Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 24, No. 3, (December 2022) pp. 223–238
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2022.24.3.223

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 24, No. 3, December 2022

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Received 17 May 2022; Revised 20 June 2022; Accepted 17 August 2022

Abstract. In this paper, a new class of fuzzy topological spaces, namely fuzzy hereditarily extremally disconnected spaces, is introduced and studied. It is obtained that fuzzy perfectly disconnected spaces are fuzzy hereditarily extremally disconnected spaces and fuzzy hereditarily extremally disconnected spaces are not fuzzy hyperconnected spaces. A condition under which fuzzy hereditarily extremally disconnected spaces become fuzzy $F^{'}$-spaces is obtained.

2020 AMS Classification: 54A40, 03E72, 54G05

Keywords: Fuzzy dense set, Fuzzy semi-open set, Fuzzy pre-open set, Fuzzy F_{σ}-set, Fuzzy perfectly disconnected space, Fuzzy hyperconnected space, Fuzzy extremally disconnected space, Fuzzy $F^{'}$-space.

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1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets, fuzzy set operations was introduced by Zadeh [1] in 1965. The concept of fuzzy topological spaces was introduced by Chang [2] in 1968. Chang's works paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. The concept of extremally disconnectedness for topological spaces has widely been studied by many mathematicians. Certain nice applications of the notion of extremally disconnected spaces are found in Hermann's work [3]. Jankovic [4] has applied the notion of extremally disconnectedness in his investigations of certain types of mappings. Sarma [5] studied the inter-relationship between fuzzy weak continuity and fuzzy semi-continuity in the context of extremal disconnectedness. In classical topology, Bezhanishvili et al. [6] introduced the notion of hereditarily extremally disconnected spaces as those topological spaces in which any two separated subsets have disjoint closures. Hereditarily extremally disconnectedness is useful in modal logic.

In the recent years, there has been a growing trend among many fuzzy topologists to introduce and study various types of fuzzy topological spaces. The concept of fuzzy extremally disconnected spaces was introduced and studied by Ghosh [7]. Nowadays, the topological space theory has been embedding in the soft set theory to obtain some interesting applications [8, 9, 10, 11]. Recently, Lee et al. [12] defined an octahedron set composed of an interval-valued fuzzy set, an intuitionistic set and a fuzzy set that will provide nice information about uncertainty and vagueness, discussed its various properties. Moreover Lee et al. [13] studied topological structures based on octahedron sets. Şenel et al. [14] applied the concept of octahedron sets to multi-criteria group decision-making problems. Also several contributions to fuzzy sets and fuzzy topologies were done by Al-Shami et al. [15, 16, 17].

The purpose of this paper is to introduce the concept of fuzzy hereditarily extremally disconnected spaces. In Section 3, several characterizations of fuzzy hereditarily extremally disconnected spaces are established. It is obtained that disjoint fuzzy open sets have disjoint closures and disjoint fuzzy sets have disjoint interiors in fuzzy hereditarily extremally disconnected spaces. It is established that fuzzy regular closed sets are fuzzy open sets and fuzzy semi-open sets are fuzzy pre-open sets in fuzzy hereditarily extremally disconnected spaces.

In Section 4, it is established that fuzzy perfectly disconnected spaces are fuzzy hereditarily extremally disconnected spaces and fuzzy hereditarily extremally disconnected spaces are fuzzy extremally disconnected spaces. It is found that fuzzy hereditarily extremally disconnected spaces are not fuzzy hyperconnected spaces. A condition under which a fuzzy hereditarily extremally disconnected space becomes a fuzzy $F^{'}$-space, is also obtained. It is established that fuzzy hereditarily extremally disconnected spaces are fuzzy hereditarily irresolvable spaces and fuzzy irresolvable spaces.

2. Preliminaries

In order to make the exposition self-contained, some basic notions and results used in the sequel, are given . In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set, I the unit interval $[0, 1]$ and J denote an index set. A fuzzy set in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$ for all $x \in X$.

Definition 2.1 ([2])**.** A *fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following conditions:

(i) $0_X \in T$, $1_X \in T$,
(ii) if $A, B \in T$, then $A \wedge B \in T$,
(iii) if $A_j \in T$ for each $j \in J$, then $\bigvee_j A_j \in T$.

The pair (X, T) is called a *fuzzy topological space* (in short fts). Every member of T is called a *T-open fuzzy set* in X.

Definition 2.2 ([2])**.** Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The *interior*, the *closure* and the *complement* of λ are defined respectively as follows:

(i) $int(\lambda) = \bigvee\{\mu \mid \mu \leq \lambda,\ \mu \in T\}$,
(ii) $cl(\lambda) = \bigwedge\{\mu \mid \lambda \leq \mu,\ 1-\mu \in T\}$,
(iii) $\lambda'(x) = 1 - \lambda(x)$ for all $x \in X$.

For a family $\{\lambda_j \mid j \in J\}$ of fuzzy sets in (X, T), the *union* $\psi = \bigvee_{j\in J} \lambda_j$ and *intersection* $\delta = \bigwedge_{j\in J} \lambda_j$ are defined respectively as:

(iv) $\psi(x) = \sup_j\{\lambda_j(x) \mid x \in X\}$,
(v) $\delta(x) = \inf_j\{\lambda_j(x) \mid x \in X\}$.

Lemma 2.3 ([18])**.** *For a fuzzy set λ of a fuzzy topological space X,*
(1) $1 - int(\lambda) = cl(1-\lambda)$,
(2) $1 - cl(\lambda) = int(1-\lambda)$.

Definition 2.4. A fuzzy set λ in a fuzzy topological space (X, T) is called a

(i) *fuzzy regular-open set*, if $\lambda = intcl(\lambda)$ and *fuzzy regular-closed set*, if $\lambda = clint(\lambda)$ [18],
(ii) *fuzzy G_δ-set*, if $\lambda = \bigwedge_{j=1}^{\infty}(\lambda_j)$, where $\lambda_j \in T$ for $j \in J$ [19],
(iii) *fuzzy F_σ-set*, if $\lambda = \bigvee_{j=1}^{\infty}(\lambda_j)$, where 1- $\lambda_j \in T$ for $j \in J$ [19],
(iv) *fuzzy pre-open set*, if $\lambda \leq intcl(\lambda)$ and *fuzzy pre-closed set*, if $clint(\lambda) \leq \lambda$ [20],
(v) *fuzzy semi-open set*, if $\lambda \leq clint(\lambda)$ and *fuzzy semi-closed set*, if $intcl(\lambda) \leq \lambda$ [18],
(vi) *fuzzy α-open set*, if $\lambda \leq intclint(\lambda)$ and *fuzzy α-closed set*, if $clintcl(\lambda) \leq \lambda$ [20].

Definition 2.5. A fuzzy set λ in a fuzzy topological space (X, T), is called a

(i) *fuzzy dense set*, if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$, i.e., $cl(\lambda) = 1$ in (X, T) [21],
(ii) *fuzzy nowhere dense set*, if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) = 0$ in (X, T) [21],
(iii) *fuzzy somewhere dense set*, if there exists a non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) \neq 0$ in (X, T) [22].

Definition 2.6 ([24])**.** Let (X, T) be a fuzzy topological space. The fuzzy sets λ and μ defined on X are said to be *fuzzy separated sets*, if $cl(\lambda) \wedge \mu = 0$ and $\lambda \wedge cl(\mu) = 0$ in (X, T).

Definition 2.7. A fuzzy topological space (X, T) is called a

(i) *fuzzy perfectly disconnected space*, if for any two non-zero fuzzy sets λ and μ defined on X with $\lambda \leq 1-\mu$, $cl(\lambda) \leq 1 - cl(\mu)$ in (X, T) [26],
(ii) *fuzzy hyperconnected space*, if every non-null fuzzy open subset of (X, T) is fuzzy dense in (X, T) [25],
(iii) *fuzzy basically disconnected space*, if the closure of every fuzzy open F_σ-set of (X, T) is fuzzy open in (X, T) [23],
(iv) *fuzzy $F^{'}$-space*, if $\lambda \leq 1-\mu$, where λ and μ are fuzzy F_σ-sets in (X, T), then $cl(\lambda) \leq 1 - cl(\mu)$ in (X, T) [27],
(v) *fuzzy hereditarily irresolvable space*, if for any two fuzzy sets λ and μ defined on X with $cl(\lambda) = cl(\mu)$ $(\neq 0)$, $\lambda \wedge \mu \neq 0$ in (X, T) [28] .

Theorem 2.8 ([22])**.** *If λ is a fuzzy somewhere dense set in a fuzzy topological space (X, T), then there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$.*

Theorem 2.9 ([18])**.** *In a fuzzy topological space,*

(1) *the closure of a fuzzy open set is a fuzzy regular closed set,*
(2) *the interior of a fuzzy closed set is a fuzzy regular open set.*

Theorem 2.10 ([7])**.** *A fuzzy topological space (X, T) is fuzzy extremally disconnected if and only if $FSO(X) \subset FPO(X)$.*

Theorem 2.11 ([7])**.** *The following are equivalent for a fuzzy topological space X:*

(1) *X is fuzzy extremally disconnected space,*
(2) *the closure of every fuzzy semi-open set in X is fuzzy open,*
(3) *the semi-closure of every fuzzy semi-open set in X is fuzzy open.*

Theorem 2.12 ([28])**.** *If a fuzzy topological space (X, T) is a fuzzy hereditarily irresolvable space, then (X, T) is a fuzzy irresolvable space.*

3. Fuzzy hereditarily extremally disconnected spaces

Definition 3.1. A fuzzy topological space (X, T) is called a *fuzzy hereditarily extremally disconnected space*, if for any two fuzzy sets λ and μ defined on X, $cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\lambda))]\}$ in (X, T).

Example 3.2. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β, γ and λ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.4$; $\alpha(b) = 0.6$; $\alpha(c) = 0.5$,
$\beta : X \to I$ is defined by $\beta(a) = 0.5$; $\beta(b) = 0.5$; $\beta(c) = 0.6$,
$\gamma : X \to I$ is defined by $\gamma(a) = 0.6$; $\gamma(b) = 0.4$; $\gamma(c) = 0.5$,
$\lambda : X \to I$ is defined by $\lambda(a) = 0.4$; $\lambda(b) = 0.3$; $\lambda(c) = 0.4$.

Then $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee (\beta \wedge \gamma), \gamma \vee (\alpha \wedge \beta), \beta \wedge (\alpha \vee \gamma), \alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that

$cl(\alpha) = 1 - \gamma = \alpha$, $int(1 - \alpha) = \gamma$,
$cl(\beta) = 1$, $int(1 - \beta) = 0$,
$cl(\gamma) = 1 - \alpha = \gamma$, $int(1 - \gamma) = \alpha$,
$cl(\alpha \vee \beta) = 1$, $int(1 - [\alpha \vee \beta]) = 0$,
$cl(\alpha \vee \gamma) = 1 - \alpha \wedge \gamma = \alpha \vee \gamma$, $int(1 - [\alpha \vee \gamma]) = \alpha \wedge \gamma$,
$cl(\beta \vee \gamma) = 1$, $int(1 - [\beta \vee \gamma]) = 0$,
$cl(\alpha \wedge \beta) = 1 - [\gamma \vee (\alpha \wedge \beta)] = \alpha \wedge \beta$, $int(1 - [\alpha \wedge \beta]) = \gamma \vee (\alpha \wedge \beta)$,
$cl(\alpha \wedge \gamma) = 1 - (\alpha \vee \gamma) = \alpha \wedge \gamma$, $int(1 - [\alpha \wedge \gamma]) = \alpha \vee \gamma$,
$cl(\beta \wedge \gamma) = 1 - (\alpha \vee [\beta \wedge \gamma]) = \beta \wedge \gamma$, $int(1 - [\beta \wedge \gamma]) = \alpha \vee (\beta \wedge \gamma)$,
$cl(\alpha \vee [\beta \wedge \gamma]) = 1 - (\beta \wedge \gamma) = \alpha \vee (\beta \wedge \gamma)$, $int(1 - [\alpha \vee [\beta \wedge \gamma]]) = \beta \wedge \gamma$,
$cl(\gamma \vee [\alpha \wedge \beta]) = 1 - (\alpha \wedge \beta) = \gamma \vee (\alpha \wedge \beta)$, $int(1 - [\gamma \vee [\alpha \wedge \beta]]) = \alpha \wedge \beta$,
$cl(\beta \wedge [\alpha \vee \gamma]) = 1 - (\beta \wedge [\alpha \vee \gamma]) = \beta \wedge [\alpha \vee \gamma]$, $int(1 - [\beta \wedge [\alpha \vee \gamma]]) = \beta \wedge [\alpha \vee \gamma]$;
$cl(\alpha \vee \beta \vee \gamma) = 1$, $int(1 - [\alpha \vee \beta \vee \gamma]) = 0$.

By computation, one can find that for any two fuzzy sets λ, μ $[= \alpha, \beta, \gamma, \lambda]$ defined on X, $cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\lambda))]\}$ in (X, T). Thus (X, T) is a fuzzy hereditarily extremally disconnected space.

Proposition 3.3. *If λ and μ are any two fuzzy sets defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then it follows that $cl[\lambda \wedge int(1-\mu)] \leq int([1-\mu] \vee cl(\lambda))$ in (X, T).*

Proof. Let λ and μ be any two fuzzy sets defined on X in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space for the fuzzy sets λ and μ, $cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\lambda))]\}$ in (X, T). Then by Lemma 2.3, $1 - cl(\lambda) = int(1-\lambda)$ and thus $cl[\lambda \wedge int(1-\mu)] \leq 1 - \{cl[\mu \wedge int(1-\lambda)]\}$ and $cl[\lambda \wedge int(1-\mu)] \leq 1 - \{cl[\mu \wedge int(1-\lambda)]\} = int(1 - [\mu \wedge int(1-\lambda)]) = int([1-\mu] \vee [1 - int(1-\lambda)]) = int([1-\mu] \vee [1-(1-cl(\lambda))]) = int([1-\mu] \vee cl(\lambda))$ in (X, T). So it follows that $cl[\lambda \wedge int(1-\mu)] \leq int([1-\mu] \vee cl(\lambda))$ in (X, T). □

Proposition 3.4. *If λ and μ are any two fuzzy sets defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then there exist a fuzzy closed set η and a fuzzy open set δ in (X, T) such that $[\lambda \wedge int(1-\mu)] \leq \eta \leq \delta \leq [cl(\lambda) \vee (1-\mu)]$ in (X, T).*

Proof. Let λ and μ be any two fuzzy sets defined on X in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.3, we have

$$cl[\lambda \wedge int(1-\mu)] \leq int([1-\mu] \vee cl(\lambda)) \text{ in } (X, T).$$

Then $\lambda \wedge int(1-\mu) \leq cl[\lambda \wedge int(1-\mu)] \leq int([1-\mu] \vee cl(\lambda)) \leq [1-\mu] \vee cl(\lambda)$. Let $\eta = cl[\lambda \wedge int(1-\mu)]$. Then η is a fuzzy closed set in (X, T). Also let $\delta = int([1-\mu] \vee cl(\lambda))$. Then δ is a fuzzy open set in (X, T). Thus there exist a fuzzy closed set η and a fuzzy open set δ in (X, T) such that $[\lambda \wedge int(1-\mu)] \leq \eta \leq \delta \leq [cl(\lambda) \vee (1-\mu)]$. □

Proposition 3.5. *If λ and μ are any two fuzzy sets defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then*

1. *$cl[\lambda \wedge int(1-\mu)] \neq 1$ in (X, T),*
2. *$int([1-\mu] \vee cl(\lambda)) \neq 0$ in (X, T).*

Proof. Let λ and μ be any two fuzzy sets defined on X in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.4, there exist a fuzzy closed set η and a fuzzy open set δ in (X, T) such that

$$[\lambda \wedge int(1-\mu)] \leq \eta \leq \delta \leq [cl(\lambda) \vee (1-\mu)].$$

(1) Since $[\lambda \wedge int(1-\mu)] \leq \eta$, $cl[\lambda \wedge int(1-\mu)] \leq cl(\eta)$. Then $cl[\lambda \wedge int(1-\mu)] \leq \eta$ [since η is fuzzy closed, $cl(\eta) = \eta$]. Thus $cl[\lambda \wedge int(1-\mu)] \neq 1$ in (X, T).

(2) Since $\delta \leq [cl(\lambda) \vee (1-\mu)]$, $int(\delta) \leq int[cl(\lambda) \vee (1-\mu)]$. Then $\delta \leq int[cl(\lambda) \vee (1-\mu)]$ [since δ is fuzzy open, $int(\delta) = \delta$]. Thus $int([1-\mu] \vee cl(\lambda)) \neq 0$ in (X, T). □

Proposition 3.6. *If λ and μ are any two fuzzy sets defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then $([1-\mu] \vee cl(\lambda))$ is a fuzzy somewhere dense set in (X, T).*

Proof. Let λ and μ be any two fuzzy sets defined on X in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.5 (2), $int([1-\mu] \vee cl(\lambda)) \neq 0$, in (X, T). Now $int([1-\mu] \vee cl(\lambda)) \leq intcl([1-\mu] \vee cl(\lambda))$, implies that $intcl([1-\mu] \vee cl(\lambda)) \neq 0$. Then $([1-\mu] \vee cl(\lambda))$ is a fuzzy somewhere dense set in (X, T). □

Corollary 3.7. *If λ and μ are any two fuzzy sets defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then $cl(\lambda) \vee cl(1-\mu)$ is a fuzzy somewhere dense set in (X, T).*

Proof. By Proposition 3.6, for the fuzzy sets λ and μ defined on X, $([1-\mu] \vee cl(\lambda))$ is a fuzzy somewhere dense set in (X, T). Then $intcl([1-\mu] \vee cl(\lambda)) \neq 0$. Now $([1-\mu] \vee cl(\lambda)) \leq cl(1-\mu) \vee cl(\lambda)$ implies that $intcl([1-\mu] \vee cl(\lambda)) \leq intcl[cl(1-\mu) \vee cl(\lambda)]$. Then $intcl[cl(1-\mu) \vee cl(\lambda)] \neq 0$. Thus $cl(\lambda) \vee cl(1-\mu)$ is a fuzzy somewhere dense set in (X, T). □

Proposition 3.8. *If λ and μ are any two fuzzy sets defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq [cl(1-\mu) \vee cl(\lambda)]$.*

Proof. The proof follows from Proposition 3.6 and Theorem 2.8. □

Proposition 3.9. *A fuzzy topological space (X, T) is a fuzzy hereditarily extremally disconnected space if and only if $cl(\lambda) \leq 1 - cl(\mu)$ for any two fuzzy separated sets λ and μ in (X, T).*

Proof. Suppose (X, T) is a fuzzy hereditarily extremally disconnected space and let λ and μ be fuzzy separated sets in (X, T). Then $cl(\lambda) \wedge \mu = 0$ and $\lambda \wedge cl(\mu) = 0$. Thus $cl(\lambda) \leq 1 - \mu$ and $\lambda \leq 1 - cl(\mu)$. Since $cl(\lambda) \leq 1 - \mu$, $\mu \leq 1 - cl(\lambda)$. So $\mu \wedge [1 - cl(\lambda)] = \mu$. Also $\lambda \leq 1 - cl(\mu)$ implies that $\lambda \wedge (1 - cl(\mu)) = \lambda$. By hypothesis, for the fuzzy sets λ and μ, $cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - [cl[\mu \wedge (1 - cl(\lambda))]]$. Hence $cl(\lambda) \leq 1 - cl(\mu)$.

Conversely, suppose $cl(\lambda) \leq 1 - cl(\mu)$ for any two fuzzy separated sets λ and μ in (X, T). Since λ and μ are fuzzy separated sets in (X, T). $cl(\lambda) \wedge \mu = 0$ and $\lambda \wedge cl(\mu) = 0$. Then $cl(\lambda) \leq 1-\mu$ and $\lambda \leq 1-cl(\mu)$. Since $cl(\lambda) \leq 1-\mu$, $\mu \leq 1-cl(\lambda)$. Thus $\mu \wedge [1 - cl(\lambda)] = \mu$. Also $\lambda \leq 1 - cl(\mu)$ implies that $\lambda \wedge (1 - cl(\mu)) = \lambda$. So $cl[\lambda \wedge (1 - cl(\mu))] = cl(\lambda)$ and $cl[\mu \wedge [1 - cl(\lambda)]] = cl(\mu)$. Moreover $cl(\lambda) \leq 1 - cl(\mu)$ implies that $cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - [cl[\mu \wedge [1 - cl(\lambda)]]]$. Hence (X, T) is a fuzzy hereditarily extremally disconnected space. □

Proposition 3.10. *If λ is a fuzzy set defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then there exists a fuzzy open set γ in (X, T) such that $clint(\lambda) \leq \gamma \leq \lambda \vee clint(\lambda)$.*

Proof. Let λ be a fuzzy set in (X, T). Then $int(\lambda) \leq \lambda$ and $int(\lambda) \leq 1 - [1 - \lambda]$. Let $\mu = 1 - \lambda$ and $\delta = int(\lambda)$. Then clearly, δ and μ are fuzzy sets in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, $cl[\delta \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\delta))]\}$. Now for the fuzzy sets δ and μ, we have

$$cl[\delta \wedge (1 - cl(\mu))] = cl[int(\lambda) \wedge (1 - cl(1 - \lambda))] = cl[int(\lambda) \wedge int(\lambda)] = clint(\lambda).$$

Also we get $1 - \{cl[\mu \wedge (1 - cl(\delta))]\} = int\{1 - [\mu \wedge (1 - cl(\delta))]\} = int\{(1 - \mu) \vee (1 - [1 - cl(\delta)])\} = int\{(1 - \mu) \vee cl(\delta)\} = int\{(1 - [1 - \lambda]) \vee cl(int(\lambda))\} = int\{\lambda \vee cl(int(\lambda))\}$. Thus $clint(\lambda) \leq int\{\lambda \vee cl(int(\lambda))\} \leq \lambda \vee cl(int(\lambda))$. It is obvious that $\gamma = int\{\lambda \vee cl(int(\lambda))\}$ is a fuzzy open set in (X, T). So for the fuzzy set λ in the fuzzy hereditarily disconnected space (X, T), there exists a fuzzy open set γ in (X, T) such that $clint(\lambda) \leq \gamma \leq \lambda \vee clint(\lambda)$. □

Proposition 3.11. *If λ is a fuzzy set defined on X in a fuzzy hereditarily extremally disconnected space (X, T), then $int(\lambda)$ is not a fuzzy dense set in (X, T).*

Proof. Let λ be a fuzzy set in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.10, for the fuzzy set λ in (X, T), there exists a fuzzy open set γ in (X, T) such that $clint(\lambda) \leq \gamma \leq \lambda \vee clint(\lambda)$. Then $cl[clint(\lambda)] \leq cl(\gamma)$. Thus $clint(\lambda) \leq cl(\gamma)$. So $clint(\lambda) \neq 1$. Hence $int(\lambda)$ is not a fuzzy dense set in (X, T). □

Proposition 3.12. *If λ and μ are any two fuzzy closed sets in a fuzzy hereditarily extremally disconnected space (X, T), then $cl[\lambda \wedge (1 - \mu)] \leq 1 - \{cl[\mu \wedge (1 - \lambda)]\}$.*

Proof. Let λ and μ be fuzzy closed sets in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, for the fuzzy sets λ and μ, we have

$$cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\lambda))]\}.$$

Since λ and μ are fuzzy closed sets in (X, T), $cl(\lambda) = \lambda$ and $cl(\mu) = \mu$. Thus we get

$$cl[\lambda \wedge (1 - \mu)] \leq 1 - \{cl[\mu \wedge (1 - \lambda)]\}.$$

□

Proposition 3.13. *If $int([1 - \mu] \vee cl(\lambda)) = 0$ for any two fuzzy sets λ and μ in a fuzzy hereditarily extremally disconnected space (X, T), then $\lambda \leq cl(\mu)$.*

Proof. Let λ and μ be fuzzy sets defined on X in (X, T) such that $int([1 - \mu] \vee cl(\lambda)) = 0$. Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.3, $cl[\lambda \wedge int(1 - \mu)] \leq int([1 - \mu] \vee cl(\lambda))$. Then by the hypothesis, $cl[\lambda \wedge int(1 - \mu)] = 0$. Thus $\lambda \wedge int(1 - \mu) = 0$ and $\lambda \leq 1 - [int(1 - \mu)]$. So by Lemma 2.3, $int(1 - \mu) = 1 - cl(\mu)$ and $\lambda \leq 1 - [1 - cl(\mu)]$. Hence $\lambda \leq cl(\mu)$. □

Corollary 3.14. *If $int([1 - \mu] \vee cl(\lambda)) = 0$, where λ is a fuzzy open set and μ is a fuzzy set in a fuzzy hereditarily extremally disconnected space (X, T), then μ is a fuzzy somewhere dense set in (X, T).*

Proof. Suppose that $int([1 - \mu] \vee cl(\lambda)) = 0$ for the two fuzzy sets λ and μ in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.13, $\lambda \leq cl(\mu)$. Then $int(\lambda) \leq intcl(\mu)$. Since λ is fuzzy open in (X, T), $int(\lambda) = \lambda$. Thus $\lambda \leq intcl(\mu)$. So $intcl(\mu) \neq 0$. Hence μ is a fuzzy somewhere dense set in (X, T). □

Proposition 3.15. *If $\lambda \leq 1 - \mu$, where λ and μ are any two fuzzy sets in a fuzzy hereditarily extremally disconnected space (X, T), then $clint(\lambda) \leq 1 - clint(\mu)$.*

Proof. Let λ and μ be any two fuzzy sets in (X, T) such that $\lambda \leq 1 - \mu$. Then $cl(\lambda) \leq cl(1 - \mu)$. By Lemma 2.3, $cl(\lambda) \leq 1 - int(\mu)$, i.e., $int(\mu) \leq 1 - cl(\lambda)$. Thus $\mu \wedge int(\mu) \leq \mu \wedge (1 - cl(\lambda))$. Since $\mu \wedge int(\mu) = int(\mu)$, $int(\mu) \leq \mu \wedge (1 - cl(\lambda))$ and $cl[int(\mu)] \leq cl[\mu \wedge (1 - cl(\lambda))]$. Then $1 - cl[\mu \wedge (1 - cl(\lambda))] \leq 1 - clint(\mu)$. Since $\lambda \leq 1 - \mu$, $\mu \leq 1 - \lambda$ and $cl(\mu) \leq cl(1 - \lambda)$. Then $cl(\mu) \leq 1 - int(\lambda)$. Thus $int(\lambda) \leq 1 - cl(\mu)$. So $\lambda \wedge int(\lambda) \leq \lambda \wedge (1 - cl(\mu))$. Since $\lambda \wedge int(\lambda) = int(\lambda)$. $int(\lambda) \leq \lambda \wedge (1 - cl(\mu))$ and $cl[int(\lambda)] \leq cl[\lambda \wedge (1 - cl(\mu))]$. Since (X, T) is a fuzzy hereditarily extremally disconnected space, $cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\lambda))]\}$.

Hence $clint(\lambda) \leq cl[\lambda \wedge (1-cl(\mu))] \leq 1-\{cl[\mu \wedge (1-cl(\lambda))]\} \leq 1-clint(\mu)$. Therefore it follows that $clint(\lambda) \leq 1 - clint(\mu)$. □

Corollary 3.16. *If $\lambda \leq 1 - \mu$, where λ and μ are any two fuzzy sets in a fuzzy hereditarily extremally disconnected space (X, T), then $int(\lambda) \leq 1 - int(\mu)$.*

Proof. By Proposition 3.15, for the fuzzy sets λ and μ with $\lambda \leq 1 - \mu$ in the fuzzy hereditarily extremally disconnected space (X, T), $clint(\lambda) \leq 1 - clint(\mu)$. Then $int(\lambda) \leq clint(\lambda) \leq 1 - clint(\mu) \leq 1 - int(\mu)$. Hence it follows that $int(\lambda) \leq 1 - int(\mu)$. □

Corollary 3.17. *If $\lambda \leq 1 - \mu$, where λ and μ are any two fuzzy sets in a fuzzy hereditarily extremally disconnected space (X, T), then*

(1) $clint(\lambda) \leq 1 - int(\mu)$,
(2) $int(\lambda) \leq 1 - clint(\mu)$.

Proof. The proof follows from Corollary 3.16 and Lemma 2.3. □

Corollary 3.18. *If $\lambda \leq 1 - \mu$, where λ and μ are fuzzy closed sets in a fuzzy hereditarily extremally disconnected space (X, T), then $intcl(\lambda) \leq 1 - intcl(\mu)$.*

Proof. By Corollary 3.16, for the fuzzy sets λ and μ with $\lambda \leq 1 - \mu$ in the fuzzy hereditarily extremally disconnected space (X, T), $int(\lambda) \leq 1 - int(\mu)$. Since λ and μ are fuzzy closed sets in (X, T), $cl(\lambda) = \lambda$ and $cl(\mu) = \mu$. Then $intcl(\lambda) \leq 1 - intcl(\mu)$. □

Proposition 3.19. *If $\lambda \leq 1 - \mu$, where λ and μ are fuzzy open sets in a fuzzy hereditarily extremally disconnected space (X, T), then $cl(\lambda) \leq 1 - cl(\mu)$.*

Proof. By Proposition 3.15, for the fuzzy sets λ and μ with $\lambda \leq 1 - \mu$ in the fuzzy hereditarily extremally disconnected space (X, T), $clint(\lambda) \leq 1 - clint(\mu)$. Since λ and μ are fuzzy open sets in (X, T), $int(\lambda) = \lambda$ and $int(\mu) = \mu$. Then $cl(\lambda) \leq 1 - cl(\mu)$. □

Proposition 3.20. *If λ is a fuzzy set in a fuzzy hereditarily extremally disconnected space (X, T), then $clint(\lambda) \leq intcl(\lambda)$.*

Proof. Suppose λ is a fuzzy set in (X, T). Then $int(\lambda) \leq \lambda$ and $int(\lambda) \leq 1 - (1 - \lambda)$. Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.15, $clint[int(\lambda)] \leq 1 - clint(1 - \lambda)$. Then $clint(\lambda) \leq 1 - (1 - intcl(\lambda))$. Thus $clint(\lambda) \leq intcl(\lambda)$. □

Proposition 3.21. *If λ is a fuzzy closed set in a fuzzy hereditarily extremally disconnected space (X, T), then $int(\lambda)$ is a fuzzy closed set in (X, T).*

Proof. Suppose λ is a fuzzy closed set in (X, T). Then $int(\lambda) \leq \lambda$ and $int(\lambda) \leq 1 - (1 - \lambda)$. It is clear that $int(\lambda)$ and $1 - \lambda$ are fuzzy open sets in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.19, $clint(\lambda) \leq 1 - cl(1 - \lambda)$. Then $clint(\lambda) \leq 1 - (1 - int(\lambda)$. Thus $clint(\lambda) \leq int(\lambda)$. But $int(\lambda) \leq clint(\lambda)$. So $clint(\lambda) = int(\lambda)$. Hence it follows that $int(\lambda)$ is a fuzzy closed set in (X, T). □

Corollary 3.22. *If λ is a fuzzy closed set in a fuzzy hereditarily extremally disconnected space (X, T), then $clint(\lambda) = intcl(\lambda)$.*

Proof. By Proposition 3.21, for the fuzzy closed set λ in a fuzzy hereditarily extremally disconnected space (X, T), $int(\lambda)$ is a fuzzy closed set in (X, T) and $clint(\lambda) = int(\lambda)$. Since λ is a fuzzy closed set in (X, T), $cl(\lambda) = \lambda$. Then $clint(\lambda) = intcl(\lambda)$. □

Corollary 3.23. *If λ is a fuzzy open set in a fuzzy hereditarily extremally disconnected space (X, T), then $cl(\lambda)$ is a fuzzy open set in (X, T).*

Proof. Suppose λ is a fuzzy open set in (X, T). Then $1 - \lambda$ is a fuzzy closed set in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.21, $int(1 - \lambda)$ is a fuzzy closed set in (X, T). Thus $1 - cl(\lambda)$ is a fuzzy closed set in (X, T). So $cl(\lambda)$ is a fuzzy open set in (X, T). □

Proposition 3.24. *If λ is a fuzzy semi-open set in a fuzzy hereditarily extremally disconnected space (X, T), then*

(1) *$clint(\lambda)$ is a fuzzy open and fuzzy regular closed set in (X, T),*
(2) *$clint(\lambda) \leq intcl(\lambda)$,*
(3) *λ is a fuzzy α-open set in (X, T).*

Proof. (1) Suppose λ is a fuzzy semi-open set in (X, T). Then $\lambda \leq clint(\lambda)$ and $\lambda \leq 1-(1-clint(\lambda))$. Let $\mu = 1-clint(\lambda)$. Then $\lambda \leq 1-\mu$. Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.15, $clint(\lambda) \leq 1 - clint(\mu)$. This implies that $clint(\lambda) \leq 1 - clint(1 - clint(\lambda)) = 1 - (1 - intcl\{clint(\lambda)\}) = 1 - (1 - intcl(int(\lambda))) = intclint(\lambda)$. Thus $clint(\lambda) \leq int[clint(\lambda)]$. But $int[clint(\lambda)] \leq clint(\lambda)$. So $clint(\lambda) = int[clint(\lambda)]$. This implies that $clint(\lambda)$ is a fuzzy open set in (X, T). Also by Theorem 2.9, $cl[int(\lambda)]$ is a fuzzy regular closed set in (X, T). Hence $clint(\lambda)$ is a fuzzy open and fuzzy regular closed set in (X, T).

(2) From (1), $clint(\lambda) = int[clint(\lambda)]$ and $int[clint(\lambda)] \leq int[cl(\lambda)]$ implies that $clint(\lambda) \leq int[cl(\lambda)]$.

(3) From (1), $clint(\lambda) \leq int[clint(\lambda)]$. Since $\lambda \leq clint(\lambda)$, $\lambda \leq intclint(\lambda)$. Then λ is a fuzzy α-open set in (X, T). □

Proposition 3.25. *If λ is a fuzzy pre-open set in a fuzzy hereditarily extremally disconnected space (X, T), then*

(1) *$clint(\lambda)$ is a fuzzy open set in (X, T),*
(2) *$clint(\lambda) \leq intcl(\lambda)$.*

Proof. (1) Suppose λ is a fuzzy pre-open set in (X, T). Then $\lambda \leq intcl(\lambda)$ and $\lambda \leq 1 - (1 - intcl(\lambda))$. Let $\mu = 1 - intcl(\lambda)$. Then $\lambda \leq 1 - \mu$. Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.15, $clint(\lambda) \leq 1 - clint(\mu)$. This implies that $clint(\lambda) \leq 1 - clint(1 - intcl(\lambda)) = 1 - (1 - intclintcl(\lambda)) \leq 1 - (1 - intclint(\lambda)) = int[clint(\lambda)]$. Thus $clint(\lambda) \leq int[clint(\lambda)]$. But $int[clint(\lambda)] \leq clint(\lambda)$. So $clint(\lambda) = int[clint(\lambda)]$. Hence $clint(\lambda)$ is a fuzzy open set in (X, T).

(2) From (1), $clint(\lambda) = int[clint(\lambda)]$ and $int[clint(\lambda)] \leq int[cl(\lambda)]$ implies that $clint(\lambda) \leq int[cl(\lambda)]$. □

Corollary 3.26. *If λ is a fuzzy pre-open set in a fuzzy hereditarily extremally disconnected space (X,T), then there exists a fuzzy open and fuzzy regular closed set δ in (X,T) such that $\delta \leq intcl(\lambda)$.*

Proof. Suppose λ is a fuzzy pre-open set in (X,T). Since (X,T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.25, $clint(\lambda)$ is a fuzzy open set in (X,T) and $clint(\lambda) \leq intcl(\lambda)$. Let $\delta = clint(\lambda)$. Then δ is a fuzzy open set in (X,T). Also by Theorem 2.9, $cl[int(\lambda)]$ is a fuzzy regular closed set in (X,T). Thus δ is a fuzzy regular closed set in (X,T). So there exists a fuzzy open and fuzzy regular closed set δ in (X,T) such that $\delta \leq intcl(\lambda)$. □

The following Proposition shows that fuzzy semi-open sets are fuzzy pre-open sets in fuzzy hereditarily extremally disconnected spaces.

Proposition 3.27. *If λ is a fuzzy semi-open set in a fuzzy hereditarily extremally disconnected space (X,T), then λ is a fuzzy pre-open set in (X,T).*

Proof. Suppose λ is a fuzzy semi-open set in (X,T). Then $\lambda \leq clint(\lambda)$ in (X,T). Since (X,T) is a fuzzy hereditarily extremally disconnected space, for the fuzzy semi-open set λ in (X,T), by Proposition 3.24, $clint(\lambda) \leq intcl(\lambda)$. Then $\lambda \leq clint(\lambda) \leq intcl(\lambda)$. Thus λ is a fuzzy pre-open set in (X,T). □

The following Propositions show that fuzzy regular closed sets are fuzzy open sets and fuzzy regular open sets are fuzzy closed sets in fuzzy hereditarily extremally disconnected spaces.

Proposition 3.28. *If λ is a fuzzy regular closed set in a fuzzy hereditarily extremally disconnected space (X,T), then λ is a fuzzy open set in (X,T).*

Proof. Suppose λ is a fuzzy regular closed set in (X,T). Then $\lambda = clint(\lambda)$, in (X,T). Moreover $clint(\lambda) \leq clintcl(\lambda) = 1-(1-clintcl(\lambda))$. Let $\mu = 1-clintcl(\lambda)$. Then $\lambda \leq 1-\mu$. Since (X,T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.15, $clint(\lambda) \leq 1-clint(\mu)$. This implies that $clint(\lambda) \leq 1-clint(1-clintcl(\lambda)) = 1-(1-intclclintcl(\lambda)) = 1-(1-intclintcl(\lambda)) \leq 1-(1-int(\lambda)) = int(\lambda)$. Thus $clint(\lambda) \leq int(\lambda)$. But $int(\lambda) \leq clint(\lambda)$. So $clint(\lambda) = int(\lambda)$, i.e., $\lambda = clint(\lambda) = int(\lambda)$. Hence λ is a fuzzy open set in (X,T). □

Proposition 3.29. *If λ is a fuzzy regular open set in a fuzzy hereditarily extremally disconnected space (X,T), then λ is a fuzzy closed set in (X,T).*

Proof. Suppose λ is a fuzzy regular open set in (X,T). Then $1-\lambda$ is a fuzzy regular closed set in (X,T). Since (X,T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.28, $1-\lambda$ is a fuzzy open set in (X,T). Thus λ is a fuzzy closed set in (X,T). □

Proposition 3.30. *A fuzzy topological space (X,T) is a fuzzy hereditarily extremally disconnected space if and only if $cl(\lambda) \leq 1-cl(\mu)$, for any two fuzzy regular open sets λ and μ such that $\lambda \leq 1-\mu$, in (X,T).*

Proof. Suppose (X,T) is a fuzzy hereditarily extremally disconnected space and let $\lambda \leq 1-\mu$, where λ and μ are regular open sets in (X,T). Then $intcl(\lambda) = \lambda$ and $intcl(\mu) = \mu$. Since (X,T) is a fuzzy hereditarily extremally disconnected space, for the fuzzy sets λ and μ, by Proposition 3.15, $clint(\lambda) \leq 1 - clint(\mu)$. Then $clint(intcl(\lambda)) \leq 1 - clint(intcl(\mu))$. This implies that $cl[intcl(\lambda)] \leq 1 - cl[intcl(\mu)]$ and $cl(\lambda) \leq 1 - cl(\mu)$.

Conversely, suppose $cl(\lambda) \leq 1 - cl(\mu)$ for any two fuzzy regular open sets λ and μ in (X, T) such that $\lambda \leq 1 - \mu$. Since λ and μ are fuzzy regular open sets in (X, T), $intcl(\lambda) = \lambda$ and $intcl(\mu) = \mu$. Now $\lambda \leq cl(\lambda) \leq 1 - cl(\mu)$ implies that $\lambda \leq 1 - cl(\mu)$ and $\lambda \wedge (1 - cl(\mu)) = \lambda$. Now $cl(\lambda) \leq 1 - cl(\mu)$ implies that $cl(\mu) \leq 1 - cl(\lambda)$. Then $\mu \leq cl(\mu) \leq 1 - cl(\lambda)$. Thus $\mu \leq 1 - cl(\lambda)$ and $\mu \wedge (1 - cl(\lambda)) = \mu$. By the hypothesis, $cl(\lambda) \leq 1 - cl(\mu)$. So $cl[\lambda \wedge (1 - cl(\mu))] \leq 1 - cl[\mu \wedge (1 - cl(\lambda))]$. Hence (X, T) is a fuzzy hereditarily extremally disconnected space. □

4. Fuzzy hereditarily extremally disconnected spaces and other fuzzy topological spaces

Proposition 4.1. *If a fuzzy topological space (X, T) is a fuzzy perfectly disconnected space, then (X, T) is a fuzzy hereditarily extremally disconnected space.*

Proof. Let λ be a fuzzy set in (X, T). Then $int(\lambda) \leq \lambda$ and $int(\lambda) \leq 1 - [1 - \lambda]$. Let $\mu = 1 - \lambda$ and $\delta = int(\lambda)$. Then clearly, δ and μ are any two fuzzy sets in (X, T) such that $\delta \leq 1 - \mu$. Since (X, T) is a fuzzy perfectly disconnected space, $\delta \leq 1 - \mu$ implies that $cl(\delta) \leq 1 - cl(\mu)$. Thus it follows that $\delta \leq cl(\delta) \leq 1 - cl(\mu) \leq 1 - \mu$. Now $\delta \leq 1 - cl(\mu)$ and $cl(\delta) \leq 1 - \mu$. Then $\delta \leq 1 - cl(\mu)$ and $\mu \leq 1 - cl(\delta)$. Now $\delta \wedge (1 - cl(\mu)) = \delta$ and $\mu \wedge (1 - cl(\delta)) = \mu$. Now $cl(\delta) \leq 1 - cl(\mu)$ in (X, T) implies that $cl[\delta \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\delta))]\}$. Thus for the fuzzy sets δ and μ, $cl[\delta \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\delta))]\}$ implies that (X, T) is a fuzzy hereditarily extremally disconnected space. □

Remark 4.2. The converse of the above Proposition need not be true. That is, fuzzy hereditarily extremally disconnected spaces need not be fuzzy perfectly disconnected spaces.

For, in Example 3.2, $\delta \leq 1 - \beta$ for the fuzzy sets δ and β in (X, T). But $cl(\delta) = 1 - (\alpha \vee \beta \vee \gamma)$ and $1 - cl(\beta) = 1 - 1 = 0$ implies that $cl(\delta) \geq 1 - cl(\beta)$. Then (X, T) is not a fuzzy perfectly disconnected space whereas (X, T) is a fuzzy hereditarily extremally disconnected space.

Proposition 4.3. *If λ is a fuzzy set in a fuzzy perfectly disconnected space (X, T), then there exists a fuzzy open set γ in (X, T) such that $clint(\lambda) \leq \gamma \leq \lambda \vee clint(\lambda)$.*

Proof. Suppose λ is a fuzzy set in (X, T). Then $int(\lambda) \leq \lambda$ and $int(\lambda) \leq 1 - [1 - \lambda]$. Let $\mu = 1 - \lambda$ and $\delta = int(\lambda)$. Then clearly, δ and μ are two fuzzy sets in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, by Proposition 4.1, (X, T) is a fuzzy hereditarily extremally disconnected space. Thus $cl[\delta \wedge (1 - cl(\mu))] \leq 1 - \{cl[\mu \wedge (1 - cl(\delta))]\}$, in (X, T). Now $cl[\delta \wedge (1 - cl(\mu))] = cl[int(\lambda) \wedge (1 - cl(1 - \lambda))] = cl[int(\lambda) \wedge int(\lambda)] = clint(\lambda)$. Also $1 - \{cl[\mu \wedge (1 - cl(\delta))]\} = int\{1 - [\mu \wedge (1 - cl(\delta))]\} = int\{(1 - \mu) \vee (1 - [1 - cl(\delta)])\} = int\{(1 - \mu) \vee cl(\delta)\} = int[(1 - \mu) \vee cl(int(\delta))] = int\{\lambda \vee cl(int(\lambda))\}$. So $clint(\lambda) \leq int\{\lambda \vee clint(\lambda)\} \leq \{\lambda \vee clint(\lambda)\}$. Let $\gamma = int\{\lambda \vee clint(\lambda)\}$. It is obvious that γ is a fuzzy open set in (X, T). Hence for the fuzzy set λ in the fuzzy perfectly disconnected space (X, T), there exists a fuzzy open set γ in (X, T) such that $clint(\lambda) \leq \gamma \leq \lambda \vee clint(\lambda)$. □

Proposition 4.4. *If a fuzzy topological space (X, T) is a fuzzy hereditarily extremally disconnected space, then (X, T) is a fuzzy extremally disconnected space.*

Proof. Let λ be a fuzzy open set in (X, T). Since (X, T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.10, for the fuzzy set λ, there exists a fuzzy open set γ in (X, T) such that $clint(\lambda) \leq \gamma \leq \lambda \vee clint(\lambda)$. Since λ is fuzzy open set in (X, T), $int(\lambda) = \lambda$. Then $cl(\lambda) \leq \gamma \leq \lambda \vee cl(\lambda)$. Thus $cl(\lambda) \leq \gamma \leq cl(\lambda)$ and $cl(\lambda) = \gamma \in T$. So $cl(\lambda)$ is a fuzzy open set in (X, T). Hence (X, T) is a fuzzy extremally disconnected space. □

Remark 4.5. The above result can also be proved by applying Theorem 2.10 and Proposition 3.27.

If λ is a fuzzy semi-open set in a fuzzy hereditarily extremally disconnected space (X, T), then by Proposition 3.27, λ is a fuzzy pre-open set in (X, T). By Theorem 2.10, (X, T) is a fuzzy extremally disconnected space.

Remark 4.6. The converse of the above Proposition need not be true. That is, fuzzy extremally disconnected spaces need not be fuzzy hereditarily extremally disconnected spaces. For, consider the following example:

Example 4.7. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β, γ, λ and μ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.5$, $\alpha(b) = 0.6$, $\alpha(c) = 0.5$,
$\beta : X \to I$ is defined by $\beta(a) = 0.6$, $\beta(b) = 0.8$, $\beta(c) = 0.5$,
$\gamma : X \to I$ is defined by $\gamma(a) = 0.5$, $\gamma(b) = 0.4$, $\gamma(c) = 0.9$,
$\lambda : X \to I$ is defined by $\lambda(a) = 0.5$, $\lambda(b) = 0.5$, $\lambda(c) = 0.4$
$\mu : X \to I$ is defined by $\mu(a) = 0.4$, $\mu(b) = 0.3$, $\mu(c) = 0.5$

Then $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that $cl(\alpha) = 1 - (\alpha \wedge \gamma) = \alpha$, $cl(\beta) = 1$, $cl(\gamma) = 1$, $cl(\alpha \vee \beta) = 1$, $cl(\alpha \vee \gamma) = 1$, $cl(\beta \vee \gamma) = 1$, $cl(\alpha \wedge \gamma) = 1 - \alpha = \alpha \wedge \gamma$, $cl(\lambda) = 1 - (\alpha \wedge \gamma) = \alpha$, $cl(\mu) = 1 - \alpha = \alpha \wedge \gamma$. Also $int(1 - \lambda) = \alpha \wedge \gamma = 1 - \alpha$, $int(1 - \beta) = 0$, $int(1 - \gamma) = \alpha$, $int(1 - [\alpha \vee \beta]) = 0$, $int(1 - [\alpha \vee \gamma]) = 0$, $int(1 - [\beta \vee \gamma]) = 0$, $int(1 - (\alpha \wedge \gamma)) = \alpha = \alpha \wedge \gamma$, $int(\lambda) = 0$, $int(\mu) = 0$. Since the closure of each fuzzy open set is fuzzy open in (X, T), (X, T) is a fuzzy extremally disconnected space. But (X, T) is not a fuzzy hereditarily extremally disconnected space.

Example 4.8. Let $X = \{a, b, c\}$. Let $I = [0, 1]$. The fuzzy sets α, β, γ, λ and μ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.4$, $\alpha(b) = 0.6$, $\alpha(c) = 0.4$,
$\beta : X \to I$ is defined by $\beta(a) = 0.6$, $\beta(b) = 0.4$, $\beta(c) = 0.6$,
$\gamma : X \to I$ is defined by $\gamma(a) = 0.5$, $\gamma(b) = 0.5$, $\gamma(c) = 0.5$

Then $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that

$cl(\alpha) = 1 - \beta = \alpha$, $cl(\beta) = 1 - \alpha = \beta$, $cl(\gamma) = 1 - \gamma = \gamma$, $cl(\alpha \vee \beta) = 1 - (\alpha \wedge \beta) = \alpha \vee \beta$, $cl(\alpha \vee \gamma) = 1 - (\beta \wedge \gamma) = \alpha \vee \gamma$, $cl(\beta \vee \gamma) = 1 - (\alpha \wedge \gamma) = \beta \vee \gamma$, $cl(\alpha \wedge \beta) = 1 - (\alpha \vee \beta) = \alpha \wedge \beta$, $cl(\alpha \wedge \gamma) = 1 - (\beta \vee \gamma) = \alpha \wedge \gamma$, $cl(\beta \wedge \gamma) = 1 - (\alpha \vee \gamma) = \beta \wedge \gamma$. Also, $int(1 - \alpha) = \beta$; $int(1 - \beta) = \alpha$, $int(1 - \gamma) = \gamma$, $int(1 - [\alpha \vee \beta]) = \alpha \wedge \beta$, $int(1 - [\alpha \vee \gamma]) = \beta \wedge \gamma$, $int(1 - [\beta \vee \gamma]) = \alpha \wedge \gamma$, $int(1 - [\alpha \wedge \beta]) = \alpha \vee \beta$, $int(1 - [\alpha \wedge \gamma]) = \beta \vee \gamma$,

$int(1-[\beta\wedge\gamma]) = \alpha\vee\gamma$. Since the closure of each fuzzy open set is fuzzy open in (X,T), (X,T) is a fuzzy extremally disconnected space. Now for the fuzzy sets α and β, $cl[\alpha\wedge(1-cl(\beta))] = cl[\alpha\wedge(1-[1-\alpha])] = cl[\alpha\wedge\alpha] = cl[\alpha] = 1-\beta = \alpha$, $cl[\beta\wedge(1-cl(\alpha))] = cl[\beta\wedge(1-[1-\beta])] = cl[\beta\wedge\beta] = cl[\beta] = 1-\alpha = \beta$ and $1-\{cl[\beta\wedge(1-cl(\alpha))]\} = 1-\beta = \alpha$. It is clear that $cl[\alpha\wedge(1-cl(\beta))] = \alpha = 1-\{cl[\beta\wedge(1-cl(\alpha))]\}$ and thus $cl[\alpha\wedge(1-cl(\beta))] \nless 1-\{cl[\beta\wedge(1-cl(\alpha))]\}$. Then (X,T) is not a fuzzy hereditarily extremally disconnected space.

The inter-relations between fuzzy perfectly disconnected spaces, fuzzy hereditarily extremally disconnected spaces and fuzzy extremally disconnected spaces are given below:

Fuzzy perfectly disconnected spaces
$\Longrightarrow$ Fuzzy hereditarily extremally disconnected spaces
$\Longrightarrow$ Fuzzy extremally disconnected spaces

Proposition 4.9. *If a fuzzy topological space (X,T) is a fuzzy hereditarily extremally disconnected space, then (X,T) is not a fuzzy hyperconnected space.*

Proof. Let λ be a fuzzy set in (X,T). Then $int(\lambda)$ is a fuzzy open set in (X,T). Since (X,T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.11, $int(\lambda)$ is not a fuzzy dense set in (X,T). Thus (X,T) is not a fuzzy hyperconnected space. □

Proposition 4.10. *If a fuzzy topological space (X,T) is a fuzzy hereditarily extremally disconnected space, then (X,T) is a fuzzy basically disconnected space.*

Proof. By Proposition 4.1, the fuzzy hereditarily extremally disconnected space (X,T) is a fuzzy extremally disconnected space. Since fuzzy extremally disconnected spaces are fuzzy basically disconnected spaces, (X,T) is a fuzzy basically disconnected space. □

Proposition 4.11. *If $\lambda \le 1-\mu$, where λ and μ are fuzzy open F_σ-sets in a fuzzy hereditarily extremally disconnected space (X,T), then $intcl(\lambda) \le 1-intcl(\mu)$.*

Proof. Let λ and μ be fuzzy open F_σ-sets in (X,T) such that $\lambda \le 1-\mu$. Since (X,T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.19, for the fuzzy open sets λ and μ with $\lambda \le 1-\mu$, $cl(\lambda) \le 1-cl(\mu)$. Then by Proposition 4.4, the fuzzy hereditarily extremally disconnected space (X,T) is a fuzzy extremally disconnected space. Thus $cl(\lambda)$ and $cl(\mu)$ are fuzzy open sets in (X,T), $intcl(\lambda) = cl(\lambda)$ and $intcl(\mu) = cl(\mu)$. So it follows that $intcl(\lambda) \le 1-intcl(\mu)$. □

Corollary 4.12. *If $\lambda \le 1-\mu$, where λ and μ are fuzzy open F_σ-sets in a fuzzy hereditarily extremally disconnected space (X,T), then $\lambda \le intcl(\lambda) \le 1-intcl(\mu) \le 1-\mu$.*

Proof. Let λ and μ be fuzzy open F_σ-sets in (X,T) such that $\lambda \le 1-\mu$. Since (X,T) is a fuzzy hereditarily extremally disconnected space, by Proposition 4.11, $intcl(\lambda) \le 1-intcl(\mu)$, in (X,T). Then $int(\lambda) \le intcl(\lambda) \le 1-intcl(\mu) \le 1-int(\mu)$. Since λ and μ are fuzzy open sets in (X,T), $int(\lambda) = \lambda$ and $int(\mu) = \mu$. Thus it follows that $\lambda \le intcl(\lambda) \le 1-intcl(\mu) \le 1-\mu$. □

The following Proposition gives a condition for fuzzy hereditarily extremally disconnected spaces to become fuzzy $F^{'}$-spaces.

Proposition 4.13. *If each fuzzy F_σ-set is a fuzzy open set in a fuzzy hereditarily extremally disconnected space (X,T), then (X,T) is a fuzzy $F^{'}$-space.*

Proof. Suppose that $\lambda \leq 1-\mu$, where λ and μ are fuzzy F_σ-sets in (X,T). Then by the hypothesis, fuzzy F_σ-sets are fuzzy open sets in (X,T). Thus λ and μ are fuzzy open sets in (X,T). Since (X,T) is a fuzzy hereditarily extremally disconnected space, by Proposition 3.19, for the fuzzy open sets λ and μ with $\lambda \leq 1-\mu$, $cl(\lambda) \leq 1-cl(\mu)$. So for the fuzzy F_σ-sets λ and μ with $\lambda \leq 1-\mu$, $cl(\lambda) \leq 1-cl(\mu)$ implies that (X,T) is a fuzzy $F^{'}$-space. □

Proposition 4.14. *If a fuzzy topological space (X,T) is a fuzzy hereditarily extremally disconnected space, then (X,T) is a fuzzy hereditarily irresolvable space.*

Proof. Let λ and μ be any two fuzzy sets in X such that $cl(\lambda) = cl(\mu)$. It is claimed that $\lambda \wedge \mu \neq 0$. Suppose that $\lambda \wedge \mu = 0$. Then $\lambda \leq 1-\mu$. This implies that $bd(\lambda) \leq bd(1-\mu) = [cl(1-\mu) \wedge cl(1-[1-\mu])] = [cl(1-\mu) \wedge cl(\mu)] = bd(\mu)$. Thus $bd(\lambda) \leq bd(\mu)$.

Now $cl[\lambda \wedge (1-cl(\mu))] = cl[\lambda \wedge (1-cl(\lambda))]$ [since $cl(\lambda) = cl(\mu)$, in (X,T)] and $cl[\lambda \wedge (1-cl(\lambda))] \leq cl(\lambda) \wedge cl(1-cl(\lambda)) \leq cl(\lambda) \wedge cl(1-\lambda) = bd(\lambda)$. Then $cl[\lambda \wedge (1-cl(\mu))] \leq bd(\lambda)$. Also $cl[\mu \wedge (1-cl(\lambda))] = cl[\mu \wedge (1-cl(\mu))]$ [since $cl(\lambda) = cl(\mu)$] and $cl[\mu \wedge (1-cl(\lambda))] \leq cl(\mu) \wedge cl(1-cl(\mu)) \leq cl(\mu) \wedge cl(1-\mu) = bd(\mu)$. Then $cl[\mu \wedge (1-cl(\lambda))] \leq bd(\mu)$. Now $cl[\lambda \wedge (1-cl(\mu))] \leq bd(\lambda) \leq bd(\mu)$ implies that $cl[\lambda \wedge (1-cl(\mu))] \wedge cl[\mu \wedge (1-cl(\lambda))] \leq bd(\mu)$. Now $0 = [\lambda \wedge \mu] \wedge (1-cl(\mu)) = \{\lambda \wedge (1-cl(\mu))\} \wedge \{[\mu \wedge (1-cl(\mu))]\} = \{\lambda \wedge (1-cl(\mu))\} \wedge \{[\mu \wedge (1-cl(\lambda))]\} \leq cl[\lambda \wedge (1-cl(\mu))] \wedge cl[\mu \wedge (1-cl(\lambda))] \leq bd(\mu)$. Then $cl[\lambda \wedge (1-cl(\mu))] \wedge cl[\mu \wedge (1-cl(\lambda))] \leq bd(\mu) \neq 0$ and $cl[\lambda \wedge (1-cl(\mu))] \nleq 1-\{cl[\mu \wedge (1-cl(\lambda))]\}$, a contradiction to (X,T) being a fuzzy hereditarily extremally disconnected space in which for the fuzzy sets λ and μ in X, $cl[\lambda \wedge (1-cl(\mu))] \leq 1-\{cl[\mu \wedge (1-cl(\lambda))]\}$. Thus $\lambda \wedge \mu \neq 0$. So for the fuzzy sets λ and μ in X with $cl(\lambda) = cl(\mu)$ $(\neq 0)$, $\lambda \wedge \mu \neq 0$ implies that (X,T) is a fuzzy hereditarily irresolvable space. □

Proposition 4.15. *If a fuzzy topological space (X,T) is a fuzzy hereditarily extremally disconnected space, then (X,T) is a fuzzy irresolvable space.*

Proof. The proof follows from Proposition 4.14 and Theorem 2.12. □

5. Conclusion

In this paper, the notion of fuzzy hereditarily extremally disconnected spaces was introduced and several characterizations of fuzzy hereditarily extremally disconnected spaces were established. It was obtained that disjoint fuzzy open sets have disjoint closures and disjoint fuzzy sets have disjoint interiors in fuzzy hereditarily extremally disconnected spaces. It was established that fuzzy regular closed sets were fuzzy open sets and fuzzy semi-open sets were fuzzy pre-open sets in fuzzy hereditarily extremally disconnected spaces. Also it was established that fuzzy perfectly disconnected spaces were fuzzy hereditarily extremally disconnected spaces and fuzzy hereditarily extremally disconnected spaces were fuzzy extremally disconnected spaces. It was found that fuzzy hereditarily extremally disconnected spaces were not fuzzy hyperconnected spaces. A condition under which a fuzzy hereditarily extremally disconnected space becomes a fuzzy $F^{'}$-space, was also obtained. It was established that fuzzy hereditarily extremally disconnected spaces were fuzzy hereditarily irresolvable spaces and fuzzy irresolvable spaces. There is a need and scope of investigation considering various types of fuzzy extremally disconnectedness so that these results may contribute to the detailed study of resolvability of fuzzy topological spaces.

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