Annals of Fuzzy Mathematics and Informatics
Volume 24, No. 3, (December 2022) pp. 239–252
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2022.24.3.239

@FMI

Estimation of degree of approximation of functions belonging to Lipschitz class by Nörlund Cesãro product summability means

J. K. Kushwaha, Laxmi Rathour, Vishnu Narayan Mishra, K. Kumar

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 24, No. 3, December 2022

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 24, No. 3, (December 2022) pp. 239–252
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2022.24.3.239

@FMI

Estimation of degree of approximation of functions belonging to Lipschitz class by Nörlund Cesãro product summability means

J. K. Kushwaha, Laxmi Rathour, Vishnu Narayan Mishra, K. Kumar

Received 19 July 2022; Revised 19 August 2022; Accepted 6 September 2022

Abstract. A number of researchers (See [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) have determined the degree of approximation of functions belonging to Lipschitz classes, using Cesãro, Euler and generalized Nörlund and various product summability means. Recently Krasniqi [15] has determined the degree of approximation of conjugate of functions using $(E, q)(C, \alpha, \beta)$ means. Mishra and Khatri [7] also determined the degree of approximation by using $(N_p.E_1)$ product means in the Hölder metric. In this paper, we have determined the degree of approximation of functions belonging to Lipschitz class and weighted class by using $(N, p)(C, \theta, \beta)$ means of Fourier series and conjugate series of Fourier series which in particular becomes $(E, q)(C, \alpha, \beta)$.

2020 AMS Classification: 42B05; 42B08

Keywords: Degree of approximation, Lipschitz class, Product summability method, Nörlund means, Fourier Series, $(N, p)(C, \theta, \beta)$ means.

Corresponding Author: Laxmi Rathour (laxmirathour817@gmail.com)

1. Introduction

The theory of approximation is a very extensive field and the study of trigonometric approximation by using summability means is of great mathematical interest and of great practical importance. Deepmala et al. [16] and Mishra et al. (See [6, 8, 10, 11]) have determined the degree of approximation of functions belonging to Lipschitz class and Weighted class by using various summability means. Krasniqi [15] has determined the degree of approximation of functions belonging to Lipschitz class by $(E, q)(C, \alpha, \beta)$ means. Now, the results of Krasniqi (See [15, 17]) have been generalized by using $(N, p)(C, \theta, \beta)$ means.

2. Preliminaries

In this section, we define Fourier series and conjugate series of Fourier series along with summability means those are widely used throughout the paper as following:

Definition 2.1. Let $f(x)$ be a 2π periodic function and integrable in Lebesgue sense and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

be its Fourier Series with n^{th} partial sum $s_n(f;x)$. The conjugate series of the Fourier series (2.1) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx). \tag{2.2}$$

Definition 2.2. Let $f : R \to R$ be a function. Then L_∞-*norm* and L_p-*norm*, denoted by $\|f\|_\infty$ and $\|f\|_p$, for f are defined respectively as follows:

$$\|f\|_\infty = sup\{|f(x)| : x \in R\} \text{ and } \|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}} \text{ for } p \geq 1.$$

The degree of approximation of a function f by a trigonometric polynomial t_n of order n under the norm $\|.\|_\infty$ is defined by Zygmund [18] with

$$\|t_n - f\|_\infty = sup\{|t_n(x) - f(x)| : x \in R\}$$

and the best approximation $E_n(f)$ of a function $f \in L_p$ is defined by the equality

$$E_n(f) = \min_{t_n} \|t_n - f\|_p.$$

A function $f \in Lip\alpha$, if $|f(x+t) - f(x)| = O\left(|t|^\alpha\right)$ for $0 < \alpha \leq 1$.

A function $f \in Lip(\alpha, p)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{\frac{1}{p}} = O\left(|t|^\alpha\right) \text{ for } 0 < \alpha \leq 1,\ p \geq 1.$$

If $\xi(t)$ is a positive increasing function and $p \geq 1$, then $f \in Lip(\xi(t), p)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{\frac{1}{p}} = O\left(\xi(t)\right).$$

Also $f \in W(L_p, \xi(t))$, if

$$\left(\int_0^{2\pi} |\{f(x+t) - f(x)\} \sin^\beta x|^p dx\right)^{\frac{1}{p}} = O\left(\xi(t)\right) \text{ for } \beta \geq 0,\ p \geq 1.$$

Definition 2.3. Let $\sum\limits_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of n^{th} partial sum $\{s_n\}$. Let $\{p_n\}$ be a non-negative sequence of constants, real or complex and let us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0,\ \forall n \geq 0,$$

$$p_{-1} = 0 = P_{-1},\ P_n \longrightarrow \infty \ as\ n \longrightarrow \infty.$$

Then the n^{th} *Cesãro mean* of order $(\theta, \beta), \theta + \beta > -1$ of the sequence $\{s_n\}$ [17] is defined by

$$C_n^{\theta,\beta} = \frac{1}{A_n^{\theta+\beta}} \sum_{k=0}^{n} A_{n-k}^{\theta-1} A_k^{\beta} s_k, \tag{2.3}$$

where $A_n^{\theta+\beta} = O(n^{\theta+\beta}),\ \theta + \beta > -1$ and $A_0^{\theta+\beta} = 1$.

The series $\sum_{n=0}^{\infty} u_n$ is said to be (C, θ, β) *summable to* s, if

$$C_n^{\theta,\beta} = \frac{1}{A_n^{\theta+\beta}} \sum_{k=0}^{n} A_{n-k}^{\theta-1} A_k^{\beta} s_k \longrightarrow s \text{ as } n \longrightarrow \infty.$$

The sequence to sequence transformation

$$t_n^N(f;x) = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu(f;x) \tag{2.4}$$

defines the sequence $\{t_n^N\}$ of *Nörlund* means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series $\sum_{n=0}^{\infty} u_n$ is said to be *summable* (N, p) *to sum* s, if $\lim_{n\to\infty} t_n^N \longrightarrow s$, particularly

$$P_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma\alpha},\ (\alpha > 0). \tag{2.5}$$

The (N, p) transform of the (C, θ, β) transform, defines $(N, p)(C, \theta, \beta)$ transform and we shall denote it by $(NC)_n^{p,\theta,\beta}$. Then we can write

$$NC = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \frac{1}{A_\nu^{\theta+\beta}} \sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1} A_k^{\beta} s_k. \tag{2.6}$$

Throughout the paper, we use the following notations:

$$V_n^{p,\theta,\beta} = \frac{1}{2\pi P_n} \left[\sum_{\nu=0}^{n} p_{n-\nu} \left\{ \frac{1}{A_\nu^{\theta+\beta}} \left(\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1} A_k^{\beta} \frac{\sin(k+\frac{1}{2})t}{\sin(\frac{t}{2})} \right) \right\} \right],$$

$$\widetilde{V}_n^{p,\theta,\beta} = \frac{1}{2\pi P_n} \left[\sum_{\nu=0}^{n} p_{n-\nu} \left\{ \frac{1}{A_\nu^{\theta+\beta}} \left(\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1} A_k^{\beta} \frac{\cos(k+\frac{1}{2})t}{\sin(\frac{t}{2})} \right) \right\} \right],$$

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

$$\varphi_x(t) = f(x+t) - f(x-t).$$

3. Major Section

We need following Lemmas to prove our Theorems.

Lemma 3.1. $|V_n^{p,\theta,\beta}(t)| = O(2n+1)$ *for* $0 \leq t \leq \frac{1}{n+1}$.

Proof. Since $0 \leq t \leq \frac{1}{n+1}$, $\sin(\frac{t}{2}) \geq (\frac{t}{\pi})$ and $\sin nt \leq n \sin t$, we get

$$\begin{aligned}
|V_n^{p,\theta,\beta}(t)| &= \frac{1}{\pi P_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\sin(k+\frac{1}{2})t}{2\sin(\frac{t}{2})}\right\}\right| \\
&\leq \frac{1}{\pi P_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{(2k+1)sint/2}{2\sin(\frac{t}{2})}\right\}\right| \\
&= \frac{1}{2\pi P_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}(2k+1)\right\}\right| \\
&= \frac{1}{2\pi P_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}(2\nu+1)\right| \\
&= O(2n+1).
\end{aligned}$$

□

Lemma 3.2. $|V_n^{p,\theta,\beta}(t)| = O\left(\frac{1}{t}\right)$ *for* $\frac{1}{n+1} \leq t < \pi$.

Proof. Since $\sin(\frac{t}{2}) \geq (\frac{t}{\pi})$, we get

$$\begin{aligned}
|V_n^{p,\theta,\beta}(t)| &= \frac{1}{\pi P_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\sin(k+\frac{1}{2})t}{2\sin(\frac{t}{2})}\right\}\right| \\
&\leq \frac{1}{2\pi P_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{1}{(t/\pi)}\right\}\right| \\
&= \frac{1}{2tP_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\right\}\right| \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

□

Lemma 3.3. $|\overline{V}_n^{p,\theta,\beta}(t)| = O\left(\frac{1}{t}\right)$ *for* $\sin(\frac{t}{2}) \geq (\frac{t}{\pi})$, $\frac{1}{n+1} \leq t < \pi$.

Proof. Since $\sin\left(\frac{t}{2}\right) \geq \left(\frac{t}{\pi}\right)$, $\frac{1}{n+1} \leq t < \pi$, we get

$$\begin{aligned}
|\overline{V}_n^{p,\theta,\beta}(t)| &= \frac{1}{\pi P_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+\frac{1}{2})t}{2\sin(\frac{t}{2})}\right\}\right| \\
&\leq \frac{1}{2tP_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{Re}{A_\nu^{\theta+\beta}}\left\{\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}e^{i(k+\frac{1}{2})t}\right\}\right| \\
&= \frac{1}{2tP_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}e^{ikt}\right|\left|e^{i\frac{t}{2}}\right| \\
&= \frac{1}{2tP_n}\left|\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}e^{ikt}\right| \\
&= \frac{1}{2tp_n}\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{A_\nu^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\left|e^{ikt}\right| \\
&= \frac{1}{2tP_n}\sum_{\nu=0}^{n} p_{n-\nu} \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

□

Now we prove following Theorems.

Theorem 3.4. *If f is a 2π-periodic function and belonging to Lipα class. Then it's degree of approximation by $(N,p)(C,\theta,\beta)$ product summability method of Fourier series is given by*

$$\|(NC)_n^{p,\theta,\beta} - f\| = O\left(\frac{1}{(n+1)^\alpha}\right),\ 0 < \alpha < 1,$$

where $(NC)_n^{p,\theta,\beta}$ denotes the $(N,p)(C,\theta,\beta)$ transform of partial sums of the Fourier series.

Proof. Let $s_k(x)$ be the partial sums of the series (2.1). Then we have

$$s_k(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi_x(t)\frac{\sin(k+\frac{1}{2})t}{\sin(\frac{t}{2})}dt.$$

Thus (C,θ,β) transform $C_\nu^{\theta,\beta}(x)$ of $s_k(x)$ is given by

$$C_\nu^{\theta,\beta}(x) - f(x) = \frac{1}{2\pi A_\nu^{\theta+\beta}}\left[\int_0^\pi \phi_x(t)\left(\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\sin(k+1/2)t}{\sin(t/2)}\right)dt\right]. \tag{3.1}$$

Now, taking (N,p) transform of above, we get

$$(NC)_n^{p,\theta,\beta} - f(x) = \frac{1}{P_n}\left[\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{2\pi A_\nu^{\theta+\beta}}\left\{\int_0^\pi \phi_x(t)\left(\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\sin(k+1/2)t}{\sin(t/2)}\right)dt\right\}\right]$$

$$= \int_0^\pi \phi_x(t) V_n^{p,\theta,\beta}(t)dt$$
$$= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi} \right) \phi_x(t) V_n^{p,\theta,\beta}(t)dt.$$

So we have

$$(NC)_n^{p,\theta,\beta} - f(x) = I_1 + I_2. \tag{3.2}$$

Now applying Lemma 3.1, we get

$$\begin{aligned}
|I_1| &\leq \int_0^{1/(n+1)} |\phi_x(t)||V_n^{p,\theta,\beta}(t)|dt \\
&= O\left(\int_0^{1/(n+1)} t^\alpha.(2n+1)dt \right) \\
&= O\left[(2n+1) \left(\frac{t^\alpha + 1)}{\alpha+1} \right)_0^{1/(n+1)} \right] \\
&= O\left[\frac{(2n+1)}{(\alpha+1)} \left(\frac{1}{(n+1)^{\alpha+1}} - 0 \right) \right] \\
&= O\left(\frac{2n+1}{(n+1)^{\alpha+1}} \right) \\
&\leq O\left[\frac{(2n+2)}{(n+1)^{\alpha+1}} \right], \text{ i.e.,}
\end{aligned}$$

$$|I_1| \leq O\left(\frac{1}{(n+1)^\alpha} \right) \text{ for } 0 < \alpha < 1. \tag{3.3}$$

Also, applying Lemma 3.2, we get

$$\begin{aligned}
|I_2| &\leq \int_{1/n+1}^{\pi} \left(|(\phi_x(t))||V_n^{p,\theta,\beta}| \right) dt \\
&= O\left(\int_{1/(n+1)}^{\pi} t^\alpha . \frac{1}{t} \right) dt \\
&= O\left(\int_{1/(n+1)}^{\pi} t^{\alpha-1} dt \right) \\
&= O\left[\frac{t^\alpha}{\alpha} \right]_{1/(n+1)}^{\pi}, \text{ i.e.,}
\end{aligned}$$

$$|I_2| \leq O\left(\frac{1}{(n+1)^\alpha} \right) \text{ for } 0 < \alpha < 1. \tag{3.4}$$

Hence by (3.3) and (3.4) in the equation (3.2), we have

$$\|(NC)_n^{p,\theta,\beta} - f\| = O\left(\frac{1}{(n+1)^{\alpha}}\right) \text{ for } 0 < \alpha < 1.$$

□

Theorem 3.5. *If f is a 2π-periodic function and belonging to $W(L_p, \xi(t))$ class, then it's degree of approximation by $(N,p)(C,\theta,\beta)$ means of Fourier series is given by*

$$\|(NC)_n^{p,\theta,\beta} - f(x)\|_p = O\left((n+1)^{r+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right)$$

provided that $\xi(t)$ satisfies the following conditions

$$\left\{\frac{\xi(t)}{t}\right\} \text{ is a decreasing sequence,} \tag{3.5}$$

$$\left\{\int_0^{1/(n+1)} \left(\frac{t|\phi_x(t)|}{\xi(t)}\right)^p \sin^{rp}(t/2)dt\right\}^{1/p} = O\left(\frac{1}{(n+1)}\right), \tag{3.6}$$

$$\left\{\int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left((n+1)^{\delta}\right), \tag{3.7}$$

where δ is an arbitrary number such that $q(1-\delta) - 1 > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < \infty$, condition (3.6) and (3.7) hold uniformly in x and $(NC)_n^{p,\theta,\beta}$ are $(N,p)(C_n^{\theta,\beta})$ means of the Fourier series (2.1).

Proof. As done in Theorem 3.4, we shall write following

$$(NC)_n^{p,\theta,\beta} - f(x) = \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi}\right)\phi_x(t)V_n^{p,\theta,\beta}(t)dt, \text{ i.e.,}$$

$$(NC)_n^{p,\theta,\beta} - f(x) = I_1^{'} + I_2^{'}. \tag{3.8}$$

Further, Using Hölder inequality and $\phi \in W(L_p, \xi(t))$, Condition (3.6) , $\sin t \geq (\frac{2t}{\pi})$, Lemma 3.1 and second mean value theorem of integral calculus, we have

$$
\begin{aligned}
|I_1^{'}| &\leq \left\{ \int_0^{1/(n+1)} \left(\frac{t|\phi_x(t)|}{\xi(t)} \right)^p \sin^{rp}(t/2)dt \right\}^{1/p} \left\{ \int_0^{1/(n+1)} \left(\frac{\xi(t)|V_n^{p;\theta,\beta}(t)|}{\sin^r(t/2)} \right)^q dt \right\}^{1/q} \\
&= O\left(\frac{1}{(n+1)}\right) \left\{ \int_0^{1/(n+1)} \left(\frac{\xi(t).(2n+1)}{t^r} \right)^q dt \right\}^{1/q} \\
&= O\left(\frac{(2n+1)}{(n+1)} \xi\left(\frac{1}{n+1}\right)\right) \left\{ \int_\epsilon^{1/(n+1)} \frac{1}{t^{rq}} dt \right\}^{1/q}, \quad \left(0 < \epsilon < \frac{1}{n+1}\right) \\
&= O\left(\frac{(2n+1)}{(n+1)} \xi\left(\frac{1}{n+1}\right)\right) \left[\left(\frac{t^{-rq+1}}{-rq+1} \right)_\epsilon^{1/(n+1)} \right]^{1/q} \\
&= O\left(\frac{(2n+1)}{(n+1)} \xi\left(\frac{1}{n+1}\right)\right) \left[\frac{(n+1)^{rq-1} - (\epsilon)^{-rq+1}}{-rq+1} \right]^{1/q} \\
&\leq O\left(\frac{\xi(\frac{1}{n+1})}{(n+1)^{\frac{1}{q}-r-1}} \right).
\end{aligned}
$$

Since $\frac{1}{p} + \frac{1}{q} = 1$,

$$
|I_1^{'}| \leq O\left((n+1)^{r+1/p} \xi\left(\frac{1}{(n+1)}\right) \right). \tag{3.9}
$$

Again, using Hölder's inequality, $|sint| \leq 1, \sin t \geq \frac{2\pi}{t}$, condition (3.5) and (3.7), Lemma 3.2 and Second mean value theorem of integral calculus, we get

$$
\begin{aligned}
|I_2^{'}| &\leq \left\{ \int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)} \right)^p \sin^{rp}(t/2)dt \right\}^{1/p} \left\{ \int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)|V_n^{p,\theta,\beta}(t)|}{t^{-\delta}\sin^r(t/2)} \right)^q dt \right\}^{1/q} \\
&= O\left((n+1)^\delta\right) \left\{ \int_{1/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{r+1-\delta}} \right)^q dt \right\}^{1/q} \\
&= O\left((n+1)^\delta\right) \left\{ \int_{1/\pi}^{(n+1)} \left(\frac{\xi(1/y)}{y^{\delta-1-r}} \right)^q \frac{dy}{y^2} \right\}^{1/q} \\
&= O\left((n+1)^\delta \xi\left(\frac{1}{n+1}\right)\right) \left\{ \int_{1/\pi}^{(n+1)} \frac{dy}{y^{q(\delta-1-r)+2}} \right\}^{1/q} \\
&= O\left((n+1)^\delta \xi\left(\frac{1}{n+1}\right)\right) \left[\left\{ \frac{y^{-q(\delta-r-1)-1}}{-q(\delta-r-1)-1} \right\}_{1/\pi}^{(n+1)} \right]^{1/q} \\
&= O\left((n+1)^\delta \xi\left(\frac{1}{n+1}\right)\right) \left[\frac{(n+1)^{(r-\delta+1)q-1} - (\pi)^{(\delta-r-1)+1}}{(r-\delta+1)q-1} \right]^{1/q}
\end{aligned}
$$

$$\leq O\left((n+1)^{\delta}\xi\left(\tfrac{1}{n+1}\right)\right)\left\{(n+1)^{(r-\delta+1)q-1)}\right\}^{1/q}$$
$$= O\left((n+1)^{\delta}\xi\left(\tfrac{1}{n+1}\right)\right)\left\{(n+1)^{r-\delta+1-\frac{1}{q}}\right\}, \text{ i.e.,}$$

$$|I_2^{'}| \leq O\left((n+1)^{r+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right) \quad \text{for} \quad \frac{1}{p}+\frac{1}{q}=1. \tag{3.10}$$

Now, combining (3.8), (3.9) and (3.10), we get

$$\|(NC)_n^{p;\theta,\beta} - f(x)\|_p = O\left((n+1)^{r+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right).$$

This Completes the proof. □

Theorem 3.6. *If $\widetilde{f}$ is a conjugate of a function which is 2π-periodic and belonging to Lipα class. Then it's degree of approximation by $(N,p)(C,\theta,\beta)$ means of conjugate series of Fourier series is given by*

$$\|(\widetilde{NC})_n^{p,\theta,\beta} - \widetilde{f}(x)\| \quad = \quad O\left(\frac{1}{(n+1)^{\alpha}}\right),\ 0<\alpha<1,$$

where $(\widetilde{NC})_n^{p,\theta,\beta}$ denotes the $(N,p)(C,\theta,\beta)$ transform of partial sums of the conjugate series of Fourier series (2.2).

Proof. Let $\widetilde{s}_k(x)$ be the partial sums of the series (2.2). Then we have

$$\widetilde{s}_k(x) - \widetilde{f}(x) \quad = \quad \frac{1}{2\pi}\left[\int_0^{\pi}\varphi_x(t)\left(\frac{\cos(k+\frac{1}{2})t}{\sin(\frac{t}{2})}\right)dt\right].$$

Thus (C,θ,β) transform $\widetilde{C}_{\nu}^{\theta,\beta}(x)$ of $\widetilde{s}_k(x)$ is given by

$$\widetilde{C}_{\nu}^{\theta,\beta}(x) - \widetilde{f}(x) = \frac{1}{2\pi A_{\nu}^{\theta+\beta}}\left[\int_0^{\pi}\varphi_x(t)\left(\sum_{k=0}^{\nu}A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+1/2)t}{\sin(t/2)}\right)dt\right]. \tag{3.11}$$

On the other hand, we have the followings:

$$\widetilde{C}_{\nu}^{\theta,\beta}(x) - \left(-\tfrac{1}{2\pi}\int_0^{\pi}\varphi_x(t)\cot(t/2)dt\right)$$
$$= \frac{1}{2\pi A_{\nu}^{\theta+\beta}}\left[\int_0^{\pi}\varphi_x(t)\left(\sum_{k=0}^{\nu}A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+1/2)t}{\sin(t/2)}\right)dt\right],$$

$$\widetilde{C}_{\nu}^{\theta,\beta}(x) - \left(-\tfrac{1}{2\pi}\int_0^{1/(n+1)}\varphi_x(t))\cot(t/2)dt - \tfrac{1}{2\pi}\int_{1/(n+1)}^{\pi}\varphi_x(t))\cot(t/2)dt\right)$$
$$= \frac{1}{2\pi A_{\nu}^{\theta+\beta}}\left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi}\right)\varphi_x(t)\sum_{k=0}^{\nu}A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+1/2)t}{\sin(t/2)}dt,$$

$$\widetilde{C}_{\nu}^{\theta,\beta}(x) - \left(-\frac{1}{2\pi}\int_{1/(n+1)}^{\pi} \varphi_x(t))\cot(t/2)dt\right)$$

$$= \frac{1}{2\pi}\left[\int_{0}^{1/(n+1)} \varphi_x(t))\left[\frac{1}{A_{\nu}^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\left(\frac{\cos(k+1/2)t}{\sin(t/2)} - \cot(t/2)\right)\right]dt\right]$$

$$+\frac{1}{2\pi A_{\nu}^{\theta+\beta}}\int_{1/(n+1)}^{\pi}\varphi_x(t)\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+1/2)t}{\sin(t/2)}dt,$$

$$\widetilde{C}_{\nu}^{\theta,\beta}(x) - \widetilde{f_n}(m)$$

$$= \frac{1}{2\pi}\left[\int_{0}^{1/(n+1)} \varphi_x(t))\left\{\frac{1}{A_{\nu}^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\left(\frac{\cos(k+1/2)t - \cos(t/2)}{\sin(t/2)}\right)\right\}dt\right]$$

$$+\frac{1}{2\pi A_{\nu}^{\theta+\beta}}\left[\int_{1/(n+1)}^{\pi}\varphi_x(t)\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+1/2)t}{\sin(t/2)}dt\right]$$

$$= \frac{1}{2\pi}\left[\int_{0}^{1/(n+1)} \varphi_x(t))\left\{\frac{1}{A_{\nu}^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\left(\frac{2\sin(k+1)t/2.sin(-kt/2)}{\sin(t/2)}\right)\right\}dt\right]$$

$$+\frac{1}{2\pi A_{\nu}^{\theta+\beta}}\left[\int_{1/(n+1)}^{\pi}\varphi_x(t)\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+1/2)t}{\sin(t/2)}dt\right].$$

Now taking (N,p) transform of above equation, we get

$$(\widetilde{NC})_{\nu}^{p,\theta,\beta}(x) - \widetilde{f_n}(m)$$

$$= \frac{-1}{p_n}\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{\pi}\left[\int_{0}^{1/(n+1)} \varphi_x(t))\left\{\frac{1}{A_{\nu}^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\left(\frac{\sin(k+1)t/2.\sin(kt/2)}{\sin(t/2)}\right)\right\}dt\right]$$

$$+\frac{1}{p_n}\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{2\pi A_{\nu}^{\theta+\beta}}\left[\int_{1/(n+1)}^{\pi}\varphi_x(t)\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\frac{\cos(k+1/2)t}{\sin(t/2)}dt\right]$$

Now applying Lemma 3.3, $|\sin nt| \leqslant n|\sin t|$, we get

$$|(\widetilde{NC})_{\nu}^{p,\theta,\beta}(x) - \widetilde{f_n}(m)|$$

$$\leq \frac{1}{p_n}\sum_{\nu=0}^{n} p_{n-\nu}\frac{1}{\pi}\left[\int_0^{1/(n+1)} |\varphi_x(t))| \left\{\frac{1}{A_\nu^{\theta+\beta}}\sum_{k=0}^{\nu} A_{\nu-k}^{\theta-1}A_k^{\beta}\left(\frac{(k+1)|\sin(t/2)|.|\sin(kt/2)|}{|\sin(t/2)|}\right)\right\} dt\right]$$

$$+ \int_{1/(n+1)}^{\pi} |\varphi_x(t)|.O\left(\frac{1}{t}\right) dt$$

$$= \frac{1}{p_n}\sum_{\nu=0}^{n} p_{n-\nu}\left[\int_0^{1/(n+1)} |\varphi_x(t)|.O(\nu+1)dt\right] + \int_{1/(n+1)}^{\pi} t^{\alpha}.O(\frac{1}{t})dt$$

$$= O(n+1)\int_0^{1/(n+1)} t^{\alpha}dt + \int_{1/(n+1)}^{\pi} t^{\alpha-1}dt$$

$$= O(n+1)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{1/(n+1)} + \left[\frac{t^{\alpha}}{\alpha}\right]_{1/(n+1)}^{\pi}$$

$$= O\left(\frac{1}{(n+1)^{\alpha}}\right) + O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

$$= O\left(\frac{1}{(n+1)^{\alpha}}\right) \text{ for } 0 < \alpha < 1, \text{ i.e.,}$$

$$\|(\widetilde{NC})_n^{p,\theta,\beta} - \widetilde{f}(x)\| = O\left(\frac{1}{(n+1)^{\alpha}}\right) \text{ for } 0 < \alpha < 1. \tag{3.12}$$

□

Theorem 3.7. *If $\widetilde{f}$ is a conjugate of a function which is 2π-periodic and belonging to $W(L_p, \xi(t))$ class , then it's degree of approximation by $(N,p)(C,\theta,\beta)$ means of conjugate series of Fourier series is given by*

$$\|(\widetilde{NC})_n^{p,\theta,\beta} - \widetilde{f}(x)\|_p = O\left((n+1)^{r+1/p}\xi\left(\frac{1}{(n+1)}\right)\right)$$

provided that $\xi(t)$ satisfies following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \tag{3.13}$$

is a decreasing sequence.

$$\left\{\int_0^{1/(n+1)} \left(\frac{t|\varphi_x(t)|}{\xi(t)}\right)^p \sin^{rp}(t/2)dt\right\}^{1/p} = O\left(\frac{1}{(n+1)}\right), \tag{3.14}$$

$$\left\{ \int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\varphi_x(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = O\left((n+1)^{\delta}\right), \tag{3.15}$$

where δ *is an arbitrary number such that* $q(1-\delta)-1>0,\ \frac{1}{p}+\frac{1}{q}=1,\ 1 \le p < \infty$, *conditions (3.14) and (3.15) hold uniformly in* x.

Proof. The proof of this theorem can be done similar to the Theorem 3.6 taking into account Lemmas 3.1 and 3.2. □

Corollary 3.8. *If* $p_n = {}^n C_{n-\nu} q^{n-\nu}$, *then* $(\widetilde{N,p})(C,\theta,\beta) = (E,q)(C,\theta,\beta)$, *which is the Case of Krasniqi* [15].

Corollary 3.9. *As*

$$\begin{aligned} P_n &= \sum_{\nu=0}^{n} p_\nu \\ &= p_0 + p_1 + \ldots \cdots + p_n \\ &= {}^n C_1 + {}^n C_2 \ldots \cdots + {}^n C_n \\ &= (1+1)^n \\ &= 2^n \text{ for } q = 1, \end{aligned}$$

we have $(N,p)(C,\theta,\beta) = (E,1)(C,\theta,\beta)$.

Corollary 3.10. *If* $\theta = 1, \beta = 0$ *and all restrictions of Theorem 3.6 hold, then*

$$\|(\widetilde{NC})_n^{P,1,0} - \widetilde{f}\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right) \ 0 < \alpha < 1.$$

Corollary 3.11. *If* $\theta = 1, \beta = 0$ *and all restrictions of Theorem 3.7 hold, then*

$$\|(\widetilde{NC})_n^{P,1,0} - \widetilde{f}\|_p = O\left((n+1)^{r+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right)\right).$$

Corollary 3.12. *If* $\theta = 1,\ \beta = 0,\ r = 0$ *and* $\xi(t) = t^\alpha$, *then weighted class reduces to* $Lip(\alpha, p)$, *then*

$$\|(\widetilde{NC})_n^{P,1,0} - \widetilde{f}\|_p = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{p}}}\right).$$

Corollary 3.13. *Let* $\theta = 1,\ \beta = 0,\ r = 0$ *and* $\xi(t) = t^\alpha$. *If* $p \to \infty$ *in above Corollary, then* $f \in Lip(\alpha, p)$ *reduces to* $Lip\alpha$ *for* $0 < \alpha < 1$ *and*

$$\|(\widetilde{NC})_n^{P,1,0} - \widetilde{f}\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right).$$

Remark 3.14. We can find similar particular results for degree of approximation of Fourier series in Theorem 3.5.

4. Conclusions

Sometimes a series is not summable by any individual summability method. But it becomes summable by taking product summability means of given series. So working in this direction we have used $(N, p)(C, \theta, \beta)$ means of Fourier series and conjugate series of Fourier series which in particular becomes $(E, q)(C, \alpha, \beta)$.Therefore, many of the known results may become particular cases of our result. On the bases of above facts we can say that our result may be useful for the coming researchers in future.

Acknowledgements. The authors are grateful to Prof. Shyam Lal, Head, Department of Mathematics, Institute of Science, Banaras Hindu University for encouragement to this work. Krishna Kumar one of the authors, is grateful to CSIR, New Delhi, India for providing financial assistance in the form of Junior Research Fellowship (JRF) Ref.No.- 18/06/2017(i) EU-V, (CSIR-UGC NET JUNE-2017).

References

[1] P. Chandra, Trigonometric approximation of function in L_p-norm, J. Math. Anal. Appl. 275 (2002) 13–26.

[2] X. Z. Krasniqi and L. N. Mishra, On the power integrability with weight of double trigonometric series, Adv. Stud. Contemp. Math. (Kyungshang) 31 (2021), DOI:http://dx.doi.org/10.17777/ascm2021.31.2.221, 22 pages.

[3] X. Z. Krasniqi, L. Rathour and L. N. Mishra, On approximation of continuous bivariate periodic functions by deferred generalized de la Vallée Poussin means of their Fourier series, Advanced Studies in Contemporary Mathematics 32 (2022) 387–402.

[4] H. H. Khan, On the degree of approximation of functions belonging to the class $Lip(\alpha, p)$, Indian J. pure Appl. Math. 5 (1974) 132–136.

[5] S. Lal, On K^{λ} summability of conjugate series of Fourier series, Bull. Calcutta Math. Soc. 89 (1997) 97–104.

[6] L. N. Mishra, M. Raiz, L. Rathour and V. N. Mishra, Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces, Yugoslav Journal of Operations Research, [S.l.], Apr. 2022. DOI: https://doi.org/10.2298/YJOR210915005M.

[7] V. N. Mishra and K. Khatri, Degree of approximation on $f \in H_w$ class by the $(N_p.E^1)$ means in the Hölder Metric, Int. J. Math. and Math. Sci. (2014) 1–9.

[8] L. N. Mishra, V. N. Mishra, K. Kharti and Deepmala, On the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1))$ class by matrix $C^1.N_p$ operator of conjugate series of its Fourier series, Applied Mathematics and Computation 237 (2014) 252–263.

[9] V. N. Mishra and L. N. Mishra, Trigonometric approximation of signals (functions) in L_p-norm, International Journal of Contemporary Mathematical Sciences 7 (2012) 909–918.

[10] V. N. Mishra, K. Khatri, L. N. Mishra and Deepmala, Trigonometric approximation of periodic signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1))$ class by Nörlund Euler $(N, p_n)(E, q)$ operator of conjugate series of its Fourier Series, Journal of Classical Analysis 5 (2) (2014) 91–105. doi: 10.7153/jca-05-08.

[11] V. N. Mishra, K. Khatri and L. N. Mishra, Using linear operators to approximate signals of $Lip(\alpha, p), (p \geq 1)$-class, Filomat 27 (2013) 353–363.

[12] V. N. Mishra, Some problems on approximations of functions in Banach spaces, Ph. D. Thesis (2007), Indian Institute of Technology, Roorkee 247 667, Uttarakhand, India.

[13] H. K. Nigam, A study on approximation of conjugate of functions belonging to Lipschitz class and generalized Lipschitz class by product means of conjugate series of Fourier series, Thai J. math. 10 (2012) 275–287.

[14] H. K. Nigam and K. Sharma, Approximation of conjugate functions belonging to $Lip(\alpha, r)$ class and to weighted $W(L_r, \xi(t))$ class by product means of conjugate Fourier series, Eur. J. Pure Appl. Math. 3 (2011) 276–286.

[15] X. Z. Krasniqi, On the degree of approximation of functions belonging to the Lipschitz class by $(E, q)(C, \alpha, \beta)$ means, Khayyam Journal of Mathematics 1 (2015) 243–252.

[16] Deepmala, L. N. Mishra and V. N. Mishra, Trigonometric approximation of signals (functions) belonging to the $W(L_r, \xi(t)), (r \geq 1)-$ class by $(E, q)(q \geq 0)$-means of the conjugate series of its Fourier series, GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13, Global Journal of Mathematical Sciences 2 (2014) 61–69.

[17] X. Z. Krasniqi, On the degree of approximation of functions of the Lipschitz class by $(E, q)(C, \alpha, \beta)$ means, J. Numer. Math. stoch. 4 (2012) 40–47.

[18] A. Zygmund, Trigonometric Series, Vol.1, 2^{nd} Rev.ed., Cambridge Univ. Press, Cambridge, New York, 1959.

Jitendra Kumar Kushwaha (k.jitendrakumar@yahoo.com)
Department of Mathematics and Statistics, Deen Dayal Upadhyaya
Gorakhpur University, Gorakhpur-273009, India

Laxmi Rathour (laxmirathour817@gmail.com, rathourlaxmi562@gmail.com)
Ward Number 16, Bhagatbandh, Anuppur-484 224, Madhya Pradesh, India

Vishnu Narayan Mishra (vishnunarayanmishra@gmail.com, vnm@igntu.ac.in)
Department of Mathematics, Indira Gandhi National Tribal University,
Lalpur, Amarkantak, Anuppur, Madhya pradesh-484 887

Krishna Kumar (kkmaths1986@gmail.com)
Department of Mathematics and Statistics, Deen Dayal Upadhyaya,
Gorakhpur University, Gorakhpur-273009, India