Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 24, No. 3, (December 2022) pp. 253–270
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2022.24.3.253

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 24, No. 3, December 2022

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Received 3 July 2022; Revised 28 July 2022; Accepted 16 September 2022

Abstract. In this paper, the concept (T, S)-bipolar Pythagorean fuzzy medial-ideals are introduced and several properties are investigated. Also, the relations between (T, S)-bipolar Pythagorean fuzzy medial-ideals and (T, S)-bipolar Pythagorean fuzzy BCI-ideals are given. The pre-image of bipolar Pythagorean (T, S)-fuzzy medial-ideals under homomorphism of BCI-algebras are defined and how the image and the pre-image of (T, S)-bipolar Pythagorean fuzzy medial ideals under homomorphism of BCI-algebras become (T, S)-bipolar Pythagorean fuzzy medial-ideals are studied. Moreover, the Cartesian product of (T, S)-bipolar Pythagorean fuzzy medial-ideals in Cartesian product BCI-algebras is established.

2020 AMS Classification: 06F35, 03G25, 08A72

Keywords: Medial BCI-algebra, Fuzzy medial ideals, (T, S)-bipolar Pythagorean fuzzy medial ideals, the pre-image of (T, S)-bipolar Pythagorean fuzzy medial ideals in BCI-algebras, Cartesian product of (T, S)-bipolar Pythagorean fuzzy medial ideals.

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1. Introduction

In 1966, Iami and Iséki [1] introduced the notion of BCK-algebras. Huang [2] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras. There is a great deal of literature which has been produced on the theory of BCK/BCI-algebras, in particular, the emphasis seems to have been put on the ideals theory of BCK/BCI-algebras. Fuzzy set theory is the concept and technique which lay a form of mathematical precision for the human thought process that in many ways is imprecise and ambiguous by the standards of classical mathematics. Fuzzy sets, intuitionistic fuzzy set, interval-valued fuzzy set, bipolar fuzzy set and other mathematical tools are often useful approaches to dealing with uncertainties. In 1965, Zadeh [3] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc. The idea of the "intuitionistic fuzzy set" was first published by Atanassov [4, 5] as a generalization of the notion of fuzzy sets. In 1991, Ougen [6] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Zhang [7] and Lee [8] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. On the other hand, triangular norm is a powerful tool in the theory research and application development of fuzzy sets (See [1, 9]). Li and Zheng [10] generalized the operators "$\wedge$" and "$\vee$" to T-norm and S-norm and defined the intuitionistic fuzzy groups of $(T-S)$-norms as a generalization of the notion of fuzzy sets. In [11]], Meng and Jun studied medial BCI-algebras. Mostafa et al. [9, 12, 13, 14, 15, 16, 17] introduced and generalized the fuzzification the notion of medial ideals in BCI-algebras. Mostafa et al. [17] used the notion of (T, S)-bipolar fuzzy sets to establish the concept of (T, S)-bipolar fuzzy medial ideals of BCI-algebras and then they obtained some related properties. Pythagorean fuzzy sets (PFSs), originally proposed by Yager [18], are a new tool to deal with vagueness with the square sum of the membership degree and the nonmembership degree equal to or less than 1, which have much stronger ability than Atanassov's intuitionistic fuzzy sets to model such uncertainty. The elements in PFS are called the Pythagorean fuzzy numbers (PFNs). Since the PFS was published so far, there have been many kinds of research on the application of PFS in various mathematical problems such as decision problem, classifier problem, and pattern recognition. Yager [19] developed various Pythagorean fuzzy aggregation operators. Zhang and Xu [20] studied extension of TOPSIS to multiple criteria decision making (MCDM) with Pythagorean fuzzy sets. Peng and Yang [21] introduced some aggregation operators on PFSs and applied them in the group decision-making problem. Zhang [22] introduce the interval-valued Pythagorean fuzzy set (IVPFS) and studied it in the MCDM. Li and Zeng [23] proposed the distance measures of PFS and applied them in MCDM.

In this paper, we introduce the concept of (T, S)-bipolar Pythagorean fuzzy medial ideals and studied its several properties. Also, we give the relations between (T, S)-bipolar Pythagorean fuzzy medial ideals and (T, S)-bipolar Pythagorean fuzzy BCI-ideals. we define the image and the pre-image of (T, S)-bipolar Pythagorean fuzzy medial ideals under homomorphism of BCI-algebras and how the preimage of (T, S)-bipolar Pythagorean fuzzy medial ideals under homomorphism of BCI-algebras become (T, S)-bipolar Pythagorean fuzzy medial ideals are studied. Moreover, the Cartesian product of (T, S)-bipolar Pythagorean fuzzy medial ideals in Cartesian product BCI-algebras is established.

2. Preliminaries

Now we review some definitions and properties that will be useful in our results

Definition 2.1 (See [2, 1]). An algebraic system of type $(2, 0)$ is called a *BCI-algebra*, if it satisfying the following conditions: for all $x,\ y,\ z \in X$,

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI-2) $(x * (x * y)) * y = x$,
(BCI-3) $x * x = 0$,
(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In a BCI-algebra X, we can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra satisfies the identity $0 * x = 0$ for each $x \in X$, then X is called a BCK*- algebra*. It is well-known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

In what follows, X will denote a BCI-algebra of type $(2, 0)$ unless otherwise specified.

Definition 2.2 (See [11]). X is called a *medial BCI-algebra*, if it satisfying the following condition: $x,\ y,\ z,\ u \in X$,

$$(x * y) * (z * u) = (x * z) * (y * u).$$

Lemma 2.3 (See Corollary 4 [11]). *X is a medial BCI-algebra if and only if it satisfies the following conditions: for all $x,\ y,\ z \in X$,*

(1) $x * (y * z) = z * (y * x)$,
(2) $x * 0 = x$,
(3) $x * x = 0$.

Lemma 2.4 (See Definition 1 (4)[11]). *Let X be a medial BCI-algebra. Then $x * (x * y) = y$ for all $x,\ y \in X$.*

Lemma 2.5. *Let X be a medial BCI-algebra. Then $0 * (y * x) = x * y$ for all $x,\ y \in X$.*

Proof. Straightforward. □

Definition 2.6. Let X be a medial BCI-algebra and let S be a non-empty subset of X. Then S is said to be *medial subalgebra* of X, if $x * y \in S$ for all $x,\ y \in S$.

Definition 2.7 (See [2, 1]). A non-empty subset I of X is called a *BCI-ideal* of X, if it satisfies the following conditions: for all $x,\ y \in X$.

(I_1) $0 \in I$,
(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.8 (See [9]). A non-empty subset M of a medial BCI-algebra X is called a *medial ideal* of X, if it satisfies the following conditions: for all $x,\ y,\ z \in X$,

(M_1) $0 \in I$,
(M_2) $z * (y * x) \in M$ and $y * z \in M$ imply $x \in M$.

For a set set X, a mapping $\mu : X \to [0, 1]$ is called a *fuzzy set* in X (See [3]Zadeh). For any $a,\ b \in [0, 1]$, we denote $min\{a, b\}$ and $max\{a, b\}$ as $a \wedge b$ and $a \vee b$ respectively.

Definition 2.9 (See [9]). Let $A\mu$ be a fuzzy set on X. Then A is called a *fuzzy BCI-subalgebra* of X, if it satisfies the following condition: for all $x,\ y \in X$,
(FS) $\mu(x * y) \geq \mu(x) \wedge \mu(y)$.

Definition 2.10 (See [9]). Let μ be a fuzzy set in X. Then A is called a *fuzzy BCI-ideal* of X, if it satisfies the following conditions: for all $x,\ y \in X$,
(FI_1) $\mu(0) \geq \mu(x)$,
(FI_2) $\mu(x) \geq \mu(x * y) \wedge \mu(y)$.

Definition 2.11 (See [9]). Let μ be a fuzzy set on a medial BCI-algebra X. Then A is called a *fuzzy medial ideal* of X, if it satisfies the following conditions: for all $x,\ y,\ z \in X$,
(FI_1) $\mu(0) \geq \mu(x)$,
(FMI_2) $\mu(x) \geq \mu(z * (y * x)) \wedge \mu(y * z)$.

Definition 2.12 (See [10]). A *triangular norm* (briefly, *t-norm*) is a mapping $T : [0,1] \times [0,1] \to [0,1]$ satisfies the following conditions: for all $x,\ y,\ z \in [0,1]$,
(T_1) (Boundary condition) $T(x, 1) = x$,
(T_2) (Commutative condition) $T(x, y) = T(y, x)$,
(T_3) (Associative condition) $T(x, T(y, z)) = T(T(x, y), z)$,
(T_4) (Monotonicity) $T(x, y) \leq T(x, z)$ whenever $y \leq z$.

A simple example of such defined t-norm is a mapping $T(a, b) = a \wedge b$ for all $a,\ b \in [0,1]$. In general, $T(a, b) \leq a \wedge b$ and $T(a, 0) = 0$ for all $a,\ b \in [0,1]$.

Definition 2.13 (See [10]). A *triangular conorm* (briefly, *s-norm*) is a mapping $S : [0,1] \times [0,1] \to [0,1]$ satisfies the following conditions: for all $x,\ y,\ z \in [0,1]$,
(S_1) $S(x, 0) = x$,
(S_2) $S(x, y) = S(y, x)$,
(S_3)) $S(x, S(y, z)) = S(S(x, y), z)$,
(S_4) $S(x, y) \leq S(x, z)$ whenever $y \leq z$.

A simple example of such defined s-norm is a mapping $S(a, b) = a \vee b$ for all $a,\ b \in [0,1]$. Every s-norm S has a useful property: $a \vee b \leq S(a, b)$ for all $a,\ b \in [0,1]$.

Definition 2.14 (See [4, 5]). Let X be a non-empty set. Then $A = (A^{\in}, A^{\notin})$ is called an *intuitinistic fuzzy set* (briefly, IFS) in X, if $A : X \to [0,1] \times [0,1]$ is a mapping satisfying the following condition: for each $x \in X$,

$$0 \leq A^{\in}(x) + A^{\notin}(x) \leq 1,$$

where the mappings $A^{\in} : X \to [0,1]$ and $A^{\notin} : X \to [0,1]$ denote the degree of membership and degree of non membership respectively.

Definition 2.15 ([18]). Let X be a non-empty set. Then $\bar{A} = (A^{\in 2}, A^{\notin 2})$ is called a *Pythagorean fuzzy set* (briefly, PFS) in X, if $\bar{A} : X \to [0,1] \times [0,1]$ is a mapping satisfying the following condition: for each $x \in X$,

$$0 \leq (A^{\in}(x))^2 + (A^{\notin}(x))^2 \leq 1,$$

where the mappings $A^{\in 2} : X \to [0,1]$ and $A^{\notin 2} : X \to [0,1]$ denote the degree of membership and degree of non membership respectively.

Definition 2.16 ([8, 3])**.** Let X be a non-empty set. Then $A = (A^N, A^P)$ is called a *bipolar valued fuzzy set* in X, if $A^N : X \to [-1, 0]$ and $A^P : X \to [0, 1]$ are mappings. For each $x \in X$, $A^N(x)$ and $A^P(x)$ denote the positive membership degree and the negative membership degree respectively.

3. (T, S)-bipolar Pythagorean fuzzy medial ideal

In this section, we introduce a new notions called (T, S)-*bipolar fuzzy ideals* and (T, S)-*bipolar fuzzy medial ideals*of a BCI-algebra and study some of their properties.

Definition 3.1 ([25])**.** Let X be a non-empty set. Then the form

$$A = ((-A^{N,\in^2}, A^{P,\in^2}), (-A^{N,\notin^2}, A^{P,\notin^2}))$$

is called a *bipolar Pythagorean fuzzy set* (briefly, BPFS) in X, if it satisfies the following conditions: for each $x \in X$,

$$- \le (-A^{N,\in^2})(x) + (-A^{N,\notin^2})(x) = -(A^{N,\in^2}(x) + (A^{N,\notin^2}(x)) \le 0,$$

$$0 \le (A^{P,\in^2})(x) + (A^{P,\notin^2})(x) = A^{P,\in^2}(x) + A^{P,\notin^2}(x) \le 1,$$

where $A^{N,\in},\ A^{N,\notin} : X \to [-1, 0]$ and $A^{P,\in},\ A^{P,\notin} : X \to [0, 1]$ are mappings.

We use the positive membership degree to denote the satisfaction degree of an element x to the property corresponding to a bipolar Pythagorean fuzzy set A and the negative membership degree to denote the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar Pythagorean fuzzy set A. Similarly, we use the positive nonmembership degree to denote the satisfaction degree of an element x to the property corresponding to a bipolar Pythagorean fuzzy set A and the negative nonmembership degree to denote the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar Pythagorean fuzzy set A.

If $A^{P,\in^2}(x) \neq 0,\ -A^{N,\in^2}(x) = 0$ and $A^{P,\notin^2}(x) = 0,\ -A^{N,\notin^2}(x) = 0$, then it is the situation that x regarded as having only the positive membership property of a bipolar Pythagorean fuzzy set A.

If $A^{P,\in^2}(x) = 0,\ -A^{N,\in^2}(x) \neq 0$ and $A^{P,\notin^2}(x) = 0,\ -A^{N,\notin^2}(x) = 0$, then it is the situation that x regarded as having only the negative membership property of a bipolar Pythagorean fuzzy set A.

If $A^{P,\in^2}(x) = 0,\ -A^{N,\in^2}(x) = 0$ and $A^{P,\notin^2}(x) \neq 0,\ -A^{N,\notin^2}(x) = 0$, then it is the situation that x regarded as having only the positive nonmembership property of a bipolar Pythagorean fuzzy set A.

If $A^{P,\in^2}(x) = 0,\ -A^{N,\in^2}(x) = 0$ and $A^{P,\notin^2}(x) = 0,\ -A^{N,\notin^2}(x) \neq 0$, then it is the situation that x regarded as having only the negative nonmembership property of a bipolar Pythagorean fuzzy set A.

It is possible for an element x to be such that $A^{P,\in^2}(x) \neq 0,\ -A^{N,\in^2}(x) \neq 0$ and $A^{P,\notin^2}(x) \neq 0,\ -A^{N,\notin^2}(x) \neq 0$ when the membership and nonmembership function of the property overlaps with its counter properties over some portion of X.

Definition 3.2. Let $A = ((-A^{N,\in^2}, A^{P,\in^2}), -A^{N,\notin^2}, A^{P,\notin^2}))$ be a bipolar Pythagorean fuzzy set in X. Then A is called a (T, S)-*bipolar Pythagorean fuzzy ideal* (briefly, (T, S)-BPFI) of X, if it satisfies the following conditions: for all $x,\ y \in X$,

(BPFI$_1$) $A^{P,\in^2}(0) \geq A^{P,\in^2}(x)$, $-A^{N,\in^2}(0) \leq -A^{N,\in^2}(x)$,
(BPFI$_2$) $A^{P,\in^2}(x) \geq T(A^{P,\in^2}(x*y), A^{P,\in^2}(x))$,
$-A^{N,\in^2}(x) \leq S(-A^{N,\in^2}(x*y), -A^{N,\in^2}(y))$,
(BPFI$_3$) $A^{P,\notin^2}(0) \geq A^{P,\notin^2}(x)$, $-A^{N,\notin^2}(0) \leq -A^{N,\notin^2}(x)$,
(BPFI$_4$) $A^{P,\notin^2}(x) \geq T(A^{P,\notin^2}(x*y), A^{P,\notin^2}(x))$,
$-A^{N,\notin^2}(x) \leq S(-A^{N,\notin^2}(x*y), -A^{N,\notin^2}(y))$.

Definition 3.3. Let $A = ((-A^{N,\in^2}, A^{P,\in^2}), -A^{N,\notin^2}, A^{P,\notin^2}))$ be a bipolar Pythagorean fuzzy set in X. Then A is called a (T,S)-*bipolar Pythagorean fuzzy medial ideal* (briefly, (T,S)-BPFMI) of X, if it satisfies the following conditions: for all $x,\ y,\ z \in X$,
(BPFI$_1$) $A^{P,\in^2}(0) \geq A^{P,\in^2}(x)$, $-A^{N,\in^2}(0) \leq -A^{N,\in^2}(x)$,
(BPFMI$_2$) $A^{P,\in^2}(x) \geq T(A^{P,\in^2}(z*(y*x)), A^{P,\in^2}(y*z))$,
$-A^{N,\in^2}(x) \leq S(-A^{N,\in^2}(z*(y*x)), -A^{N,\in^2}(y*z))$,
(BPFI$_3$) $A^{P,\notin^2}(0) \geq A^{P,\notin^2}(x)$, $-A^{N,\notin^2}(0) \leq -A^{N,\notin^2}(x)$,
(BPFMI$_4$) $A^{P,\notin^2}(x) \geq T(A^{P,\notin^2}(z*(y*x)), A^{P,\notin^2}(y*z))$,
$-A^{N,\notin^2}(x) \leq S(-A^{N,\notin^2}(z*(y*x)), -A^{N,\notin^2}(y*z))$.

Example 3.4. Let $X = \{0,1,2,3\}$ be a BCI-algebra with a binary operation $*$ defined by the following Table 3.1:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	2	0
2	2	0	0	0
2	3	2	1	0

Table 3.1

Consider the bipolar Pythagorean fuzzy set $A = ((-A^{N,\in^2}, A^{P,\in^2}), (-A^{N,\notin^2}, A^{P,\notin^2}))$ in X given by Table 3.2:

$*$	0	1	2	3
$A^{N,\in}$	-0.7	-0.7	-0.4	-0.3
$A^{P,\in}$	0.6	0.5	0.3	0.3
$A^{N,\notin}$	-0.2	-0.3	-0.5	-0.7
$A^{P,\notin}$	0.1	0.2	0.3	.40

Table 3.2

Let $T,\ S : [0,1]\times[0,1] \to [0,1]$ be the mappings defined as follows: for all $a,\ b \in [0,1]$,

$$T(a,b) = (a+b-1) \vee 0,\ \ S(a,b) = [1-(a+b)] \wedge 1.$$

Then we can easily calculate that A is a (T,S)-bipolar Pythagorean fuzzy medial ideal of X.

Proposition 3.5. *Every (T,S)-BPFMI of X is a (T,S)-BPFI of X.*

Proof. The proof is straightforward from Definitions 3.2 and 3.3. □

Proposition 3.6. *Let A be a (T,S)-BPFMI of X and let $x,\ y \in X$. If $x \leq y$, then the following conditions hold:*

(1) $-A^{N,\in^2}(x) \leq -A^{N,\in^2}(y)$,
(2) $A^{P,\in^2}(x) \geq A^{P,\in^2}(y)$,
(3) $-A^{N,\notin^2}(x) \leq -A^{N,\notin^2}(y)$,
(4) $A^{P,\notin^2}(x) \geq A^{P,\notin^2}(y)$.

Proof. Suppose $x \leq y$. Then clearly, $x * y = 0$. Thus we have

$$\begin{aligned} -A^{N,\in^2}(x) &\leq S(-A^{N,\in^2}(0*(y*x)), -A^{N,\in^2}(y*0)) \text{ [By ((BPFMI}_2)] \\ &= S(-A^{N,\in^2}(x*y), -A^{N,\in^2}(y)) \text{ [By Lemmas 2.5 and 2.3 ((3)]} \\ &= S(-A^{N,\in^2}(0), -A^{N,\in^2}(y)) \text{ [Since } x*y=0] \\ &= -A^{N,\in^2}(y), \text{ [By ((BPFI}_1) \text{ and the definition of } S] \end{aligned}$$

$$\begin{aligned} -A^{P,\in^2}(x) &\geq T(A^{P,\in^2}(0*(y*x)), A^{P,\in^2}(y*0)) \text{ [By ((BPFMI}_2)] \\ &= T(A^{P,\in^2}(x*y), A^{P,\in^2}(y)) \text{ [By Lemmas 2.5 and 2.3 ((3)]} \\ &= T(A^{P,\in^2}(0), A^{P,\in^2}(y)) \text{ [Since } x*y=0] \\ &= -A^{P,\in^2}(y). \text{ [By ((BPFI}_3) \text{ and the definition of } T] \end{aligned}$$

So the conditions ((1) and ((2) hold. Similarly, we can prove that the conditions ((3) and ((4) hold. □

Definition 3.7. Let A be a bipolar Pythagorean fuzzy set in X. Then A is called a (T,S)-*bipolar Pythagorean fuzzy subalebra* (briefly, (T,S)-BPFSA) of X, if it satisfies the following conditions: for all $x,\ y \in X$,

(BPFSA$_1$) $-A^{N,\in^2}(x*y) \leq S(-A^{N,\in^2}(x), -A^{N,\in^2}(y))$,
(BPFSA$_2$) $A^{P,\in^2}(x*y) \geq T(A^{P,\in^2}(x), A^{P,\in^2}(y))$,
(BPFSA$_3$) $-A^{N,\notin^2}(x*y) \leq S(-A^{N,\notin^2}(x), -A^{N,\notin^2}(y))$,
(BPFSA$_4$) $A^{P,\notin^2}(x*y) \geq T(A^{P,\notin^2}(x), A^{P,\notin^2}(y))$.

Example 3.8. Let A be the bipolar Pythagorean fuzzy set in X given in Example 3.4. Then we can easily check that a (T,S)-bipolar Pythagorean fuzzy sub algebra of X.

Proposition 3.9. *Every (T,S)-BPFMI of a BCK-algebra X is a (T,S)-bipolar Pythagorean fuzzy BCI-subalgebra of X.*

Proof. Let A be a (T,S)-BPFMI of a BCK-algebra X and let $x,\ y \in X$. Then we have

$$\begin{aligned} (x*y)*x &= (x*x)*y \text{ [Since } X \text{ is a } BCK\text{-algebra]} \\ &= 0*y \text{ [By (BCI-1))]} \\ &= 0. \text{ [Since } X \text{ is a } BCK\text{-algebra]} \end{aligned}$$

Thus $x*y \leq x$. By Proposition 3.6, we get

$$-A^{N,\in^2}(x*y) \leq -A^{N,\in^2}(x),\ A^{P,\in^2}(x*y) \geq A^{P,\in^2}(x), \tag{3.1}$$

$$-A^{N,\notin^2}(x*y) \leq -A^{N,\notin^2}(x),\ A^{P,\notin^2}(x*y) \geq A^{P,\notin^2}(x). \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}-A^{N,\in^2}(x) &\leq S(-A^{N,\in^2}(0*(y*x)), -A^{N,\in^2}(y*0)) \text{ [By (BPFMI}_2\text{)]}\\ &= S(-A^{N,\in^2}(x*y), -A^{N,\in^2}(y*0)) \text{ [By Lemma 2.5]}\\ &= S(-A^{N,\in^2}(x*y), -A^{N,\in^2}(y)) \text{ [Since } X \text{ is a } BCK\text{-algebra]}\\ &\leq S(-A^{N,\in^2}(x), -A^{N,\in^2}(y)), \text{ [By (3.1) and (S}_4\text{)]}\end{aligned}$$

$$\begin{aligned}-A^{P,\in^2}(x) &\geq T(A^{P,\in^2}(0*(y*x)), A^{P,\in^2}(y*0)) \text{ [By (BPFMI}_2\text{)]}\\ &= T(A^{P,\in^2}(x*y), A^{P,\in^2}(y*0)) \text{ [By Lemma 2.5]}\\ &= T(A^{P,\in^2}(x*y), A^{P,\in^2}(y)) \text{ [Since } X \text{ is a } BCK\text{-algebra]}\\ &\geq T(A^{P,\in^2}(x), A^{P,\in^2}(y)). \text{ [By (3.1) and (T}_4\text{)]}\end{aligned}$$

So A satisfies the conditions (BPFSA$_2$) and (BPFSA$_2$). Similarly, from (3.2), we can prove that A satisfies the conditions (BPFSA$_3$) and (BPFSA$_4$). Hence A is a (T, S)-BPFSA of X. □

Proposition 3.10. *Let A be a (T, S)-BPFMI of X and let $x,\ y,\ z \in X$. If $x*y \leq z$, then the following conditions hold:*

(1) $-A^{N,\in^2}(x) \leq S(-A^{N,\in^2}(y), -A^{N,\in^2}(z))$,
(2) $A^{P,\in^2}(x) \geq T(A^{P,\in^2}(y), A^{P,\in^2}(z))$,
(3) $-A^{N,\notin^2}(x) \leq S(-A^{N,\notin^2}(y), -A^{N,\notin^2}(z))$,
(4) $A^{P,\notin^2}(x) \geq T(A^{P,\notin^2}(y), A^{P,\notin^2}(z))$.

Proof. Let $x,\ y,\ z \in X$ such that $x * y \leq z$. Then

$$\begin{aligned}-A^{N,\in^2}(x) &\leq S(-A^{N,\in^2}(0*(y*x)), -A^{N,\in^2}(y*0)) \text{ [By (BPFMI}_2\text{)]}\\ &= S(-A^{N,\in^2}(x*y), -A^{N,\in^2}(y)) \text{ [By Lemmas 2.5 and 2.3]}\\ &\leq S(-A^{N,\in^2}(z), -A^{N,\in^2}(y)), \text{ [By Proposition 3.6 and (S}_4\text{)]}\end{aligned}$$

$$\begin{aligned}A^{P,\in^2}(x) &\geq T(A^{P,\in^2}(0*(y*x)), A^{P,\in^2}(y*0)) \text{ [By (BPFMI}_2\text{)]}\\ &= T(A^{P,\in^2}(x*y), A^{P,\in^2}(y)) \text{ [By Lemmas 2.5 and 2.3]}\\ &\geq T(A^{P,\in^2}(z), A^{P,\in^2}(y)). \text{ [By Proposition 3.6 and (T}_4\text{)]}\end{aligned}$$

Thus (1) and (2) hold. Similarly, we can prove that (3) and (4) hold. □

We give the converse of Proposition 3.10.

Proposition 3.11. *Let A be a (T, S)-BPFSA of X. Suppose the following conditions hold: for all $x,\ y,\ z \in X$ such that $x * y \leq z$,*

(1) $-A^{N,\in^2}(x) \leq S(-A^{N,\in^2}(y), -A^{N,\in^2}(z))$,
(2) $A^{P,\in^2}(x) \geq T(A^{P,\in^2}(y), A^{P,\in^2}(z))$,
(3) $-A^{N,\notin^2}(x) \leq S(-A^{N,\notin^2}(y), -A^{N,\notin^2}(z))$,
(4) $A^{P,\notin^2}(x) \geq T(A^{P,\notin^2}(y), A^{P,\notin^2}(z))$.

Then A be a (T, S)-BPFMI of X.

Proof. Let A be a (T, S)-BPFSA of X. Recall that: for all $x \in X$,

$$-A^{N,\in^2}(0) \leq -A^{N,\in^2}(x),\ \ A^{P,\in^2}(0) \geq A^{P,\in^2}(x)$$

and

$$-A^{N,\notin^2}(0) \leq -A^{N,\notin^2}(x),\ \ A^{P,\notin^2}(0) \geq A^{P,\notin^2}(x).$$

Then the conditions (BPFI$_1$) and (BPFI$_3$) hold.

Now let $x,\ y,\ z \in X$. Then by Lemma 2.3 (1) and Proposition 3.10,

$$x * (z * (y * x)) = (y * x) * (z * x) \leq y * z.$$

From the hypothesis and the above inequality, it is necessary to prove that (BPFMI$_2$) and (BPFMI$_4$) hold. It follows that

$$-A^{N,\in^2}(x) \leq S(-A^{N,\in^2}(z * (y * x)), -A^{N,\in^2}(y * z)),$$
$$A^{P,\in^2}(x) \geq T(A^{P,\in^2}(z * (y * x)), A^{P,\in^2}(y * z))$$

and

$$-A^{N,\notin^2}(x) \leq S(-A^{N,\notin^2}(z * (y * x)), -A^{N,\notin^2}(y * z)),$$
$$A^{P,\notin^2}(x) \geq T(A^{P,\notin^2}(z * (y * x)), A^{Pnot\in^2}(y * z)).$$

Thus A be a (T,S)-BPFMI of X. □

The form $a = ((a^{N,\in}, a^{P,\in}), (a^{N,\notin}, a^{P,\notin}))$ is called a *bipolar Pythagorean fuzzy number* (briefly, BPFN), if it satisfies the following conditions

(i) $a^{N,\in}, a^{P,\in}),\ (a^{N,\notin}, a^{P,\notin})) \in [-1,0] \times [0,1]$,

(ii) $-1 \leq -(a^{N,\in^2} + a^{N,\notin^2}) \leq 0$ and $0 \leq a^{P,\in^2} + a^{P,\notin^2}) \leq 1$.

For any two BOFNs $a,\ b,\ a \leq_{BP} b$ if and only if $a^{N,\in} \geq b^{N,\in}$, $a^{N,\notin} \geq b^{N,\notin}$ and $a^{P,\in} \leq b^{P,\in}$, $a^{P,\notin} \geq b^{P,\notin}$.

Definition 3.12. Let X be a non-empty set, let A be a BPFS in X and let a be a BPFN. Then the *a-cut* or *a-level set* of A, denoted by $[A]_a$, is a subset of X defined as follows:

$$[A]_a = \{x \in X : -A^{N,\in^2}(x) \leq a^{N,\in},\ A^{P,\in^2}(x) \geq a^{P,\in},$$
$$-A^{N,\notin^2}(x) \leq a^{N,\notin},\ A^{P,\notin^2}(x) \geq a^{P,\notin}\}.$$

We obtain a relationship between (T,S)-BPFMIs of X and medial ideals of X.

Theorem 3.13. *A is a (T,S)-BPFMI of X if and only if for each BPFN a, either $[A]_a = \varnothing$ or $[A]_a$ is a medial ideal of X.*

Proof. Suppose A is a (T,S)-BPFMI of X and assume that $[A]_a \neq \varnothing$ for each BPFN a. Then there is $x \in [A]_a$. By the definition of $[A]_a$, we get

$$-A^{N,\in^2}(x) \leq a^{N,\in},\ A^{P,\in^2}(x) \geq a^{P,\in},\ -A^{N,\notin^2}(x) \leq a^{N,\notin},\ A^{P,\notin^2}(x) \geq a^{P,\notin}.$$

By (BPFI$_1$) and (BPFI$_3$), we have

$$-A^{N,\in^2}(0) \leq a^{N,\in},\ A^{P,\in^2}(0) \geq a^{P,\in},\ -A^{N,\notin^2}(0) \leq a^{N,\notin},\ A^{P,\notin^2}(0) \geq a^{P,\notin}.$$

Thus $0 \in [A]_a$. So $[A]_a$ satisfies the condition (M$_3$).

Now let $x,\ y,\ z \in X$ such that $z * (y * x) \in [A]_a$ and $y * z \in [A]_a$. Then clearly,

$$-A^{N,\in^2}(x * (y * z)) \leq a^{N,\in},\ -A^{N,\in^2}(y * z) \leq a^{N,\in},$$
$$A^{P,\in^2}(x * (y * z)) \geq a^{P,\in},\ A^{P,\in^2}(y * z) \geq a^{P,\in},$$
$$-A^{N,\notin^2}(x * (y * z)) \leq a^{N,\notin},\ -A^{N,\notin^2}(y * z) \leq a^{N,\notin},$$
$$A^{P,\notin^2}(x * (y * z)) \geq a^{P,\notin},\ A^{P,\notin^2}(y * z) \geq a^{P,\notin}.$$

Thus by the hypothesis and the conditions (S$_4$), (T$_4$), we have

$$-A^{N,\in^2}(x) \leq S(-A^{N,\in^2}(x*(y*z)), -A^{N,\in^2}(y*z)) \leq a^{N,\in},$$
$$A^{P,\in^2}(x) \geq T(A^{P,\in^2}(x*(y*z)), A^{P,\in^2}(y*z)) \geq a^{P,\in},$$
$$-A^{N,\notin^2}(x) \leq S(-A^{N,\notin^2}(x*(y*z)), -A^{N,\notin^2}(y*z)) \leq a^{N,\notin},$$
$$A^{N,\notin^2}(y*z)),\ A^{P,\notin^2}(x) \geq T(A^{P,\notin^2}(x*(y*z)), A^{P,\notin^2}(y*z)) \geq a^{P,\notin}.$$

So $x \in [A]_a$. Hence $[A]_a$ is a medial ideal of X.

Conversely, suppose the necessary condition holds and for each BPFN a, let

$$-A^{N,\in^2}(x) = a^{N,\in},\ A^{P,\in^2}(x) = a^{P,\in},\ -A^{N,\notin^2}(x) = a^{N,\notin},\ A^{P,\notin^2}(x) = a^{P,\notin}.$$

Then clearly, $[A]_a \neq \varnothing$. Since $[A]_a$ is a medial ideal of X, $0 \in [A]_a$. Thus we get: for each $x \in X$,

$$-A^{N,\in^2}(0) \leq a^{N,\in} = -A^{N,\in^2}(x),\ A^{P,\in^2}(0) \geq a^{P,\in} = A^{P,\in^2}(x),$$
$$-A^{N,\notin^2}(0) \leq a^{N,\notin} = -A^{N,\notin^2}(x),\ A^{P,\notin^2}(0) \geq a^{P,\notin} = -A^{P,\notin^2}(x).$$

So A satisfies the conditions (BPFI$_1$) and (BPFI$_3$).

Assume that there are $x,\ y,\ z \in X$ such that

$$A^{P,\in^2}(x) < T(A^{P,\in^2}(z*(y*x)), A^{P,\in^2}(y*z)).$$

Let a_0 be the BPFN such that

$$a_0^{P,\in} = \frac{1}{2}[A^{P,\in^2}(x) + T(A^{P,\in^2}(z*(y*x)), A^{P,\in^2}(y*z))].$$

Then clearly, $A^{P,\in^2}(x) < a_0^{P,\in} < T(A^{P,\in^2}(z*(y*x)), A^{P,\in^2}(y*z))$. Thus $x \notin [A]_{a_0}$ but $z*(y*x),\ y*z \in [A]_{a_0}$. So $[A]_{a_0}$ is not a medial ideal of X. This contradicts the hypothesis.

Assume that there are $a,\ b,\ c \in X$ such that

$$-A^{N,\in^2}(a) > S(-A^{N,\in^2}(c*(b*a)), -A^{N,\in^2}(b*c)).$$

Let α_0 be the BPFN such that

$$\alpha_0^{N,\in} = \frac{1}{2}[-A^{N,\in^2}(a) + S(-A^{N,\in^2}(c*(b*a)), -A^{N,\in^2}(b*c))].$$

Then clearly, $S(-A^{N,\in^2}(c*(b*a)), -A^{N,\in^2}(b*c))] < \alpha_0^{N,\in} < -A^{N,\in^2}(a)$. Thus $a \notin [A]_{\alpha_0}$ but $c*(b*a),\ b*c \in [A]_{\alpha_0}$. So $[A]_{\alpha_0}$ is not a medial ideal of X. This contradicts the hypothesis.

Assume that there are $x',\ y',\ z' \in X$ such that

$$A^{P,\notin^2}(x') < T(A^{P,\notin^2}(z'*(y'*x')), A^{P,\notin^2}(y'*z')).$$

Let b_0 be the BPFN such that

$$b_0^{P,\notin} = \frac{1}{2}[A^{P,\notin^2}(x') + T(A^{P,\notin^2}(z'*(y'*x')), A^{P,\notin^2}(y'*z'))].$$

Then clearly, $A^{P,\notin^2}(x') < b_0^{P,\notin} < T(A^{P,\notin^2}(z'*(y'*x')), A^{P,\notin^2}(y'*z'))$. Thus $x' \notin [A]_{b_0}$ but $z'*(y'*x'),\ y'*z' \in [A]_{b_0}$. So $[A]_{b_0}$ is not a medial ideal of X. This contradicts the hypothesis.

Assume that there are $a^{'},\ b^{'},\ c^{'} \in X$ such that

$$-A^{N,\notin^2}(a^{'}) > S(-A^{N,\notin^2}(c^{'} * (b^{'} * a^{'})), -A^{N,\notin^2}(b^{'} * c^{'})).$$

Let β_0 be the BPFN such that

$$\beta_0^{N,\notin} = \frac{1}{2}[-A^{N,\notin^2}(a^{'}) + S(-A^{N,\notin^2}(c^{'} * (b^{'} * a^{'})), -A^{N,\notin^2}(b^{'} * c^{'}))].$$

Then clearly, $S(-A^{N,\notin^2}(c^{'} * (b^{'} * a^{'})), -A^{N,\notin^2}(b^{'} * c^{'}))] < \beta_0^{N,\notin} < -A^{N,\notin^2}(a^{'})$. Thus $a^{'} \notin [A]_{\beta_0}$ but $c^{'} * (b^{'} * a^{'}),\ b^{'} * c^{'} \in [A]_{\beta_0}$. So $[A]_{\beta_0}$ is not a medial ideal of X. This contradicts the hypothesis. This completes the proof. □

4. The preimage of a (T, S)-bipolar Pythagorean fuzzy medial ideal

Definition 4.1. Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be BCI-algebras. Then a mapping $f : X \to Y$ is called a *homomorphism*, if $f(x * y) = f(x) *^{'} f(y)$ for any $x,\ y \in X$.

Note that if $f : X \to Y$ is a homomorphism of BCI-algebras, then $f(0) = 0^{'}$.

Definition 4.2. Let $X,\ Y$ be two sets, let $f : X \to Y$ be a mapping and let B be a BPFS in Y. Then the *preimage of B under f*, denoted by

$$f^{-1}(B) = ((f^{-1}(-B^{N,\in^2}), f^{-1}(B^{P,\in^2})), (f^{-1}(-B^{N,\notin^2}), f^{-1}(B^{P,\notin^2}))),$$

is a BPFS in X defined as follows: for each $x \in X$,

$$f^{-1}(-B^{N,\in^2})(x) = -B^{N,\in^2}(f(x)), f^{-1}(B^{P,\in^2})(x) = B^{P,\in^2}(f(x)),$$
$$f^{-1}(-B^{N,\notin^2})(x) = -B^{N,\notin^2}(f(x)), f^{-1}(B^{P,\in^2})(x) = B^{P,\notin^2}(f(x)).$$

Proposition 4.3. *Let $f : X \to Y$ is a homomorphism of BCI-algebras. if B is a BPFMI of Y, then $f^{-1}(B)$ is a BPFMI of X.*

Proof. Let $x \in X$. Then we get

$$\begin{aligned} f^{-1}(-B^{N,\in^2})(x) &= -B^{N,\in^2}(f(x)) \\ &\geq -B^{N,\in^2}(0) \text{ [By (BPFI}_1\text{)]} \\ &= -B^{N,\in^2}(f(0)) \text{ [Since } f \text{ is a homomorphism]} \\ &= f^{-1}(-B^{N,\in^2})(0), \end{aligned}$$

$$\begin{aligned} f^{-1}(B^{P,\in^2})(x) &= B^{P,\in^2}(f(x)) \\ &\leq B^{P,\in^2}(0) \\ &= B^{P,\in^2}(f(0)) \\ &= f^{-1}(B^{P,\in^2})(0). \end{aligned}$$

Thus $f^{-1}(B)$ satisfies the condition (BPFI$_1$). Similarly, we can see that $f^{-1}(B)$ satisfies the condition (BPFI$_3$).

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} f^{-1}(B^{P,\in^2})(x) &= B^{P,\in^2}(f(x)) \\ &\geq T(B^{P,\in^2}(f(z) * (f(y) * f(x))), B^{P,\in^2}(f(y) * f(z)))) \\ &\quad \text{[By (BPFMI}_2\text{)]} \\ &= T(B^{P,\in^2}(f(z * (y * x))), B^{P,\in^2}(f(y * z))) \\ &\quad \text{[Since } f \text{ is a homomorphism]} \end{aligned}$$

$$= T(f^{-1}(B^{P,\in^2})(z*(y*x)), f^{-1}(B^{P,\in^2})(y*z)),$$

$$\begin{aligned} f^{-1}(-B^{N,\in^2})(x) &= -B^{N,\in^2}(f(x)) \\ &\leq S(-B^{N,\in^2}(f(z)*(f(y)*f(x))), -B^{N,\in^2}(f(y)*f(z)))) \\ &= S(-B^{N,\in^2}(f(z*(y*x))), -B^{N,\in^2}(f(y*z))) \\ &= S(f^{-1}(-B^{N,\in^2})(z*(y*x)), f^{-1}(-B^{N,\in^2})(y*z)). \end{aligned}$$

Thus $f^{-1}(B)$ satisfies the condition ($BPFMI_2$). Similarly, we can show that $f^{-1}(B)$ satisfies the condition ($BPFMI_4$). So $f^{-1}(B)$ is a (T,S)-BPFMI of X. □

Proposition 4.4. *Let $f : X \to Y$ is an epimorphism of BCI-algebras and let B be a BPFS in Y. if $f^{-1}(B)$ is a BPFMI of X, then B is a BPFMI of Y.*

Proof. Let $a \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = a$. Then we get

$$\begin{aligned} -B^{N,\in^2}(a) &= -B^{N,\in^2}(f(x)) \\ &= f^{-1}(-B^{N,\in^2})(x) \\ &\geq f^{-1}(-B^{N,\in^2})(0) \text{ [By (BPFI}_1\text{)]} \\ &= -B^{N,\in^2}(f(0)) \\ &= -B^{N,\in^2}(0), \text{ [Since } f \text{ is a homomorphism]} \end{aligned}$$

$$\begin{aligned} B^{P,\in^2}(a) &= B^{P,\in^2}(f(x)) \\ &= f^{-1}(B^{P,\in^2})(x) \\ &\leq f^{-1}(-B^{P,\in^2})(0) \\ &= -B^{P,\in^2}(f(0)) \\ &= -B^{P,\in^2}(0). \end{aligned}$$

Thus B satisfies the condition ($BPFI_1$). Similarly, we can easily see that B satisfies the condition ($BPFI_3$).

Now let $a,\ b,\ c \in Y$. Then clearly, there are $x,\ y,\ z \in X$ such that $f(x) = a$, $f(y) = b$, $f(z) = c$. Thus we have

$$\begin{aligned} -B^{N,\in^2}(a) &= -B^{N,\in^2}(f(x)) \\ &= f^{-1}(-B^{N,\in^2})(x) \\ &\leq S(f^{-1}(-B^{N,\in^2})(z*(y*x)), f^{-1}(-B^{N,\in^2})(y*z)) \\ &\quad \text{[By (BPFI}_2\text{)]} \\ &= S(-B^{N,\in^2}(f(z*(y*x))), -B^{N,\in^2}(f(y*z))) \\ &= S(-B^{N,\in^2}(f(z)*(f(y)*f(x))), -B^{N,\in^2}(f(y)*f(z))) \\ &\quad \text{[Since } f \text{ is a homomorphism]} \\ &= S(-B^{N,\in^2}(c*(b*a)), -B^{N,\in^2}(b*c)), \end{aligned}$$

$$\begin{aligned} B^{P,\in^2}(a) &= B^{P,\in^2}(f(x)) \\ &= f^{-1}(B^{P,\in^2})(x) \\ &\geq T(f^{-1}(B^{P,\in^2})(z*(y*x)), f^{-1}(B^{P,\in^2})(y*z)) \\ &= T(B^{P,\in^2}(f(z*(y*x))), B^{P,\in^2}(f(y*z))) \\ &= T(B^{N,\in^2}(f(z)*(f(y)*f(x))), B^{P,\in^2}(f(y)*f(z))) \end{aligned}$$

$$= T(B^{P,\in^2}(c * (b * a)), B^{P,\in^2}(b * c)).$$

So B satisfies the condition ($BPFMI_2$). Similarly, we can prove that B satisfies the condition ($BPFMI_4$). Hence B is a (T, S)-BPFMI of X. □

5. The Cartesian product of (T, S)-bipolar Pythagorean medial ideals

Definition 5.1. Let X be a nonempty set and let λ, μ be fuzzy sets in X. Then the *Cartesian product* of λ and μ, denoted by $\lambda \times \mu$, is a fuzzy in $X \times X$ defined as follows: for each $(x, y) \in X \times X$,

$$(\lambda \times \mu)(x, y) = \lambda(x) \wedge \mu(y).$$

Definition 5.2. Let A, B be two BPFSs in X. Then the (T, S)-*Cartesian product* of A and B, denoted by

$$A \times B = ((-A^{N,\in^2} \times -B^{N,\in^2}, A^{P,\in^2} \times B^{P,\in^2}), (-A^{N,\notin^2} \times -B^{N,\notin^2}, A^{P,\notin^2} \times B^{P,\notin^2})),$$

is a BPFS in $X \times X$ defined as follows: for all x, $y \in X$,

$$-A^{N,\in^2} \times -B^{N,\in^2}(x, y) = S(-A^{N,\in^2}(x), -B^{N,\in^2}(y)),$$
$$A^{P,\in^2} \times B^{P,\in^2}(x, y) = T(A^{P,\in^2}(x), B^{P,\in^2}(y)),$$
$$-A^{N,\notin^2} \times -B^{N,\notin^2}(x, y) = S(-A^{N,\notin^2}(x), -B^{N,\notin^2}(y)),$$
$$A^{P,\notin^2} \times B^{P,\notin^2}(x, y) = T(A^{P,\notin^2}(x), B^{P,\notin^2}(y)),$$

where $-A^{N,\in} \times -B^{N,\in}$, $-A^{N,\notin} \times -B^{N,\notin} : X \times X \to [-1, 0]$,
$A^{P,\in} \times B^{P,\in}$, $A^{P,\notin} \times B^{P,\notin} : X \times X \to [0, 1]$.

Proposition 5.3. *If A, B be a (T, S)-BPFMI of X, then $A \times B$ is a (T, S)-BPFMI of $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Then we have

$$\begin{aligned}(-A^{N,\in^2} \times -B^{N,\in^2})(x, y) &= S(-A^{N,\in^2}(x), -B^{N,\in^2}(y))\\ &\geq S(-A^{N,\in^2}(0), -B^{N,\in^2}(0)) \text{ [By the hypothesis and (BPFI}_1\text{)]}\\ &= (-A^{N,\in^2} \times -B^{N,\in^2})(0, 0),\end{aligned}$$

$$\begin{aligned}(A^{P,\in^2} \times B^{P,\in^2})(x, y) &= T(A^{P,\in^2}(x), B^{P,\in^2}(y))\\ &\leq T(A^{P,\in^2}(0), B^{P,\in^2}(0))\\ &= (A^{P,\in^2} \times B^{P,\in^2})(0, 0)\end{aligned}$$

Thus $A \times B$ satisfies the condition $BPFI_1$). Similarly, $A \times B$ satisfies the condition $BPFI_3$).

Now let (x_1, x_2), (y_1, y_2), $(z_1, z_2) \in X \times X$. Then we get

$$\begin{aligned}&S((-A^{N,\in^2} \times -B^{N,\in^2})((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))),\\ &\quad (-A^{N,\in^2} \times -B^{N,\in^2})((y_1, y_2) * (z_1, z_2)))\\ &= S((-A^{N,\in^2} \times -B^{N,\in^2})((z_1, z_2) * ((y_1 * x_1, y_2 * x_2)),\\ &\quad (-A^{N,\in^2} \times -B^{N,\in^2})(y_1 * z_1, y_2 * z_2))\\ &= S((-A^{N,\in^2} \times -B^{N,\in^2})((z_1 * (y_1 * x_1)), z_2 * (y_2 * x_2)),\\ &\quad (-A^{N,\in^2} \times -B^{N,\in^2})(y_1 * z_1, y_2 * z_2))\\ &= S(S(-A^{N,\in^2}(z_1 * (y_1 * x_1)), -B^{N,\in^2}(z_2 * (y_2 * x_2)),\end{aligned}$$

$$\begin{aligned}
&S(-A^{N,\in^2}(y_1 * z_1), -B^{N,\in^2}(y_2 * z_2)))\\
&= S(S(-A^{N,\in^2}(z_1 * (y_1 * x_1)), -A^{N,\in^2}(y_1 * z_1)),\\
&\quad S(-B^{N,\in^2}(z_2 * (y_2 * x_2)), -B^{N,\in^2}(y_2 * z_2)))\\
&\geq S(-A^{N,\in^2}(x_1), -B^{N,\in^2}(x_2)) \text{ [By the hypothesis and (BPFMI}_2\text{)]}\\
&= (-A^{N,\in^2} \times -B^{N,\in^2})(x_1, x_2),
\end{aligned}$$

$$\begin{aligned}
&T((A^{P,\in^2} \times B^{P,\in^2})((z_1, z_2) * ((y_1, y_2) * (x_1, x_2))),\\
&\quad (A^{P,\in^2} \times B^{P,\in^2})((y_1, y_2) * (z_1, z_2)))\\
&= T((A^{P,\in^2} \times B^{P,\in^2})((z_1, z_2) * ((y_1 * x_1, y_2 * x_2)),\\
&\quad (A^{P,\in^2} \times B^{P,\in^2})(y_1 * z_1, y_2 * z_2))\\
&= T((A^{P,\in^2} \times B^{P,\in^2})((z_1 * (y_1 * x_1)), z_2 * (y_2 * x_2)),\\
&\quad (A^{P,\in^2} \times B^{P,\in^2})(y_1 * z_1, y_2 * z_2))\\
&= T(T(A^{P,\in^2}(z_1 * (y_1 * x_1)), B^{P,\in^2}(z_2 * (y_2 * x_2)),\\
&\quad T(A^{P,\in^2}(y_1 * z_1), B^{P,\in^2}(y_2 * z_2)))\\
&= T(T(A^{P,\in^2}(z_1 * (y_1 * x_1)), A^{P,\in^2}(y_1 * z_1)),\\
&\quad T(B^{P,\in^2}(z_2 * (y_2 * x_2)), B^{P,\in^2}(y_2 * z_2)))\\
&\leq T(A^{P,\in^2}(x_1), B^{P,\in^2}(x_2))\\
&= (A^{P,\in^2} \times B^{P,\in^2})(x_1, x_2).
\end{aligned}$$

Thus $A \times B$ satisfies the condition BPFI$_2$). Similarly, $A \times B$ satisfies the condition BPFI$_4$). So $A \times B$ is (T, S)-BPFMI of $X \times X$. □

6. Conclusions and future works

we have studied the (T, S)-bipolar Pythagorean fuzzy of medial-ideal in BCI-algebras. Also we discussed few results of bipolar fuzzy of medial ideal in BCI-algebras under homomorphism and the preimage of (T, S)-bipolar Pythagorean fuzzy of medial ideal under homomorphism of BCI-algebras are defined. How the pre-image of (T, S)-bipolar Pythagorean fuzzy of medial ideal under homomorphism of BCI-algebras become (T, S)-bipolar Pythagorean fuzzy of medial ideal are studied. Moreover, the (T, S)-Cartesian product of (T, S)-bipolar Pythagorean fuzzy of medial ideals was established.

In the future, The main purpose of our future work is to investigate the foldedness of other types of Pythagorean fuzzy ideals with special properties such as a bipolar Pythagorean (interval valued) Pythagorean fuzzy n-fold of ideals in some algebras.

7. Some algorithms

Algorithm for BCI-algebras

Input (X: set with 0 element, $*$: Binary operation)
Output ("X is a BCI-algebra or not")
Begin If $X = \varnothing$, then go to (1)
End if
If $0 \notin X$, then go to (1);
End if
Stop: = false

$i = 1$;
While $i \leq \mid X \mid$ and not (Stop) do
If $x_i * x_i \neq 0$, then
Stop: = true
End if
$j = 1$;
While $j \leq \mid X \mid$ and not (Stop) do
If $(x_i * (x_i * y_j) * y_j \neq 0$, then
Stop: = true
End if
End if
$k = 1$;
While $k \leq \mid X \mid$ and not (stop) do
If $((x_i * y_j) * (x_i * z_k)) * (z_k * y_j) \neq 0$, then
Stop: = true
End if
End while
End while
End while
If stop then
(1) Output "(X is not a BCI-algebra")
Else
Output ("X is not a QS-algebra")
End if
End.

Algorithm for fuzzy sets
Input (X: BCI-algebra, $J\mu : X \to [0, 1]$)
Output ("μ is a fuzzy set in X or not")
Begin
Stop: = false
$i = 1$;
While $i \leq \mid X \mid$ and not (stop) do
If $\mu(x) < 0$ or $\mu(x) > 1$, then
Stop: = true
End if
End if while
If stop then
Output ("μ is a fuzzy set in X")
Else
Output ("μ is not a fuzzy set in X")
Else
Output ("J is a QS-ideal of X")
End if
End.

Algorithm for medial ideals

Input (X : BCI-algebra, I: subset of X);
Output (" I is a medial ideal of X or not")
Begin
If $I = \varnothing$, then go to (1);
End if
If $0 \notin I$, then go to (1);
End if
Stop: false;
$i = 1$;
While $i \leq\mid X \mid$ and not (Stop) do
$j = 1$;
While $j \leq\mid X \mid$ and not (Stop) do
$k = 1$;
While $k \leq\mid X \mid$ and not (Stop) do
If $x_k * (y_j * x_i) \in I$ and $y_j * z_k \in I$, then If $x_i \notin I$, then Stop: = true
End If
End If
End while
End while
End while
If stop, then
Output ("I is a medial ideal of X")
Else
(1) Output ("I is not a medial ideal of X")
End If
End

Algorithm for Bipolar Pythagorean medial ideal

Input (X : BCI-algebra, $-A^{N,\in^2}$, $-A^{N,\notin^2} \in [-1, 0]$ and $A^{P,\in^2}$, $A^{P,\notin^2} \in [0, 1]$ fuzzy sets in X);
Output (" $A = ((-A^{N,\in^2}, A^{P,\in^2}), (-A^{N,\notin^2}, A^{P,\notin^2}))$ is a BPFMI of X or not")
Begin
Stop: false;
$i = 1$;
If $-A^{N,\in^2}(0) > -A^{N,\in^2}(x)$, $A^{P,\in^2}(0) < A^{P,\in^2}(x)$, $-A^{N,\notin^2}(0) > -A^{N,\notin^2}(x)$, $A^{P,\notin^2}(0) < A^{P,\notin^2}(x)$, then
Stop: true
End if
$j = 1$;
While $j \leq\mid X \mid$ and not (Stop) do
$k = 1$;
While $k \leq\mid X \mid$ and not (Stop) do
If $-A^{N,\in^2}(x) > S(-A^{N,\in^2}(z * (y * x)), -A^{N,\in^2}(y * z))$, $A^{P,\in^2}(x) < T(A^{P,\in^2}(z * (y * x)), A^{P,\in^2}(y * z))$ or $-A^{N,\notin^2}(x) > S(-A^{N,\notin^2}(z * (y * x)), -A^{N,\notin^2}(y * z))$,

$A^{P,\notin^2}(x) < T(A^{P,\notin^2}(z*(y*x)), A^{P,\notin^2}(y*z))$, then
Stop: = true
End If
End while
End while
End while
If stop, then
Output ("$A = ((-A^{N,\in^2}, A^{P,\in^2}), -A^{N,\notin^2}, A^{P,\notin^2})$) is not a BPFM of X")
Else
Output ("$A = ((-A^{N,\in^2}, A^{P,\in^2}), -A^{N,\notin^2}, A^{P,\notin^2})$) is a BPFM of X")
End If
End

Acknowledgements. The authors are greatly appreciate the referees for their valuable comments and suggestions for improving the paper.

Conflicts of Interest

State any potential conflicts of interest here or "the authors declare no conflict of interest".

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