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Interval-valued relations and their application to category theory

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Abstract. In this paper, we define an interval-valued relation and an interval-valued partition, and study some of their properties. Also we define interval-valued relational spaces and obtain some properties of the category (denoted by **IVRel**), which is class of interval-valued relational spaces and the morphisms between them. Furthermore, we prove that the full subcategory $\mathbf{IVRel_R}$ of the category **IVRel** is a topological universe over **Set**.

2020 AMS Classification: 18A05

Keywords: Interval-valued set, Interval-valued (vanishing) point, Interval-valued relation, Interval-valued partition, Interval-valued relational space, Topological (co-topological) category, Cartesian closed category, Topological universe.

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1. Introduction

In 2020, Kim et al. [1] studied various topological structures via interval-valued sets proposed by Yao [2]. Recently, Han et al. [3] introduced the notions of interval-valued ideals, interval-valued positive implicative ideals, interval-valued implicative ideals and interval-valued commutative ideals in BCK-algebras, and discussed some of their properties.

A (binary) relation play an important role in congruence, graph theory, and computer science, etc. The category is applied to many fields of mathematics including abstract algebra. Moreover, it has an important relevance in the study of theoretical computer science, mathematical fundamentals, and mathematical physics. In particular, it has already been known ([4, 5, 6, 7]) that the concept of the topological universe proposed by Nell [8] can be effectively used in various fields of mathematics. Recently, Lee et al. [9] constructed the category $CRel_P(H)$ [resp. $CRel_R(H)$] of cubic H-relational spaces and P-preserving [resp. R-preserving] mappings between them, and discussed their categorical structures in the sense of a topological universe (See [10, 11, 12, 13] for the further researches).

It is our aim to study the category of relations based on interval-valued sets in a viewpoint of a topological universe. To do this, we study in two directions. First, we define an interval-valued relation and obtain its various properties. Second, we form the category (denoted by **IVRel**), which is class of interval-valued relational spaces and the morphisms between them, and find some of its properties. Moreover, we prove that the full subcategory $\mathbf{IVRel_R}$ of the category **IVRel** is the topological universe over **Set**.

2. Preliminaries

We list basic definitions and two results interval-valued sets needed in next sections.

Definition 2.1 (See [2]). Let X be an nonempty set. Then the form

$$[A^-, A^+] = \{B \subset X : A^- \subset B \subset A^+\}$$

is called an *interval-valued set* (briefly, IVS) or *interval set* in X, if $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In this case, A^- [resp. A^+] represents the set of minimum [resp. maximum] memberships of elements of X to A. In fact, A^- [resp. A^+] is a minimum [resp. maximum] subset of X agreeing or approving for a certain opinion, view, suggestion or policy. $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the *interval-valued empty* [resp. *whole*] *set* in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$]. We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in IVS(X)$, then

$$\chi_A = [\chi_{A^-}, \chi_{A^+}]$$

is an interval-valued fuzzy set in X introduced by Zadeh [14]. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

Definition 2.2 (See [2]). Let X be a nonempty set and let $A,\ B \in IVS(X)$. Then

(i) we say that A is *contained in* B, denoted by $A \subset B$, if $A^- \subset B^-$ and $A^+ \subset B^+$,
(ii) we say that A is *equal to* B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,
(iii) the *complement* of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = [(A^+)^c, (A^-)^c],$$

(iv) the *union* of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = [A^- \cup B^-, A^+ \cup B^+],$$

(v) the *intersection* of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = [A^- \cap B^-, A^+ \cap B^+].$$

Definition 2.3 ([1]). Let X be a nonempty set, let $a \in X$ and let $A \in IVS(X)$. Then the form $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an *interval-valued* [resp. *vanishing*] *point* in X and denoted by a_1 [resp. a_0]. We will denote the set of all interval-valued points in X as $IVP(X) = IV_P(X) \cup IV_{VP}(X)$, where $IV_P(X)$ [resp. $IV_{VP}(X)$] denotes the set of all interval-valued [resp. vanishing] points in X.

(i) We say that a_1 *belongs to* A, denoted by $a_1 \in A$, if $a \in A^-$.
(ii) We say that a_0 *belongs to* A, denoted by $a_0 \in A$, if $a \in A^+$.

Result 2.4 (Theorem 3.14 [1]). *Let $(A_j)_{j \in J} \subset IVS(X)$ and let $a \in X$.*

(1) $a_1 \in \bigcap A_j$ [resp. $a_0 \in \bigcap A_j$] *if and only if $a_1 \in A_j$ [resp. $a_0 \in A_j$], for each $j \in J$.*
(2) $a_1 \in \bigcup A_j$ [resp. $a_0 \in \bigcup A_j$] *if and only if there exists $j \in J$ such that* $a_1 \in A_j$ [resp. $a_0 \in A_j$].

Result 2.5 (Theorem 3.15 [1]). *Let $A, B \in IVS(X)$. Then*

(1) *$A \subset B$ if and only if $a_1 \in A \Rightarrow a_1 \in B$* [resp. $a_0 \in A \Rightarrow a_0 \in B$] *for each $a \in X$.*
(2) *$A = B$ if and only if $a_1 \in A \Leftrightarrow a_1 \in B$* [resp. $a_0 \in A \Leftrightarrow a_0 \in B$] *for each $a \in X$.*

3. Interval-valued relations

We define an interval-valued relation from X to Y, and study some of its properties, where $X$and Y are nonempty sets. Also we define an interval-valued equivalence relation on X and an interval-valued partition of X and study some of its properties.

Definition 3.1. Let X, Y be two nonempty sets and let $A \in IVS(X)$, $B \in IVS(Y)$. Then the *Cartesian product* of A and B, denoted by $A \times B$, is an interval-valued set in $X \times Y$ defined as follows:

$$A \times B = [A^- \times B^-, A^+ \times B^+].$$

Example 3.2. Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$ be sets. Let $A \in IVS(X)$, $B \in IVS(Y)$ given by:

$$A = [\{a, b\}, \{a, b, d\}],\ B = [\{1\}, \{1, 2\}].$$

Then clearly, $A \times B = [\{(a, 1), (b, 1)\}, \{(a, 1), (a, 2), (b, 1), (b, 2), (d, 1), (d, 2)\}]$.

Proposition 3.3. *Let X, Y be two nonempty sets. Then the followings hold:*

$$a_1 \times b_1 = [\{(a, b)\}, \{(a, b)\}] \text{ and } a_0 \times b_0 = [\varnothing, \{(a, b)\}]$$

for any $a_1,\ a_0 \in \widetilde{X}$ and $b_1,\ b_0 \in \widetilde{Y}$.

In Proposition 3.3, we write $a_1 \times b_1$ [resp. $a_0 \times b_0$] as $(a, b)_1$ [resp. $(a, b)_0$] and it will be called an *interval-valued* [resp. *vanishing*] *ordered point* in $X \times Y$. Then by Definition 2.3, $IVP(X \times Y) = IV_P(X \times Y) \cup IV_{VP}(X \times Y)$. In this case, $IVP(X \times Y)$ will be called the *interval-valued Cartesian product* of X and Y, and denoted by $\widetilde{X} \times \widetilde{Y}$.

Example 3.4. Let $X = \{a, b, c\},\ Y = \{1, 2\}$. Then clearly,

$$\widetilde{X} \times \widetilde{Y} = \{(a,1)_1, (a,2)_1, (b,1)_1, (b,2)_1, (c,1)_1, (c,2)_1, (a,1)_0, (a,2)_0, (b,1)_0, (b,2)_0, (c,1)_0, (c,2)_0\}.$$

Remark 3.5. (1) Let $X_1,\ X_2,\ \cdots,\ X_n$ be nonempty set. Then we can define the *interval-valued Cartesian product* of $X_1,\ X_2,\ \cdots,\ X_n$ as follows:

$$\prod_{i=1}^{n} \widetilde{X_i} = IVP(\prod_{i=1}^{n} X_i) = IV_P(\prod_{i=1}^{n} X_i) \cup IV_{VP}(\prod_{i=1}^{n} X_i).$$

Each member of $IV_P(\prod_{i=1}^{n} X_i)$ [resp. $IV_{VP}(\prod_{i=1}^{n} X_i)$] is called an *interval-valued* [resp. *vanishing*] *n-ordered point* in $\prod_{i=1}^{n} X_i$.

(2) Let $(X_j)_{j\in J}$ be a collection of arbitrary sets. Then we can define the *interval-valued Cartesian product* of $(X_j)_{j\in J}$ as follows:

$$\prod_{j\in J} \widetilde{X_j} = IVP(\prod_{j\in J} X_j) = IV_P(\prod_{j\in J} X_j) \cup IV_{VP}(\prod_{j\in J} X_j).$$

Each member of $IV_P(\prod_{j\in J} X_j)$ [resp. $IV_{VP}(\prod_{j\in J} X_j)$] is called an *interval-valued* [resp. *vanishing*] *arbitrary-ordered point* in $\prod_{j\in J} X_j$.

Proposition 3.6. *Let X be a nonempty set. Then $A \times \widetilde{\varnothing} = \widetilde{\varnothing} = \widetilde{\varnothing} \times A$ for each $A \in IVS(X)$.*

Proposition 3.7. *Let X be a nonempty set and let $A,\ B,\ C \in IVS(X)$. Then the followings hold:*

(1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$,
(2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Definition 3.8. Let $X,\ Y$ be two nonempty sets. Then $\rho = [\rho^-, \rho^+]$ is called an *interval-valued relation from X to Y*, if $\rho \in IVS(X \times Y)$, i.e., $\rho^-,\ \rho^+ \subset X \times Y$. If $\rho \in IVS(X \times X)$, then ρ is called an *interval-valued relation on X*. We will denote the *interval-valued empty* [resp. *whole*] *relation on X* as $\dot{\varnothing}$ [resp. $\dot{X}$] and the set of all interval-valued relations on X as $IVR(X)$.

Example 3.9. Let $X = \{a, b, c, d, e\}$. Consider the interval-valued set ρ in $X \times X$ given by:

$$\rho = [\{(a,a), (a,b), (b,c), (d,e), (e,e)\}, \{(a,a), (a,b), (n,a), (b,c), (c,d), (d,e), (e,e)\}].$$

Then clearly, $\rho \in IVR(X)$.

Remark 3.10. (1) If ρ is a classical relation on a set X, then $[\varnothing, \rho],\ [\rho, \rho] \in IVR(X)$.

(2) If $\rho = [\rho^-, \rho^+]$ is an interval-valued relation on a set X, then $\chi_\rho = [\chi_{\rho^-}, \chi_{\rho^+}]$ is an interval-valued fuzzy relation on X in the sense of Roy and Biswas [15]. Moreover, ρ^- and ρ^+ are classical relations on X.

(3) If $\rho \in IVR(X)$, then $\rho^-,\ \rho^+ \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the set of all classical relations on X (See [16]).

From (1) and (2), we can consider an interval-valued relation as a generalization of a classical relation and a special case of an interval-valued fuzzy relation.

Theorem 3.11. *Let X, Y be two nonempty sets. Then ρ is an interval-valued relation from X to Y if and only if there is $\mathcal{R} \subset \widetilde{X} \times \widetilde{Y}$ such that $\rho = \bigcup \mathcal{R}$.*

Proof. The proof is obvious from Proposition 3.3 and Definition 3.8. □

Example 3.12. Let $X = \{a, b, c\}$. Then we have

$$\widetilde{X} \times \widetilde{X} = \{(a,a)_1, (a,b)_1, (a,c)_1, (b,a)_1, (b,b)_1, (b,c)_1, (c,a)_1, (c,b)_1, (c,c)_1, (a,a)_0, (a,b)_0, (a,c)_0, (b,a)_0, (b,b)_0, (b,c)_0, (c,a)_0, (c,b)_0, (c,c)_0\}.$$

Consider $\mathcal{R} \subset \widetilde{X} \times \widetilde{X}$ given by:

$$\mathcal{R} = \{(a,a)_1, (b,a)_1, (c,a)_0, (c,c)_0\}.$$

Then we easily check that

$$\bigcup \mathcal{R} = [\{(a,a), (b,a)\}, \{(a,a), (b,a), (c,a), (c,c)\}].$$

Thus $\bigcup \mathcal{R} \in IVR(X)$. Furthermore, $\dot{X} = \bigcup(\widetilde{X} \times \widetilde{X}) = [X \times X, X \times X]$.

Since an interval-valued relation from a set X to a set Y is an interval-valued set in $X \times Y$, we can define the inclusion $\rho \subset \sigma$, the intersection $\rho \cap \sigma$, the union $\rho \cup \sigma$ and the complement ρ^c for any interval-valued relations ρ and σ same as Definition 2.2. Then we obtain the following results.

Proposition 3.13 (See (i1)–(i3) and (k1)–(k3), [2])**.** *Let X be a nonempty set and let ρ, σ, $\tau \in IVR(X)$. Then*

(1) $\dot{\varnothing} \subset \rho \subset \dot{X}$,
(2) *if* $\rho \subset \sigma$ *and* $\sigma \subset \tau$, *then* $\rho \subset \tau$,
(3) $\rho \subset \rho \cup \sigma$ *and* $\sigma \subset \rho \cup \sigma$,
(4) $\rho \cap \sigma \subset \rho$ *and* $\rho \cap \sigma \subset \sigma$,
(5) $\rho \subset \sigma$ *if and only if* $\rho \cap \sigma = \rho$,
(6) $\rho \subset \sigma$ *if and only if* $\rho \cup \sigma = \sigma$.

Proposition 3.14 (See (I1)–(I8), [2])**.** *Let X be a non-empty set and let ρ, σ, $\tau \in IVR(X)$. Then*

(1) (Idempotent laws) $\rho \cup \rho = \rho$, $\rho \cap \rho = \rho$,
(2) (Commutative laws) $\rho \cup \sigma = \sigma \cup \rho$, $\rho \cap \sigma = \sigma \cap \rho$,
(3) (Associative laws) $\rho \cup (\sigma \cup \tau) = (\rho \cup \sigma) \cup \tau$, $\rho \cap (\sigma \cap \tau) = (\rho \cap \sigma) \cap \tau$,
(4) (Distributive laws) $\rho \cup (\sigma \cap \tau) = (\rho \cup \sigma) \cap (\rho \cup \tau)$,
$\rho \cap (\sigma \cup \tau) = (\rho \cap \sigma) \cup (\rho \cap \tau)$,
(5) (Absorption laws) $\rho \cup (\rho \cap \sigma) = \rho$, $\rho \cap (\rho \cup \sigma) = \rho$,
(6) (DeMorgan's laws) $(\rho \cup \sigma)^c = \rho^c \cap \sigma^c$, $(\rho \cap \sigma)^c = \rho^c \cup \sigma^c$,
(7) $(\rho^c)^c = \rho$,
(8) (8_a) $\rho \cup \dot{\varnothing} = \rho$, $\rho \cap \dot{\varnothing} = \dot{\varnothing}$,
(8_b) $\rho \cup \dot{X} = \dot{X}$, $\rho \cap \dot{X} = \rho$,
(8_c) $\dot{X}^c = \dot{\varnothing}$, $\dot{\varnothing}^c = \dot{X}$,
(8_d) $\rho \cup \rho^c \neq \dot{X}$, $\rho \cap \rho^c \neq \dot{\varnothing}$ *in general* (See Example 3.7, [1]).

Example 3.15. Let $X = \{a, b, c\}$. Consider the interval-valued relation on X given by:

$$\rho = [\{(a,a), (b,c)\}, \{(a,a), (a,b), (b,c)\}] \in IVR(X).$$

Then clearly, we have

$\rho^c = [\{(a,c),(b,a),(b,b),(c,a),(c,b),(c,c)\}, \{(a,b),(a,c),(b,a),(b,b),(c,a),(c,b),(c,c)\}]$.

Thus we get

$$\rho \cap \rho^c = [\varnothing, \{(a,b)\}] \neq \dot{\varnothing},$$
$$\rho \cup \rho^c = [\{(a,a),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}, X] \neq \dot{X}.$$

Definition 3.16. Let X, Y be two nonempty sets, let $(a,b)_1$, $(a,b)_0 \in \widetilde{X} \times \widetilde{Y}$ and let ρ be an interval-valued relation from X to Y.

(i) We say that $(a,b)_1$ *belongs to* ρ, *denoted by* $(a,b)_1 \in \rho$, if $(a,b) \in \rho^-$.
(ii) We say that $(a,b)_0$ *belongs to* ρ, *denoted by* $(a,b)_0 \in \rho$, if $(a,b) \in \rho^+$.

Proposition 3.17 (See Proposition 3.11 [1])**.** *Let X, Y be two nonempty sets, let ρ be an interval-valued relation from X to Y and let*

$$\rho_{IVP} = \bigcup_{(a,b)_1 \in \rho} (a,b)_1,\ \rho_{IVVP} = \bigcup_{(a,b)_0 \in \rho} (a,b)_0.$$

Then $\rho = \rho_{IVP} \cup \rho_{IVVP}$. In fact, $\rho_{IVP} = [\rho^-, \rho^-]$ and $\rho_{IVVP} = [\varnothing, \rho^+]$.

Theorem 3.18 (See Proposition 3.14 [1])**.** *Let X be a nonempty set, let $(\rho_j)_{j\in J} \subset IVR(X)$ and let $(x,y) \in X \times X$.*

(1) $(x,y)_1 \in \bigcap_{j\in J} \rho_j$ [resp. $(x,y)_0 \in \bigcap_{j\in J} \rho_j$] *if and only if* $(x,y)_1 \in \rho_j$ [resp. $(x,y)_0 \in \rho_j$] *for each* $j \in J$.
(2) $(x,y)_1 \in \bigcup_{j\in J} \rho_j$ [resp. $(x,y)_0 \in \bigcup_{j\in J} \rho_j$] *if and only if* $(x,y)_1 \in \rho_j$ [resp. $(x,y)_0 \in \rho_j$] *for some* $j \in J$.

Theorem 3.19 (See Proposition 3.16 [1])**.** *Let X be a nonempty set and let ρ, $\sigma \in IVR(X)$. Then $\rho \subset \sigma$ if and only if $(x,y)_1 \in \rho \Rightarrow (x,y)_1 \in \sigma$* [resp. $(x,y)_0 \in \rho \Rightarrow (x,y)_0 \in \sigma$] *for each $(x,y) \in X \times X$.*

Let ρ be an interval-valued relation from a set X to a set Y. Then we write $(a,b)_1$, $(a,b)_0 \in \rho$ as $a_1 \rho b_1$ and $a_0 \rho b_0$, and may be read: "a_1 is ρ-related to a_1" and "a_0 is ρ-related to a_0".

Definition 3.20. Let ρ be an interval-valued relation from a set X to a set Y. Then the *domain* and the *image* of ρ, denoted by $Dom(\rho)$ and $Im(\rho)$, are an interval-valued set in X and Y defined as follows:

$$Dom(\rho) = \bigcup\{a_1 \in \widetilde{X} : (a,b)_1 \in \rho \text{ for some } b_1 \in \widetilde{Y}\}$$
$$\cup \bigcup\{a_0 \in \widetilde{X} : (a,b)_0 \in \rho \text{ for some } b_0 \in \widetilde{Y}\},$$

$$Im(\rho) = \bigcup\{b_1 \in \widetilde{Y} : (a,b)_1 \in \rho \text{ for some } a_1 \in \widetilde{X}\}$$
$$\cup \bigcup\{b_0 \in \widetilde{Y} : (a,b)_0 \in \rho \text{ for some } a_0 \in \widetilde{X}\}.$$

Remark 3.21. Let ρ be the interval-valued relation from a set X to a set Y. Then from Definition 3.20, we can easily see that

$$Dom(\rho) = Dom(\rho^-) \cup Dom(\rho^+),\ Im(\rho) = Im(\rho^-) \cup Im(\rho^+),$$

where

$Dom(\rho^-) = \{a \in X : (\exists b \in Y)(a,b) \in \rho^-\}$, $Dom(\rho^+) = \{a \in X : (\exists b \in Y)(a,b) \in \rho^+\}$,

$Im(\rho^-) = \{b \in Y : (\exists a \in X)(a,b) \in \rho^-\},\ Im(\rho^+) = \{b \in Y : (\exists a \in Y)(a,b) \in \rho^+\}.$

Example 3.22. Let ρ be the interval-valued relation from $X = \{a,b,c\}$ to $Y = \{1,2,3\}$ given by:

$$\rho = [\{(a,1),(b,2)\},\{(a,1),(b,1),(b,2),(c,1)\}.$$

Then clearly, $Dom(\rho) = [\{a,b\},\{a,b,c\}]$ and $Im(\rho) = [\{1,2\},\{1,2\}]$.

Proposition 3.23. *Let $\rho,\ \sigma \in IVR(X)$. If $\rho \subset \sigma$, then $Dom(\rho) \subset Dom(\sigma)$ and $Im(\rho) \subset Im(\sigma)$*

Proof. The proof is obvious from Theorem 3.19 and Definition 3.20. □

Definition 3.24. Let $\rho,\ \sigma \in IVR(X)$. Then the *product* of ρ and σ, denoted by $\sigma \circ_{IV} \rho$, is an interval-valued relation on X defined as follows:

$$\sigma \circ_{IV} \rho = \bigcup\{(x,y)_1 \in \widetilde{X} \times \widetilde{X} : (\exists z_1 \in \widetilde{X})\ (x,z)_1 \in \rho,\ (z,y)_1 \in \sigma\}$$
$$\cup \bigcup\{(x,y)_0 \in \widetilde{X} \times \widetilde{X} : (\exists z_0 \in \widetilde{X})\ (x,z)_0 \in \rho,\ (z,y)_0 \in \sigma\}.$$

In fact, we can easily see that

$$\sigma \circ_{IV} \rho = [\sigma^- \circ \rho^-, \sigma^+ \circ \rho^+],$$

where $\sigma^- \circ \rho^-$ and $\sigma^+ \circ \rho^+$ denote the product of classical relations.

Proposition 3.25. *Let $\rho,\ \sigma,\ \tau \in IVR(X)$. If $\rho \subset \sigma$, then $\rho \circ_{IV} \tau \subset \sigma \circ_{IV} \tau$ and $\tau \circ_{IV} \rho \subset \tau \circ_{IV} \sigma$.*

Proposition 3.26 (See Proposition 4.4, [16])**.** *$(IVR(X), \circ_{IV})$ is a semigroup.*

Definition 3.27. Let $\rho = [\rho^-, \rho^+]$ be an interval-valued relation from a set X to a set Y. Then the *inverse* of ρ, denoted by $\rho^{-1} = [(\rho^-)^{-1}, (\rho^+)^{-1}]$, is an interval-valued relation from Y to X such that $b_1 \rho^{-1} a_1,\ b_0 \rho^{-1} a_0$ if and only if $a_1 \rho b_1,\ a_0 \rho b_0$, i.e.,

$$\rho^{-1} = \bigcup\{(b,a)_1 \in \widetilde{Y} \times \widetilde{X} : (a,b)_1 \in \rho\} \cup \bigcup\{(b,a)_0 \in \widetilde{Y} \times \widetilde{X} : (a,b)_0 \in \rho\}.$$

In fact, we can easily have

$$\rho^{-1} = \rho^{-1}_{IVP} \cup \rho^{-1}_{IVVP}.$$

Proposition 3.28. *Let ρ be an interval-valued relation from a set X to a set Y. Then $Dom(\rho) = Im(\rho^{-1})$ and $Im(\rho) = Dom(\rho^{-1})$.*

Proposition 3.29. *Let $\rho,\ \sigma,\ \rho_1, \cdots,\ \rho_n \in IVR(X)$.*

(1) *$\rho^{-1} \in IVR(X)$ and $(\rho^{-1})^{-1} = \rho$.*
(2) *If $\rho \subset \sigma$, then $\rho^{-1} \subset \sigma^{-1}$.*
(3) *$\rho_1 \circ_{IV} \rho_2 \circ_{IV} \cdots \circ_{IV} \rho_n)^{-1} = \rho_n^{-1} \circ_{IV} \cdots \circ_{IV} \rho_2^{-1} \circ_{IV} \rho_1^{-1}$.*

Definition 3.30. Let X be a nonempty set, let $\rho \in IVR(X)$ and let $x \in X$, $A \in IVS(X)$.

(i) ρx_1, ρx_0 and ρx are interval-valued sets in X defined respectively as follows:

$$\rho x_1 = \bigcup_{(x,y)_1 \in \rho} y_1,\ \rho x_0 = \bigcup_{(x,y)_0 \in \rho} y_0,\ \rho x = \rho x_1 \cup \rho x_0,$$

where $y_1,\ y_0 \in \widetilde{X}$. In fact, we can easily see that

$$\rho x = \rho^{-} x \cup \rho^{+} x,$$

where $\rho^{-}x = \{y \in X : (x, y) \in \rho^{-}\},\ \rho^{+}x = \{y \in X : (x, y) \in \rho^{-}\}$.

(ii) ρA is the interval-valued set in X defined by:

$$\rho A = \bigcup_{a_1 \in A} \rho a_1 \cup \bigcup_{a_0 \in A} \rho a_0.$$

In fact, $(\rho A)^{-} = \bigcup_{a \in A^{-}} (\rho)^{-} a,\ (\rho)^{+} A = \bigcup_{a \in A^{+}} (\rho)^{+} a$.

It is obvious that $\rho x \neq \widetilde{\varnothing}$ if and only if $\rho x_1 \neq \widetilde{\varnothing}$ or $\rho x_0 \neq \widetilde{\varnothing}$, i.e., $x_1 \in Dom(\rho)$ or $x_0 \in Dom(\rho)$.

Remark 3.31. Let X be a nonempty set, let $\rho \in IVR(X)$ and let $x \in X$. Then $\rho^{-1}x$ is the interval-valued set in X similarly defined as Definition 3.30 (i) and $\rho^{-1}x \neq \widetilde{\varnothing}$ if and only if $x_1 \in Im(\rho)$ or $x_0 \in Im(\rho)$.

Definition 3.32. Let X be a nonempty set. Then φ is called an *interval-valued partial mapping* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,

(i) if $(x, y)_1 \in \varphi$ and $(x, z)_1 \in \varphi$, then $y_1 = z_1$,
(ii) if $(x, y)_0 \in \varphi$ and $(x, z)_0 \in \varphi$, then $y_0 = z_0$.

We will denote the set of all interval-valued partial mappings of X as $IVPM(X)$. If $\varphi \in IVPM(X)$ and $(x, y)_1 \in \varphi$ [resp. $(x, y)_0 \in \varphi$], then we will write

$$\varphi(x_1) = y_1 \text{ [resp.} \varphi(x_0) = y_0].$$

Remark 3.33. (1) If $\varphi \in \mathcal{PT}(X)$, then $[\varphi, \varphi] \in IVPM(X)$, where $\mathcal{PT}(X)$ denotes the set of all partial mappings of X (See [16]).
(2) If $\varphi \in IVPM(X)$, then $\varphi^{-},\ \varphi^{+} \in \mathcal{PT}(X)$.
(3) Let $\varphi \in IVPM(X)$. Then φ^{-1} need not be an interval-valued partial mapping of X (See Example 3.34).

Example 3.34. Let $X = \{a, b, c\}$. Consider three interval-valued relations ρ, σ and φ on X given by:

$$\rho = [\{(a, b), (a, c), (b, c)\}, \{(a, a), (a, b), (a, c), (b, b), (b, c)\}],$$

$$\sigma = [\{(a, a), (b, a)\}, \{(a, a), (b, a), (c, a), (c, b)\}],$$

$$\varphi = [\{(a, a), (b, a)\}, \{(a, a), (b, a), (c, b)\}].$$

Then we can easily check that $\rho \notin IVPM(X)$, $\sigma \notin IVPM(X)$ but $\varphi \in IVPM(X)$. By Remark 3.33 (2), we can see that

$$\varphi^{-} = \{(a, a), (b, a)\},\ \varphi^{+} = \{(a, a), (b, a), (c, b)\} \in \mathcal{PT}(X).$$

Furthermore, $\varphi^{-1} \notin IVPM(X)$ since $(\varphi^{-})^{-1} = \{(a, a), (a, b)\} \notin \mathcal{PT}(X)$.

Proposition 3.35 (See Proposition 4.16, [16])**.** *$IVPM(X)$ is a subsemigroup of $(IVR(X), \circ_{IV})$.*

Proof. It is sufficient to prove that $\psi \circ_{IV} \varphi \in IVPM(X)$ for any $\varphi,\ \psi \in IVPM(X)$. Let $\varphi,\ \psi \in IVPM(X)$ and suppose $(x,y)_1,\ (x,z)_1 \in \psi \circ_{IV} \varphi$. Then there are $u_1,\ v_1 \in \widetilde{X}$ such that

$$(x,u)_1 \in \psi,\ (u,y)_1 \in \varphi,\ (x,v)_1 \in \psi,\ (v,z)_1 \in \varphi.$$

Thus by Definition 3.32 (i), $u_1 = v_1$ and $y_1 = z_1$. So $\psi \circ_{IV} \varphi$ satisfies the condition Definition 3.32 (i). Similarly, we can show that $\psi \circ_{IV} \varphi$ satisfies the condition Definition 3.32 (ii). Hence $\psi \circ_{IV} \varphi \in IVPM(X)$. □

Proposition 3.36 (See Proposition 4.17, [16])**.** *Suppose* $\varphi,\ \psi \in IVPM(X)$. *Then*

(1) $Dom(\psi \circ_{IV} \varphi) = \varphi^{-1}[Im(\varphi) \cap Dom(\psi)]$,
(2) $Im(\psi \circ_{IV} \varphi) = \psi[Im(\varphi) \cap Dom(\psi)]$,
(3) $(\psi \circ_{IV} \varphi)(x_1) = \varphi(\psi(x_1))$ *and* $(\psi \circ_{IV} \varphi)(x_0) = \varphi(\psi(x_0))$ *for any* $x_1,\ x_0 \in Dom(\psi \circ_{IV} \varphi)$.

Proof. (1) Let $x_1 \in Dom(\psi \circ_{IV} \varphi)$. Then there are $y_1,\ z_1 \in \widetilde{X}$ such that $(x,y)_1 \in \varphi,\ (y,z)_1 \in \psi$. Thus $y_1 \in [Im(\varphi) \cap Dom(\psi)]$ and $(y,x)_1 \in \varphi^{-1}$. So by Definition 3.30, we have

$$x_1 \in \varphi^{-1}y_1 \subset \varphi^{-1}[Im(\varphi) \cap Dom(\varphi)].$$

Similarly, we get $x_0 \in \varphi^{-1}[Im(\varphi) \cap Dom(\psi)]$ for each $x_0 \in Dom(\psi \circ_{IV} \varphi)$. Hence $Dom(\psi \circ_{IV} \varphi) \subset \varphi^{-1}[Im(\varphi) \cap Dom(\psi)]$.

Conversely, let $x_1 \in \varphi^{-1}[Im(\varphi) \cap Dom(\psi)]$. Then there is $z_1 \in Im(\varphi) \cap Dom(\psi)$ such that $x_1 \in \varphi^{-1}z_1$, i.e., $(x,z)_1 \in \varphi$. Since $z_1 \in Dom(\psi)$, there is $y_1 \in \widetilde{X}$ such that $(z,y)_1 \in \psi$. Thus $(x,y)_1 \in \psi \circ_{IV} \varphi$, i.e., $x_1 \in Dom(\psi \circ_{IV} \varphi)$. Similarly, we have $x_0 \in Dom(\psi \circ_{IV} \varphi)$. So $\varphi^{-1}[Im(\varphi) \cap Dom(\psi)] \subset Dom(\psi \circ_{IV} \varphi)$. Hence the result holds.

(2) The proof is similar to (1).

(3) $(x,z)_1 \in \psi \circ_{IV} \varphi$ if and only if there is $y_1 \in \widetilde{X}$ such that $(x,y)_1 \in \varphi$ and $(y,z)_1 \in \psi$. Since $\varphi,\ \psi,\ \psi \circ_{IV} \varphi \in IVPM(X)$, Definition 3.32, we have $y_1 = \varphi(x_1)$ and $z_1 = \psi(y_1)$. Thus $(\psi \circ_{IV} \varphi)(x_1) = \varphi(\psi(x_1))$. Similarly, we have $(\psi \circ_{IV} \varphi)(x_0) = \varphi(\psi(x_0))$. □

Definition 3.37. Let $\widetilde{f} \in IVPM(X)$. Then $\widetilde{f}$ is called an *interval-valued mapping from* X *into* X (in briefly, *of* X), if $D(\widetilde{f}) = \widetilde{X}$. We will denote the set of all interval-valued mappings of X as $IVM(X)$. It is clear that $\widetilde{f} \in IVM(X)$, then $f^-,\ f^+ \in \mathcal{T}(X)$, where $\mathcal{T}(X)$ denotes the set of all classical mappings of X (See [16]).

It is obvious that $\widetilde{f} \in IVM(X) \not\Rightarrow \widetilde{f}^{-1} \in IVM(X)$ in general.

Proposition 3.38. *If* $\widetilde{f},\ \widetilde{g} \in IVM(X)$, *then* $\widetilde{g} \circ_{IV} \widetilde{f} \in IVM(X)$.

The following is an immediate consequence of Propositions 3.35 and 3.38.

Corollary 3.39 (See Proposition 4.18, [16])**.** $(IVM(X), \circ_{IV})$ *is a subsemigroup of* $(IVPM, \circ_{IV})$ *and then a subsemigroup of* $(IVR(X), \circ_{IV})$.

Definition 3.40. The *interval-valued diagonal relation* on X, denoted by $\widetilde{\triangle}_X$ or $\widetilde{\triangle}$, is defined by:

$$\widetilde{\triangle}_X = [\bigcup_{x\in X}(x,x)_1, \bigcup_{x\in X}(x,x)_1].$$

From Proposition 3.17, we can easily see that

$$\widetilde{\triangle} = \widetilde{\triangle}_{IVP} = [\{(x,x) : x \in X\}, \{(x,x) : x \in X\}].$$

Moreover, $\widetilde{\triangle} \in IVM(X)$, and it will be called an interval-valued identity mapping and denoted by $\widetilde{id}_X$. In fact, $\widetilde{id}_X = [id_X, id_X]$, where id_X denotes the classical identity mapping.

Equivalence relations are very important in modern mathematics, for example, factor groups in algebra, quotient spaces in topology and modular number systems in number theory, etc. Now we give the definition for an equivalence relation in terms of interval-valued sets.

Definition 3.41. Let X be a nonempty set and let $\rho \in IVR(X)$. Then R is said to be *interval-valued*

(i) *reflexive*, if $x_1\rho x_1$, $x_0\rho x_0$ for each $x \in X$, i.e., $\widetilde{\triangle} \subset \rho$,
(ii) *symmetric*, if $x_1\rho y_1$, $x_0\rho y_0$ imply $y_1\rho x_1$, $y_0\rho x_0$ for any x, $y \in X$, i.e., $\rho = \rho^{-1}$,
(iii) *transitive*, if $x_1\rho y_1$, $y_1\rho z_1$ and $x_0\rho y_0$, $y_0\rho z_0$ imply $x_1\rho z_1$ and $x_0\rho z_0$ for any x, y, $z \in X$, i.e., $\rho \circ_{IV} \rho \subset \rho$,
(iv) an *equivalence relation* on X, if it is reflexive, symmetric and transitive.

We will denote the set of all interval-valued equivalence relations on X as $IVR_E(X)$.

From Definitions 3.40 and 3.41, it is obvious that $\widetilde{\triangle} \in IVR_E(X)$.

Remark 3.42. (1) If $\rho \in IVR_E(X)$, then $Dom(\rho) = Im(\rho) = \widetilde{X}$.
(2) If $\rho \in IVR_E(X)$, then ρ^-, $\rho^+ \in R_E(X)$, where $R_E(X)$ denotes the set of all classical equivalence relations on X.

Example 3.43. Let $X = \{a, b, c, d, e\}$ and let $\rho = [\rho^-, \rho^+]$ be the interval-valued relation on X given by:

$$\rho^- = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (b,a)\},$$

$$\rho^+ = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (b,a), (c,a), (a,c), (b,c), (c,b)\}.$$

Then clearly, $\rho \in IVR_E(X)$.

Definition 3.44. Let $\Sigma = (A_j)_{j\in J}$ be a family of interval-valued sets in a set X. Then Σ is called an *interval-valued partition* of X, if it satisfies the following conditions:

(i) $\Sigma^- = (A_j^-)_{j\in J}$ and $\Sigma^+ = (A_j^+)_{j\in J}$
(ii) Σ^- is a classical partition of X.

Example 3.45. Let $X = \{a, b, c, d, e\}$ and consider two families $\Sigma_1 = \{A_1, A_2, A_3\}$ and $\Sigma_2 = \{B_1, B_2, B_3\}$of interval-valued sets given by:

$$A_1 = [\{a, b\}, \{a, b, c\}],\ A_2 = [\{c, d\}, \{c, d, e\}],\ A_3 = [\{e\}, \{b, e\}],$$

$$B_1 = [\{a, b\}, \{a, b\}],\ B_2 = [\{c, d\}, \{c, d\}],\ B_3 = [\{e\}, \{e\}].$$

Then we can easily check that Σ_1 is not an interval-valued partition but Σ_2 is an interval-valued partition of X.

Remark 3.46. Let $\Sigma = (A_j)_{j\in J}$ be a family of interval-valued sets in a set X. If there is $j_0 \in J$ such that $A^-_{j_0} \subsetneqq A^+_{j_0}$, then $\Sigma^+ = (A^+_j)_{j\in J}$ cannot a partition of X. Thus Σ is not an interval-valued partition of X. So for Σ to be an interval-valued partition of X, the sufficient condition "$A^-_j = A^+_j$ for each $j \in J$" must be added. Hence in this case, to prove that Σ is an interval-valued partition of X, it is sufficient to show that Σ^- is an partition of X.

In Definition 3.30 (i), if $\rho \in IVR_E(X)$, then for each $x \in X$, ρx is called an *interval-valued equivalence class determined by* x *and* ρ. The set of all interval-valued classes in X is denoted by X/ρ, i.e., $X/\rho = \{\rho x \in IVS(X) : x \in X\}$ and X/ρ is called the *interval-valued quotient set of* X *by* ρ.

Proposition 3.47. *Let* X *be a nonempty set and let* $\rho \in IVR_E(X)$. *Suppose* $\rho^- x = \rho^+ x$ *for each* $x \in X$. *Then* X/ρ *is an interval-valued partition of* X.

Proof. The proof is similar to the classical case. □

In Proposition 3.47, if the condition "$\rho^- x = \rho^+ x$ for each $x \in X$" is omitted, then X/ρ can be not an interval-valued partition of X (See Example 3.48).

Example 3.48. Let ρ be the interval-valued equivalence relation on X given in Example 3.43. Then we can easily calculate that

$$\rho a = [\{a, b\}, \{a, b, c\}],\ \rho c = [\{c\}, \{a, b, c\}],\ \rho d = [\{d\}, \{d\}],\ \rho e = [\{e\}, \{e\}].$$

Since $(X/\rho)^+ = \{\rho^+ a, \rho^+ c, \rho^+ d, \rho^+ e\}$ is not a partition of X, X/ρ is not an interval-valued partition of X, where $X/\rho = \{\rho a, \rho c, \rho d, \rho e\}$.

Proposition 3.49. *Let* X *be a nonempty set and let* $\rho \in IVR_E(X)$. *Suppose* $\rho^- x = \rho^+ x$ *for each* $x \in X$. *Then* X/ρ *is an interval-valued partition of* X.

Proof. The proof is similar to the classical case. □

The following is the converse corresponding to Proposition 3.49.

Proposition 3.50. *Let* X *be a nonempty set and let* $\Sigma = (A_j)_{j\in J}$ *be an interval-valued partition of* X. *We define an interval-valued relation* X/Σ *on* X *as follows: for each* $(x, y) \in X \times X$,

$$x(X/\Sigma)^- y \text{ if and only if } (\exists j \in J)\ x,\ y \in A^-_j.$$

Then $X/\Sigma \in IVR_E(X)$. *Moreover,* $X/(X/\Sigma) = \Sigma$.

Proof. The proof is similar to the classical case. □

Example 3.51. Let us consider the interval-valued equivalence relation X/Σ_2 on X given in Example 3.45. Then clearly by Proposition 3.49, $X/\Sigma_2 \in IVR_E(X)$. In fact, $(X/\Sigma_2)^- = (X/\Sigma_2)^+ = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d), (e, e)\}$. Furthermore, we can confirm that $X/(X/\Sigma_2) = \Sigma_2$.

4. The category of interval-valued relations

We construct the category of interval-valued relational spaces and the morphisms between them, and study it in the sense of a topological universe. From Definitions 3.32 and 3.37, we have the following definition.

Definition 4.1. Let $X,\ Y$ be nonempty sets and let $\widetilde{f} = [f^-, f^+] \in IVR(X \times Y)$. Then $\widetilde{f}$ is called an *interval-valued mapping from* X *into* Y, denoted by $\widetilde{f} : X \to Y$, if $f^-,\ f^+ : X \to Y$ are classical mappings. If $f^-,\ f^+$ are injective [resp. surjective, bijective], then $\widetilde{f}$ is said to be *interval-valued injective* [resp. *surjective, bijective*].

Example 4.2. Consider two mappings $f^-,\ f^+ : \mathbb{R} \to \mathbb{R}$ defined by: for each $x \in \mathbb{R}$,

$$f^-(x) = x + 1,\ f^+(x) = 2x + 1.$$

Then clearly, $\widetilde{f} = [f^-, f^+]$ is an interval-valued mapping. Moreover, $\widetilde{f}$ is bijecrive.

Definition 4.3. Let $X,\ Y$ be two non-empty sets, let $\widetilde{f} : X \to Y$ be an interval-valued mapping and let $A \in IVS(X),\ B \in IVS(Y)$.

(i) The *image of* A *under* $\widetilde{f}$, denoted by $\widetilde{f}(A)$, is an IVS in Y defined as:

$$\widetilde{f}(A) = [f^-(A^-), f^+(A^+)].$$

(ii) The *preimage of* B *under* $\widetilde{f}$, denoted by $\widetilde{f}^{-1}(B)$, is an IVS in X defined as:

$$\widetilde{f}^{-1}(B) = [(f^-)^{-1}(B^-), (f^+)^{-1}(B^+)].$$

It is obvious that $\widetilde{f}(a_{_1}) = \widetilde{f}(a)_{_1}$ and $\widetilde{f}(a_{_0}) = \widetilde{f}(a)_{_0}$ for each $a \in X$.

Proposition 4.4. *Let $X,\ Y$ be two non-empty sets, let $\widetilde{f} : X \to Y$ be an interval-valued mapping, let $A,\ A_1,\ A_2 \in IVS(X),\ (A_j)_{j\in J} \subset IVS(X)$ and let $B,\ B_1,\ B_2 \in IVS(Y),\ (A_j)_{j\in J} \subset IVS(Y)$. Then*

(1) *if $A_1 \subset A_2$, then $\widetilde{f}(A_1) \subset \widetilde{f}(A_2)$,*
(2) *if $B_1 \subset B_2$, then $\widetilde{f}^{-1}(B_1) \subset \widetilde{f}^{-1}(B_2)$,*
(3) *$A \subset \widetilde{f}^{-1}(\widetilde{f}(A))$ and if $\widetilde{f}$ is injective, then $A = \widetilde{f}^{-1}(\widetilde{f}(A))$,*
(4) *$\widetilde{f}(\widetilde{f}^{-1}(B)) \subset B$ and if $\widetilde{f}$ is surjective, $\widetilde{f}(\widetilde{f}^{-1}(B)) = B$,*
(5) *$\widetilde{f}^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} \widetilde{f}^{-1}(B_j)$,*
(6) *$\widetilde{f}^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} \widetilde{f}^{-1}(B_j)$,*
(7) *$\widetilde{f}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \widetilde{f}(A_j)$,*
(8) *$f(\bigcap_{j\in J} A_j) \subset \bigcap_{j\in J} \widetilde{f}(A_j)$ and if $\widetilde{f}$ is injective, then $\widetilde{f}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \widetilde{f}(A_j)$,*
(9) *if $\widetilde{f}$ is surjective, then $\widetilde{f}(A)^c \subset \widetilde{f}(A^c)$.*
(10) *$\widetilde{f}^{-1}(B^c) = \widetilde{f}^{-1}(B)^c$.*
(11) *$\widetilde{f}^{-1}(\widetilde{\varnothing}) = \widetilde{\varnothing},\ \widetilde{f}^{-1}(\widetilde{X}) = \widetilde{X}$,*
(12) *$\widetilde{f}(\widetilde{\varnothing}) = \widetilde{\varnothing}$ and if $\widetilde{f}$ is surjective, then $\widetilde{f}(\widetilde{X}) = \widetilde{X}$,*
(13) *if $\widetilde{g} : Y \to Z$ is an interval-valued mapping, then $(\widetilde{g} \circ_{_{IV}} \widetilde{f})^{-1}(C) = \widetilde{f}^{-1}(\widetilde{g}^{-1}(C))$ for each $C \in IVS(Z)$.*

Proof. The proof is straightforward. □

Definition 4.5. Let $\widetilde{f}: X \to Y$ and $\widetilde{g}: Z \to W$ be any interval-valued mappings. Then the *product* of $\widetilde{f}$ and $\widetilde{g}$, denoted by $\widetilde{f} \times \widetilde{g} = [f^- \times g^-, f^+ \times g^+]$, is an interval-valued mapping from $X \times Z$ into $Y \times W$ defined as follows:

$$f^- \times g^-,\ f^+ \times g^+ : X \times Z \to Y \times W \text{ are classical product mappings.}$$

In particular, we write the product mapping $\widetilde{f} \times \widetilde{f} : X \times X \to Y \times Y$ as $\widetilde{f}^2$.

Definition 4.6. Let X be a nonempty set and let $\rho \in IVR(X)$. Then (X, ρ) is called an *interval-valued relational space* (in briefly, IVRS).

It is clear that if (X, ρ) is an IVRS, then (X, ρ^-), (X, ρ^+) are relational spaces in the sense of Chung [17].

Definition 4.7. Let (X, ρ), (Y, σ) be two IVRSs and let $\widetilde{f}: X \to Y$ be an interval-valued mapping. Then $\widetilde{f} : (X, \rho) \to (Y, \sigma)$ is called an *interval-valued relation preserving mapping* (briefly, IVRPreM), if $\widetilde{f}^2(\rho) \subset \sigma$.

Remark 4.8. From Result 2.4, Definition 4.7 and Proposition 4.4 (3), we can easily see that:

$$
\begin{aligned}
&\widetilde{f} : (X, \rho) \to (Y, \sigma) \text{ is an IVRPreM}\\
\text{if and only if } &(\widetilde{f}(x), \widetilde{f}(y))_1 \in \sigma \text{ and } (\widetilde{f}(x), \widetilde{f}(y))_0 \in \sigma\\
&\text{for any } (x, y)_1,\ (x, y)_0 \in \rho\\
\text{if and only if } &\rho \subset (\widetilde{f}^{-1})^2(\sigma), \text{ i.e., } \rho^- \subset ({f^-}^{-1})^2(\sigma^-),\ \rho^+ \subset ({f^+}^{-1})^2(\sigma^+).
\end{aligned}
$$

In fact, $f^- : (X, \rho^-) \to (Y, \sigma^-)$ and $f^+ : (X, \rho^+) \to (Y, \sigma^+)$ are relation preserving mappings (See [17]).

The followings are immediate consequences of Definitions 3.40 and 4.7, and Proposition 4.4 (13).

Proposition 4.9. *Let (X, ρ), (Y, σ), (Z, δ) be IVRSs.*

(1) *The interval-valued identity mapping $\widetilde{id}_X : (X, \rho) \to (X, \rho)$ is an IVPreM.*
(2) *If $\widetilde{f} : (X, \rho) \to (Y, \sigma)$ and $\widetilde{g} : (Y, \sigma) \to (Z, \delta)$ are IVPreMs, then $\widetilde{g} \circ_{IV} \widetilde{f}$ is an IVPreM.*

From Proposition 4.9, we can see that **IVRel** forms a concrete category, where **IVRel** denotes the the class of all IVRSs and IVRPreMs between them. Every **IVRel**-morphism will be called an **IVRel**-*mapping*.

Remark 4.10. (1) From Definition 3.32, Proposition 4.5 and Theorem 1.4 in [17], we can think that the category **IVRel** composes of two categories consisting of all relational spaces and relation preserving mappings between them, say **IVRel**$^-$ and **IVRel**$^+$. Then we will write

$$\mathbf{IVRel} = [\mathbf{IVRel}^-, \mathbf{IVRel}^+].$$

(2) It is clear that every singleton set has two different interval-valued relations. Then the category **IVRel** is not properly fibred. However **IVRel** is well-powered and cowell-powered.

Proposition 4.11. *The category* **IVRel** *is topological over* **Set***, where* **Set** *is the category consisting of all sets and mappings between them. That is, for each set*

X, each family $((X_j, \rho_j))_{j\in J}$ of IVRSs and for each family $(\widetilde{f}_j : X \to X_j)_{j\in J}$ of interval-valued mapping (called a source of interval-valued mappings), there is a $\rho \in IVR(X)$ which initial with respect to $(X, (\widetilde{f}_j)_{j\in J}, ((X_j, \rho_j))_{j\in J})$ (See [18, 19] for the definition of a topological category).

Proof. Let X be a set and let $((X_j, \rho_j))_{j\in J}$ be a family of IVRSs.

(i) Suppose $(\widetilde{f}_j : X \to X_j)_{j\in J}$ is a source of interval-valued mappings. We define an interval-valued relation ρ on X defined as follows:

$$\rho = \bigcap_{j\in J} (\widetilde{f}_j^{-1})^2(\rho_j) = [\bigcap_{j\in J} ({f_j^-}^{-1})^2 \rho_j^-, \bigcap_{j\in J} ({f_j^+}^{-1})^2 \rho_j^+].$$

Then it is well-known (Theorem 1.6 (1) [17]) that $f_j^- : (X, \rho^-) \to (X_j, \rho_j^-)$ and $f_j^+ : (X, \rho^+) \to (X_j, \rho_j^+)$ are relation preserving mappings. Thus $\widetilde{f}_j : (X, \rho) \to (X_j, \rho_j)$ is an **IVRel**-mapping.

(ii) Suppose (Y, σ) is any IVRS and $\widetilde{g} : Y \to X$ is an interval-valued mapping such that $\widetilde{g} \circ \widetilde{f} : (Y, \sigma) \to (X_j, \rho_j)$ is an **IVRel**-mapping for each $j \in J$. Then from (Theorem 1.6 (1) [17]), $g^- : (Y, \sigma^-) \to (X, \rho^-)$ and $g^+ : (Y, \sigma^+) \to (X, \rho^+)$ are relation preserving mappings. Thus $\widetilde{g} : (Y, \sigma) \to (X, \rho)$ is an **IVRel**-mapping. So ρ is the initial interval-valued relation on X with respect to $(\widetilde{f}_j)_{j\in J}$. This completes the proof.

□

The following is the dual of Proposition 4.12.

Corollary 4.12. *The category* **IVRel** *is cotopological over* **Set**. *In fact, the final interval-valued relation ρ on X with respect to a sink $(\widetilde{f}_j : X_j \to X)_{j\in J}$ is*

$$\rho = \bigcup_{j\in J} \widetilde{f}_j^2(\rho_j) = [\bigcup_{j\in J} {f_j^-}^2 \rho_j^-, \bigcup_{j\in J} {f_j^+}^2 \rho_j^+].$$

Proof. From (Theorem 1.5 [19]), it is clear that **IVRel** is cotopological over **Set**. However, by (Theorem 1.6 (1) [17]), we can prove directly that ρ is the final interval-valued relation on X with respect to $(\widetilde{f}_j)_{j\in J}$. □

Remark 4.13. (1) **IVRel** is complete and cocomplete (See Proposition [18] and Theorem 1.6 [19]).

(2) **IVRel** is well-powered and co-well-powered (See Proposition [18]).

(3) **IVRel** is Cartesian closed over **Set** (See Examples 1.9 (a) [20]).

Proposition 4.14. *Final episinks in* **IVRel** *are preserved by pullbacks.*

Proof. From Remark 4.6 (1), it is sufficient to prove that final episinks in $\mathbf{IVRel}^-$ and $\mathbf{IVRel}^+$ are preserved by pullbacks. Let $(g_j^- : (X, \rho_j^-) \to (Y, \rho_Y^-))_{j\in J}$ be any final episink in $\mathbf{IVRel}^-$ and let $f^- : (W, \rho_W^-) \to (Y, \rho_Y^-)$ be any $\mathbf{IVRel}^-$-mapping. For each $j \in J$, let $U_j^- = \{(w, x_j) \in W \times X_j : f^-(w) = g_j^-(x_j)\}$, let $\rho_{U_j^-}^- = \rho_W^- \times \rho_j^- \mid_{U_j^- \times U_j^-}$ and let e_j and p_j denote the usual projections of U_j^-. Then $e_j : (U_j^-, \rho_{U_j^-}^-) \to (W, \rho_W^-)$ and $p_j : (U_j^-, \rho_{U_j^-}^-) \to (X_j, \rho_j^-)$ are $\mathbf{IVRel}^-$-mappings, and the following diagram is a pullback square in $\mathbf{IVRel}^-$:

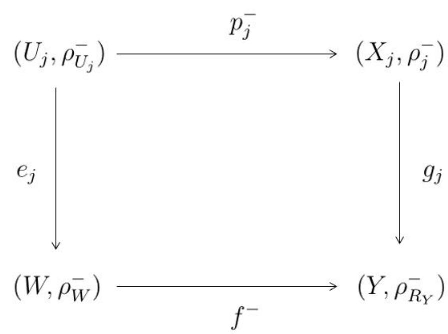

FIGURE 1. A pullback square in **IVRel**$^-$

Now we prove that $(e_j : (U^-_j, \rho^-_{U^-_j}) \to (W, \rho^-_W))_{j\in J}$ is a final episink in **IVRel**$^-$. Let $w \in W$. Since $(g^-_j)_{j\in J}$ is an episink, there are $j \in J$ and $x_j \in X_j$ such that $g^-_j(x_j) = f^-(w)$. Thus $(w, x_j) \in U^-_j$ and $w = e_j(w, x_j)$. So $(e_j)_{j\in J}$ is an episink.

Suppose $\rho^{*,-}$is the final relation on W with respect to $(e_j)_{j\in J}$ and let $(w, w^{'}) \in \rho^-_W$. Since $f^- : (W, \rho^-_W) \to (Y, \rho^-_Y)$ is an **IVRel**$^-$-mapping, we have

$$(f^-(w), f^-(w^{'})) \in \rho^-_Y.$$

Since $(g^-_j : (X_j, \rho^-_j) \to (Y, \rho^-_Y)$ is an episink, by Corollary 4.12, we get

$$(f^-(w), f^-(w^{'})) \in \bigcup_{j\in J}(g^-_j \times g^-_j)(\rho^-_j).$$

By Result 2.4 (2), there are $j \in J$ and $(x_j, x^{'}_j) \in \rho^-_j$ such that

$$(f^-(w), f^-(w^{'})) = (g^-_j(x_j), g^-_j(x^{'}_j)) = (g^-_j \times g^-_j)(x_j, x^{'}_j).$$

Then $(x_j, x^{'}_j) = ((g^-_j)^{-1} \times (g^-_j)^{-1})(f^-(w), f^-(w^{'}))$. Thus we have

$$((w, x_j), (w^{'}, x^{'}_j)) = (e^{-1}_j \times e^{-1}_j)(w, w^{'}) \text{ and } ((w, x_j), (w^{'}, x^{'}_j)) \in \rho^-_{U^-_j}.$$

Since $(w, w^{'}) \in \rho^-_W$, $(w, w^{'}) \in \rho^{*,-}$. So $\rho^-_W \subset \rho^{*,-}$. On the other hand, since $(e_j : (U^-_j, \rho^-_{U^-_j}) \to (W, \rho^{*,-})$ is final, $id^-_W : (W, \rho^{*,-}) \to (W, \rho^-_W)$ is a relational preserving mapping. Hence $\rho^{*,-} \subset \rho^-_W$, i.e., $\rho^-_W = \rho^{*,-}$. Therefore final episinks in **IVRel**$^-$ are preserved by pullbacks. Similarly, we can show that final episinks in **IVRel**$^+$ are preserved by pullbacks. This completes the proof. □

Now we consider a subcategory of **IVRel** which is a topological universe over **Set**.

Definition 4.15. Let X be a nonempty set and let $\rho \in IVR(X)$. Then (X, ρ) is called a *reflexive relational space*, if ρ is reflexive, i.e., $\widetilde{\triangle} \subset \rho$.

It is obvious that the class of all interval-valued reflexive relational spaces and **IVRel**-mappings between them forms a full subcategory of **IVRel** and denoted by $\mathbf{IVRel_R}$.

Definition 4.16 ([8, 18, 19, 21]). Let **A** be a concrete category.

(i) The **A**-*fibre* of a set X is the class of all **A**-structures on X.

(ii) **A** is said to be *properly fibred over* **Set**, if it satisfies the following conditions:

(ii$_a$) (Fibre-smallness) for each set X, the **A**-fibre is a set,

(ii$_b$) (Terminal separator property) for each singleton set X, the **A**-fibre of X has precisely one element,

(ii$_c$) if ξ and η are **A**-structures on a set X such that $id_X : (X, \xi) \to (X, \eta)$ and $id_X : (X, \eta) \to (X, \xi)$ are **A**-morphisms, then $\xi = \eta$.

The following is an immediate consequence of Definitions 4.15 and 4.16.

Lemma 4.17. $\mathbf{IVRel_R}$ *is properly fired over* **Set**.

Lemma 4.18. $\mathbf{IVRel_R}$ *is closed under the formation of initial sources in* **IVRel**.

Proof. Let $(\widetilde{f}_j : (X, \rho) \to (X_j, \rho_j)_{j\in J}$ be an initial source in **IVRel** such that $(X_j, \rho_j) \in \mathbf{IVRel_R}$ for each $j \in J$. Then by Proposition 4.11,

$$\rho = \bigcap_{j\in J} (\widetilde{f}_j^{-1})^2(\rho_j) = [\bigcap_{j\in J} ({f_j^-}^{-1})^2\rho_j^-, \bigcap_{j\in J} ({f_j^+}^{-1})^2\rho_j^+].$$

Since ρ_j is reflexive for each $j \in J$, $(x, x) \in \rho_j^-$ and $(x, x) \in \rho_j^+$ for each $j \in J$ and each $x \in X$. Thus $(x, x) \in \bigcap_{j\in J} ({f_j^-}^{-1})^2\rho_j^-$ and $(x, x) \in \bigcap_{j\in J} ({f_j^+}^{-1})^2\rho_j^+$ for each $x \in X$. So $(x, x) \in \rho^-$ and $(x, x) \in \rho^+$. Hence ρ is reflexive. This completes the proof. □

From Remark 4.13 (2), Theorems 2.5 and 2.6 in [19], we have the following.

Corollary 4.19. (1) $\mathbf{IVRel_R}$ *is a bireflective subcategory of* **IVRel**.
(2) $\mathbf{IVRel_R}$ *is topological over* **Set**.

The following is an immediate consequence of Corollary 4.19 and Theorem 1.5 in [19].

Corollary 4.20. $\mathbf{IVRel_R}$ *is cotopological over* **Set**.

Proposition 4.21. $\mathbf{IVRel_R}$ *is Cartesian closed over* **Set**.

Proof. From Remark 4.10, we write $\mathbf{IVRel_R} = [\mathbf{IVRel_R}^-, \mathbf{IVRel_R}^+]$. Then by Theorem 1.12 in [17], $\mathbf{IVRel_R}^-$ and $\mathbf{IVRel_R}^+$ are Cartesian closed over **Set**. Thus $\mathbf{IVRel_R}$ is Cartesian closed over **Set**. □

Proposition 4.22. *Final covering families in* $\mathbf{IVRel_R}$ *are preserved by pullbacks.*

Proof. In order to prove this, it is sufficient to show that final covering families in $\mathbf{IVRel_R}^-$ [resp. $\mathbf{IVRel_R}^+$] are preserved by pullbacks. Let $(g_j^- : (X_j^-, \rho_j^-) \to (Y, \rho_Y^-))_{j\in J}$ be any final covering in $\mathbf{IVRel_R}^-$ and let $f^- : (W, \rho_W^-) \to (Y, \rho_Y^-)$ be any $\mathbf{IVRel}^-$-mapping, where (W, ρ_W^-) is a reflexive relational space. For each $j \in J$, let us take U_j, $\rho_{U_j^-}^-$, e_j and p_j as in the proof of Proposition 4.14. Since $\mathbf{IVRel_R}^-$

is closed under the formation of pullbacks in **IVRel**$^-$ by Theorem 2.4 in [19], it is enough to prove that $(e_j)_{j\in J}$ is final.

Suppose $\rho^{*,-}$ is the final relation on W with respect to $(e_j)_{j\in J}$ and let $(w, w^{'}) \in (W \times W - \widetilde{\triangle}_W)$. Then by similar argument in the proof of Proposition 4.14, we get

$$(w, w^{'}) \in \rho^-_W \Longleftrightarrow (w, w^{'}) \in \rho^{*,-}, \text{ i.e., } \rho^-_W = \rho^{*,-} \text{ on } W \times W - \widetilde{\triangle}_W.$$

If $(w, w) \in \widetilde{\triangle}_W$, then clearly we have

$$(w, w) \in \rho^-_W \Longleftrightarrow (w, w) \in \rho^{*,-}, \text{ i.e., } \rho^-_W = \rho^{*,-} \text{ on } \widetilde{\triangle}_W.$$

Thus in all, $\rho^-_W = \rho^{*,-}$ on $W \times W$. Similarly, we can see that $\rho^+_W = \rho^{*,+}$ on $W \times W$. This completes the proof. □

Proposition 4.23. $\mathbf{IVRel_R}$ *is a topological universe over* **Set**.

Proof. From Lemma 4.17, Corollary 4.20 and Proposition 4.22, we can easily see that $\mathbf{IVRel_R}$ satisfies all the conditions of a topological universe. □

Remark 4.24. (1) (The redefinition of Definition 4.3) Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping and let $A \in IVS(X),\ B \in IVS(Y)$.

(i) The *image of* A *under* f, denoted by $f(A)$, is an IVS in Y defined as:

$$f(A) = [f(A^-), f(A^+)].$$

(ii) The *preimage of* B *under* f, denoted by $f^{-1}(B)$, is an IVS in X defined as:

$$f^{-1}(B) = [f^{-1}(B^-), f^{-1}(B^+)].$$

It is obvious that $f(a_1) = f(a)_1$ and $f(a_0) = f(a)_0$ for each $a \in X$.
Then we can see that the image and preimage under mappings have almost similar properties in Proposition 4.4.

(2) (The redefinition of Definition 4.7) Let $(X, \rho),\ (Y, \sigma)$ be two IVRSs and let $f : X \to Y$ be a mapping. Then $f : (X, \rho) \to (Y, \sigma)$ is called an *interval-valued relation preserving mapping* (briefly, IVRPreM), if $f^2(\rho) \subset \sigma$, where $f^2 = f \times f$. Then we can easily see that $\mathbf{IVRel}^*$ forms a concrete category, where $\mathbf{IVRel}^*$ denotes the the class of all IVRSs and IVRPreMs between them. Every $\mathbf{IVRel}^*$-morphism will be called an $\mathbf{IVRel}^*$*-mapping*. Furthermore, we can prove that the full subcategory $\mathbf{IVRel_R}^*$ of $\mathbf{IVRel}^*$ is a topological universe over **Set**.

5. Conclusions

By defining an (equivalence) relation and a partition via interval-valued sets, we could discussed their various properties. Furthermore, we formed the concrete category **IVRel** of interval-valued relational spaces and the morphisms between them, and investigated it in the view-point of a topological universe. In particular, we proved that the full subcategory $\mathbf{IVRel_R}$ of the category **IVRel** is the topological universe over **Set**. In the future, we will apply the concept of interval-valued sets to group theory, graph theory and various logic algebras.

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References

[1] J. Kim, Y. B. Jun, J. G. Lee, K. Hur, Topological structures based on interval-valued sets, Ann. Fuzzy Math. Inform. 20 (3) (2020) 273–295.

[2] Y. Yao, Interval sets and interval set algebras, Proc. 8th IEEE Int. Conf. on Cognitive Intormatics (ICCI'09) (2009) 307–314.

[3] S. H. Han, S. M. Mostafa, K. Hur, Jong-Il Baek, Interval-valued ideals in BCK-algebras, To be accepted to AFMI.

[4] A. Kriegle and L. D. Nel, A conveniant setting for holomorphy, Cahiers de Topology et Géométrie Différentielle Catégoriques 26 (1985) 273–309.

[5] A. Kriegle and L. D. Nel, Convenient vector spaces of smooth functions, Mathematische 147 (1) (1990) 39–45.

[6] L. D. Nel, Enriched locally convex structures, differential calculus and Riesz representations, J. Pure Appl. Algebra 42 (1986) 165–184.

[7] L. D. Nel, Categorical differential calculus for complete convex spaces, Carleton Univ. Canada KIS 5 B 6 (1987).

[8] L. D. Nel, Topological universes and smooth Gelfand-Naimark duallity, Mathematical Applications of Category Theory, Proc. A. M. S. Spec. Session Denver, Contemporary Mathematics 30 (1984) 224–276.

[9] Jeong-Gon Lee, Kul Hur and Xueyou Chen, A study on cubic H-relations in a topological universe viewpoint, Mathematics 2020, 8, 482; doi:10.3390/math8040482.

[10] K. Hur, H-fuzzy relations (I): A topological universe viewpoint. Fuzzy Sets Syst. 61 (1994) 239–244.

[11] K. Hur, H-fuzzy relations (II): A topological universe viewpoint. Fuzzy Sets Syst. 63 (1995) 73–79.

[12] K. Hur, S. Y. Jang and H. W. Kang, Intuitionistic H-fuzzy relations. Int. J. Math. Math. Sci. 17 (2005) 2723–2734.

[13] P. K. Lim, S. H. Kim and K. Hur, The category VRel(H). Int. Math. Forum 5 (2010) 1443–1462.

[14] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I. Inf. Sci. 8 (1975) 199–249.

[15] M. K. Roy and R. Biswas, I-v fuzzy relations and Sanchez's approach for medical diagnosis, Fuzzy Sets and Systems 47 (1992) 35–38.

[16] J. M. Howie, An Introduction To Semigroup Theory, Academic Press 1976.

[17] Se Hwa Chung, On relational spaces, Journal of the Korea Society of Mathematical Education XXIV (1) (1985) 35–39.

[18] H. Herrlich, Cartesian closed topological categories, Math. Coll. Univ. Cape Twon 9 (1974) 1–16.

[19] C. Y. Kim, S. S. Hong, Y. H. Hong and P. U. Park, Algebras in Cartesian Closed Topological Categories, Lecture Notes Series in Seminars Math. 1 Yonsei Univ. Seoul 1979.

[20] Commentations Mathematicae Universitatis Carolinae 27 (2) (1986) 235–257.

[21] H. Herrlich and G. E. Streker, Category Theory, Allyn and Bacon 1973.

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