residuated lattice and $(L, \odot)$ is a complete co-residuated lattice. Kim and Ko [9] introduced the concepts of fuzzy join and meet complete lattices by using distance spaces instead of fuzzy partially ordered spaces on complete co-residuated lattices. Moreover, Oh and Kim [10, 11, 12, 13, 14, 15] obtained some properties of Alexandrov fuzzy topologies, distance functions, various fuzzy connection and fuzzy concept lattices by using distance functions instead of fuzzy partially orders on complete co-residuated lattices.

If $f : (X, S_X) \to (Y, S_Y)$ is an embedding map, then f is an injective map and is a structure-preserving map. The precise meaning of structure-preserving depends on the kind of mathematical structures S_X and S_Y. In this paper, we study embedding maps by using distance functions based on complete co-residuated lattices. In Theorem 3.3, for a distance space (X, d_X), we construct two embedding maps $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ and $g : (X, d_X^{-1}) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ by $f(x) = (d_X)_x$ and $g(x) = (d_X)^x$ where $(d_X)_x(y) = d_X(x, y)$ and $(d_X)^x(y) = d_X(y, x)$ for all $x, y \in X$. In Theorems 3.4 and 3.5, we study the various maps induced by two maps. In Theorem 3.6, as the topological representation, we investigate the embedding map $h : (X, d_X) \to (\tau_{d^{-1}_{\tau_{d_X^{-1}}}}, d_{\tau_{d^{-1}_{\tau_{d_X^{-1}}}}})$. In Example 3.8, we give an example for an information.

2. Preliminaries

Definition 2.1 ([7, 8, 9, 10])**.** An algebra $(L, \wedge, \vee, \oplus, \perp, \top)$ is called a *complete co-residuated lattice*, if it satisfies the following conditions:

(C1) $L = (L, \vee, \wedge, \perp, \top)$ is a complete lattice where $\perp$ is the bottom element and $\top$ is the top element,

(C2) $a = a \oplus \perp$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$,

(C3) $(\bigwedge_{i\in\Gamma} a_i) \oplus b = \bigwedge_{i\in\Gamma}(a_i \oplus b)$.

Let $(L, \leq, \oplus)$ be a complete co-residuated lattice. For all $x, y \in L$, define

$$x \ominus y = \bigwedge\{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ if and only if $x \geq (z \ominus y)$.

For $\alpha \in L$ and $A \in L^X$, we define $(\alpha \ominus A)$, $(\alpha \oplus A)$ and $\alpha_X \in L^X$ by: for each $x \in L$,

$$(\alpha \ominus A)(x) = \alpha \ominus A(x),\ (\alpha \oplus A)(x) = \alpha \oplus A(x) \text{ and } \ \alpha_X(x) = \alpha \text{ respectively.}$$

Let $n(x) = \top \ominus x$. The condition $n(n(x)) = x$ for all $x \in L$ is called a *double negative law.*

Lemma 2.2 ([7, 8, 9, 10])**.** *Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. Then for all $x, y, z, x_i, y_i \in L$, the following properties hold.*

(1) *If $y \leq z$, $x \oplus y \leq x \oplus z$, $y \ominus x \leq z \ominus x$ and $x \ominus z \leq x \ominus y$.*

(2) $(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ *and* $x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$.

(3) $(\bigwedge_{i\in\Gamma} x_i) \ominus y \leq \bigwedge_{i\in\Gamma}(x_i \ominus y)$.

(4) $x \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \ominus y_i)$.

(5) $x \ominus x = \perp$, $x \ominus \perp = x$ *and* $\perp \ominus x = \perp$. *Moreover,* $x \ominus y = \perp$ *iff* $x \leq y$.

(6) $y \oplus (x \ominus y) \geq x$, $y \geq x \ominus (x \ominus y)$ *and* $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.

(7) $x \ominus (y \oplus z) = (x \ominus y) \ominus z = (x \ominus z) \ominus y$.

(8) $x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$, $x \ominus y \geq (x \ominus z) \ominus (y \ominus z)$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$ *and* $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.

(9) $x \oplus y = \perp$ *iff* $x = \perp$ *and* $y = \perp$.

(10) $(x \oplus y) \ominus z \leq x \oplus (y \ominus z)$ *and* $(x \ominus y) \oplus z \geq x \ominus (y \ominus z)$.

(11) $(\bigvee_{i\in\Gamma} x_i) \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigvee_{i\in\Gamma}(x_i \ominus y_i)$.

(12) $(\bigwedge_{i\in\Gamma} x_i) \ominus (\bigwedge_{i\in\Gamma} y_i) \leq \bigvee_{i\in\Gamma}(x_i \ominus y_i)$.

(13) *If L satisfies a double negative law and $n(x) = \top \ominus x$, then $n(x \oplus y) = n(x) \ominus y = n(y) \ominus x$ and $x \ominus y = n(y) \ominus n(x)$.*

Definition 2.3 ([9, 10]). A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X, if it satisfies the following conditions:

(A1) if $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau$,
(A2) if $A \in \tau$ and $\alpha \in L$, then $\alpha_X, A \ominus \alpha, A \oplus \alpha \in \tau$.
The pair (X, τ) is called an *Alexandrov topological space* on X.

Definition 2.4 ([9, 10]). Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \to L$ is called a *distance function*, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(M1) $d_X(x, x) = \perp$,
(M2) $d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z)$,
(M3) If $d_X(x, y) = d_X(y, x) = \perp$, then $x = y$.
The pair (X, d_X) is called a *distance space.*

Theorem 2.5 ([9, 10]). *Let (X, d_X) be a distance space. Define*

$$\tau_{d_X} = \{A \in L^X \mid A(x) \oplus d_X(x, y) \geq A(y)\},$$
$$\tau_{d_X^{-1}} = \{A \in L^X \mid A(x) \oplus d_X(y, x) \geq A(y)\}.$$

Then τ_{d_X} and $\tau_{d_X^{-1}}$ are Alexandrov topologies such that

$$\tau_{d_X} = \{\bigwedge_{x\in X} A(x)\oplus d_X(x, -) \mid A \in L^X\} \text{ and } \tau_{d_X^{-1}} = \{\bigwedge_{x\in X} A(x)\oplus d_X(-, x) \mid A \in L^X\}.$$

Remark 2.6 ([9, 10]). (1) Let $d_X : X \times X \to [0, \infty]$ be a distance function. Then (X, d_X) is called a *pseudo-quasi-metric space.*

(2) Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. Define a function

$$d_L : L \times L \to L \text{ by } d_L(x, y) = x \ominus y.$$

Then by Lemma 2.3 (5) and (6), (L, d_L) is a distance space.

Also for $\tau_X \subset L^X$, define a function

$$d_{\tau_X} : \tau_X \times \tau_X \to L \text{ by } d_{\tau_X}(A, B) = \bigvee_{x\in X} (A(x) \ominus B(x)).$$

Then (τ_X, d_{τ_X}) is a distance space.

In this paper, we assume that $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ is a complete co-residuated lattice.

3. Some properties of embedding maps in complete co-residuated lattices

Definition 3.1. Let (X, d_X) and (Y, d_Y) be distance spaces. Then a map $f : X \to Y$ is called *embedding*, if $d_X(x, y) = d_X(f(x), f(y))$ for all $x, y \in X$.

Remark 3.2. Let $f : X \to Y$ be an embedding map. For $f(x) = f(y)$, we have $d_X(f(x), f(y)) = \perp = d_X(x, y)$. By (M3), $x = y$.

Theorem 3.3. *Let (X, d_X) be a distance space. Define $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ and $g : (X, d_X^{-1}) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ by $f(x) = (d_X)_x$ and $g(x) = (d_X)^x$ where $(d_X)_x(y) = d_X(x, y)$ and $(d_X)^x(y) = d_X(y, x)$ for all $x, y \in X$. Then the following properties hold.*

(1) *Two maps f and g are well-defined and are embedding.*
(2) *$A \in \tau_{d_X}$ if and only if $d_{\tau_{d_X}}(A, f(-)) = A(-) = \bigwedge_{z\in X}(A(z) \oplus f(z)(-))$.*
(3) *$B \in \tau_{d_X^{-1}}$ if and only if $d_{\tau_{d_X^{-1}}}(B, g(-)) = B(-) = \bigwedge_{z\in X}(B(z) \oplus g(z)(-))$.*

Proof. (1) Let $x,\ y,\ z \in X$. Since

$$\begin{aligned} f(x)(y) \oplus d_X(y, z) = d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z) = f(x)(z),\\ g(x)(y) \oplus d_X^{-1}(y, z) = d_X(y, x) \oplus d_X(z, y) \geq d_X(z, x) = g(x)(z), \end{aligned}$$

we have $f(x) \in \tau_{d_X}$ and $g \in \tau_{d_X^{-1}}$. Then f and g are well-defined.

Let $x,\ y \in X$. Since

$$\begin{aligned} d_{\tau_{d_X}}(f(x), f(y)) &= \bigvee_{z\in X}(f(x)(z) \ominus f(y)(z))\\ &= \bigvee_{z\in X}(d_X(x, z) \ominus d_X(y, z)) = d_X(x, y),\\ d_{\tau_{d_X^{-1}}}(g(x), g(y)) &= \bigvee_{z\in X}(g(x)(z) \ominus g(y)(z))\\ &= \bigvee_{z\in X}(d_X(z, x) \ominus d_X(z, y)) = d_X(y, x) = d_X^{-1}(x, y), \end{aligned}$$

f and g are embedding.

(2) Suppose $A \in \tau_{d_X}$. Then $A(x) \oplus d_X(x, z) \geq A(z)$ implies $A(x) \geq A(z) \ominus d_X(x, z)$ and $\bigvee_{z\in X}(A(z) \ominus d_X(x, z)) \geq A(x) \ominus d_X(x, x) = A(x)$. Thus

$$\begin{aligned} d_{\tau_{d_X}}(A, f(x)) &= \bigvee_{z\in X}(A(z) \ominus d_X(x, z)) = \bigvee_{z\in X}(A(z) \ominus g(z)(x)) = A(x)\\ &= \bigwedge_{z\in X}(A(z) \oplus d_X(z, x)) = \bigwedge_{z\in X}(A(z) \oplus f(z)(x)). \end{aligned}$$

Conversely, suppose $A = \bigwedge_{z\in X}(A(z) \oplus f(z)(-))$. Then by (1), $f(z) \in \tau_{d_X}$ and $(A(z) \oplus f(z)(-)) \in \tau_{d_X}$. Thus $A \in \tau_{d_X}$.

(3) Suppose $B \in \tau_{d_X^{-1}}$. Then

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(B, g(x)) &= \bigvee_{z\in X}(B(z) \ominus f(z)(x)) = B(x)\\ &= \bigwedge_{z\in X}(B(z) \oplus d_X(x, z)) = \bigwedge_{z\in X}(B(z) \oplus g(z)(x)). \end{aligned}$$

Conversely, suppose $B = \bigwedge_{z\in X}(B(z) \oplus g(z)(-))$. Then by (1), we have

$$g(z) \in \tau_{d_X^{-1}} \text{ and } (B(z) \oplus g(z)(-)) \in \tau_{d_X^{-1}}.$$

Thus $B \in \tau_{d_X^{-1}}$. □

Theorem 3.4. *Let (X, d_X) be a distance space. Define $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ by $f(x) = (d_X)_x$. Then the following properties hold.*

(1) *Define $f^{s\oslash} : \tau_{d_X} \to \tau_{d_{\tau_{d_X}}}$ and $f^{\leftarrow}_{\oplus} : \tau_{d_{\tau_{d_X}}} \to \tau_{d_X}$ by*

$$\begin{aligned} f^{s\oslash}(A)(B) &= \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X}}(B, f(x))),\\ f^{\leftarrow}_{\oplus}(\alpha)(x) &= \bigwedge_{z\in X}(\alpha(f(z)) \oplus d_X(z, x)). \end{aligned}$$

Then $f^{s\oslash}$ and $f^{\leftarrow}_{\oplus}$ are well-defined and

$$f^{s\oslash}(A)(f(-)) = A,\ f^{\leftarrow}_{\oplus}(f^{s\oslash}(A)) = A.$$

(2) *Define* $f^{\oplus} : \tau_{d_X} \to \tau_{d_{\tau_{d_X}}}$ *and* $f^{s\leftarrow}_{\oslash} : \tau_{d_{\tau_{d_X}}} \to \tau_{d_X}$ *by*

$$\begin{array}{ll} f^{\oplus}(A)(B) & = \bigwedge_{x\in X}(d_{\tau_{d_X}}((f(x),B) \oplus A(x)), \\ f^{s\leftarrow}_{\oslash}(\alpha)(-) & = \bigvee_{z\in X}(\alpha(f(z)) \ominus d_X(-,z)). \end{array}$$

Then $f^{\oplus}$ *and* $f^{s\leftarrow}_{\oslash}$ *are well-defined and*

$$f^{\oplus}(A)(f(-)) = A,\ f^{s\leftarrow}_{\oslash}(f^{\oplus}(A)) = A.$$

Moreover, if $A \in \tau_{d_X}$*, then* $f^{s\leftarrow}_{\oslash}(f^{\oplus}(A)) = f^{\leftarrow}_{\oplus}(f^{s\oslash}(A))$.

(3) *Define* $(d_{\tau_{d_X}})_{f(y)} : \tau_{d_X} \to L$ *by* $(d_{\tau_{d_X}})_{f(y)}(C) = d_{\tau_{d_X}}(f(y),C)$. *Then* $(d_{\tau_{d_X}})_{f(y)} \in \tau_{d_{\tau_{d_X}}}$ *and* $f^{\leftarrow}_{\oplus}((d_{\tau_{d_X}})_{f(y)}) = f(y) = f^{s\leftarrow}_{\oslash}((d_{\tau_{d_X}})_{f(y)})$. *Moreover,*

$$\begin{array}{ll} d_{\tau_{d_{\tau_{d_X}}}}(f^{s\oslash}(A),(d_{\tau_{d_X}})_{f(y)}) & = A(y) = d_{\tau_{d_X}}(A, f^{\leftarrow}_{\oplus}((d_{\tau_{d_X}})_{f(y)})), \\ d_{\tau_{d_{\tau_{d_X}}}}(d_{\tau_{d_X}})_{f(y)}, f^{\oplus}(A)) & = d_{\tau_{d_X}}(f(y),A) = d_{\tau_{d_X}}(f^{s\leftarrow}_{\oslash}((d_{\tau_{d_X}})_{f(y)}, A). \end{array}$$

Proof. (1) Let $A,\ B,\ C \in \tau_{d_X}$. Then we have

$$\begin{array}{l} f^{s\oslash}(A)(B) \oplus d_{\tau_{d_X}}(B,C) \oplus d_{\tau_{d_X}}(C,f(x)) \\ \geq f^{s\oslash}(A)(B) \oplus d_{\tau_{d_X}}(B,f(x)) \\ = \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X}}(B,f(x))) \oplus d_{\tau_{d_X}}(B,f(x)) \\ \geq A(x). \end{array}$$

Thus $f^{s\oslash}(A)(B) \oplus d_{\tau_{d_X}}(B,C) \geq A(x) \ominus d_{\tau_{d_X}}(C,f(x))$. So $f^{s\oslash}(A) \in \tau_{d_{\tau_{d_X}}}$. Hence $f^{s\oslash}$ is well-defined.

Let $\alpha \in \tau_{d_{\tau_{d_X}}}$. Then we get

$$\begin{array}{ll} f^{\leftarrow}_{\oplus}(\alpha)(x) \oplus d_X(x,y) & = \bigwedge_{z\in X}(\alpha(f(z)) \oplus d_X(z,x)) \oplus d_X(x,y) \\ & \geq \bigwedge_{z\in X}(\alpha(f(z)) \oplus d_X(z,y)) = f^{\leftarrow}_{\oplus}(\alpha)(y). \end{array}$$

Thus $f^{\leftarrow}_{\oplus}(\alpha) \in \tau_{d_X}$. So $f^{\leftarrow}_{\oplus}$ is well-defined.

Now let $A \in \tau_{d_X}$. Then we have

$$\begin{array}{ll} f^{s\oslash}(A)(f(-)) & = \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X}}(f(-),f(x))) \\ & = \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X}}(f(-),f(x))) \\ & = \bigvee_{x\in X}(A(x) \ominus d_X(-,x)) = A(-). \end{array}$$

Thus $f^{\leftarrow}_{\oplus}(f^{s\oslash}(A))(-) = \bigwedge_{z\in X}(f^{s\oslash}(A)(f(z)) \oplus d_X(z,-)) = A(-)$.

(2) Let $A,\ B,\ C \in \tau_{d_X}$. Then we get

$$\begin{array}{ll} f^{\oplus}(A)(B) \oplus d_{\tau_{d_X}}(B,C) & = \bigwedge_{x\in X}(d_{\tau_{d_X}}((f(x),B) \oplus A(x)) \oplus d_{\tau_{d_X}}(B,C) \\ & \geq \bigwedge_{x\in X}(d_{\tau_{d_X}}((f(x),C) \oplus A(x)) = f^{\oplus}(A)(C). \end{array}$$

Thus $f^{\oplus}(A) \in \tau_{d_{\tau_{d_X}}}$. So $f^{\oplus}$ is well-defined.

Let $\alpha \in \tau_{d_{\tau_{d_X}}}$. Then we have

$$f^{s\leftarrow}_{\oslash}(\alpha)(x) \oplus d_X(x,y) \oplus d_X(y,z) \geq (\alpha(f(z)) \ominus d_X(x,z)) \oplus d_X(x,z) \geq \alpha(f(z))$$

and

$$f^{s\leftarrow}_{\oslash}(\alpha)(x) \oplus d_X(x,y) \geq \alpha(f(z)) \ominus d_X(x,z).$$

Thus $f^{s\leftarrow}_{\oslash}(\alpha)(x) \oplus d_X(x,y) \geq f^{s\leftarrow}_{\oslash}(\alpha)(y)$. So $f^{s\leftarrow}_{\oslash}(\alpha) \in \tau_{d_X}$. Hence $f^{s\leftarrow}_{\oslash}$ is well-defined.

Now let $A \in \tau_{d_X}$. Then we have

$$f^{\oplus}(A)(f(-)) = \bigwedge_{x\in X}(d_{\tau_{d_X}}((d_X)_x, f(-)) \oplus A(x) = A(-),$$
$$f^{s\leftarrow}_{\oslash}(f^{\oplus}(A))(-) = \bigvee_{z\in X}(f^{\oplus}(A)(f(z)) \ominus d_X(-,z)) = A(-).$$

(3) Let $y \in X$. Then we have

$$\begin{aligned}(d_{\tau_{d_X}})_{f(y)}(C) \oplus d_{\tau_{d_X}}(C,D) &= d_{\tau_{d_X}}(f(y),C) \oplus d_{\tau_{d_X}}(C,D)\\ &\geq d_{\tau_{d_X}}(f(y),D)\\ &= (d_{\tau_{d_X}})_{f(y)}(D).\end{aligned}$$

Thus $(d_{\tau_{d_X}})_{f(y)} \in \tau_{d_{\tau_{d_X}}}$.

Let $x,\ y \in X$. Then we get

$$\begin{aligned}f^{\leftarrow}_{\oplus}((d_{\tau_{d_X}})_{f(y)})(x) &= \bigwedge_{z\in X}((d_{\tau_{d_X}})_{f(y)}(f(z)) \oplus d_X(z,x))\\ &= \bigwedge_{z\in X}(d_{\tau_{d_X}}(f(y),f(z)) \oplus d_X(z,x))\\ &= \bigwedge_{z\in X}(d_X(y,z) \oplus d_X(z,x))\\ &= d_X(y,x) = f(y)(x) \text{ [By Theorem 3.3]},\end{aligned}$$

$$\begin{aligned}f^{s\leftarrow}_{\oslash}((d_{\tau_{d_X}})_{f(y)})(x) &= \bigvee_{z\in X}((d_{\tau_{d_X}})_{f(y)}(f(z)) \ominus d_X(x,z))\\ &= \bigvee_{z\in X}(d_{\tau_{d_X}}(f(y),f(z)) \ominus d_X(x,z))\\ &= \bigvee_{z\in X}(d_X(y,z) \ominus d_X(x,z))\\ &= d_X(y,x) = f(y)(x).\end{aligned}$$

Moreover, for all $A \in \tau_{d_X}$, we have

$$\begin{aligned}&d_{\tau_{d_{\tau_{d_X}}}}(f^{s\oslash}(A), (d_{\tau_{d_X}})_{f(y)})\\ &= \bigvee_{C\in\tau_{d_X}}(f^{s\oslash}(A)(C) \ominus (d_{\tau_{d_X}})_{f(y)}(C))\\ &= \bigvee_{C\in\tau_{d_X}}((\bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X}}(C,f(x)))) \ominus d_{\tau_{d_X}}(f(y),C))\\ &= \bigvee_{x\in X}(A(x) \ominus \bigwedge_{C\in\tau_{d_X}}(d_{\tau_{d_X}}(C,f(x)) \oplus d_{\tau_{d_X}}(f(y),C)))\\ &\quad \text{[By Lemma 2.2 (2) and (7)]}\\ &= \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X}}(f(y),f(x)))\\ &= \bigvee_{x\in X}(A(x) \ominus d_X(y,x)) = A(y),\end{aligned}$$

$$\begin{aligned}&d_{\tau_{d_X}}(A, f^{\leftarrow}_{\oplus}((d_{\tau_{d_X}})_{f(y)}))\\ &= \bigvee_{x\in X}(A(x) \ominus f(y)(x))\\ &= \bigvee_{x\in X}(A(x) \ominus d_X(y,x)) = A(y).\end{aligned}$$

Thus $d_{\tau_{d_{\tau_{d_X}}}}(f^{s\oslash}(A), (d_{\tau_{d_X}})_{f(y)}) = A(y) = d_{\tau_{d_X}}(A, f^{\leftarrow}_{\oplus}((d_{\tau_{d_X}})_{f(y)}))$.

For all $A \in \tau_{d_X}$ and $y \in X$, we get

$$\begin{aligned}&d_{\tau_{d_{\tau_{d_X}}}}((d_{\tau_{d_X}})_{f(y)}, f^{\oplus}(A))\\ &= \bigvee_{C\in\tau_{d_X}}((d_{\tau_{d_X}})_{f(y)}(C) \ominus f^{\oplus}(A)(C))\\ &= \bigvee_{C\in\tau_{d_X}}(d_{\tau_{d_X}}(f(y),C) \ominus \bigwedge_{x\in X}(A(x) \oplus d_{\tau_{d_X}}(f(x),C)))\\ &\quad \text{[By Lemma 2.2 (2) and (7)]}\\ &= \bigvee_{x\in X}(\bigvee_{C\in\tau_{d_X}}(d_{\tau_{d_X}}(f(y),C) \ominus d_{\tau_{d_X}}(f(x),C)) \ominus A(x))\\ &= \bigvee_{x\in X}(d_{\tau_{d_X}}(f(y),f(x)) \ominus A(x)) = \bigvee_{x\in X}(d_X(y,x) \ominus A(x))\\ &= d_{\tau_{d_X}}(f(y),A),\end{aligned}$$

$$\begin{aligned}&d_{\tau_{d_X}}(f^{\leftarrow}_{\oplus}((d_{\tau_{d_X}})_{f(y)}), A)\\ &= \bigvee_{x\in X}(d_X(y,x) \ominus A(x))\end{aligned}$$

$$= d_{\tau_{d_X}}(f(y), A).$$

So $d_{\tau_{d_{\tau_{d_X}}}}((d_{\tau_{d_X}})_{f(y)}, f^{\oplus}(A)) = d_{\tau_{d_X}}(f(y), A) = d_{\tau_{d_X}}(f^{s\leftarrow}_{\oslash}((d_{\tau_{d_X}})_{f(y)}, A).$ □

Theorem 3.5. *Let (X, d_X) be a distance space. Define $g : (X, d_X^{-1}) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ by $g(x) = (d_X)^x$ where $(d_X)^x(y) = d_X(y, x)$ for all $x,\ y \in X$.*

(1) *Define $g^{\oslash} : \tau_{d_X^{-1}} \to \tau_{d_{\tau_{d_X^{-1}}}}$ and $g^{s\leftarrow}_{\oplus} : \tau_{d_{\tau_{d_X^{-1}}}} \to \tau_{d_X^{-1}}$ by*

$$g^{\oslash}(A)(B) = \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X^{-1}}}(B, g(x))),$$
$$g^{s\leftarrow}_{\oplus}(\beta)(x) = \bigwedge_{z\in X}(\beta(g(z)) \oplus d_X^{-1}(z, x)).$$

Then $g^{\oslash}$ and $g^{s\leftarrow}_{\oplus}$ are well-defined, and

$$g^{\oslash}(B)(g(-)) = B,\ g^{s\leftarrow}_{\oplus}(g^{\oslash}(B))(-) = B.$$

(2) *Define $g^{s\oplus} : \tau_{d_X^{-1}} \to \tau_{d_{\tau_{d_X^{-1}}}}$ and $g^{\leftarrow}_{\oslash} : \tau_{d_{\tau_{d_X^{-1}}}} \to \tau_{d_X^{-1}}$ by*

$$g^{s\oplus}(A)(B) = \bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}((g(x), B) \oplus A(x)),$$
$$g^{\leftarrow}_{\oslash}(\beta)(x) = \bigvee_{z\in X}(\beta(g(z)) \ominus d_X^{-1}(x, z)).$$

Then $g^{s\oplus}$ and $g^{\leftarrow}_{\oslash}$ are well-defined, and

$$g^{s\oplus}(A)(g(-)) = A,\ g^{\leftarrow}_{\oslash}(g^{s\oplus}(A))(-) = A(-).$$

Moreover, if $A \in \tau_{d_X^{-1}}$, then $g^{s\leftarrow}_{\oplus}(g^{\oslash}(A)) = g^{\leftarrow}_{\oslash}(g^{s\oplus}(A))$.

(3) *Define $(d_{\tau_{d_X^{-1}}})_{g(y)} : \tau_{d_X^{-1}} \to L$ by $(d_{\tau_{d_X^{-1}}})_{g(y)}(C) = d_{\tau_{d_X^{-1}}}(g(y), C)$. Then $(d_{\tau_{d_X^{-1}}})_{g(y)} \in \tau_{d_{\tau_{d_X^{-1}}}}$ and $g^{s\leftarrow}_{\oplus}((d_{\tau_{d_X^{-1}}})_{g(y)}) = g(y) = g^{\leftarrow}_{\oslash}((d_{\tau_{d_X^{-1}}})_{g(y)})$. Moreover,*

$$d_{\tau_{d_{\tau_{d_X^{-1}}}}}(g^{\oslash}(A), (d_{\tau_{d_X^{-1}}})_{g(y)}) = A(y) = d_{\tau_{d_X^{-1}}}(A, g^{s\leftarrow}_{\oplus}((d_{\tau_{d_X^{-1}}})_{g(y)}))$$

and

$$\begin{aligned} d_{\tau_{d_{\tau_{d_X^{-1}}}}}((d_{\tau_{d_X^{-1}}})_{g(y)}, g^{s\oplus}(A)) &= d_{\tau_{d_X^{-1}}}(g(y), A) \\ &= d_{\tau_{d_X^{-1}}}(g^{s\leftarrow}_{\oplus}((d_{\tau_{d_X^{-1}}})_{g(y)}), A). \end{aligned}$$

Proof. (1) Let $A,\ B,\ C \in \tau_{d_X^{-1}}$. Then we have

$$\begin{aligned} g^{\oslash}(A)(B) \oplus d_{\tau_{d_X^{-1}}}(B, C) \quad &\oplus d_{\tau_{d_X^{-1}}}(C, g(x)) \geq g^{\oslash}(A)(B) \oplus d_{\tau_{d_X^{-1}}}(B, g(x)) \\ &= \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X^{-1}}}(B, g(x))) \oplus d_{\tau_{d_X^{-1}}}(g(x), B) \\ &\geq A(x). \end{aligned}$$

Thus $g^{\oslash}(A)(B) \oplus d_{\tau_{d_X^{-1}}}(B, C) \geq A(x) \ominus d_{\tau_{d_X^{-1}}}(B, g(x))$. So $g^{\oslash}(A) \in \tau_{d_{\tau_{d_X^{-1}}}}$. Hence $g^{\oslash}$ is well-defined.

For all $\beta \in \tau_{d_{\tau_{d_X^{-1}}}}$, we get

$$\begin{aligned} g^{s\leftarrow}_{\oplus}(\beta)(x) \oplus d_X(y, x) \quad &= \bigwedge_{z\in X}(\beta(g(z)) \oplus d_X(x, z)) \oplus d_X(y, x) \\ &\geq \bigwedge_{z\in X}(\beta(g(z)) \oplus d_X(y, z)) = g^{s\leftarrow}_{\oplus}(\beta)(y). \end{aligned}$$

Then $g^{s\leftarrow}_{\oplus}(\beta) \in \tau_{d_X^{-1}}$. Thus $g^{s\leftarrow}_{\oplus}$ is well-defined.

Now let $A,\ B \in \tau_{d_X^{-1}}$. Then we get

$$\begin{aligned} g^{\oslash}(A)(g(-)) &= \bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X^{-1}}}(g(-), g(x))) \\ &= \bigvee_{x\in X}(A(x) \ominus d_X^{-1}(-, x)) = \bigvee_{x\in X}(A(x) \ominus d_X(x, -)) = A, \\ g_{\oplus}^{s\leftarrow}(g^{\oslash}(B))(-) &= \bigwedge_{z\in X}(g^{\oslash}(B)(g(z)) \oplus d_X(-, z)) \\ &= \bigwedge_{z\in X}(B(z) \oplus d_X(-, z)) = B. \end{aligned}$$

(2) Let $A,\ B \in \tau_{d_X^{-1}}$. Then we have

$$\begin{aligned} g^{s\oplus}(A)(B) \oplus d_{\tau_{d_X^{-1}}}(B, C) &= \bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}((g(x), B) \oplus A(x)) \oplus d_{\tau_{d_X^{-1}}}(B, C) \\ &\geq \bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}((g(x), C) \oplus A(x)) = g^{s\oplus}(A)(C). \end{aligned}$$

Thus $g^{s\oplus}(A) \in \tau_{d_{\tau_{d_X^{-1}}}}$. So $g^{s\oplus}$ is well-defined.

For all $\beta \in \tau_{d_{\tau_{d_X^{-1}}}}$, we get

$$g_{\oslash}^{\leftarrow}(\beta)(x) \oplus d_X(y, x) \oplus d_X(z, y) \geq (\beta(g(z)) \ominus d_X(z, x)) \oplus d_X(z, x) \geq \beta(g(z))$$

and

$$g_{\oslash}^{\leftarrow}(\beta)(x) \oplus d_X(y, x) \geq \beta(g(z)) \ominus d_X(z, y).$$

Then $g_{\oslash}^{\leftarrow}(\beta)(x) \oplus d_X(y, x) \geq g_{\oslash}^{\leftarrow}(\beta)(y)$. Thus $g_{\oslash}^{\leftarrow}(\beta) \in \tau_{d_X^{-1}}$. So $g_{\oslash}^{\leftarrow}$ is well-defined.

For all $A \in \tau_{d_X^{-1}}$,

$$\begin{aligned} g^{s\oplus}(A)(g(-)) &= \bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}(g(x), g(-)) \oplus A(x)) \\ &= \bigwedge_{x\in X}(d_X^{-1}(x, -) \oplus A(x)) = A, \\ g_{\oslash}^{\leftarrow}(g^{s\oplus}(A))(-) &= \bigvee_{z\in X}(g^{s\oplus}(A)(g(z)) \ominus d_X^{-1}(-, z)) \\ &= \bigvee_{z\in X}(A(z) \ominus d_X(z, -)) = A(-). \end{aligned}$$

(3) Let $y \in X$. Then we get

$$\begin{aligned} &(d_{\tau_{d_X^{-1}}})_{g(y)}(C) \oplus d_{\tau_{d_X^{-1}}}(C, D) = d_{\tau_{d_X^{-1}}}(g(y), C) \oplus d_{\tau_{d_X^{-1}}}(C, D) \\ &\geq d_{\tau_{d_X^{-1}}}(g(y), D) = (d_{\tau_{d_X^{-1}}})_{g(y)}(D). \end{aligned}$$

Thus $(d_{\tau_{d_X^{-1}}})_{g(y)} \in \tau_{d_{\tau_{d_X^{-1}}}}$.

For all $x,\ y \in X$,

$$\begin{aligned} g_{\oplus}^{s\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)})(x) &= \bigwedge_{z\in X}((d_{\tau_{d_X^{-1}}})_{g(y)}(g(z)) \oplus d_X(x, z)) \\ &= \bigwedge_{z\in X}(d_X^{-1}(y, z) \oplus d_X(x, z)) = d_X(x, y) = g(y)(x), \\ g_{\oslash}^{\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)})(x) &= \bigvee_{z\in X}((d_{\tau_{d_X^{-1}}})_{g(y)}(g(z)) \ominus d_X(z, x)) \\ &= \bigvee_{z\in X}(d_X(z, y) \ominus d_X(z, x)) = d_X(x, y) = g(y)(x). \end{aligned}$$

So $g_{\oplus}^{s\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)}) = g(y) = g_{\oslash}^{\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)})$.

For all $A \in \tau_{d_X^{-1}}$,

$$\begin{aligned} &d_{\tau_{d_{\tau_{d_X^{-1}}}}}(g^{\oslash}(A), (d_{\tau_{d_X^{-1}}})_{g(y)}) \\ &= \bigvee_{C\in\tau_{d_X^{-1}}}(g^{\oslash}(A)(C) \ominus (d_{\tau_{d_X^{-1}}})_{g(y)}(C)) \\ &= \bigvee_{C\in\tau_{d_X^{-1}}}((\bigvee_{x\in X}(A(x) \ominus d_{\tau_{d_X^{-1}}}^{-1}((d_X)^x, C))) \ominus d_{\tau_{d_X^{-1}}}((d_X)^y, C)) \\ &= \bigvee_{C\in\tau_{d_X^{-1}}}\bigvee_{x\in X}(A(x) \ominus (d_{\tau_{d_X^{-1}}}(C, (d_X)^x) \oplus d_{\tau_{d_X^{-1}}}((d_X)^y, C)) \end{aligned}$$

$$= \bigvee_{x\in X}(A(x)\ominus \bigwedge_{C\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(C,(d_X)^x)\oplus d_{\tau_{d_X^{-1}}}((d_X)^y,C))$$
$$= \bigvee_{x\in X}(A(x)\ominus d_{\tau_{d_X^{-1}}}((d_X)^y,(d_X)^x))$$
$$= \bigvee_{x\in X}(A(x)\ominus d_X(x,y)) = A(y),$$

$$d_{\tau_{d_X^{-1}}}(A, g_{\oplus}^{s\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)}))$$
$$= \bigvee_{x\in X}(A(x)\ominus d_X(x,y)) = A(y).$$

For all $A\in\tau_{d_X^{-1}}$ and $y\in X$,

$$d_{\tau_{d_{\tau_{d_X^{-1}}}}}((d_{\tau_{d_X^{-1}}})_{g(y)}, g^{s\oplus}(A))$$
$$= \bigvee_{C\in\tau_{d_X^{-1}}}((d_{\tau_{d_X^{-1}}})_{g(y)}(C)\ominus g^{s\oplus}(A)(C))$$
$$= \bigvee_{C\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(g(y),C)\ominus \bigwedge_{x\in X}(A(x)\oplus d^{-1}_{\tau_{d_X^{-1}}}(C,g(x))))$$
$$= \bigvee_{x\in X}(\bigvee_{C\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(g(y),C)d^{-1}_{\tau_{d_X^{-1}}}(C,g(x)))\ominus A(x))$$
$$= \bigvee_{x\in X}(d_{\tau_{d_X^{-1}}}(g(y),g(x))\ominus A(x))$$
$$= \bigvee_{x\in X}(d_X(x,y)\ominus A(x)) = d_{\tau_{d_X^{-1}}}(g(y),A),$$

$$d_{\tau_{d_X^{-1}}}(g_{\oplus}^{s\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)}),A)$$
$$= \bigvee_{x\in X}(d_X(x,y)\ominus A(x)) = d_{\tau_{d_X^{-1}}}(g(y),A).$$ □

Theorem 3.6. *Let* $\tau_{d^{-1}_{\tau_{d_X^{-1}}}} = \{\alpha\in L^{\tau_{d_X^{-1}}} \mid \alpha(A)\oplus d^{-1}_{\tau_{d_X^{-1}}}(A,B)\geq \alpha(B)\}$. *Define a map* $h: X\to \tau_{d^{-1}_{\tau_{d_X^{-1}}}}$ *by* $h(x)(A)=\hat{x}(A)=A(x)$. *Then* $h:(X,d_X)\to (\tau_{d^{-1}_{\tau_{d_X^{-1}}}}, d_{\tau_{d^{-1}_{\tau_{d_X^{-1}}}}})$ *is an embedding map.*

Proof. Let $A,\ B,\ C\in\tau_{d_X^{-1}}$. Then we get

$$\begin{aligned}\hat{x}(A)\oplus d^{-1}_{\tau_{d_X^{-1}}}(A,B) &= \hat{x}(A)\oplus d_{\tau_{d_X^{-1}}}(B,A)\\ &= \hat{x}(A)\oplus \bigvee_{y\in X}(B(y)\ominus A(y))\\ &\geq A(x)\oplus(B(x)\ominus A(x))\\ &\geq B(x)=\hat{x}(B).\end{aligned}$$

Thus $h(x)=\hat{x}\in\tau_{d^{-1}_{\tau_{d_X^{-1}}}}$. So h is well-defined.

Let $A\in\tau_{d_X^{-1}}$. Since $A(y)\oplus d_X^{-1}(y,x)=A(y)\oplus d_X(x,y)\geq A(x)$, we have

$$\begin{aligned}d_X(x,y) &\geq \bigvee_{A\in\tau_{d_X^{-1}}}(A(x)\ominus A(y))\\ &= \bigvee_{A\in\tau_{d_X^{-1}}}(\hat{x}(A)\ominus\hat{y}(A))\\ &= d_{\tau_{d^{-1}_{\tau_{d_X^{-1}}}}}(\hat{x},\hat{y}).\end{aligned}$$

Let $g(z)(x) = d_X(x,z)$. Since $g(z)(x) \oplus d_X(y,x) \geq g(z)(y)$ for all $z \in X$, we have $g(z) \in \tau_{d_X^{-1}}$. For $g(z) \in \tau_{d_X^{-1}}$ with $z \in X$,

$$\begin{array}{ll} d_{\tau_{d_{\tau_{d_X^{-1}}}^{-1}}}(\hat{x},\hat{y}) & = \bigvee_{A \in \tau_{d_X^{-1}}}(A(x) \ominus A(y)) \\ & \geq \bigvee_{g(z) \in \tau_{d_X^{-1}}}(g(z)(x) \ominus g(z)(y)) \\ & = \bigvee_{z \in X}(d_X(x,z) \ominus d_X(y,z)) = d_X(x,y). \end{array}$$

Hence $d_{\tau_{d_{\tau_{d_X^{-1}}}^{-1}}}(\hat{x},\hat{y}) = d_X(x,y)$. □

Example 3.7. Let $([0,1], \leq, \vee, \wedge, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice defined by $n(x) = 1 - x$, $x \oplus y = (x+y) \wedge 1$, $x \ominus y = (x-y) \vee 0$. Let $X = \{x, y, z\}$ be a set. Define $d_X \in L^{X \times X}$ by

$$d_X = \begin{pmatrix} 0 & 0.5 & 0.8 \\ 0.7 & 0 & 0.6 \\ 0.4 & 0.6 & 0 \end{pmatrix}.$$

One can show that d_X is a distance function.

(1) Define $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ by $f(x) = (d_X)_x$. For $f(x) = (d_X)_x = (0, 0.5, 0.8) \in \tau_{d_X}$ and $f(y) = (d_X)_y = (0.7, 0, 0.6) \in \tau_{d_X}$, we have

$$d_{\tau_{d_X}}(f(x), f(y)) = \bigvee_{z \in X}(d_X(x,z) \ominus d_X(y,z)) = d_X(x,y) = 0.5.$$

Since $0.4 = f(x)(x) \oplus d_X^{-1}(x,z) \not\geq f(x)(z) = 0.8$ and $0.5 = f(y)(y) \oplus d_X^{-1}(y,x) \not\geq f(y)(x) = 0.7$, we have $f(x) \notin \tau_{d_X^{-1}}, f(y) \notin \tau_{d_X^{-1}}$.

For $g(x) = (d_X)^x = (0, 0.7, 0.4) \in \tau_{d_X^{-1}}$ and $g(y) = (d_X)^y = (0.5, 0, 0.6) \in \tau_{d_X^{-1}}$, we have

$$d_{\tau_{d_X^{-1}}}(g(x), g(y)) = \bigvee_{z \in X}(d_X(z,x) \ominus d_X(z,y)) = d_X^{-1}(x,y) = d_X(y,x) = 0.7.$$

Since $0.5 = g(x)(x) \oplus d_X(x,y) \not\geq g(x)(y) = 0.7$ and $\bigwedge_{z \in X}(g(y)(z) \oplus d_X(z,-)) = f(y)$, we have $g(x) \notin \tau_{d_X}, g(y) \in \tau_{d_X}$.

Let $A,\ B,\ C \in [0,1]^X$ with

$$\begin{array}{l} A(x) = 0.3, A(y) = 0.2, A(z) = 0.5, B(x) = 0.6, B(y) = 0.3, B(z) = 0.5, \\ C(x) = 0.7, C(y) = 0.8, C(z) = 0.1. \end{array}$$

Then

$$\begin{array}{ll} A & = \bigwedge_{x \in X}(A(x) \oplus d_X(x,-)) = \bigwedge_{x \in X}(A(x) \oplus d_X(-,x)) \\ & = \bigwedge_{x \in X}(A(x) \oplus f(x)) = \bigwedge_{x \in X}(A(x) \oplus g(x)) \\ & = \bigvee_{x \in X}(A(x) \ominus f(x)) = \bigwedge_{x \in X}(A(x) \ominus g(x)), \\ B & = \bigwedge_{x \in X}(B(x) \oplus d_X(x,-)) = \bigwedge_{x \in X}(B(x) \oplus d_X(-,x)) \\ & = \bigwedge_{x \in X}(B(x) \oplus f(x)) = \bigwedge_{x \in X}(B(x) \oplus g(x)) \\ & = \bigvee_{x \in X}(B(x) \ominus f(x)) = \bigwedge_{x \in X}(B(x) \ominus g(x)), \end{array}$$

$$\begin{array}{l} C \neq \bigwedge_{x \in X}(C(x) \oplus d_X(x,-)) = (0.5, 0.7, 0.1), \\ C \neq \bigwedge_{x \in X}(C(x) \oplus d_X(-,x)) = (0.7, 0.7, 0.1), \\ C \neq \bigvee_{x \in X}(C(x) \ominus d_X(x,-)) = (0.7, 0.8, 0.2), \\ C \neq \bigvee_{x \in X}(C(x) \ominus d_X(-,x)) = (0.7, 0.8, 0.3). \end{array}$$

By Theorem 3.3, we have $A, B \in \tau_{d_X}$, $C \notin \tau_{d_X}$, $A, B \in \tau_{d_X^{-1}}$ and $C \notin \tau_{d_X^{-1}}$. By Theorem 3.4, we have $f^{s\oslash} : \tau_{d_X} \to \tau_{d_{\tau_{d_X}}}$ and $f_{\oplus}^{\leftarrow} : \tau_{d_{\tau_{d_X}}} \to \tau_{d_X}$ such that

$$\begin{array}{ll} f^{s\oslash}(B)(A) & = \bigvee_{x\in X}(B(x) \ominus d_{\tau_{d_X}}(A, (d_X)_x)) \\ & = (0.6 \ominus 0.3) \vee (0.3 \ominus 0.2) \vee (0.5 \ominus 0.4) = 0.3 \\ f^{s\oslash}(B)(f(-)) & = \bigvee_{x\in X}(B(x) \ominus d_{\tau_{d_X}}(f(-), (d_X)_x)) \\ & = \bigvee_{x\in X}(B(x) \ominus d_X(-, x)) = B = (0.6, 0.3, 0.5) \\ f_{\oplus}^{\leftarrow}(f^{s\oslash}(B))(-) & = \bigwedge_{z\in X}(f^{s\oslash}(B)(f(z)) \oplus d_X(z, -)) \\ & = \bigwedge_{z\in X}(B(z) \oplus d_X(z, -)) = B = (0.6, 0.3, 0.5). \end{array}$$

Also by Theorem 3.4, we have $f^{\oplus} : \tau_{d_X} \to \tau_{d_{\tau_{d_X}}}$ and $f_{\oslash}^{s\leftarrow} : \tau_{d_{\tau_{d_X}}} \to \tau_{d_X}$ such that

$$\begin{array}{ll} f^{\oplus}(A)(B) & = \bigwedge_{x\in X}(d_{\tau_{d_X}}((d_X)_x, B) \oplus A(x)) \\ & = (0.3 \oplus 0.3) \wedge (0.1 \oplus 0.2) \wedge (0.3 \oplus 0.5) = 0.3, \\ f^{\oplus}(A)(f(-)) & = \bigwedge_{x\in X}(d_{\tau_{d_X}}((d_X)_x, f(-)) \oplus A(x) = A = (0.3, 0.2, 0.5), \\ f_{\oslash}^{s\leftarrow}(f^{\oplus}(A))(-) & = \bigvee_{z\in X}(f^{\oplus}(A)(f(z)) \ominus d_X(-, z)) = A = (0.3, 0.2, 0.5). \end{array}$$

For all $y \in X$,

$$\begin{array}{ll} d_{\tau_{d_{\tau_{d_X}}}}(f^{s\oslash}(A), (d_{\tau_{d_X}})_{f(y)}) & = \bigvee_{C\in\tau_{d_X}}(f^{s\oslash}(A)(C) \ominus (d_{\tau_{d_X}})_{f(y)}(C)) \\ & = \bigvee_{x\in X}(A(x) \ominus d_X(y, x)) = A(y) = 0.2, \\ f_{\oplus}^{\leftarrow}((d_{\tau_{d_X}})_{f(y)})(x) & = \bigwedge_{z\in X}((d_{\tau_{d_X}})_{f(y)}(f(z)) \oplus d_X(x, z)) \\ & = d_X(y, x) = (0.7, 0, 0.6), \\ d_{\tau_{d_X}}(A, f_{\oplus}^{\leftarrow}((d_{\tau_{d_X}})_{f(y)})) & = \bigvee_{x\in X}(A(x) \ominus d_X(y, x)) = A(y) = 0.2. \end{array}$$

(2) Define $g : (X, d_X) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ by $g(x) = (d_X)^x$. By Theorem 3.5, we have

$$g^{\oslash} : \tau_{d_X^{-1}} \to \tau_{d_{\tau_{d_X^{-1}}}} \text{ and } g_{\oplus}^{s\leftarrow} : \tau_{d_{\tau_{d_X^{-1}}}} \to \tau_{d_X^{-1}} \text{ such that}$$

$$\begin{array}{ll} g^{\oslash}(B)(A) & = \bigvee_{x\in X}(B(x) \ominus d_{\tau_{d_X^{-1}}}(A, g(x))) \\ & = \bigvee_{x\in X}(B(x) \ominus d_{\tau_{d_X^{-1}}}(A, (d_X)^x)) \\ & = (0.6 \ominus 0.3) \vee (0.3 \ominus 0.2) \vee (0.5 \ominus 0.5) = 0.3, \\ g^{\oslash}(B)(g(-)) & = \bigvee_{x\in X}(B(x) \ominus d_{\tau_{d_X^{-1}}}(g(-), g(x))) \\ & = \bigvee_{x\in X}(B(x) \ominus d_X^{-1}(-, x)) = \bigvee_{x\in X}(B(x) \ominus d_X(x, -)) \\ & = B = (0.6, 0.3, 0.5), \\ g_{\oplus}^{s\leftarrow}(g^{\oslash}(B))(-) & = \bigwedge_{z\in X}(g^{\oslash}(B)(g(z)) \oplus d_X(-, z)) \\ & = \bigwedge_{z\in X}(B(z) \oplus d_X(-, z)) = B = (0.6, 0.3, 0.5). \end{array}$$

Also by Theorem 3.5, we have $g^{s\oplus} : \tau_{d_X^{-1}} \to \tau_{d_{\tau_{d_X^{-1}}}}$ and $g_{\oslash}^{\leftarrow} : \tau_{d_{\tau_{d_X^{-1}}}} \to \tau_{d_X^{-1}}$ such that

$$\begin{array}{ll} g^{s\oplus}(A)(B) & = \bigwedge_{x\in X}(d^{-1}_{\tau_{d_X^{-1}}}(B, (d_X)^x) \oplus A(x)) \\ & = \bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}((d_X)^x, B) \oplus A(x)) \\ & = (0.4 \oplus 0.3) \wedge (0.1 \oplus 0.2) \wedge (0.3 \oplus 0.5) = 0.3, \\ g^{s\oplus}(A)(g(-)) & = \bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}(g(-), (d_X)^x) \oplus A(x)) \\ & = \bigwedge_{x\in X}(d_X(x, -) \oplus A(x)) = A = (0.3, 0.2, 0.5), \\ g_{\oslash}^{\leftarrow}(g^{s\oplus}(A))(-) & = \bigvee_{z\in X}(g^{s\oplus}(A)(g(z)) \ominus d_X(z, x)) = A = (0.3, 0.2, 0.5) \\ & = \bigvee_{z\in X}(A(z) \ominus d_X(z, -)) = A(-). \end{array}$$

For all $y \in X$,

$$\begin{array}{rl} d_{\tau_{d_{\tau_{d_X^{-1}}}}}(g^{\oslash}(A), (d_{\tau_{d_X^{-1}}})_{g(y)}) & = \bigvee_{x\in X}(A(x) \ominus d_X(x,y)) = A(y) = 0.2, \\ g_{\oplus}^{s\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)})(-) & = d_X(-,y) = (0.5, 0, 0.6), \\ d_{\tau_{d_X^{-1}}}(A, g_{\oplus}^{s\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)})) & = \bigvee_{x\in X}(A(x) \ominus d_X(x,y)) = A(y) = 0.2, \\ d_{\tau_{d_{\tau_{d_X^{-1}}}}}((d_{\tau_{d_X^{-1}}})_{g(y)}, g^{s\oplus}(A)) & = \bigvee_{x\in X}(d_X(x,y) \ominus A(x)) = 0.2 \\ & = d_{\tau_{d_X^{-1}}}(g_{\oplus}^{s\leftarrow}((d_{\tau_{d_X^{-1}}})_{g(y)}), A). \end{array}$$

Example 3.8. Let $X = \{h_i \mid i = \{1,2,3\}\}$ and $Y = \{e, b, w, c, i\}$ be sets with h_i=house, e=expensive, b= beautiful, w=wooden, c= creative, i=in the green surroundings. Let $([0,1], \oplus, \ominus, n, 0, 1)$ be a complete co-residuated lattice defined in Example 3.7. Let $R \in [0,1]^{X\times Y}$ be the fuzzy information as follows:

R	e	b	w	c	i
h_1	0.9	0.3	0.7	0.9	0.2
h_2	0.5	0.8	0.4	0.3	0.5
h_3	0.4	0.9	0.8	0.7	0.6

Define $d_X : X \times X \to [0,1]$ by

$$d_X(x,y) = \bigvee_{a\in Y}(R(x,a) \ominus R(y,a)).$$

Then

d_X	h_1	h_2	h_3
h_1	0	0.6	0.5
h_2	0.5	0	0
h_3	0.6	0.4	0

Define $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ by $f(x) = (d_X)_x$ and $g : (X, d_X) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ by $g(x) = (d_X)^x$. Then f and g are two embedding maps. By a similar method used in Example 3.7, one can investigate various maps.

4. Conclusion

Let (X, d_X) be a distance space. We have constructed two embedding maps $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ and $g : (X, d_X^{-1}) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ by $f(x) = (d_X)_x$ and $g(x) = (d_X)^x$ where $(d_X)_x(y) = d_X(x,y)$ and $(d_X)^x(y) = d_X(y,x)$ for all $x, y \in X$. We have studied their properties and have given their examples. As a topological representation, we have investigated the embedding map $h : (X, d_X) \to (\tau_{d^{-1}_{\tau_{d_X^{-1}}}}, d_{\tau_{d^{-1}_{\tau_{d_X^{-1}}}}})$. Moreover, we have suggested Example 3.8 for an information system.

In the future, by using the concepts of embedding maps, information systems and decision rules can be investigated on co-residuated lattices.

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